Journal of Global Optimization (2019) 75:273–339
https://doi.org/10.1007/s10898-019-00817-7

Fractional 0–1 programs: links between mixed-integer linear and conic quadratic formulations

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Received: 22 February 2019 / Revised: 20 July 2019 / Accepted: 30 July 2019 / Published online: 6 September 2019

Abstract
This paper focuses on methods that improve the performance of solution approaches for multiple-ratio fractional 0–1 programs (FPs) in their general structure. In particular, we explore the links between equivalent mixed-integer linear programming and conic quadratic programming reformulations of FPs. Thereby, we show that integrating the ideas behind these two types of reformulations of FPs allows us to push further the limits of the current state-of-the-art results and tackle larger-size problems. We perform extensive computational experiments to compare the proposed approaches against the current reformulations from the literature.

Keywords Fractional 0–1 programming · Conic quadratic programming · Mixed-integer linear programming · Polymatroid cuts · Binary-expansion · Assortment optimization

1 Introduction

We consider the general structure of fractional (hyperbolic) 0–1 programs

$$\text{(FP)} \qquad \min_{x \in X} \sum_{i \in I} \frac{a_{i0} + \sum_{j \in J} a_{ij} x_j}{b_{i0} + \sum_{j \in J} b_{ij} x_j},$$

where $I = \{1, \ldots, m\}$, $J = \{1, \ldots, n\}$ and $X \subseteq \mathbb{B}^n$ for $\mathbb{B} = \{0, 1\}$. In addition to the assumption that FP is in minimization form, we also assume that all data are non-negative integers, i.e., $a_{i0}, a_{ij}, b_{i0}, b_{ij} \in \mathbb{Z}_+$ for all $i \in I$, $j \in J$. Both assumptions are without loss of

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generality provided that the weaker (and commonly used) assumption $b_{i0} + \sum_{j \in J} b_{ij} x_j > 0$ for all $i \in I$ and $x \in \mathbb{B}^n$ holds, see "Appendix A" for a discussion.

Fractional binary optimizations arise naturally in many contexts that involve optimization of efficiency measures (e.g., maximizing the ratio of return/investment or profit/time and minimizing the ratio of cost/time, see [10,27,32,34]), averages, probabilities and percentages, among others. Fractional optimization models can be found in diverse application areas including problems in data mining (such as feature selection [17,26] and biclustering [12, 37]), scheduling [31], retail assortment [13,25,35], set covering [2,3], facility location [36], stochastic service systems [15], finding alternative solutions to binary linear programs [38], clinical trials [7], and so on. For an overview of applications and solution methods for FPs we refer to a recent survey in [10].

Problem FP is well-known to be NP-hard whenever $m \geqslant 2$, see [20,28,29]. To solve fractional binary programs, several *mixed-integer linear programming* (MILP) reformulations of FPs have been proposed (see, e.g., [22,36,41,42]) consisting of linearizing bilinear terms by introducing additional $O(nm)$ continuous variables and big-M constraints. The MILP formulations are commonly used, but they do not handle well large-scale multiple-ratio ($m \geqslant 2$) FPs, see, e.g., [11,16,25], due in part to the weak relaxations caused by the big-M constraints, and also due to the large number of newly added variables and constraints.

Borrero et al. [9] recently proposed an alternative MILP reformulation based on performing *binary expansions* of certain integer-valued expressions. The formulation can substantially reduce the number of bilinear terms that require linearization, thus requiring much fewer variables and constraints than the original MILP formulations. As a consequence, the binarized formulation scales better to large instances; however, binary expansion also leads to weaker continuous (convex) relaxations, which in turn can hurt performance in branch-and-bound.

Noting that $x_j = x_j^2$ for $x_j \in \mathbb{B}$, recently Şen et al. [33] proposed a *mixed-integer conic quadratic programming* (MICQP) reformulation for assortment optimization. Additionally, Atamtürk and Gómez [4] proposed another MICQP reformulation for FPs by explicitly involving submodular functions, and used extended polymatroid cuts [5,23] to exploit the submodular structure and strengthen the formulations. Both the aforementioned conic quadratic reformulations result in stronger convex relaxations than the standard MILP counterparts, as the latter requires linearization of bilinear terms with big-M constraints. Furthermore, thanks to recent advances in commercial MICQP optimization softwares such as CPLEX [21] and Gurobi [18], small- and medium-sized FPs can be solved efficiently. However, the solvers still struggle with large-scale mixed-integer nonlinear optimization problems, and hence the performance of the conic quadratic reformulations degrades considerably in larger instances.

Our contributions and the structure of the paper The main goal in this paper is to develop formulations for generally structured fractional 0–1 programs that perform well for all instance sizes, with special focus on large instances where current methods fail. Specifically, our contribution is threefold:

(i) We perform a comprehensive review of MILP and MICQP formulations of FPs given in the literature and explore the relationships between them.

(ii) We demonstrate how to integrate MICQP and MILP formulations to obtain novel formulations that simultaneously have strong convex relaxations, and a limited number of variables and constraints.

(iii) By means of computational experiments, we show that the proposed formulations outperform existing alternative formulations.

In order to achieve (i), in Sect. 2 we study the links between the classical MILP formulations LF and LEF, originally proposed in [41] and [22,42], respectively; the binary-expansion

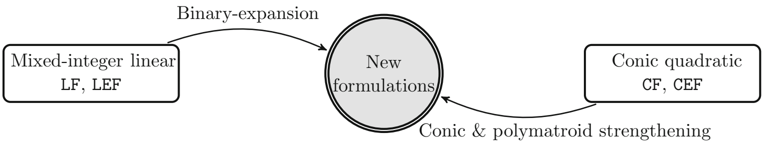


Fig. 1 Schematic representation of the ideas in this study. We exploit binary-expansion technique (from MILP) and conic and polymatroid strengthening (from MICQP) to develop enhanced formulations for $\mathtt{FP}$s

Table 1 Formulations studied in this paper

Formulation	Version	Linear-based		Conic	
		Without cuts	With cuts	Without cuts	With cuts
Compact	Basic	$\mathtt{LF}$ [41]	$\mathbf{LF}^{\mathbf{P}}$	$\mathtt{CF}$ [4]	$\mathtt{CF}^{\mathtt{P}}$ (+) [4]
	Binary expansion	$\mathtt{LF}_{\mathtt{log}}$ [9]	$\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$ ($\star$)	–	–
Extended	Basic	$\mathtt{LEF}$ [22]	$\mathbf{LEF}^{\mathbf{P}}$ (+)	$\mathtt{CEF}$ (+) [33]	$\mathbf{CEF}^{\mathbf{P}}$
	Binary expansion	$\mathtt{LEF}_{\mathtt{log}}$ [9]	$\mathbf{LEF}^{\mathbf{P}}_{\mathbf{log}}$	$\mathbf{CEF}_{\mathbf{log}}$	$\mathbf{CEF}^{\mathbf{P}}_{\mathbf{log}}$ ($\star$)

No citation is given for new formulations represented in **bold**. The symbols "+" and "$\star$" denote that the corresponding formulation has a superior performance in medium- and large-size instances of our computations, respectively

MILP formulation $\mathtt{LF}_{\mathtt{log}}$ developed in [9]; the MICQP formulations $\mathtt{CF}$ and $\mathtt{CEF}$ given in [4] and [33], respectively, as well as the MICQP formulation strengthened using polymatroids $\mathtt{CF}^{\mathtt{P}}$, also given in [4].

In order to attain (ii), in Sect. 3 we show how to use binary expansions (emanated from MILPs) in MICQP formulations; and how to use conic strengthening (originally proposed in the context of $\mathtt{CEF}$) and polymatroid cuts (originated from $\mathtt{CF}^{\mathtt{P}}$) to strengthen the formulations. More importantly, we show how to incorporate binary expansions and polymatroid strengthening in a single (either MILP or MICQP) formulation. Figure 1 shows the schematic representation of these ideas.

To achieve (iii), in Sect. 4, we conduct extensive computational results by using benchmark test instances and observe that the incorporation of improvements leads to formulations that perform better than the existing formulations in the literature.

In addition to the aforementioned formulations for $\mathtt{FP}$s, several new formulations are developed in this paper. We use the following naming conventions: names starting with "L" correspond to linear formulations, while names starting with "C" correspond to conic quadratic formulations; the letter "F" following the first letter indicates a compact formulation while the letters "EF" following the first letter indicate an *extended formulation*, i.e., a (usually stronger) formulation with additional variables and/or constraints; the subscript "log" indicates a formulation using binary expansions; finally, the superscript "P" indicates a strengthened formulation using polymatroid cuts. Table 1 provides a short summary of all formulations discussed in the paper, and Fig. 2 depicts the relationships between the convex relaxations of the formulations. In order to better distinguish the formulations developed in this paper from the existing ones in the literature we represent new formulations in **bold** throughout the paper.

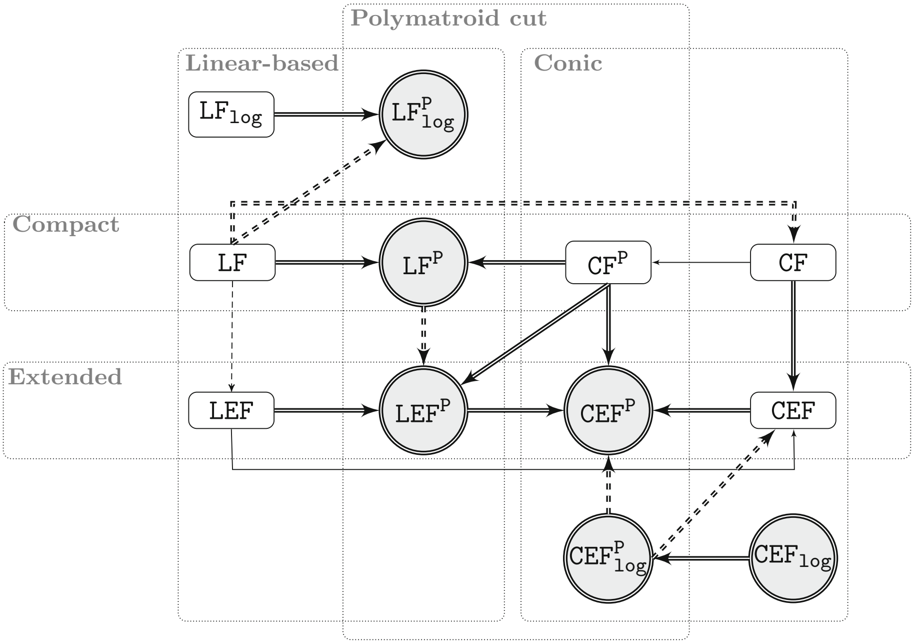


Fig. 2 Relationships between the strengths of the convex relaxations of the formulations studied in this paper. Single rectangular frames and single lines indicate existing formulations and shown relations in the literature, respectively. Double circle frames indicate new formulations, and double lines indicate relations shown in this paper. The symbol S1 $\Rightarrow$ S2 (or S1 $\rightarrow$ S2) indicates that formulation S2 has a stronger convex relaxation than formulation S1; this type of relations are demonstrated analytically in Sects. 2 and 3. Additionally, the symbol S1 $\Longrightarrow$ S2 (or S1 $\dashrightarrow$ S2) indicates that S2 resulted in smaller root gaps than S1 in most of our computations; this type of relations are shown experimentally by performing computational results in Sect. 4

2 Formulations

Herein, we review the MICQP and the (best-known) MILP reformulations of FPs existing in the literature, and describe their interrelatedness. Toward this goal, following our naming convention, in Sect. 2.1 we consider the compact formulations LF, CF and the strengthened version of CF with polymatroid cuts, i.e., CF^{P}. Then in Sect. 2.2 we discuss the extended formulations LEF and CEF involving more variables and/or constraints than LF and CF, respectively. Finally, in Sect. 2.3 we study the binary-expansion reformulations of MILPs.

2.1 Compact formulations

For each $i \in I$ let

$$t_i := \frac{a_{i0} + \sum_{j \in J} a_{ij} x_j}{b_{i0} + \sum_{j \in J} b_{ij} x_j}. \tag{1}$$

Then the substitution of variable t_i for all $i \in I$ in FP yields

$$\min_{x \in X, t \geqslant 0} \quad \sum_{i \in I} t_i \tag{2a}$$

$$\text{s.t.} \quad b_{i0} t_i + \sum_{j \in J} b_{ij} x_j t_i \geqslant a_{i0} + \sum_{j \in J} a_{ij} x_j \qquad \forall i \in I \tag{2b}$$

in which (2b) holds at equality at any optimal solution. Observe that constraint (2b) is nonlinear and non-convex (for $x \in [0, 1]^n$) due to the presence of bilinear terms $x_j t_i$. In the following, we take two convexification procedures. The first uses a concave over-estimator of the left-hand side of inequality (2b), resulting in a MILP; see Sect. 2.1.1. The second uses a convex underestimator of the right-hand side of inequality (2b) chosen to ensure convexity of the ensuing constraint, resulting in a MICQP; see Sect. 2.1.2.

2.1.1 Compact MILP formulation (LF)

The first approach is based on the linearization of $x_j t_i$, which can be accomplished by including additional variables and linear constraints [1,36,42]. Specifically, the concave envelope of $x_j t_i$, where $x_j \in \mathbb{B}$ and t_i is bounded, can be described with the bound constraints and the linear constraints $z_{ij} \leqslant t_i^U x_j$ and $z_{ij} \leqslant t_i + t_i^L(x_j - 1)$, where z_{ij} is a variable representing the hypograph of the bilinear term, and t_i^U and t_i^L are an upper bound and a lower bound on t_i, respectively. Note that under the data non-negativity assumption (see "Appendix A") the presence of the concave envelope of $x_j t_i$ is sufficient for this linearization. Thus, problem FP can be formulated as the MILP [36,41]:

$$(\mathtt{LF}) \qquad \min \sum_{i \in I} t_i \tag{3a}$$

$$\text{s.t. } b_{i0} t_i + \sum_{j \in J} b_{ij} z_{ij} = a_{i0} + \sum_{j \in J} a_{ij} x_j \qquad \forall i \in I \tag{3b}$$

$$z_{ij} \leqslant t_i^U x_j, \; z_{ij} \leqslant t_i + t_i^L(x_j - 1) \qquad \forall i \in I, j \in J \tag{3c}$$

$$x \in X, t, z \geqslant 0. \tag{3d}$$

Formulation LF exploits the integrality restriction on x ($x \in \mathbb{B}^n$) to construct the concave overestimator of the left-hand side of (2b), but may be weak due to the used big-M constraints (3c). Classical big-M values used are $t_i^U = (a_{i0} + \sum_{j \in J} a_{ij})/b_{i0}$ and $t_i^L = a_{i0} / \left(b_{i0} + \sum_{j \in J} b_{ij}\right)$. Thus, LF is especially weak if either the entries a_{ij} and b_{ij} or the number of variables (n) are large.

2.1.2 Compact MICQP formulations (CF and $\mathtt{CF}^{\mathrm{P}}$)

An alternative approach to resolve the non-convexity of (2b) is using conic quadratic programming. For each $i \in I$, we define

$$r_i = b_{i0} + \sum_{j \in J} b_{ij} x_j, \tag{4}$$

and

$$R_i = \left\{ x \in \mathbb{B}^n, (r_i, t_i) \in \mathbb{R}_+^2 \mid t_i r_i \geqslant a_{i0} + \sum_{j \in J} a_{ij} x_j \right\}.$$

Thus, problem (2) is equivalent to $\min_{x \in X, t, r \geqslant 0} \left\{ \sum_{i \in I} t_i \mid (4) \text{ and } (x, r_i, t_i) \in R_i, \; \forall i \in I \right\}$, that is still non-convex due to R_i.

A simple convex relaxation of R_i can be obtained by squaring the binary variables (and relaxing the integrality constraints), i.e., constraint (2b) can be written as $t_i r_i \geqslant a_{i0} +$

$\sum_{j \in J} a_{ij} x_j = a_{i0} + \sum_{j \in J} a_{ij} x_j^2$, where the equality holds for $x_j \in \mathbb{B}$. Thus, problem (2) can be posed as the MICQP [4]:

$$(\texttt{CF}) \qquad \min_{\substack{x \in X,\\ t,r \geqslant 0}} \sum_{i \in I} t_i \tag{5a}$$

$$\text{s.t.} \quad t_i r_i \geqslant a_{i0} + \sum_{j \in J} a_{ij} x_j^2 \qquad \forall i \in I \tag{5b}$$

$$r_i = b_{i0} + \sum_{j \in J} b_{ij} x_j \qquad \forall i \in I. \tag{5c}$$

The nonlinear constraint (5b) is a rotated cone constraint, which can be directly used with off-the-shelf solvers for MICQP. Observe that, unlike LF, formulation CF does not involve big-M constraints. On the other hand, since $x_j^2 \leqslant x_j$ for $x_j \in [0, 1]$, we see that squaring the variables may also lead to a weak relaxation. In fact, formulation CF only uses the upper bounds on x to construct the relaxation, but does not exploit the integrality constraints to derive stronger formulations.

A better convex relaxation of R_i can be obtained by using the strongest convex relaxation of R_i, i.e., the convex hull of R_i, denoted by $conv(R_i)$, see [4]:

$$(\texttt{CF}^{\text{P}}) \qquad \min_{\substack{x \in X,\\ t,r \geqslant 0}} \sum_{i \in I} t_i$$

$$\text{s.t.} \quad (x, r_i, t_i) \in conv(R_i) \qquad \forall i \in I$$

$$r_i = b_{i0} + \sum_{j \in J} b_{ij} x_j \qquad \forall i \in I.$$

Obviously, $\texttt{CF}^{\text{P}}$ has a tighter convex relaxation than CF. However, formulation $\texttt{CF}^{\text{P}}$ is much larger than CF, as it requires a factorial number of constraints to construct $conv(R_i)$. Specifically, let Σ denote the set of all permutations for set $\{1, \ldots, n\}$. For a given permutation $\sigma := (\sigma(1), \ldots, \sigma(n)) \in \Sigma$, $i \in I$ and $j \in J$, define

$$\pi_{i,\sigma(j)} = \sqrt{\sum_{k=0}^{j} a_{i,\sigma(k)}} - \sqrt{\sum_{k=0}^{j-1} a_{i,\sigma(k)}},$$

where $a_{i,\sigma(0)} = a_{i0}$, and consider the *nonlinear extended polymatroid inequalities*

$$t_i r_i \geqslant \left(\sqrt{a_{i0}} + \sum_{j=1}^{n} \pi_{i,\sigma(j)} x_{\sigma(j)} \right)^2 \qquad \forall \sigma \in \Sigma, i \in I. \tag{6}$$

Proposition 1 ([4]) *The extended polymatroid inequalities and bound constraints describe* $conv(R_i)$, *i.e.,* $conv(R_i) = \left\{ x \in [0, 1]^n, (r_i, t_i) \in \mathbb{R}_+^2 \mid (6) \right\}$.

Remark 1 In order to avoid adding all $m \cdot (n!)$ constraints of the form (6), Atamtürk and Gómez [4] add constraint (5b)—which is redundant for $\texttt{CF}^{\text{P}}$—to the formulation, and add a small number of constraints (6) in a cutting surface fashion. The separation of such constraints can be done in $O(n \log n)$ using the greedy algorithm for optimization over polymatroids [14]. □

Remark 2 For each $i \in I$, inequalities (6) can be implemented in a lifted formulation using a single three-dimensional rotated cone inequality and $n!$ linear inequalities—which can be added as cutting planes. Specifically, $(x, r_i, t_i) \in conv(R_i)$ if and only if there exists $s_i \in \mathbb{R}_+$ such that

$$t_i r_i \geqslant s_i^2, \qquad \text{and} \qquad \sqrt{a_{i0}} + \sum_{j=1}^{n} \pi_{i,\sigma(j)} x_{\sigma(j)} \leqslant s_i, \qquad \forall \sigma \in \Sigma.$$

Such a representation is preferable when using current off-the-shelf MICQP solvers, see [4] for further discussions. □

2.2 Extended formulations

Unlike compact formulations, which are based on convexifications of either the right-hand side or the left-hand side of (2b), extended formulations simultaneously consider both sides of (2b). Let

$$y_i := \frac{1}{b_{i0} + \sum_{j \in J} b_{ij} x_j} = \frac{1}{r_i} \qquad \forall i \in I,$$

where r_i is given by (4). Then the substitution of variable y_i for all $i \in I$ in FP yields

$$\min_{x \in X, t, y \geqslant 0} \sum_{i \in I} t_i \tag{7a}$$

$$\text{s.t.} \quad t_i \geqslant a_{i0} y_i + \sum_{j \in J} a_{ij} x_j y_i \qquad \forall i \in I \tag{7b}$$

$$b_{i0} y_i + \sum_{j \in J} b_{ij} x_j y_i \geqslant 1 \qquad \forall i \in I, \tag{7c}$$

where t_i is given by (1). Both constraints (7b) and (7c) hold at equality at any optimal solution.

Observe that (7b) and (7c) use non-convex bilinear terms $x_j y_i$. In order to resolve the non-convexity, we first review LEF, a classical MILP formulation based on formulation (7), see Sect. 2.2.1. Then we review the conic quadratic formulation CEF, which is a strengthening of the LEF. Moreover, we demonstrate that CEF is also a strengthening of CF, see Sect. 2.2.2—in contrast, although LEF has been observed to be stronger than LF in practice, it does not theoretically dominate LF.

2.2.1 Extended MILP formulation (LEF)

The first approach is based on the linearization of $x_j y_i$. Unlike the approach discussed in Sect. 2.1.1, both the concave and convex envelopes of the bilinear terms need to be constructed, requiring four linear inequalities per term. Letting y_i^U and y_i^L be upper and lower bounds on variable y_i, and letting $\bar{z}_{ij} := x_j y_i$, we find the MILP formulation [22]:

$$(\text{LEF}) \quad \min \sum_{i \in I} t_i \tag{8a}$$

$$\text{s.t.} \; t_i = a_{i0} y_i + \sum_{j \in J} a_{ij} \bar{z}_{ij} \qquad \forall i \in I \tag{8b}$$

$$b_{i0} y_i + \sum_{j \in J} b_{ij} \bar{z}_{ij} = 1 \qquad \forall i \in I \tag{8c}$$

$$\bar{z}_{ij} \leqslant y_i^U x_j, \bar{z}_{ij} \geqslant y_i^L x_j, \bar{z}_{ij} \leqslant y_i + y_i^L(x_j - 1),$$
$$\bar{z}_{ij} \geqslant y_i + y_i^U(x_j - 1) \quad \forall i \in I, j \in J \tag{8d}$$
$$x \in X, t, y, \bar{z} \geqslant 0. \tag{8e}$$

Classical big-M values used are $y_i^U = 1/b_{i0}$ and $y_i^L = 1/(b_{i0} + \sum_{j \in J} b_{ij})$. Thus, LEF is especially weak if either the entries b_{ij} or the number of variables (n) are large (but is not sensitive to the values a_{ij}).

2.2.2 Extended conic formulation (CEF)

Şen et al. [33] propose a conic strengthening of LEF in the context of the assortment problem under mixed multinomial logit choice model, but we show that the strengthening can be used for generally structured fractional binary programs. In particular, since $\bar{z}_{ij} = x_j y_i$ for $x_j \in \mathbb{B}$ and $r_i = 1/y_i$, it follows that the constraint $\bar{z}_{ij} r_i \geqslant x_j$ is valid for LEF; squaring the binary variables, one obtains a convex (rotated cone) constraint that can be used to strengthen the formulations. Moreover, constraint (7c) is in fact conic quadratic representable ($y_i r_i \geqslant 1$). Thus, we obtain the formulation:

$$(\texttt{CEF}) \quad \min \sum_{i \in I} t_i \tag{9a}$$
$$\text{s.t. } t_i = a_{i0} y_i + \sum_{j \in J} a_{ij} \bar{z}_{ij} \quad \forall i \in I \tag{9b}$$
$$b_{i0} y_i + \sum_{j \in J} b_{ij} \bar{z}_{ij} = 1 \quad \forall i \in I \tag{9c}$$
$$\bar{z}_{ij} \leqslant y_i^U x_j, \bar{z}_{ij} \geqslant y_i^L x_j, \bar{z}_{ij} \leqslant y_i + y_i^L(x_j - 1),$$
$$\bar{z}_{ij} \geqslant y_i + y_i^U(x_j - 1) \quad \forall i \in I, j \in J \tag{9d}$$
$$r_i = b_{i0} + \sum_{j \in J} b_{ij} x_j \quad \forall i \in I \tag{9e}$$
$$\bar{z}_{ij} r_i \geqslant x_j^2 \quad \forall i \in I, j \in J \tag{9f}$$
$$y_i r_i \geqslant 1 \quad \forall i \in I \tag{9g}$$
$$x \in X, t, y, r, \bar{z} \geqslant 0. \tag{9h}$$

Formulation CEF generalizes the conic quadratic formulation of [33] - developed for the assortment problem under mixed multinomial logit choice model - for the general fractional binary program FP. Formulation CEF is stronger than LEF as it includes additional constraints. As we now show, formulation CEF is also stronger than CF.

Proposition 2 *The natural convex relaxation of* CEF *is stronger than the relaxation of* CF.

Proof We start from formulation CF. For each $i \in I$ divide both sides of (5b) by $r_i > 0$, leading to the equivalent representation

$$t_i \geqslant \frac{a_{i0}}{r_i} + \sum_{j \in J} a_{ij} \frac{x_j^2}{r_i}.$$

Using the substitutions $y_i \geqslant \frac{1}{r_i}$ and $\bar{z}_{ij} \geqslant \frac{x_j^2}{r_i}$ for all $i \in I$, $j \in J$ we can write CF as

$$\min_{\substack{x_j \in X,\\ t_i, r_i, y_i, \bar{z}_{ij} \geqslant 0}} \quad \sum_{i \in I} t_i \tag{10a}$$

$$\text{s.t.} \quad t_i \geqslant a_{i0} y_i + \sum_{j \in J} a_{ij} \bar{z}_{ij} \qquad \forall i \in I \tag{10b}$$

$$y_i r_i \geqslant 1 \qquad \forall i \in I \tag{10c}$$

$$\bar{z}_{ij} r_i \geqslant x_j^2 \qquad \forall i \in I, j \in J \tag{10d}$$

$$r_i = b_0 + b_{ij} x_j \qquad \forall i \in I. \tag{10e}$$

Observe that none of the transformations discussed exploit the integrality constraints, thus formulation (10) above has the same continuous relaxation as CF. If formulation (10) is strengthened using constraints (9c) and (9d), then one obtains precisely CEF, thus proving the proposition. □

Remark 3 (Extended formulation of CF) Formulations CF and (10) are equivalent, in the sense that their natural convex relaxations (by relaxing integrality constraints in x) coincide. However, formulation (10) requires $m + nm$ additional variables. Moreover, (10) has $m + nm$ three-dimensional rotated cone constraints, while formulation CF has m $(n+2)$-dimensional rotated cone constraints. The extended formulation (10) is preferable in the context of branch-and-bound, as the corresponding linear outer approximations are stronger, see [40]. In fact, modern conic quadratic branch-and-bound solvers will automatically reformulate CF into a form similar to (10) in the presolve process. □

2.3 MILP binary-expansion formulation ($\mathtt{LF}_{\mathtt{log}}$)

Under the data integrality assumption, the binary-expansion technique attempts to reduce the number of bilinear terms ($x_j t_i$ or $x_j y_i$) that need to be linearized in LF or LEF. Specifically, for the binary-expansion reformulation of LF, let $\theta_i^b := \lfloor \log_2 \, (\sum_{j \in J} b_{ij}) \rfloor + 1$, then by using the substitution $\sum_{j \in J} b_{ij} x_j = \sum_{k=1}^{\theta_i^b} 2^{k-1} w_{ik}^b$ in problem (2) we get

$$\min \quad \sum_{i \in I} t_i \tag{11a}$$

$$\text{s.t.} \quad b_{i0} t_i + \sum_{k=1}^{\theta_i^b} 2^{k-1} w_{ik}^b t_i = a_{i0} + \sum_{j \in J} a_{ij} x_j \qquad \forall i \in I \tag{11b}$$

$$\sum_{j \in J} b_{ij} x_j = \sum_{k=1}^{\theta_i^b} 2^{k-1} w_{ik}^b \qquad \forall i \in I \tag{11c}$$

$$x \in X, w_{ik}^b \in \mathbb{B}, t_i \geqslant 0 \qquad \forall i \in I, k \in \{1, \ldots, \theta_i^b\}. \tag{11d}$$

Observe that, since $x_j \in \mathbb{B}$, the left-hand side of constraint (11c) is integer for any feasible solution of (11), and thus constraint (11c) can always be satisfied at equality. Using a similar linearization as the one described in Sect. 2.1.1 to linearize the product terms $w_{ik}^b t_i$, we obtain the MILP formulation [9]:

$$(\texttt{LF}_{\texttt{log}}) \quad \min \quad \sum_{i\in I} t_i \tag{12a}$$

$$\text{s.t.} \quad b_{i0}t_i + \sum_{k=1}^{\theta_i^b} 2^{k-1} z_{ik}^b = a_{i0} + \sum_{j\in J} a_{ij}x_j, \quad \forall i \in I \tag{12b}$$

$$\sum_{j\in J} b_{ij}x_j = \sum_{k=1}^{\theta_i^b} 2^{k-1} w_{ik}^b, \quad \forall i \in I \tag{12c}$$

$$z_{ik}^b \leqslant t_i^U w_{ik}^b,\ z_{ik}^b \leqslant t_i + t_i^L(w_{ik}^b - 1) \quad \forall i \in I, k \in \{1,\ldots,\theta_i^b\} \tag{12d}$$

$$x \in X, w_{ik}^b \in \mathbb{B}, z_{ik}^b \geqslant 0, t_i \geqslant 0 \quad \forall i \in I, k \in \{1,\ldots,\theta_i^b\}. \tag{12e}$$

When $\theta_i^b << n$, which is the case when n is large and the coefficients b_{ij} are small, formulation $\texttt{LF}_{\texttt{log}}$ requires substantially less (continuous) variables and big-M constraints than LF, but the strength of the continuous relaxation of $\texttt{LF}_{\texttt{log}}$ is weaker. Nonetheless, by performing computational experiments, see Sect. 4, we observe that for large instances formulation $\texttt{LF}_{\texttt{log}}$ results in much more branch-and-bound nodes explored and better performance overall.

Remark 4 It is also possible to develop a binary-expansion reformulation for LEF. However, based on the results in [9,24] such a formulation performs poorly. Thus, we omit $\texttt{LEF}_{\texttt{log}}$ from Fig. 2 and the discussion in this paper for the sake of brevity. □

In Example 1 below, we evaluate the formulations discussed in Sect. 2 for a specific instance.

Example 1 Consider unconstrained ($X = \mathbb{B}^n$) two-ratio ($m = 2$) five-variate ($n = 5$) fractional 0–1 program

$$\min_{x\in\mathbb{B}^5} \left\{ \frac{1 + x_1 + x_2 + 2x_3 + 2x_4 + x_5}{2 + x_1 + x_2 + x_3 + x_4 + x_5} + \frac{2 + 2x_1 + 3x_2 + x_3 + x_4}{1 + 2x_1 + 2x_2 + 3x_3} \right\}, \tag{13}$$

which has the optimal objective function value 1.75.

(i) The objective function values of convex relaxations, computed by CPLEX solver 12.7.1 [21], for the basic reformulations of (13), i.e., LF, CF, LEF, and CEF are: 0.482, 1.236, 1.484, and 1.639, respectively.

(ii) For permutation $\sigma = (1,2,3,4,5)$, polymatroid inequalities (6) for the first and second ratios are, respectively,

$$t_1r_1 \geqslant \left(1 + (\sqrt{2}-1)x_1 + (\sqrt{3}-\sqrt{2})x_2 + (\sqrt{5}-\sqrt{3})x_3 + (\sqrt{7}-\sqrt{5})x_4 + (\sqrt{8}-\sqrt{7})x_5\right)^2, \tag{14a}$$

$$t_2r_2 \geqslant \left(2 + (\sqrt{4}-\sqrt{2})x_1 + (\sqrt{7}-\sqrt{4})x_2 + (\sqrt{8}-\sqrt{7})x_3 + (\sqrt{9}-\sqrt{8})x_4 + 0x_5\right)^2. \tag{14b}$$

If we add (14a) and (14b) to CF (without (5b)), then the objective function value of the convex relaxation of the resulting formulation is improved to 1.349. Additionally, if inequalities (6) for all 5! and 4! permutations of the first and second ratios' numerators indices (in total 144 rotated cone constraints) are added to CF (without (5b)), then the resulting formulation is $\texttt{CF}^{\texttt{P}}$ with an improved relaxation objective function value equal to 1.697. Thus, $\texttt{CF}^{\texttt{P}}$ results in the best convex relaxation among the formulations of Sect. 2 in this particular instance.

(iii) By using the binary-expansion technique, constraint (2b) in model (2) for the first and second ratios, i.e.,

$$2t_1 + (x_1 + x_2 + x_3 + x_4 + x_5)t_1 \geqslant 1 + x_1 + x_2 + 2x_3 + 2x_4 + x_5, \text{ and} \tag{15a}$$

$$t_2 + (2x_1 + 2x_2 + 3x_3)t_2 \geqslant 2 + 2x_1 + 3x_2 + x_3 + x_4, \tag{15b}$$

can be replaced, respectively, by

$$2t_1 + (2^0 w^b_{11} + 2^1 w^b_{12} + 2^2 w^b_{13})t_1 \geqslant 1 + x_1 + x_2 + 2x_3 + 2x_4 + x_5, \text{ and} \tag{16a}$$

$$t_2 + (2^0 w^b_{21} + 2^1 w^b_{22} + 2^2 w^b_{23})t_2 \geqslant 2 + 2x_1 + 3x_2 + x_3 + x_4. \tag{16b}$$

Note that instead of linearizing 8 bilinear terms ($x_j t_i$) in the left-hand sides of (15a) and (15b), which results in `LF`, only 6 bilinear terms ($w^b_{ik} t_i$) are required to be linearized in the left-hand sides of (16a) and (16b), which lead to formulation $\texttt{LF}_{\texttt{log}}$. Recall that fewer linearizations implies fewer number of additional continuous variables and big-M constraints. However, $\texttt{LF}_{\texttt{log}}$ has a weaker convex relaxation objective value than `LF` (0.405 vs. 0.482). Thus, $\texttt{LF}_{\texttt{log}}$ results in the worst convex relaxation in this particular instance, but also in the smallest and easiest to solve convex relaxation. □

3 Enhancements

None of the formulations presented in Sect. 2 consistently outperforms the others. MICQP formulations are in general stronger and perform best in small- and medium-size problems; however, due to the difficulties of optimization solvers to handle the nonlinear convex relaxations, they may fail to adequately process the root node in larger instances. In contrast, the binarized MILPs tend to perform better than MICQPs in larger instances thanks to the reduced formulation size and linear convex relaxations; however, they do not perform as well in small instances. Finally, MILP formulations perform somewhat in between the MICQPs and binarized MILPs.

In this section, we aim to further improve the performance of the existing formulations for `FP`s. First, from the analysis in Sect. 2, it becomes apparent how to "mix" the ideas behind these formulations to improve their performance, see Sect. 3.1. Then, in Sect. 3.2, we develop binary-expansion techniques for conic quadratic formulations. By using the proposed improvements, we obtain strong formulations of moderate sizes, which perform well across all problem sizes and are particularly effective in larger instances.

3.1 "Mixing" formulations ($\mathbf{CEF^P}$, $\mathbf{LF^P}$, $\mathbf{LEF^P}$, and $\mathbf{LF^P_{log}}$)

Herein, we employ polymatroid cuts in `CEF`. Then, more interestingly, we make MILP formulations `LF`, `LEF`, and $\texttt{LF}_{\texttt{log}}$ able to benefit from polymatroid strengthening, as well.

First, note that neither `CEF` nor $\texttt{CF}^{\text{P}}$ theoretically dominates the other in terms of strength of the continuous relaxations. Moreover, in our computations (see Appendix 4), neither consistently dominates the other. Nonetheless, we can obtain a stronger new formulation simply by adding the nonlinear extended polymatroid inequalities to `CEF`, i.e.,

$$(\mathbf{CEF^P}): \min_{x,y,\bar{z},t,r} \left\{ \sum_{i\in I} t_i \mid (9b) - (9h),\ (x, r_i, t_i) \in conv(R_i)\ \forall i \in I \right\}.$$

Clearly, $\mathbf{CEF^P}$ is stronger than `CEF` and based on Proposition 2, it is also stronger than $\texttt{CF}^{\text{P}}$. Indeed, formulation $\mathbf{CEF^P}$ results in the best convex relaxations among the formulations presented in this paper. However, due to its size, it is impractical in larger instances. We address this issue by using the binary-expansion idea in Sect. 3.2 to moderate its size. We also point out that several approaches to strengthen the MILP formulations have been proposed in the literature, see, e.g., [25,36]. Clearly, such approaches can naturally be used with any of the formulations presented in Sect. 2, or the new formulations introduced in this section.

Second, as pointed out in Remark 1, previous implementations of $\mathtt{CF}^{\mathrm{P}}$ also added large-dimensional conic quadratic constraints (5b), which substantially increases the computational burden of solving the convex relaxations despite the recent advances in off-the-shelf optimization solvers. An alternative is to use the nonlinear extended polymatroid constraints with formulation $\mathtt{LF}$, i.e.,

$$(\mathbf{LF^P}): \min_{x,y,z,t,r} \left\{ \sum_{i\in I} t_i \mid (3b)-(3d),\ r_i = b_{i0} + \sum_{j\in J} b_{ij}x_j,\ (x, r_i, t_i) \in conv(R_i)\ \forall i \in I \right\}.$$

Clearly, $\mathbf{LF^P}$ dominates both $\mathtt{LF}$ and $\mathtt{CF}^{\mathrm{P}}$ in terms of the strength of the convex relaxation[1]. More importantly, using the extended formulation described in Remark 2, $\mathbf{LF^P}$ requires only m three-dimensional rotated cone constraints, which are much easier to handle than m $(n+2)$-dimensional conic constraints of $\mathtt{CF}^{\mathrm{P}}$. Alternatively, efficient polyhedral outer-approximations of the rotated cone constraint can be easily constructed [6,39], and $\mathbf{LF^P}$ can be implemented in a pure MILP framework.

Similarly, one can use the nonlinear extended polymatroid constraints with formulations $\mathtt{LEF}$ and $\mathtt{LF}_{\mathtt{log}}$, yielding

$$(\mathbf{LEF^P}): \min_{x,\bar{z},t,y,r} \left\{ \sum_{i\in I} t_i \mid (8b)-(8e),\ r_i = b_{i0} + \sum_{j\in J} b_{ij}x_j,\ (x, r_i, t_i) \in conv(R_i)\ \forall i \in I \right\}, \text{ and}$$

$$(\mathbf{LF^P_{log}}): \min_{x,w^b,z^b,t,r} \left\{ \sum_{i\in I} t_i \mid (12b)-(12e),\ r_i = b_{i0} + \sum_{j\in J} b_{ij}x_j,\ (x, r_i, t_i) \in conv(R_i)\ \forall i \in I \right\}.$$

Formulation $\mathbf{LEF^P}$ has (in our computations) a stronger convex relaxation than $\mathbf{LF^P}$ while maintaining easy-to-solve convex relaxations in small and medium instances. Formulation $\mathbf{LF^P_{log}}$ has a small size, but has weaker convex relaxations than $\mathbf{LF^P}$ and $\mathbf{LEF^P}$.

By comparing $\mathbf{LF^P}$ and $\mathbf{LEF^P}$ with $\mathtt{CF}^{\mathrm{P}}$ we can conclude that the former both are stronger than $\mathtt{CF}^{\mathrm{P}}$ as depicted in Fig. 2. Based on the discussion given in Sect. 2.2.2, we also conclude that $\mathbf{CEF^P}$ is stronger than $\mathbf{LEF^P}$.

Standout formulation Formulation $\mathbf{LF^P_{log}}$ is one of the best formulations in our computations. It was observed in [9] (and corroborated in our experiments) that while the continuous relaxation of $\mathtt{LF}_{\mathtt{log}}$ is weaker than $\mathtt{LF}$, which in turn is much weaker than $\mathtt{LEF}$, it may result in better performance due to the faster exploration of the branch-and-bound tree. With the inclusion of the nonlinear polymatroid inequalities, formulation $\mathbf{LF^P_{log}}$ has a convex relaxation strength similar to $\mathtt{CF}^{\mathrm{P}}$, which is substantially stronger than $\mathtt{LF}$ and was also observed to be stronger than $\mathtt{LEF}$ [4]. Moreover, using $\mathbf{LF^P_{log}}$ results in small formulations with a few nonlinearities, thus allowing for a much faster exploration of the branch-and-bound tree than $\mathtt{CF}^{\mathrm{P}}$, and performing well across all instance sizes. Intuitively, formulation $\mathbf{LF^P_{log}}$ benefits both from the advantages of the conic formulations (strength) and binarization ideas (speed).

Remark 5 We need to point out that $conv(R_i)$ is implemented in this paper using rotated cone constraints instead of explicit polyhedral outer approximations. Hence, $\mathbf{LF^P}$, $\mathbf{LEF^P}$ and $\mathbf{LF^P_{log}}$ are in fact MICQPs, see also Remarks 1 and 2; however, in contrast to other MICQPs in the paper, they involve only a small number of "easy" 3-dimensional rotated cone constraints. □

[1] The second domination statement holds only if all inequalities (6) are added. Nonetheless, $\mathbf{LF^P}$ is able to achieve excellent convex relaxations with a modest number of cuts.

3.2 Enhancements on CEF

Next, we develop a binary-expansion reformulation for the conic quadratic program CEF, which we call $\mathbf{CEF_{log}}$, see Sect. 3.2.1. Then we extend the notion of polymatroid cuts to the binary-expansion space in order to further strengthen $\mathbf{CEF_{log}}$, see Sect. 3.2.2.

3.2.1 MICQP binary-expansion formulation ($\mathbf{CEF_{log}}$)

As pointed out earlier, the MICQP reformulations of FPs do not require the linearization of bilinear terms. Nevertheless, we demonstrate that binarization technique—developed in Sect. 2.3 for MILPs—still can be employed to reduce the number of variables and rotated quadratic cone constraints in CEF as shown below.

Let $\theta_i^a := \lfloor \log_2(\sum_{j\in J} a_{ij}) \rfloor + 1$ and, by using the substitution $\sum_{j\in J} a_{ij}x_j = \sum_{k=1}^{\theta_i^a} 2^{k-1} w_{ik}^a$, we can rewrite (7) as

$$\min \quad \sum_{i\in I} t_i \tag{17a}$$

$$\text{s.t.} \quad t_i \geqslant a_{i0} y_i + \sum_{k=1}^{\theta_i^a} 2^{k-1} w_{ik}^a y_i \qquad \forall i \in I \tag{17b}$$

$$\sum_{j\in J} a_{ij}x_j = \sum_{k=1}^{\theta_i^a} 2^{k-1} w_{ik}^a \qquad \forall i \in I \tag{17c}$$

$$r_i y_i \geqslant 1 \qquad \forall i \in I \tag{17d}$$

$$r_i = b_{i0} + \sum_{j\in J} b_{ij}x_j \qquad \forall i \in I \tag{17e}$$

$$x \in X,\, y_i \geqslant 0,\, w_{ik}^a \in \mathbb{B} \qquad \forall i \in I, k \in \{1,\dots,\theta_i^a\}. \tag{17f}$$

Then by introducing variables $z_{ik}^a := w_{ik}^a y_i = w_{ik}^a / r_i$ and exploiting the fact that $\left(w_{ik}^a\right)^2 = w_{ik}^a$ for $w_{ik}^a \in \mathbb{B}$, problem (17) can be convexified as

$$\min \quad \sum_{i\in I} t_i \tag{18a}$$

$$\text{s.t.} \quad t_i \geqslant a_{i0} y_i + \sum_{k=1}^{\theta_i^a} 2^{k-1} z_{ik}^a \qquad \forall i \in I \tag{18b}$$

$$z_{ik}^a r_i \geqslant \left(w_{ik}^a\right)^2 \qquad \forall i \in I, k \in \{1,\dots,\theta_i^a\} \tag{18c}$$

$$\sum_{j\in J} a_{ij}x_j = \sum_{k=1}^{\theta_i^a} 2^{k-1} w_{ik}^a \qquad \forall i \in I \tag{18d}$$

$$r_i y_i \geqslant 1 \qquad \forall i \in I \tag{18e}$$

$$r_i = b_{i0} + \sum_{j\in J} b_{ij}x_j \qquad \forall i \in I \tag{18f}$$

$$x \in X,\, y_i \geqslant 0,\, w_{ik}^a \in \mathbb{B},\, z_{ik}^a \geqslant 0 \qquad \forall i \in I, k \in \{1,\dots,\theta_i^a\}. \tag{18g}$$

Formulation (18) can be further strengthened by adding the linearization constraints $z_{ij}^a \geqslant y_i^L w_{ik}^a$, and $z_{ij}^a \geqslant y_i + y_i^U(w_{ik}^a - 1)$. The resulting conic quadratic binary-expansion reformulation is

$$
\begin{aligned}
(\mathbf{CEF_{log}}) \quad & \min \sum_{i \in I} t_i && && \text{(19a)}\\
& \text{s.t.} \quad t_i \geqslant a_{i0} y_i + \sum_{k=1}^{\theta_i^a} 2^{k-1} z_{ik}^a && \forall i \in I && \text{(19b)}\\
& r_i = b_{i0} + \sum_{j \in J} b_{ij} x_j && \forall i \in I && \text{(19c)}\\
& z_{ik}^a r_i \geqslant (w_{ik}^a)^2 && \forall i \in I, k \in \{1, \ldots, \theta_i^a\} && \text{(19d)}\\
& y_i r_i \geqslant 1 && \forall i \in I && \text{(19e)}\\
& \sum_{j \in J} a_{ij} x_j = \sum_{k=1}^{\theta_i^a} 2^{k-1} w_{ik}^a && \forall i \in I && \text{(19f)}\\
& z_{ik}^a \geqslant y_i^L w_{ik}^a,\ z_{ik}^a \geqslant y_i + y_i^U(w_{ik}^a - 1) && \forall i \in I, k \in \{1, \ldots, \theta_i^a\} && \text{(19g)}\\
& w_{ik}^a \in \mathbb{B},\ z_{ik}^a \geqslant 0 && \forall i \in I, k \in \{1, \ldots, \theta_i^a\} && \text{(19h)}\\
& x \in X, t, y, r \geqslant 0. && && \text{(19i)}
\end{aligned}
$$

Formulation $\mathbf{CEF_{log}}$ requires $m + \sum_{i \in I} \theta_i^a$ rotated cone constraints, which can be significantly less than the $m + mn$ rotated cone constraints required by CEF.

Remark 6 It is also possible to develop binary-expansion reformulations for CF and CF^{P}. However, since these formulations do not include any product term of a binary and a continuous variables, the binary expansion does not allow us to reduce neither the number of their variables nor constraints. Therefore, we have excluded CF_{log} and $\text{CF}^{\text{P}}_{\text{log}}$ from Table 1, Fig. 2 and the discussion in this paper. □

3.2.2 Polymatroid cuts in the binary-expansion space ($\mathbf{CEF^P_{log}}$)

Formulation $\mathbf{CEF_{log}}$ can be further strengthened by exploiting submodularity. Specifically, observe that by multiplying constraint (17b) by r_i and exploiting that $y_i r_i = 1$ in optimal solutions of (17), we find that the constraints $(w_i^a, r_i, t_i) \in R_i^{\log}$ can be added, where

$$
R_i^{\log} = \left\{ w_i^a \in \mathbb{B}^{\theta_i^a}, (r_i, t_i) \in \mathbb{R}_+^2 \mid t_i r_i \geqslant a_{i0} + \sum_{k=1}^{\theta_i^a} 2^{k-1} (w_{ik}^a)^2 \right\}.
$$

An ideal formulation of $R_i^{\log}$ can be found using polymatroid cuts, similarly to the approach in Sect. 2.1.2, i.e.,

$$
conv(R_i^{\log}) = \left\{ (w_i^a, r_i, t_i) \in [0,1]^{\theta_i^a} \times \mathbb{R}_+^2 \mid t_i r_i \geqslant (\sqrt{a_{i0}} + \lambda_i' w_i^a)^2,\ \forall \lambda_i \in \Lambda_i \right\}, \quad (20)
$$

where

$$\Lambda_i = \Big\{ \lambda_i \in \mathbb{R}_+^{\theta_i^a} \mid \lambda_{i,\sigma(k)} = \sqrt{\gamma_{i,\sigma(k)}} - \sqrt{\gamma_{i,\sigma(k-1)}},$$
$$\text{where } \gamma_{i,\sigma(k)} = 2^{\sigma(k)-1} + \gamma_{i,\sigma(k-1)} \text{ and } \gamma_{i,\sigma(0)} = a_{i0},$$
$$\text{for all permutations } \sigma \in [\theta_i^a], k \in \{1, \dots, \theta_i^a\} \Big\}.$$

Observe that $\theta_i^a \ll n$ (for all $i \in I$) for large size problems with sufficiently small values for a_{ij}. Consequently, we have $(\theta_i^a)! \ll n!$, for each $i \in I$, and thus, $conv(R_i^{\log})$ can be constructed using significantly fewer polymatroid cuts than $conv(R_i)$. Adding $(w_i^a, r_i, t_i) \in conv(R_i^{\log})$ to $\mathbf{CEF_{log}}$ allows this binarized formulation to benefit from polymatroid cuts, that is given by

$$(\mathbf{CEF^P_{log}}): \min_{x,y,z,t,r,w^a} \Big\{ \sum_{i \in I} t_i \mid (19b) - (19i),\ (w_i^a, r_i, t_i) \in conv(R_i^{\log}),\ \forall i \in I \Big\}.$$

Standout formulation Formulation $\mathbf{CEF^P_{log}}$ is another of the best formulations in our computations. Similarly to $\mathbf{LF^P_{log}}$, formulation $\mathbf{CEF^P_{log}}$ is able to strike a good balance between the size and the strength of the convex relaxation by incorporating binary-expansion and polymatroid cuts, resulting in a similar performance as `CEF` in small instances, but scales much better to larger problems.

Example 1 (continued). Next, we evaluate the reformulations of (13) for the models proposed in Sect. 3.

(iv) In order to take the advantage of polymatroid strengthening, we add to `LF`, $\mathtt{LF_{log}}$, `LEF` constraints of the form (4), i.e., $r_1 = 2+x_1+x_2+x_3+x_4+x_5$ and $r_2 = 1+2x_1+2x_2+3x_3$. Additionally, we add 144 rotated cone constraints of the form (6) to the aforementioned formulations and `CEF`. Then we obtain $\mathbf{LF^P}$, $\mathbf{LF^P_{log}}$, $\mathbf{LEF^P}$, and $\mathbf{CEF^P}$, that have improved relaxation objective function values of 1.697 (vs. 0.482 of `LF`), 1.697 (vs. 0.405 of $\mathtt{LF_{log}}$), 1.702 (vs. 1.484 of `LEF`), and 1.702 (vs. 1.639 of `CEF`), respectively, and close most of the gap to the optimal objective function value 1.75.

(v) By using the binary-expansion technique, constraint (7b) in model (7) for the first and second ratios, i.e.,

$$t_1 \geqslant y_1 + (x_1 + x_2 + 2x_3 + 2x_4 + x_5)y_1, \text{ and} \tag{21a}$$
$$t_2 \geqslant 2y_2 + (2x_1 + 3x_2 + x_3 + x_4)y_2, \tag{21b}$$

can be replaced, respectively, by

$$t_1 \geqslant y_1 + (2^0 w_{11}^a + 2^1 w_{12}^a + 2^2 w_{13}^a)y_1, \text{ and} \tag{22a}$$
$$t_2 \geqslant 2y_2 + (2^0 w_{21}^a + 2^1 w_{22}^a + 2^2 w_{23}^a)y_2. \tag{22b}$$

In order to obtain `CEF` we need to convexify 9 bilinear terms $x_j y_i$ in the right-hand sides of (21a) and (21b) as rotated cone constraints $\bar{z}_{ij} r_i \geqslant x_j^2$. In comparison, in order to achieve $\mathbf{CEF_{log}}$ only 6 bilinear terms $w_{ik}^a y_i$ in the right-hand sides of (22a) and (22b) are required to be convexified as $z_{ik}^a r_i \geqslant (w_{ik}^a)^2$. Although $\mathbf{CEF_{log}}$ has 3 fewer rotated cone constraints than `CEF`, it has a worse relaxation objective function value (1.244 vs. 1.639). Next, we improve its relaxation by using polymatroid cuts in the binary-expansion space.

(vi) For permutation $\sigma = (1, 2, 3)$ inequalities $t_i r_i \geqslant (\sqrt{a_{i0}} + \lambda_i' w_i^a)^2$ in (20) for the first and second ratios are, respectively,

Table 2 The reformulation sizes (number of variables and constraints), where n and m are defined as in FP, q is the number of constraints defining X, $\theta_i^a = \lfloor \log_2(\sum_{j\in J} a_{ij}) \rfloor + 1$ and $\theta_i^b = \lfloor \log_2(\sum_{j\in J} b_{ij}) \rfloor + 1$

Formulation	Variables		Constraints	
	Continuous	Binary	Linear	Rotated cone
MILP-based reformulations				
LF	$m(n+1)$	n	$m(2n+1)+q$	–
LF$^{\mathbf{P}}$	$m(n+2)$	n	$m(2n+2)+q+\text{cuts}^*$	m
LF$_{\text{log}}$	$m+\sum_{i\in I}\theta_i^b$	$n+\sum_{i\in I}\theta_i^b$	$2m+2\sum_{i\in I}\theta_i^b+q$	–
LF$^{\mathbf{P}}_{\mathbf{log}}$	$2m+\sum_{i\in I}\theta_i^b$	$n+\sum_{i\in I}\theta_i^b$	$3m+2\sum_{i\in I}\theta_i^b+q+\text{cuts}$	m
LEF	$m(n+2)$	n	$m(4n+2)+q$	–
LEF$^{\mathbf{P}}$	$m(n+3)$	n	$m(4n+3)+q+\text{cuts}$	m
MICQP reformulations				
CF	$m(n+3)$	n	$2m+q$	$m(n+1)^{**}$
CF$^{\text{P}}$	$m(n+3)$	n	$2m+q+\text{cuts}$	$m(n+2)$
CEF	$m(n+3)$	n	$m(4n+3)+q$	$m(n+1)$
CEF$^{\mathbf{P}}$	$m(n+3)$	n	$m(4n+3)+q+\text{cuts}$	$m(n+2)$
CEF$_{\mathbf{log}}$	$3m+\sum_{i\in I}\theta_i^a$	$n+\sum_{i\in I}\theta_i^a$	$3m+2\sum_{i\in I}\theta_i^a+q$	$m+\sum_{i\in I}\theta_i^a$
CEF$^{\mathbf{P}}_{\mathbf{log}}$	$3m+\sum_{i\in I}\theta_i^a$	$n+\sum_{i\in I}\theta_i^a$	$3m+2\sum_{i\in I}\theta_i^a+q+\text{cuts}$	$2m+\sum_{i\in I}\theta_i^a$

Subscript "log" and superscript "P" are reserved for binary-expansion and polymatroid cuts, respectively
*Polymatroid cuts are added on the fly, implemented as discussed in Remark 2
**Formulations CF and CF$^{\text{P}}$ are based on extended formulation (10)

$$t_1 r_1 \geqslant \left(1+(\sqrt{2}-1)w^a_{11}+(\sqrt{4}-\sqrt{2})w^a_{12}+(\sqrt{8}-\sqrt{4})w^a_{13}\right)^2, \text{ and} \tag{23a}$$

$$t_2 r_2 \geqslant \left(2+(\sqrt{3}-\sqrt{2})w^a_{12}+(\sqrt{5}-\sqrt{3})w^a_{22}+(\sqrt{9}-\sqrt{5})w^a_{32}\right)^2. \tag{23b}$$

If we add (23a) and (23b) to **CEF$_{\mathbf{log}}$**, then its relaxation objective function value from 1.244 is improved to 1.311. If we add all $2\cdot 3! = 12$ polymatroid inequalities to **CEF$_{\mathbf{log}}$**, then the resulting formulation is **CEF$^{\mathbf{P}}_{\mathbf{log}}$** with a better relaxation objective function value of 1.446. Note that the number of cuts added to obtain **CEF$^{\mathbf{P}}_{\mathbf{log}}$** is significantly fewer than the number of cuts added in order to obtain any of the other formulations strengthened with polymatroid cuts (12 vs. 144 cuts).

Therefore, from this example, we observe that there is a trade-off between using polymatroid cuts and binarization. The former improves the relaxation objective function value at the expense of a larger problem, and the latter reduces the number of (continuous) variables and (either linear or rotated cone) constraints at the cost of a weaker relaxation. However, the incorporation of these ideas leads to moderate size formulations, i.e., **CEF$^{\mathbf{P}}_{\mathbf{log}}$** and **LF$^{\mathbf{P}}_{\mathbf{log}}$**, that benefit from strong convex relaxations. □

3.3 Problems sizes

Table 2 shows the number of continuous and binary variables as well as the number of linear and rotated cone constraints for MILP and MICQP formulations discussed in Sects. 2 and 3. By comparing each binarized formulation with the corresponding basic formulation, it is seen that the binary-expansion technique can potentially decrease the number of continuous variables and also the number of linear/rotated cone constraints—especially for large values of n—with a moderate increase in the number of binary variables. We also observe that adjusting the formulations to enable them to use polymatroid cuts only slightly increases the number of variables or constraints.

4 Computational Experiments

We perform extensive computational experiments to evaluate the performances of the currently existing formulations in the literature presented in Sect. 2 and to compare them versus the enhancements developed in Sect. 3. We outline the structure and parameters of the computational experiments in Sect. 4.1. We discuss the obtained results in Sect. 4.3 and "Appendix B".

4.1 Computational environment and test instances

All of the computational instances are solved using CPLEX 12.7.1 [21] on a 32-core CPU (2.90 GHz) with 160 GB of RAM; we allocate a single thread and 8 GB of RAM for each individual experiment, and use a time limit of 1 h (3600 s). To avoid running-out-of-memory difficulties we use the "node-file storage-feature" of CPLEX to store some parts of the branch-and-cut tree on disk when the size of the tree exceeds the allocated memory. The polymatroid inequalities are added at the root node by using callback functions of CPLEX as described in Remarks 1 and 2.

Test instances We consider three classes of instances: "small" ($n \in \{25, 50, 100\}$) and "medium" ($n \in \{200, 500, 1000\}$) size instances with $m = \lfloor 10\% \cdot n \rfloor$, and "large" size instances ($n \in \{2000, 5000, 10000\}$) with $m = 100$. For each choice of n and each of the following data generation settings five instances are sampled and the results are averaged.

• *Assortment data set*. For the first setting, we consider the assortment optimization problems that naturally arise in many applications such as online advertising, retailing, and revenue management [30]. Under the mixed multinomial logit model (see, e.g., [25,33,35]) we are given $I = \{1, 2, \ldots, m\}$ classes of customers and $J = \{1, 2, \ldots, n\}$ available products. Then the assortment optimization problem is defined as the problem of deciding which assortment of products $S \subseteq J$ must be offered to customers in order to maximize the expected revenue. In particular, let r_{ij} and μ_{ij} denote the revenue and customer preference weight associated with selling product j to customer class i, respectively, and μ_{i0} is the no-purchase preference in class i. Then, for a given assortment S, the probability that customer class i chooses product $j \in S$ is $\mu_{ij}/(\mu_{i0} + \sum_{j\in S} \mu_{ij})$. Thus, the problem of maximizing the expected revenue for all classes of customers under the mixed multinomial logit model can be formulated as the multiple-ratio fractional binary program of the form

$$\max_{x\in X} \sum_{i\in I} \frac{\sum_{j\in J} r_{ij}\mu_{ij}x_j}{\mu_{i0} + \sum_{j\in J} \mu_{ij}x_j}. \tag{24}$$

In (24) variable x_j is 1 if and only if the decision maker offers product j. Note that (24) is a special case of the generally structured FPs, since in each ratio $i \in I$ the coefficient of x_j, for all $j \in J$ in the numerator, i.e., $a_{ij} = r_{ij}\mu_{ij}$, is proportional to its coefficient in the denominator, $b_{ij} = \mu_{ij}$; moreover, $a_{i0} = 0$ and $b_{i0} = \mu_{i0}$.

Problem (24) can be transformed into an equivalent minimization problem. Specifically, based on equation (26) and the related discussion in "Appendix A", for each customer class $i \in I$ we have

$$\frac{\sum_{j\in J} r_{ij}\mu_{ij}x_j}{\mu_{i0}+\sum_{j\in J}\mu_{ij}x_j} = \frac{k_i\mu_{i0}+\sum_{j\in J}(r_{ij}\mu_{ij}+k_i\mu_{ij})x_j}{\mu_{i0}+\sum_{j\in J}\mu_{ij}x_j} - k_i,$$

for any $k_i \in \mathbb{R}$. Let $k_i = -\overline{r}_i = -\max_{j\in J} r_{ij}$, then

$$\begin{aligned}&\max_{x\in X}\sum_{i\in I}\frac{-\overline{r}_i\mu_{i0}+\sum_{j\in J}(r_{ij}\mu_{ij}-\overline{r}_i\mu_{ij})x_j}{\mu_{i0}+\sum_{j\in J}\mu_{ij}x_j}+\overline{r}_i\\ &\quad= -\min_{x\in X}\sum_{i\in I}\frac{\overline{r}_i\mu_{i0}+\sum_{j\in J}\mu_{ij}(\overline{r}_i-r_{ij})x_j}{\mu_{i0}+\sum_{j\in J}\mu_{ij}x_j}+\overline{r}_i.\end{aligned} \tag{25}$$

Transformation (25) is precisely the transformation used in [33] and satisfies the data non-negativity assumption. To satisfy the data integrality assumption, we multiply by 10 each of the terms $\mu_{i0}\overline{r}_i$, $\mu_{ij}(\overline{r}_i - r_{ij})$, μ_{i0}, and μ_{ij}, for all $i \in I$ and $j \in J$, and round them down to the nearest integer values.

For our test instances, we generate the data as in the assortment optimization problem considered in [33]. Specifically, the product prices are the same across the customer classes, i.e., $r_{ij} = r_j$ for all $i \in I$ and drawn from a $U[1, 3]$ distribution. Moreover, the preferences μ_{ij} are drawn from a $U[0, 1]$ distribution, and $\mu_{i0} = 5$ for all $i \in I$.

Moreover, Şen et al. [33] consider $X = \left\{x \in \mathbb{B}^n \mid \sum_{j=1}^n x_j \leqslant \kappa\right\}$. We let $\kappa \in \{10\% \cdot n,\ 20\% \cdot n,\ n\}$. The cardinality constraints: $\kappa = 10\% \cdot n$ and $\kappa = 20\% \cdot n$ correspond to a "small" and "large" retailer, respectively, where there is a physical limitation on the number of products that can be offered to customers. Additionally, $\kappa = n$ indicates the unconstrained case, i.e., $X = \mathbb{B}^n$, and it corresponds to an online retailer with the ability to sell many products [24].

Şen et al. [33] consider only the combinations $n = 200,\ m = 20$ and $n = 500,\ m = 50$. For these combinations we use the the same data (now part of the conic benchmark library, CBLIB) available at `http://cblib.zib.de`. For the other combinations of n and m tested in the paper we generate the data randomly in the aforementioned fashion.

• *Uniformly generated data set*. For the second setting, we use data generated similarly to [9,24]. Specifically, the coefficients a_{ij} and b_{ij} are each sampled from a (discrete) $U[0, 20]$ distribution, except for b_{i0} which is sampled from a $U[1, 20]$. The feasible region is given by $X = \left\{x \in \mathbb{B}^n \mid \sum_{j=1}^n x_j = \kappa\right\}$ with $\kappa \in \{10\% \cdot n,\ 20\% \cdot n\}$; we also consider the unconstrained case ($X = \mathbb{B}^n$).

For constrained instances, since in both settings X contains a single cardinality constraint, the number of variables added in the binary-expansion formulations can be reduced by setting $\theta_i^a := \lfloor \log_2 \left(\sum_{j=1}^{\kappa} a_{i[j]}\right)\rfloor + 1$ and $\theta_i^b := \lfloor \log_2 \left(\sum_{j=1}^{\kappa} b_{i[j]}\right)\rfloor + 1$, for all $i \in I$, where $a_{i[j]}$ and $b_{i[j]}$ denote the j-th largest element of a_i and b_i, respectively. For all the formulations—except LF, $\mathtt{LF}_{\mathtt{log}}$, and $\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$—we use $y_i^L = 1/(b_{i0} + \sum_{j=1}^{\kappa} b_{i[j]})$ and $y_i^U = 1/b_{i0}$ as valid lower and upper bounds for linearization, respectively. For LF, $\mathtt{LF}_{\mathtt{log}}$, and $\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$ we use $t_i^L = 0$ and $t_i^U = (a_{i0} + \sum_{j=1}^{\kappa} a_{i[j]})/b_{i0}$ as valid bounds.

Metrics For each of the formulations we define, $z^\star$: the objective function value of an optimal integer solution (or the best-found integer solution if an optimal solution could not be found by the formulation within the time limit), $z^{\texttt{Rlx}}$: the optimal objective function value of the continuous relaxation, $z^{\texttt{Ron}}$: the objective function value obtained after processing the root node (i.e., after adding polymatroid cuts and considering other strengthening techniques used by CPLEX)[2], and $z^{\texttt{Bbn}}$: the best lower-bound at the termination of the solver. Moreover, we define $Z^\star$ as the objective function value of the best-known integer solution over all solution methods.

Then, in our experiments, we report the following metrics of interest: the continuous relaxation gap, `Rlx-gap` $= \frac{|Z^\star - z^{\texttt{Rlx}}|}{Z^\star} \times 100\%$; the root node gap, `Ron-gap` $= \frac{|Z^\star - z^{\texttt{Ron}}|}{Z^\star} \times 100\%$; the end gap, `End-gap` $= \frac{|z^\star - z^{\texttt{Bbn}}|}{z^\star} \times 100\%$; the best bound gap, `Bbn-gap`$= \frac{|Z^\star - z^{\texttt{Bbn}}|}{Z^\star} \times 100\%$; and the optimality gap, `Opt-gap`$= \frac{|Z^\star - z^\star|}{Z^\star} \times 100\%$. In addition, we report the `Time` in seconds required to solve the problems, and the number of branch-and-bound `Nodes` explored. In all cases we report the averages over five instances generated with the same parameters (n, m, κ).

4.2 Preliminary analysis

Here, we briefly analyze the results for the MILP and MICQP formulations outlined in Sect. 2. More detailed results are omitted from the current discussion for the sake of brevity and are reported in "Appendix B".

In particular, the extended formulations `LEF` and `CEF` are stronger (they have better `Rlx-gap`) than the corresponding compact formulations `LF` and `CF`, respectively. The extended formulations also have better `time` and `End-gap` than the corresponding compact formulations; see Tables 7 and 8 for the results and "Appendix B.1" for an additional discussion.

Although `LF` has a poor performance even for small instances, its "binarization", i.e., $\texttt{LF}_{\texttt{log}}$, leads to significant improvements in the running time due to the reduction in the size of the formulation, see Tables 9 and 10 and the discussion in "Appendix B.2". These results are consistent with the previous results in the literature (see, e.g., [9,24]) that $\texttt{LF}_{\texttt{log}}$ has a superior performance over `LF` and $\texttt{LEF}_{\texttt{log}}$.

Additionally, recall that among the existing formulations in the literature the polymatroid cuts have been employed only for the strengthening of `CF` and the resulting formulation, i.e., $\texttt{CF}^{\texttt{P}}$ significantly outperforms `CF` with respect to the metrics `time`, `End-gap`, and `Ron-gap`. See [4] and our results presented in Tables 13 and 14; we also refer to "Appendix B.3" for an additional discussion.

4.3 Standout versus the state-of-the-art formulations

In this section, we further compare the performance of the state-of-the-art formulations available in the literature identified in Sect. 4.2, i.e., the extended MILP formulation `LEF` and the compact binary-expansion formulation $\texttt{LF}_{\texttt{log}}$ as well as the extended MICQP formulation `CEF` and the compact MICQP formulation with polymatroid cuts $\texttt{CF}^{\texttt{P}}$. In addition, we report the results of the two standout formulations derived in Sect. 3: the binary-expansion MILP and

[2] For MILP formulations, $z^{\texttt{Rlx}} \leqslant z^{\texttt{Ron}}$ as additional constraints are added at the root node. For MICQP formulations, this is not necessarily the case: $z^{\texttt{Rlx}}$ is found via interior point methods, while $z^{\texttt{Ron}}$ is obtained after solving a linear outer approximation—which may have a weaker continuous relaxation.

MICQP formulations strengthened with polymatroid cuts, i.e., $\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$ and $\mathbf{CEF}^{\mathbf{P}}_{\mathbf{log}}$, respectively. In “Appendix B”, we present additional computational results and discuss in detail our extensive experiments to evaluate the individual and combined effects of the enhancements developed in the paper.

Tables 3 and 4 show the results for the assortment and the uniformly generated instances, respectively, and for different values of n, m and κ with respect to the running time and the end gap. A detailed comparison of the standout and the state-of-the-art formulations with respect to all the metrics defined in Sect. 4.1 is provided in Tables 5 and 6 of “Appendix B”. In the tables, we use the “†” symbol to denote that CPLEX was unable to fully process the root node of the branch-and-bound tree within the time limit of 1 h for a given formulation.

Observe that, overall, the uniformly generated instances used in [9], see Table 4, are much more difficult to solve than the assortment instances used in [33], see Table 3. In particular, only uniformly generated instances with $n \leqslant 50$ can be solved to optimality (by any formulation), while assortment instances with $n \leqslant 500$ can in general be handled well by MICQP formulations.

Figure 3 shows the number of continuous and binary variables as well as the number of linear and rotated cone constraints of the formulations as a function of dimension (n). Figure 4 depicts the performance profile of solution methods and can be used to evaluate the effectiveness of each formulation in *easy* instances (the instances that are solved to optimality by at least one solution method). Figure 5 portrays the end gaps across all instances as a function of the dimension and can be utilized to explore the effectiveness of each formulation in *hard*, larger, instances (the instances that are not solved to optimality by any solution method in the time limit). Figures 6 and 7 show the relaxation gaps and the root node gaps, respectively, across all instances as a function of the dimension and can be used to evaluate the strengths of the convex relaxations.

In the *easy* instances, we see from Fig. 4 that `CEF` performs best. Formulation `CEF` also has the best relaxation strength among the formulations presented (Figs. 6 and 7). In fact, in most of the instances that `CEF` solves to optimality, `Ron-gap` is nearly 0 and optimality is proven with a few branch-and-bound nodes (see Table 5 with $n \leqslant 500$).

However, when *hard* instances are also taken into account `CEF` is not necessarily the best formulation, mainly due to the fact that its large size (Fig. 3) hampers its performance, and other formulations match or improve upon the end gaps of `CEF` even for $100 \leqslant n \leqslant 500$; see Fig. 5. Indeed, in the uniformly generated instances (Table 6), `CEF` is not able to fully close the root node gap, and the performance in branch-and-bound is substantially impaired due to the difficulty of solving the large, nonlinear convex subproblems. Additionally, existing conic formulations $\mathtt{CF}^{\mathtt{P}}$ and `CEF` scale the worst among the formulations presented, and CPLEX is unable to process the root node for those formulations in large settings with $n \geqslant 1000$.

On the other hand, $\mathtt{LF}_{\mathtt{log}}$ has the best scaling properties among the previously proposed formulations in the literature. Notably, unlike `LEF`, `CEF` and $\mathtt{CF}^{\mathtt{P}}$, it is able to fully process the root node in all instances with $n \geqslant 1000$ and explore thousands of branch-and-bound nodes or more. Moreover, it is competitive with the other formulations in terms of end gaps for $n \leqslant 100$ and outperforms other existing formulations at $n = 100$; see Fig. 5. However, it has substantially weaker convex relaxations than all the other formulations (see Figs. 6 and 7), and as a consequence it struggles on the *easy* instances (Fig. 4) and has worse end gaps for $200 \leqslant n \leqslant 500$ than the other previously proposed formulations.

The new formulations $\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$ and $\mathbf{CEF}^{\mathbf{P}}_{\mathbf{log}}$, which combine the binary-expansion technique, conic strengthening and polymatroid strengthening, perform well across all dimen-

Table 3 Computational results to evaluate the best existing methods in the literature against the standout formulations for the *assortment data set* [33]

n, m	κ	$10\% \cdot n$		$20\% \cdot n$		Unconstrained	
	Ref.	`Time`	`End-gap`	`Time`	`End-gap`	`Time`	`End-gap`
25,2*	$\mathtt{LF}_{\mathtt{log}}$	**0**	0.0%	**0**	0.0%	1	0.0%
	`LEF`	**0**	0.0%	**0**	0.0%	**0**	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	**0**	0.0%	**0**	0.0%	1	0.0%
	`CEF`	**0**	0.0%	**0**	0.0%	**0**	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	**0**	0.0%	**0**	0.0%	**0**	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	1	0.0%	**0**	0.0%	2	0.0%
50,5*	$\mathtt{LF}_{\mathtt{log}}$	1	0.0%	2	0.0%	18	0.0%
	`LEF`	**0**	0.0%	**1**	0.0%	**0**	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	1	0.0%	2	0.0%	4	0.0%
	`CEF`	1	0.0%	**1**	0.0%	1	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	**0**	0.0%	**1**	0.0%	6	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	**0**	0.0%	2	0.0%	21	0.0%
100,10*	$\mathtt{LF}_{\mathtt{log}}$	979	0.0%	3155	0.4%	3600	1.6%
	`LEF`	3357	1.6%	2190	0.2%	**1**	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	10	0.0%	20	0.0%	25	0.0%
	`CEF`	6	0.0%	**4**	0.0%	6	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	**1**	0.0%	6	0.0%	3600	0.8%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	2	0.0%	22	0.0%	3600	0.3%
200,20*	$\mathtt{LF}_{\mathtt{log}}$	3600	6.7%	3600	8.7%	3600	24.1%
	`LEF`	3600	8.6%	3600	1.1%	**29**	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	**27**	0.0%	64	0.0%	1562	0.2%
	`CEF`	73	0.0%	**40**	0.0%	59	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	710	0.0%	3400	0.3%	3600	6.3%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	2353	0.5%	3600	2.2%	3600	6.4%
500,50*	$\mathtt{LF}_{\mathtt{log}}$	3600	39.8%	3600	54.0%	3600	55.7%
	`LEF`	3600	8.3%	**2520**	0.2%	3501	**0.4%**
	$\mathtt{CF}^{\mathtt{P}}$	**1194**	0.0%	3452	0.3%	3600	7.7%
	`CEF`	3611	0.2%	2620	0.0%	3604	0.5%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	0.8%	3600	3.3%	3600	15.2%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	4.7%	3600	12.2%	3601	26.1%
1000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	55.9%	3600	62.7%	3600	76.5%
	`LEF`	3600	13.9%	3722	**0.9%**	3600	**1.7%**
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	3600	†	3600	†
	`CEF`	3605	†	3600	†	3600	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3601	†	3601	20.9%	3601	26.1%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3601	**10.0%**	3600	22.6%	3600	33.8%

Table 3 continued

n, m	κ / Ref.	$10\% \cdot n$ Time	$10\% \cdot n$ End-gap	$20\% \cdot n$ Time	$20\% \cdot n$ End-gap	Unconstrained Time	Unconstrained End-gap
2000,100**	$\texttt{LF}_{\texttt{log}}$	3600	57.8%	3600	70.5%	3600	78.3%
	LEF	3601	†	3600	†	3601	†
	$\texttt{CF}^{\texttt{P}}$	3600	†	3600	†	3600	†
	CEF	3600	†	3600	†	3600	†
	$\texttt{LF}^{\texttt{P}}_{\texttt{log}}$	3601	†	3600	41.4%	3601	**33.1%**
	$\texttt{CEF}^{\texttt{P}}_{\texttt{log}}$	3600	**16.1%**	3600	**30.7%**	3600	53.4%
5000,100**	$\texttt{LF}_{\texttt{log}}$	3600	78.1%	3600	80.6%	3601	83.5%
	LEF	7807	†	8155	†	7241	†
	$\texttt{CF}^{\texttt{P}}$	3600	†	3600	†	3600	†
	CEF	3600	†	3600	†	3600	†
	$\texttt{LF}^{\texttt{P}}_{\texttt{log}}$	3601	**29.2%**	3601	49.0%	3601	**50.7%**
	$\texttt{CEF}^{\texttt{P}}_{\texttt{log}}$	3600	39.3%	3600	**40.6%**	3600	58.4%
10000,100**	$\texttt{LF}_{\texttt{log}}$	3600	88.4%	3600	83.1%	3602	93.0%
	LEF	4225	†	4026	†	3603	†
	$\texttt{CF}^{\texttt{P}}$	3600	†	3600	†	3600	†
	CEF	3600	†	3600	†	3600	†
	$\texttt{LF}^{\texttt{P}}_{\texttt{log}}$	3601	55.4%	3601	53.2%	3601	**54.7%**
	$\texttt{CEF}^{\texttt{P}}_{\texttt{log}}$	3600	**33.4%**	3601	**45.4%**	3601	†

For each combination of n, m, κ and each formulation, we present averages over five instances for time (Time) in seconds and end gap (End-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

* *easy* instances

** *hard* instances

sions. Binarization leads to a significant size reduction especially in larger instances, e.g., for $n = 10,000$ the number of rotated cone constraints of 1,000,100 (corresponding to CEF) reduces to 1750 (corresponding to $\mathbf{CEF^P_{log}}$); see Fig. 3d. On the other hand, polymatroid cuts improve the convex relaxation quality of the formulations. In particular, from Fig. 7 we observe that $\mathbf{LF^P_{log}}$ and $\mathbf{CEF^P_{log}}$ are able to achieve a substantial root node strengthening over the simple binary-expansion formulation $\texttt{LF}_{\texttt{log}}$, and approximately match the strength of LEF. As a consequence, in the *easy* instances (Fig. 4), they also match the performance of LEF and consistently outperform $\texttt{LF}_{\texttt{log}}$, but still lag behind the stronger conic formulations CEF and $\texttt{CF}^{\texttt{P}}$.

However, once *hard* instances are also taken into account, we see from Fig. 5 that they achieve the best performance overall. Notably, they match the performance of the best formulations for $n \leqslant 500$, but they scale to instances with n in the thousands and consistently outclass $\texttt{LF}_{\texttt{log}}$ (the only other formulation that scales to those instances).

Table 4 Computational results to evaluate the best existing methods in the literature against the standout formulations for the *uniformly generated data set* [9]

n, m	κ	$10\% \cdot n$		$20\% \cdot n$		Unconstrained	
	Ref.	Time	End-gap	Time	End-gap	Time	End-gap
25,2*	$\mathtt{LF_{log}}$	**0**	0.0%	1	0.0%	1	0.0%
	LEF	**0**	0.0%	**0**	0.0%	**0**	0.0%
	$\mathtt{CF^P}$	3	0.0%	4	0.0%	4	0.0%
	CEF	**0**	0.0%	**0**	0.0%	1	0.0%
	$\mathtt{LF^P_{log}}$	**0**	0.0%	1	0.0%	1	0.0%
	$\mathtt{CEF^P_{log}}$	1	0.0%	1	0.0%	6	0.0%
50,5*	$\mathtt{LF_{log}}$	3	0.0%	20	0.0%	52	0.0%
	LEF	**2**	0.0%	**13**	0.0%	**43**	0.0%
	$\mathtt{CF^P}$	78	0.0%	3601	6.5%	2903	3.0%
	CEF	3	0.0%	18	0.0%	100	0.0%
	$\mathtt{LF^P_{log}}$	9	0.0%	27	0.0%	85	0.0%
	$\mathtt{CEF^P_{log}}$	6	0.0%	26	0.0%	86	0.0%
100,10**	$\mathtt{LF_{log}}$	3600	**5.0%**	3600	**5.0%**	3600	11.2%
	LEF	3600	12.3%	3600	17.1%	3600	38.5%
	$\mathtt{CF^P}$	3600	43.5%	3600	44.3%	3600	42.0%
	CEF	3600	10.7%	3600	15.5%	3600	40.1%
	$\mathtt{LF^P_{log}}$	3600	7.5%	3600	6.1%	3600	17.2%
	$\mathtt{CEF^P_{log}}$	3600	7.2%	3603	5.2%	3600	**10.9%**
200,20**	$\mathtt{LF_{log}}$	3600	41.7%	3600	37.7%	3600	58.2%
	LEF	3600	**30.0%**	3600	31.1%	3600	70.6%
	$\mathtt{CF^P}$	3600	65.8%	3600	61.6%	3600	70.9%
	CEF	3600	30.9%	3600	**30.0%**	3600	76.4%
	$\mathtt{LF^P_{log}}$	3600	41.6%	3600	35.6%	3600	58.0%
	$\mathtt{CEF^P_{log}}$	3600	35.5%	3600	34.3%	3600	**54.4%**
500,50**	$\mathtt{LF_{log}}$	3600	48.7%	3600	48.7%	3600	87.0%
	LEF	3600	**42.8%**	3600	**41.1%**	3600	90.3%
	$\mathtt{CF^P}$	3600	†	3600	†	3600	84.9%
	CEF	3603	**42.8%**	3604	41.8%	3603	93.4%
	$\mathtt{LF^P_{log}}$	3600	48.4%	3600	48.1%	3600	**82.9%**
	$\mathtt{CEF^P_{log}}$	3600	46.3%	3600	43.1%	3600	86.7%
1000,100**	$\mathtt{LF_{log}}$	3600	50.3%	3600	50.1%	3600	96.6%
	LEF	3601	†	3601	†	3601	†
	$\mathtt{CF^P}$	3600	†	3600	†	3600	95.6%
	CEF	3600	†	3600	†	3600	†
	$\mathtt{LF^P_{log}}$	3600	50.2%	3600	50.2%	3600	**91.9%**
	$\mathtt{CEF^P_{log}}$	3600	**48.0%**	3600	**44.5%**	3600	92.2%

Table 4 continued

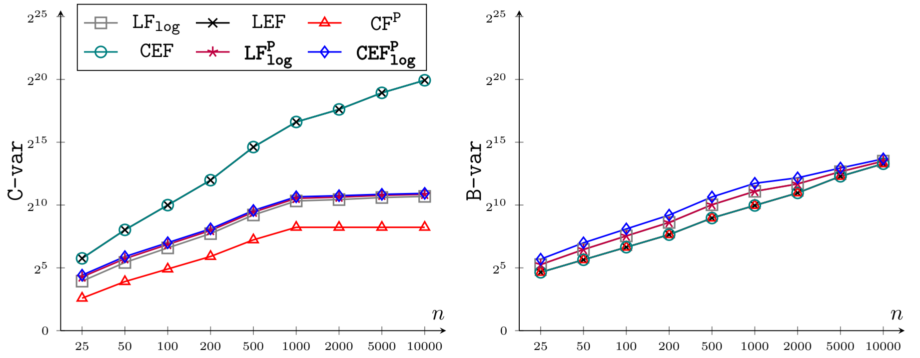
n, m	κ	$10\% \cdot n$		$20\% \cdot n$		Unconstrained	
	Ref.	Time	End-gap	Time	End-gap	Time	End-gap
2000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	50.7%	3600	50.6%	3600	97.8%
	LEF	3601	†	3602	†	3601	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	3600	†	3600	†
	CEF	3600	†	3600	†	3600	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	50.8%	3600	50.7%	3600	**94.8%**
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	**47.8%**	3600	**44.6%**	3600	96.6%
5000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	67.9%	3600	65.0%	3601	98.8%
	LEF	4755	†	3938	†	3603	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	3600	†	3600	†
	CEF	3600	†	3600	†	3600	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	68.8%	3600	67.9%	3601	**96.9%**
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	**46.7%**	3601	**45.2%**	3601	98.3%
10000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	68.6%	3600	68.2%	3601	99.4%
	LEF	9500	†	6022	†	5619	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	3600	†	3600	†
	CEF	3600	†	3600	†	3600	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3601	68.5%	3601	68.4%	3601	**97.8%**
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3601	**47.5%**	3600	**44.8%**	3600	†

For each combination of n, m, κ and each formulation, we present averages over five instances for time (Time) in seconds and end gap (End-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**
*$easy$ instances
**$hard$ instances

5 Conclusions

Fractional 0–1 programming problems have traditionally been tackled by reformulating the problems as MILPs a large number of variables and constraints. However, new techniques have recently been proposed to improve upon the classical MILP formulations. This paper focuses on two such recent enhancements: a binary-expansion technique that decreases the number of variables and constraints at the expense of weak convex relaxations; and conic and submodular strengthenings, which improve the convex relaxations at the expense of even larger and harder to solve convex relaxations. Naturally, these two ideas are at odds with each other, and which enhancement is preferable largely depends on each particular instance.

In this paper, we develop formulations that combine both enhancement ideas. The new formulations are compact and require a modest number of variables and constraints, yet retain the relaxation strength of formulations of much larger sizes. As a consequence, the new formulations are able to perform well across all instance classes. Specifically, in our computations using benchmark instances, we observe that the new formulations perform as well as the best existing methods in small and easy problems, and vastly outperform existing methods in larger and harder instances.

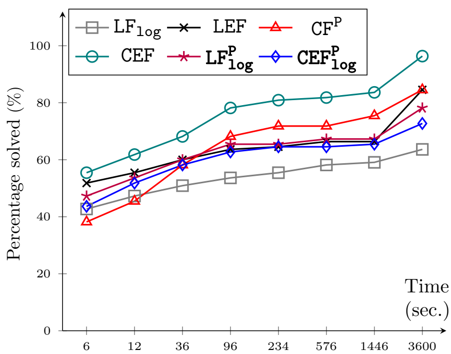


(a) Avg. No. of continuous variables (C-var) **(b)** Avg. No. of binary variables (B-var)

(c) Avg. No. of linear constraints (L-const) **(d)** Avg. No. of rotated cone constraints (C-const)

Fig. 3 The average sizes (numbers of continuous and binary variables as well as numbers of linear and rotated cone constraints) of formulations as a function of dimension (n). The averages are over five test instances of both the assortment [33] and the uniformly generated [9] data sets and capacity sizes $\kappa \in \{10\% \cdot n,\ 20\% \cdot n\}$ as well as the unconstrained case

Fig. 4 Performance profile for *easy* instances, that are the instances solved to optimality by at least one formulation. They include 80 instances of the assortment data (all instances with $n \leqslant 500$ and five instances with $n = 1000$), and 30 instances of the uniformly generated data (all instances with $n \leqslant 50$). We depict the percentage of such instances that could be solved as a function of the time (in logarithmic scale) for each formulation



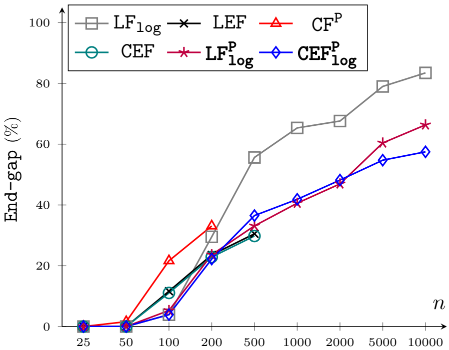


Fig. 5 Average end gap (End-gap) for all instances as a function of dimension. No gap is reported when a given formulation is unable to solve the root node within the time limit

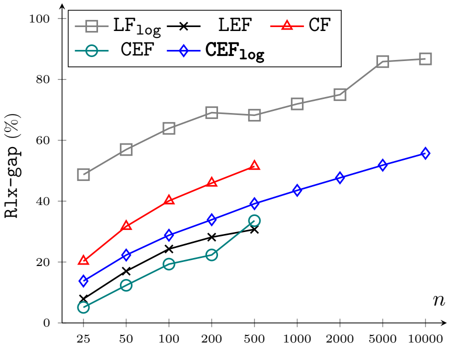


Fig. 6 Average relaxation gap (Rlx-gap) for all instances as a function of dimension. Observe that Rlx-gap does not account for the effect of polymatroid cuts; thus, we consider formulations without the impact of the polymatroid strengthening. No gap is reported when a given formulation is unable to solve the root node within the time limit

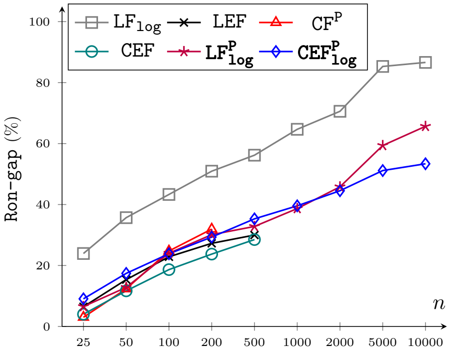


Fig. 7 Average root node gap (Ron-gap) for all instances as a function of dimension. Observe that Ron-gap accounts for the strengthening from polymatroid cuts, but it is also impacted unfavorably by the use of (possibly weak) linear outer approximations. No gap is reported when a given formulation is unable to solve the root node within the time limit

Acknowledgements The authors thank the Review Team for their insightful and constructive comments. This paper is based upon work supported by the National Science Foundation under Grant No. 1818700.

A Assumption justifications

We make the following assumptions in the paper.

Assumption 1 All data are integers, i.e., $a_{ij}, b_{ij} \in \mathbb{Z}$ for all $i \in I$, $j \in J \cup \{0\}$.

Assumption 2 All data are non-negative, i.e., $a_{ij}, b_{ij} \geqslant 0$ for all $i \in I$ and $j \in J \cup \{0\}$.

The first assumption is without loss of generality, as otherwise rational coefficients can be scaled. Assumption 2 is naturally satisfied in most application settings, as the data typically represents probabilities, prices, weights, utilities etc.—see, e.g., [10] and the applications described therein.

Nonetheless, Assumption 2 is without loss of generality provided that (the weaker and commonly made assumption in the FP literature, see, e.g., [8,9,19]) $b_{i0} + \sum_{j\in J} b_{ij}x_j > 0$ for all $x \in \mathbb{B}^n$ holds. In each ratio $i \in I$, for every $j \in J$ such that $b_{ij} < 0$ and every j such that $b_{ij} = 0$ and $a_{ij} < 0$, replace x_j with $\bar{x}_j = 1 - x_j$, resulting in a problem satisfying $b_{ij} \geqslant 0$ (possibly with at most n additional variables and constraints). Then observe that for any $k_i \in \mathbb{R}$

$$\frac{a_{i0} + \sum_{j\in J} a_{ij}x_j}{b_{i0} + \sum_{j\in J} b_{ij}x_j} = \frac{(a_{i0} + k_i b_{i0}) + \sum_{j\in J}(a_{ij} + k_i b_{ij})x_j}{b_{i0} + \sum_{j\in J} b_{ij}x_j} - k_i. \tag{26}$$

Thus, by letting k_i sufficiently large for each $i \in I$, we find a problem where all coefficients are non-negative.

Finally, note that if a fractional program is in maximization form and satisfies $b_{i0} + \sum_{j\in J} b_{ij}x_j > 0$ for all $x \in \mathbb{B}^n$, then it can be transformed into an equivalent problem in minimization form (by negating all coefficients a_{i0} and a_{ij}), and then applying the process above to obtain a problem satisfying Assumption 2.

B Additional computational results

In this appendix, we compare the performance of the formulations presented in the paper (not restricted to those discussed in Sect. 4.2 and Sect. 4.3 and presented in Tables 3 and 4 as well as their extended versions, i.e., Tables 5 and 6) to evaluate the individual and combined effects of the enhancements. In order to have a better comparison of the results, we repeat the results for some of the formulations in different subsections.

In particular, first, in Appendix B.1, we compare the basic MILP and the basic MICQP formulations without using additional enhancements. Then in Appendix B.2, we focus on the effect of the binary-expansion technique on the basic formulations. Next, in Appendix B.3, we focus on the impact of polymatroid cuts. In Appendix B.4, we test the formulations that benefit from the integration of the binary-expansion technique with the polymatroid cuts. Recall that, in the following tables, the "†" symbol is used if CPLEX is unable to fully process the root node of the branch-and-bound tree within the time limit for a given formulation.

B.1 Linear versus conic formulations

Here, we evaluate the basic MILP (LF, LEF) and the basic MICQP (CF, CEF) reformulations, see Tables 7 and 8. Observe that, in most cases, LEF, CF, and CEF are stronger than LF, i.e., they have better Rlx-gap. Additionally, as expected, the extended formulations LEF and CEF are stronger than compact formulations, i.e., LF and CF, respectively. The extended formulations also shows better running time and end gap than the corresponding compact formulations. In general, CEF performs better than LEF for low values of the parameter κ, while LEF is comparatively better for high values of κ. Moreover, none of the formulations except CF (with a very poor performance) are able to scale to $n = 1000$ for all instances. These results justify the development of enhanced formulations for the medium and large size instances.

B.2 Binary-expansion

Here, we explore the individual impact of binary-expansion technique on the performance and size of the basic formulations. Specifically, we compare LF and CEF versus their binarized versions, i.e., $\mathtt{LF}_{\mathtt{log}}$ and $\mathbf{CEF}_{\mathbf{log}}$, respectively. We do not consider the binarized formulations of LEF and CF as discussed in Remarks 4 and 6, respectively.

In Tables 9 and 10, we observe that LF has a poor performance even for $n = 100$. In contrast, its binarization leads to noteworthy improvements in the results due to the reduction in its size. These results are consistent with the previous results in the literature that $\mathtt{LF}_{\mathtt{log}}$ outperforms LF and $\mathtt{LEF}_{\mathtt{log}}$—see [9,24].

On the other hand, for $n \leqslant 500$ formulation CEF has a superior performance over $\mathbf{CEF}_{\mathbf{log}}$ with respect to either time or the considered gaps; e.g., for $n = 500$ and $\kappa = 10\% \cdot n$ in Table 9, CEF reports the 0.2% average End-gap, compared to 5.1% for $\mathbf{CEF}_{\mathbf{log}}$. Nonetheless, $\mathbf{CEF}_{\mathbf{log}}$ is able to scale to problems with $n \geqslant 1000$ while formulation CEF is not. Additionally, for the instances with $n \geqslant 2000$ we observe that in most cases $\mathbf{CEF}_{\mathbf{log}}$ outperforms (the superior MILP formulation) $\mathtt{LF}_{\mathtt{log}}$, as well.

Tables 11 and 12 show the impact of binarization in the reduction of the number of continuous variables and the numbers of linear as well as rotated cone constraints for the assortment and the uniformly generated data sets, respectively. It can be seen that the binary-expansion technique substantially reduces the number of (continuous) variables and constraints with a slight increase in the number of binary variables; the percent of these reductions gets larger as n grows. For example, in Table 11 for $n = 1000$, $\mathtt{LF}_{\mathtt{log}}$ and $\mathbf{CEF}_{\mathbf{log}}$ have at least 97,900 and 391,500 fewer continuous variables and linear constraints, respectively, than LF and CEF with the cost of at most 2100 more binary variables. The binary-expansion technique also leads to a reduction of 97,900 rotated cone constraints for CEF.

B.3 Polymatroid cuts

Next, we explore the individual impact of polymatroid cuts on the basic formulations, namely, LF, LEF, CF, and CEF. Notably, for $n \leqslant 500$ in Tables 13 and 14, we observe that polymatroid cuts have a significant improvement effect on the performance (time and End-gap) of compact formulations LF and CF. However, the cuts are not that effective for LEF and CEF, as these extended formulations are much stronger and the cuts provide only a marginal improvement in the relaxation quality while increasing the sizes of the formulations.

Additionally, for $n \geqslant 1000$ polymatroid cuts are not beneficial and employing them makes the results worse, see, e.g., in Table 13 and $n = 1000$ that End-gap of LEF from 13.9% increases to 81% after employing the cuts. The reason is that CPLEX consumes the allocated time only to manage the cuts and process the root node.

B.4 Integration of binary-expansion and polymatroid cuts

Here, we explore the effect of simultaneous usage of both techniques, i.e., the impact of the incorporation of polymatroid cuts with binary expansion on LF and CEF. Tables 15 and 16 present the results and we make the following observations. Formulation $\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$ either outperforms LF, $\mathbf{LF}^{\mathbf{P}}$, and $\mathrm{LF}_{\mathrm{log}}$ or (in a few cases) has a competitive performance with $\mathbf{LF}^{\mathbf{P}}$. On the other hand, for the small- and medium-size instances CEF and $\mathbf{CEF}^{\mathbf{P}}$ are competitive and they have better performances than $\mathbf{CEF}_{\mathbf{log}}$ and $\mathbf{CEF}^{\mathbf{P}}_{\mathbf{log}}$. However, for large instances $\mathbf{CEF}^{\mathbf{P}}_{\mathbf{log}}$ outperforms CEF, $\mathbf{CEF}_{\mathbf{log}}$ and $\mathbf{CEF}^{\mathbf{P}}$. These observations imply that—specially in large instances—the integration of binarization and polymatroid cuts in both MILPs and MICQPs leads to superior formulations. Specifically, $\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$ and $\mathbf{CEF}^{\mathbf{P}}_{\mathbf{log}}$ perform better than the corresponding basic formulations and the enhanced ones that only use one of the improving techniques.

Additionally, it appears that for instances up to 500 variables, in general, CEF and $\mathbf{CEF}^{\mathbf{P}}$ are the most efficient formulations. For instances with $n \geqslant 1000$, $\mathbf{CEF}^{\mathbf{P}}_{\mathbf{log}}$ and $\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$ outperform the others. Finally, we observe that, in general, $\mathbf{CEF}^{\mathbf{P}}_{\mathbf{log}}$ has a better performance in the constrained instances, while $\mathbf{LF}^{\mathbf{P}}_{\mathbf{log}}$ is superior in the unconstrained instances.

Table 5 Computational results to evaluate the best existing methods in the literature against the standout formulations for the assortment data set [33]

n, m	κ	$10\% \cdot n$							$20\% \cdot n$						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
25,2*	$\mathtt{LF}_{\mathtt{log}}$	**0**	5	0.0%	15.3%	1.1%	0.0%	0.0%	**0**	27	0.0%	28.1%	4.3%	0.0%	0.0%
	LEF	**0**	2	0.0%	0.7%	0.7%	0.0%	0.0%	**0**	9	0.0%	0.9%	0.8%	0.0%	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	**0**	0	0.0%	1.5%	0.8%	0.0%	0.0%	**0**	0	0.0%	2.8%	1.9%	0.0%	0.0%
	CEF	**0**	0	0.0%	0.0%	0.0%	0.0%	0.0%	**0**	1	0.0%	0.1%	0.1%	0.0%	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	**0**	0	0.0%	0.5%	0.3%	0.0%	0.0%	**0**	1	0.0%	0.9%	0.7%	0.0%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	1	1	0.0%	0.1%	0.1%	0.0%	0.0%	**0**	0	0.0%	0.6%	0.1%	0.0%	0.0%
50,5*	$\mathtt{LF}_{\mathtt{log}}$	1	233	0.0%	30.0%	5.7%	0.0%	0.0%	2	2109	0.0%	44.4%	14.5%	0.0%	0.0%
	LEF	**0**	123	0.0%	3.0%	3.0%	0.0%	0.0%	**1**	1111	0.0%	3.2%	3.2%	0.0%	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	1	0	0.0%	2.3%	0.1%	0.0%	0.0%	2	0	0.0%	5.5%	0.0%	0.0%	0.0%
	CEF	1	1	0.0%	0.0%	0.0%	0.0%	0.0%	**1**	7	0.0%	0.2%	0.2%	0.0%	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	**0**	0	0.0%	0.9%	0.3%	0.0%	0.0%	**1**	19	0.0%	2.4%	0.1%	0.0%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	**0**	4	0.0%	0.5%	0.3%	0.0%	0.0%	2	250	0.0%	1.9%	0.9%	0.0%	0.0%
100,10*	$\mathtt{LF}_{\mathtt{log}}$	979	364,141	0.0%	42.7%	14.5%	0.0%	0.0%	3155	1,732,777	0.4%	55.3%	23.7%	0.4%	0.0%
	LEF	3357	345,641	1.6%	8.3%	8.3%	1.6%	0.0%	2190	361,599	0.2%	5.0%	5.0%	0.2%	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	10	0	0.0%	3.8%	0.0%	0.0%	0.0%	20	0	0.0%	7.8%	0.0%	0.0%	0.0%
	CEF	6	14	0.0%	0.7%	0.2%	0.0%	0.0%	**4**	23	0.0%	0.6%	0.2%	0.0%	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	**1**	86	0.0%	1.9%	0.1%	0.0%	0.0%	6	2434	0.0%	4.4%	0.1%	0.0%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	2	215	0.0%	1.2%	0.3%	0.0%	0.0%	22	3199	0.0%	3.7%	2.1%	0.0%	0.0%

Table 5 continued

n, m	κ	$10\% \cdot n$							$20\% \cdot n$						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
200,20*	$\mathtt{LF}_{\mathtt{log}}$	3600	549,079	6.7%	52.9%	24.7%	6.1%	0.6%	3600	383,827	8.7%	64.7%	31.9%	7.7%	1.1%
	LEF	3600	42,413	8.6%	10.6%	10.6%	8.6%	0.0%	3600	129,049	1.1%	3.0%	3.0%	1.0%	0.1%
	$\mathtt{CF}^{\mathtt{P}}$	**27**	5	0.0%	5.6%	0.0%	0.0%	0.0%	64	131	0.0%	13.1%	0.0%	0.0%	0.1%
	CEF	73	190	0.0%	1.5%	0.4%	0.0%	0.0%	**40**	31	0.0%	1.5%	0.1%	0.1%	0.1%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	710	158,569	0.0%	4.1%	0.2%	0.0%	0.0%	3400	715,941	0.3%	10.9%	0.4%	0.2%	0.1%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	2353	74,047	0.5%	2.6%	2.2%	0.4%	0.1%	3600	112,151	2.2%	8.2%	6.8%	1.8%	0.4%
500,50*	$\mathtt{LF}_{\mathtt{log}}$	3600	102,004	39.8%	53.0%	35.4%	31.8%	16.0%	3600	84,392	54.0%	68.6%	35.0%	31.4%	137.4%
	LEF	3600	2548	8.3%	8.7%	8.7%	8.2%	0.1%	**2520**	8118	0.2%	0.8%	0.2%	0.6%	0.8%
	$\mathtt{CF}^{\mathtt{P}}$	**1194**	311	0.0%	8.9%	0.1%	0.1%	0.1%	3452	707	0.3%	22.2%	0.7%	0.7%	1.0%
	CEF	3611	779	0.2%	32.6%	0.3%	0.1%	0.1%	2620	842	0.0%	38.4%	0.5%	0.7%	0.8%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	110,452	0.8%	8.8%	0.3%	0.3%	0.5%	3600	57,797	3.3%	24.8%	1.3%	1.3%	2.1%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	55,687	4.7%	5.4%	5.0%	4.3%	0.4%	3600	63,450	12.2%	15.8%	14.0%	11.5%	0.8%
1000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	55,776	55.9%	60.6%	51.0%	46.3%	25.3%	3600	58,847	62.7%	77.4%	61.2%	59.1%	9.5%
	LEF	3600	215	13.9%	4.4%	4.4%	4.3%	16.8%	3722	488	**0.9%**	1.2%	0.0%	0.1%	1.0%
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3605	†	†	†	†	†	†	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3601	†	†	†	†	†	†	3601	6378	20.9%	39.4%	15.8%	15.6%	6.7%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3601	32,326	**10.0%**	9.9%	9.5%	9.1%	1.0%	3600	26,843	22.6%	23.9%	22.5%	21.8%	1.1%

Table 5 continued

n, m	κ	$10\% \cdot n$							$20\% \cdot n$						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
2000,100**	LF_{log}	3600	58,217	57.8%	68.0%	62.7%	55.1%	6.3%	3600	56,546	70.5%	84.0%	79.1%	67.9%	9.7%
	LEF	3601	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CF^P	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	LF^P_{log}	3601	†	†	†	†	†	†	3600	32,898	41.4%	48.9%	41.2%	39.5%	3.2%
	CEF^P_{log}	3600	38,716	**16.1%**	15.8%	15.4%	14.7%	1.6%	3600	28,575	**30.7%**	32.5%	31.9%	30.6%	0.1%
5000,100**	LF_{log}	3600	23,558	78.1%	86.8%	84.7%	77.3%	3.9%	3600	37,298	80.6%	93.4%	93.4%	80.2%	1.9%
	LEF	7807	†	†	†	†	†	†	8155	†	†	†	†	†	†
	CF^P	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	LF^P_{log}	3601	7220	**29.2%**	50.1%	25.1%	25.0%	5.9%	3601	15,186	49.0%	59.9%	47.6%	45.8%	6.3%
	CEF^P_{log}	3600	13,966	39.3%	27.5%	30.7%	28.7%	402.8%	3600	13,736	**40.6%**	33.9%	40.1%	39.1%	2.6%
10000,100**	LF_{log}	3600	13,230	88.4%	90.0%	90.0%	82.0%	224.1%	3600	8857	83.1%	94.7%	94.7%	80.6%	39.2%
	LEF	4225	†	†	†	†	†	†	4026	†	†	†	†	†	†
	CF^P	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	LF^P_{log}	3601	5481	55.4%	58.6%	54.7%	53.2%	5.4%	3601	9440	53.2%	61.4%	49.2%	48.2%	10.5%
	CEF^P_{log}	3600	9979	**33.4%**	5.0%	34.9%	31.0%	3.7%	3601	7247	**45.4%**	22.0%	37.6%	35.6%	17.9%

Table 5 continued

n, m	κ	Unconstrained						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
25,2*	$\mathtt{LF}_{\mathtt{log}}$	1	386	0.0%	56.1%	18.1%	0.0%	0.0%
	LEF	**0**	0	0.0%	0.0%	0.0%	0.0%	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	1	0	0.0%	8.2%	1.2%	0.0%	0.0%
	CEF	**0**	0	0.0%	0.0%	0.0%	0.0%	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	**0**	20	0.0%	1.8%	0.1%	0.0%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	2	379	0.0%	1.8%	1.5%	0.0%	0.0%
50,5*	$\mathtt{LF}_{\mathtt{log}}$	18	35,496	0.0%	65.6%	28.2%	0.0%	0.0%
	LEF	**0**	2	0.0%	0.0%	0.0%	0.0%	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	4	4	0.0%	13.4%	0.9%	0.0%	0.0%
	CEF	1	2	0.0%	0.0%	0.0%	0.0%	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	6	6721	0.0%	5.9%	0.3%	0.0%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	21	11,132	0.0%	5.9%	5.2%	0.0%	0.0%
100,10*	$\mathtt{LF}_{\mathtt{log}}$	3600	1,543,428	1.6%	75.9%	38.5%	1.4%	0.1%
	LEF	**1**	35	0.0%	0.1%	0.0%	0.1%	0.1%
	$\mathtt{CF}^{\mathtt{P}}$	25	370	0.0%	17.8%	0.1%	0.1%	0.1%
	CEF	6	17	0.0%	0.3%	0.0%	0.1%	0.1%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	1,535,465	0.8%	12.1%	1.5%	0.7%	0.1%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	411,139	0.3%	12.1%	8.4%	0.2%	0.1%

Table 5 continued

n, m	κ	Unconstrained						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
200,20*	$\mathtt{LF}_{\mathtt{log}}$	3600	300,111	24.1%	82.7%	49.9%	21.9%	2.9%
	LEF	**29**	1327	0.0%	0.3%	0.3%	0.1%	0.1%
	$\mathtt{CF}^{\mathtt{P}}$	1562	26,768	0.2%	23.6%	0.5%	0.1%	0.1%
	CEF	59	332	0.0%	1.1%	0.1%	0.1%	0.1%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	374,382	6.3%	22.3%	6.0%	5.6%	0.7%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	144,453	6.4%	22.2%	13.8%	4.8%	1.7%
500,50*	$\mathtt{LF}_{\mathtt{log}}$	3600	92,414	55.7%	91.8%	74.7%	53.1%	5.9%
	LEF	3501	12,727	**0.4%**	1.8%	0.4%	0.9%	1.3%
	$\mathtt{CF}^{\mathtt{P}}$	3600	440	7.7%	26.7%	0.5%	0.4%	10.0%
	CEF	3604	272	0.5%	23.9%	0.7%	0.8%	1.3%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	65,850	15.2%	33.6%	13.4%	13.0%	2.6%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3601	129,520	26.1%	33.5%	20.6%	16.2%	13.5%
1000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	55,641	76.5%	94.3%	79.1%	73.8%	11.5%
	LEF	3600	351	**1.7%**	1.7%	1.1%	1.0%	0.7%
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3601	30,129	26.1%	43.6%	23.9%	23.4%	3.6%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	4283	33.8%	42.9%	25.0%	24.6%	14.0%

Table 5 continued

n, m	κ	Unconstrained						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
2000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	39,585	78.3%	96.0%	81.6%	77.7%	2.7%
	LEF	3601	†	†	†	†	†	†
	$\mathtt{CF}^{\mathrm{P}}$	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathrm{P}}_{\mathtt{log}}$	3601	8660	**33.1%**	52.2%	31.2%	31.1%	3.0%
	$\mathtt{CEF}^{\mathrm{P}}_{\mathtt{log}}$	3600	931	53.4%	48.2%	33.0%	33.0%	268.5%
5000,100**	$\mathtt{LF}_{\mathtt{log}}$	3601	12,870	83.5%	96.8%	96.8%	83.0%	3.6%
	LEF	7241	†	†	†	†	†	†
	$\mathtt{CF}^{\mathrm{P}}$	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathrm{P}}_{\mathtt{log}}$	3601	6818	**50.7%**	61.1%	49.3%	49.2%	3.1%
	$\mathtt{CEF}^{\mathrm{P}}_{\mathtt{log}}$	3600	3257	58.4%	50.8%	47.5%	47.5%	26.4%
10000,100**	$\mathtt{LF}_{\mathtt{log}}$	3602	5082	93.0%	97.6%	97.6%	91.6%	22.0%
	LEF	3603	†	†	†	†	†	†
	$\mathtt{CF}^{\mathrm{P}}$	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathrm{P}}_{\mathtt{log}}$	3601	5482	**54.7%**	65.0%	54.3%	54.3%	1.0%
	$\mathtt{CEF}^{\mathrm{P}}_{\mathtt{log}}$	3601	†	†	†	†	†	†

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (`Time`) in seconds, number of nodes (`Nodes`) processed, end gap (`End-gap`), continuous relaxation gap (`Rlx-gap`), root node gap (`Ron-gap`), best-bound gap (`Bbn-gap`), and optimality gap (`Opt-gap`). For each choice of n, m, and κ, among the solution methods, the best average time and the best average `End-gap` (if `Time`$\geqslant$ 3600 s) are in **bold**

**easy* instances

***hard* instances

Table 6 Computational results to evaluate the best existing methods in the literature against the standout formulations for the uniformly generated data set [9]

n, m	κ	$10\% \cdot n$							$20\% \cdot n$						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
25,2*	$\mathrm{LF}_{\mathrm{log}}$	**0**	23	0.0%	49.8%	28.0%	0.0%	0.0%	1	95	0.0%	50.1%	32.3%	0.0%	0.0%
	LEF	**0**	1	0.0%	6.8%	5.4%	0.0%	0.0%	**0**	24	0.0%	10.8%	9.2%	0.0%	0.0%
	CF^{P}	3	2	0.0%	31.4%	4.6%	0.0%	0.0%	4	179	0.0%	40.5%	6.3%	0.0%	0.0%
	CEF	**0**	2	0.0%	4.0%	3.8%	0.0%	0.0%	**0**	14	0.0%	7.9%	7.0%	0.0%	0.0%
	$\mathrm{LF}^{\mathrm{P}}_{\mathrm{log}}$	**0**	13	0.0%	29.5%	11.5%	0.0%	0.0%	1	222	0.0%	42.3%	12.0%	0.0%	0.0%
	$\mathrm{CEF}^{\mathrm{P}}_{\mathrm{log}}$	1	4	0.0%	10.3%	6.9%	0.0%	0.0%	1	106	0.0%	19.9%	18.7%	0.0%	0.0%
50,5*	$\mathrm{LF}_{\mathrm{log}}$	3	3364	0.0%	50.7%	43.7%	0.0%	0.0%	20	21,061	0.0%	54.2%	45.0%	0.0%	0.0%
	LEF	**2**	381	0.0%	18.5%	16.4%	0.0%	0.0%	**13**	9831	0.0%	20.9%	19.9%	0.0%	0.0%
	CF^{P}	78	22,068	0.0%	56.2%	23.9%	0.0%	0.0%	3601	1,058,360	6.5%	56.3%	25.4%	6.4%	0.1%
	CEF	3	172	0.0%	14.6%	13.7%	0.0%	0.0%	18	6093	0.0%	15.6%	15.3%	0.0%	0.0%
	$\mathrm{LF}^{\mathrm{P}}_{\mathrm{log}}$	9	5014	0.0%	45.9%	21.3%	0.0%	0.0%	27	29,157	0.0%	51.4%	30.0%	0.0%	0.0%
	$\mathrm{CEF}^{\mathrm{P}}_{\mathrm{log}}$	6	2477	0.0%	24.4%	22.6%	0.0%	0.0%	26	8630	0.0%	29.9%	29.1%	0.0%	0.0%
100,10**	$\mathrm{LF}_{\mathrm{log}}$	3600	2,079,337	**5.0%**	54.5%	48.7%	4.8%	0.2%	3600	2,153,102	**5.0%**	56.4%	49.8%	4.8%	0.3%
	LEF	3600	480,988	12.3%	29.4%	27.5%	12.3%	0.0%	3600	616,551	17.1%	30.4%	29.7%	17.1%	0.0%
	CF^{P}	3600	462,737	43.5%	72.8%	50.8%	41.0%	4.3%	3600	166,635	44.3%	66.9%	47.3%	42.1%	4.1%
	CEF	3600	221,990	10.7%	25.6%	24.8%	10.3%	0.4%	3600	275,594	15.5%	25.0%	25.7%	14.9%	0.8%
	$\mathrm{LF}^{\mathrm{P}}_{\mathrm{log}}$	3600	2,588,756	7.5%	54.1%	45.5%	6.8%	0.8%	3600	2,821,692	6.1%	56.3%	48.3%	5.3%	0.9%
	$\mathrm{CEF}^{\mathrm{P}}_{\mathrm{log}}$	3600	482,188	7.2%	34.8%	34.5%	6.3%	0.9%	3603	463,914	5.2%	36.6%	36.6%	3.6%	1.7%

Table 6 continued

n, m	κ	$10\% \cdot n$							$20\% \cdot n$						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
200,20**	$\mathtt{LF}_{\mathtt{log}}$	3600	612,063	41.7%	56.8%	54.5%	37.5%	7.3%	3600	490,278	37.7%	58.0%	54.7%	33.9%	6.0%
	LEF	3600	47,486	**30.0%**	36.2%	35.4%	30.0%	0.0%	3600	58,945	31.1%	36.0%	35.6%	31.0%	0.1%
	$\mathtt{CF}^{\mathtt{P}}$	3600	25,375	65.8%	80.4%	66.0%	63.7%	6.0%	3600	1113	61.6%	72.0%	58.9%	58.8%	7.1%
	CEF	3600	20,677	30.9%	30.9%	33.3%	28.9%	2.9%	3600	22,387	**30.0%**	23.0%	32.7%	28.9%	1.5%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	1,104,491	41.6%	56.7%	53.5%	36.5%	8.7%	3600	938,882	35.6%	57.9%	54.5%	31.5%	6.3%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	174,404	35.5%	40.1%	39.9%	31.2%	6.6%	3600	113,509	34.3%	40.0%	39.9%	30.8%	5.4%
500,50**	$\mathtt{LF}_{\mathtt{log}}$	3600	81,055	48.7%	49.0%	48.9%	47.0%	3.2%	3600	60,815	48.7%	47.2%	47.1%	45.0%	7.2%
	LEF	3600	636	**42.8%**	43.0%	42.7%	42.4%	0.7%	3600	1324	**41.1%**	38.6%	38.5%	38.0%	5.2%
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3603	17	**42.8%**	20.4%	41.3%	41.2%	2.8%	3604	26	41.8%	5.7%	37.3%	37.2%	7.8%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	108,291	48.4%	49.0%	49.0%	46.8%	3.1%	3600	70,193	48.1%	47.2%	47.1%	44.6%	6.7%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	29,818	46.3%	45.2%	44.9%	42.7%	7.1%	3600	24,696	43.1%	40.7%	40.7%	39.1%	7.0%
1000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	52,994	50.3%	48.7%	48.7%	48.0%	4.7%	3600	41,825	50.1%	50.9%	50.8%	50.0%	0.0%
	LEF	3601	†	†	†	†	†	†	3601	†	†	†	†	†	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	48,719	50.2%	48.7%	48.7%	48.0%	4.4%	3600	24,225	50.2%	50.7%	50.8%	50.2%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	10,436	**48.0%**	45.3%	45.3%	44.5%	6.9%	3600	9445	**44.5%**	44.7%	45.0%	44.5%	0.1%

Table 6 continued

n, m	κ	$10\% \cdot n$							$20\% \cdot n$						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
2000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	41,092	50.7%	51.2%	51.2%	50.6%	0.1%	3600	30,062	50.6%	50.8%	50.7%	50.2%	0.8%
	LEF	3601	†	†	†	†	†	†	3602	†	†	†	†	†	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	15,925	50.8%	51.1%	51.2%	50.8%	0.0%	3600	14,228	50.7%	50.8%	50.8%	50.3%	0.8%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	9576	**47.8%**	48.3%	48.2%	47.7%	0.1%	3600	7815	**44.6%**	45.1%	45.0%	44.6%	0.1%
5000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	18,499	67.9%	68.6%	68.6%	67.1%	2.8%	3600	34,661	65.0%	69.5%	69.5%	65.0%	0.0%
	LEF	4755	†	†	†	†	†	†	3938	†	†	†	†	†	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	9434	68.8%	68.6%	68.6%	68.0%	2.7%	3600	12,867	67.9%	69.5%	69.2%	67.9%	0.1%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	4295	**46.7%**	47.2%	47.0%	46.7%	0.0%	3601	3406	**45.2%**	45.7%	45.5%	45.1%	0.0%
10000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	15,052	68.6%	69.0%	69.0%	68.1%	1.6%	3600	11,855	68.2%	69.2%	69.2%	67.9%	1.0%
	LEF	9500	†	†	†	†	†	†	6022	†	†	†	†	†	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3601	5732	68.5%	69.0%	69.0%	68.0%	1.6%	3601	6058	68.4%	69.2%	68.8%	68.0%	1.2%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3601	896	**47.5%**	47.9%	47.7%	47.5%	0.0%	3600	1165	**44.8%**	45.1%	45.0%	44.8%	0.0%

Table 6 continued

n, m	κ	Unconstrained						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
25,2*	$\mathtt{LF}_{\mathtt{log}}$	1	199	0.0%	93.0%	59.8%	0.0%	0.0%
	LEF	**0**	21	0.0%	28.1%	22.7%	0.0%	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	4	24	0.0%	37.1%	3.4%	0.0%	0.0%
	CEF	1	16	0.0%	18.6%	13.2%	0.0%	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	1	142	0.0%	45.5%	14.6%	0.0%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	6	133	0.0%	42.8%	27.1%	0.0%	0.0%
50,5*	$\mathtt{LF}_{\mathtt{log}}$	52	55,437	0.0%	96.9%	77.1%	0.0%	0.0%
	LEF	**43**	35,334	0.0%	56.3%	49.6%	0.0%	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	2903	1,043,778	3.0%	56.6%	23.9%	3.0%	0.0%
	CEF	100	35,410	0.0%	43.6%	40.9%	0.0%	0.0%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	85	81,310	0.0%	60.3%	25.0%	0.0%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	86	25,435	0.0%	59.4%	46.4%	0.0%	0.0%
100,10*	$\mathtt{LF}_{\mathtt{log}}$	3600	2,487,103	11.2%	98.6%	84.6%	10.7%	0.6%
	LEF	3600	654,126	38.5%	72.5%	66.6%	38.5%	0.0%
	$\mathtt{CF}^{\mathtt{P}}$	3600	330,256	42.0%	71.5%	50.4%	40.4%	2.9%
	CEF	3600	130,787	40.1%	63.7%	61.1%	39.3%	1.5%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	1,928,384	17.2%	74.5%	48.6%	16.1%	1.4%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	417,221	**10.9%**	73.6%	61.2%	10.9%	0.0%

Table 6 continued

n, m	κ	Unconstrained						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
200,20*	$\mathtt{LF}_{\mathtt{log}}$	3600	519,981	58.2%	99.3%	89.9%	54.9%	7.9%
	LEF	3600	63,610	70.6%	83.0%	78.8%	70.5%	0.3%
	$\mathtt{CF}^{\mathtt{P}}$	3600	7872	70.9%	81.0%	66.3%	64.8%	21.0%
	CEF	3600	4559	76.4%	76.0%	75.8%	71.3%	21.6%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	434,136	58.0%	82.1%	65.9%	52.7%	13.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	279,263	**54.4%**	81.4%	73.4%	50.1%	9.4%
500,50*	$\mathtt{LF}_{\mathtt{log}}$	3600	139,697	87.0%	99.9%	96.1%	86.1%	7.1%
	LEF	3600	113	90.3%	91.4%	89.3%	89.2%	11.5%
	$\mathtt{CF}^{\mathtt{P}}$	3600	1208	84.9%	89.8%	83.1%	82.5%	15.9%
	CEF	3603	1	93.4%	80.5%	90.9%	90.9%	51.3%
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	181,247	**82.9%**	90.5%	85.9%	81.3%	9.9%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	23,870	86.7%	89.7%	86.6%	84.6%	20.1%
1000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	48,644	96.6%	99.9%	97.3%	96.3%	8.5%
	LEF	3601	†	†	†	†	†	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	406	95.6%	83.4%	95.5%	95.1%	27.6%
	CEF	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	108,734	**91.9%**	93.1%	92.0%	91.2%	8.4%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	476	92.2%	92.5%	90.2%	90.2%	35.7%

Table 6 continued

n, m	κ	Unconstrained						
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Bbn-gap	Opt-gap
2000,100**	$\mathtt{LF}_{\mathtt{log}}$	3600	35,408	97.8%	100.0%	98.2%	97.7%	3.4%
	LEF	3601	†	†	†	†	†	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3600	69,565	**94.8%**	95.5%	95.1%	94.7%	2.6%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	339	96.6%	95.1%	93.7%	93.7%	101.2%
5000,100**	$\mathtt{LF}_{\mathtt{log}}$	3601	13,907	98.8%	100.0%	98.8%	98.6%	22.2%
	LEF	3603	†	†	†	†	†	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3601	16,900	**96.9%**	96.7%	96.5%	96.4%	14.4%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3601	34	98.3%	96.4%	96.0%	96.0%	132.1%
10000,100**	$\mathtt{LF}_{\mathtt{log}}$	3601	2471	99.4%	100.0%	99.3%	99.3%	25.6%
	LEF	5619	†	†	†	†	†	†
	$\mathtt{CF}^{\mathtt{P}}$	3600	†	†	†	†	†	†
	CEF	3600	†	†	†	†	†	†
	$\mathtt{LF}^{\mathtt{P}}_{\mathtt{log}}$	3601	6595	**97.8%**	98.0%	97.9%	97.8%	0.0%
	$\mathtt{CEF}^{\mathtt{P}}_{\mathtt{log}}$	3600	†	†	†	†	†	†

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (Time) in seconds, number of nodes (Nodes) processed, end gap (End-gap), continuous relaxation gap (Rlx-gap), root node gap (Ron-gap), best-bound gap (Bbn-gap), and optimality gap (Opt-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

**easy* instances

***hard* instances

Table 7 Computational results to compare basic MILP and MICQP formulations for the assortment data set [33]

n, m	κ	$10\% \cdot n$						$20\% \cdot n$						Unconstrained					
	Ref.	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap
25,2	LF	**0**	5	30	0.0%	14.1%	4.7%	2	5	1395	0.0%	42.6%	24.5%	2	5	1538	0.0%	35.0%	12.1%
	LEF	**0**	5	2	0.0%	0.7%	0.7%	**0**	5	9	0.0%	0.9%	0.8%	**0**	5	0	0.0%	0.0%	0.0%
	CF	1	5	13	0.0%	1.5%	1.5%	1	5	70	0.0%	2.8%	2.8%	2	5	1278	0.0%	8.2%	8.2%
	CEF	**0**	5	0	0.0%	0.0%	0.0%	**0**	5	1	0.0%	0.1%	0.1%	**0**	5	0	0.0%	0.0%	0.0%
50,5	LF	55	5	40,100	0.0%	51.7%	41.3%	3600	0	935,667	29.4%	60.1%	49.8%	3600	0	1,495,669	19.3%	52.9%	41.4%
	LEF	**0**	5	123	0.0%	3.0%	3.0%	**1**	5	1111	0.0%	3.2%	3.2%	**0**	5	2	0.0%	0.0%	0.0%
	CF	2	5	157	0.0%	2.3%	2.2%	7	5	3016	0.0%	5.5%	5.5%	2916	2	802,844	1.0%	13.4%	13.4%
	CEF	1	5	1	0.0%	0.0%	0.0%	**1**	5	7	0.0%	0.2%	0.2%	1	5	2	0.0%	0.0%	0.0%
100,10	LF	3600	0	170,481	70.1%	78.3%	77.7%	3600	0	473,158	61.7%	72.1%	70.2%	3600	0	730,892	51.5%	67.5%	64.2%
	LEF	3357	1	345,641	1.6%	8.3%	8.3%	2190	3	361,599	0.2%	5.0%	5.0%	**1**	5	35	0.0%	0.1%	0.0%
	CF	67	5	7786	0.0%	3.8%	3.6%	3371	1	231,729	3.5%	7.8%	8.6%	3600	0	94,567	11.9%	17.8%	17.9%
	CEF	**6**	5	14	0.0%	0.7%	0.2%	**4**	5	23	0.0%	0.6%	0.2%	6	5	17	0.0%	0.3%	0.0%
200,20	LF	3600	0	28,528	82.3%	84.4%	84.4%	3600	0	47,569	77.0%	79.4%	79.3%	3600	0	49,188	73.4%	78.3%	77.9%
	LEF	3600	0	42,413	8.6%	10.6%	10.6%	3600	1	129,049	1.1%	3.0%	3.0%	**29**	5	1327	0.0%	0.3%	0.3%
	CF	3600	0	55,709	32.9%	5.6%	36.0%	3600	0	29,821	62.3%	13.1%	63.7%	3600	0	14,939	65.2%	23.6%	73.1%
	CEF	**73**	5	190	0.0%	1.5%	0.4%	**40**	5	31	0.0%	1.5%	0.1%	59	5	332	0.0%	1.1%	0.1%

Table 7 continued

n, m	κ	$10\% \cdot n$						$20\% \cdot n$						Unconstrained					
	Ref.	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap
500,50	LF	3600	0	1620	90.3%	89.0%	89.0%	3600	0	2097	86.7%	86.2%	86.2%	3601	0	5755	86.2%	86.4%	86.4%
	LEF	3600	0	2548	8.3%	8.7%	8.7%	**2520**	2	8118	0.2%	0.8%	0.2%	3501	1	12,727	**0.4%**	1.8%	0.4%
	CF	3600	0	11,986	96.4%	8.9%	100.0%	3600	0	11,367	100.0%	22.2%	100.0%	3600	0	4110	96.1%	26.7%	100.0%
	CEF	3611	0	779	**0.2%**	32.6%	0.3%	2620	5	842	0.0%	38.4%	0.5%	3604	0	272	0.5%	23.9%	0.7%
1000,100	LF	3601	0	3	99.2%	93.1%	93.1%	3600	0	10	99.0%	90.4%	90.4%	3601	0	33	99.0%	90.5%	90.5%
	LEF	3600	0	215	**13.9%**	4.4%	4.4%	3722	0	488	**0.9%**	1.2%	0.0%	3600	0	351	**1.7%**	1.7%	1.1%
	CF	3600	0	4962	100.0%	15.7%	100.0%	3605	0	4612	100.0%	29.7%	100.0%	3600	0	1882	100.0%	30.2%	100.0%
	CEF	3605	0	†	†	†	†	3600	0	†	†	†	†	3600	0	†	†	†	†

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (Time) in seconds, number (#) of instances solved to optimality, number of nodes (Nodes) processed, end gap (End-gap), continuous relaxation gap (Rlx-gap), root-node gap (Ron-gap), best bound gap (Bbn-gap) the optimality gap (Opt-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

Table 8 Computational results to compare basic MILP and MICQP formulations for the uniformly generated data set [9]

n, m	κ	$10\% \cdot n$						$20\% \cdot n$						Unconstrained					
	Ref.	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap
25,2	LF	**0**	5	38	0.0%	81.5%	48.1%	1	5	433	0.0%	70.4%	41.5%	1	5	497	0.0%	89.7%	46.9%
	LEF	**0**	5	1	0.0%	6.8%	5.4%	**0**	5	24	0.0%	10.8%	9.2%	**0**	5	21	0.0%	28.1%	22.7%
	CF	1	5	114	0.0%	31.4%	24.8%	1	5	1198	0.0%	40.5%	40.5%	1	5	1657	0.0%	37.1%	37.1%
	CEF	**0**	5	2	0.0%	4.0%	3.8%	**0**	5	14	0.0%	7.9%	7.0%	1	5	16	0.0%	18.6%	13.2%
50,5	LF	1554	5	589,909	0.0%	85.7%	82.8%	3600	0	2,006,223	46.9%	75.2%	69.9%	3600	0	2,791,336	44.9%	96.0%	91.5%
	LEF	**2**	5	381	0.0%	18.5%	16.4%	**13**	5	9831	0.0%	20.9%	19.9%	**43**	5	35,334	0.0%	56.3%	49.6%
	CF	75	5	31,692	0.0%	56.2%	56.2%	3606	0	1,112,962	6.0%	56.3%	56.3%	2594	2	677,850	7.3%	56.6%	56.6%
	CEF	3	5	172	0.0%	14.6%	13.7%	18	5	6093	0.0%	15.6%	15.3%	100	5	35,410	0.0%	43.6%	40.9%
100,10	LF	3600	0	1,100,713	82.1%	87.5%	87.1%	3600	0	1,252,405	71.1%	77.1%	76.0%	3600	0	1,308,606	90.2%	98.3%	96.7%
	LEF	3600	0	480,988	12.3%	29.4%	27.5%	3600	0	616,551	17.1%	30.4%	29.7%	3600	0	654,126	**38.5%**	72.5%	66.6%
	CF	3600	0	223,083	52.3%	72.8%	72.8%	3600	0	154,439	54.9%	66.9%	66.9%	3600	0	220,110	53.2%	71.5%	71.4%
	CEF	3600	0	221,990	**10.7%**	25.6%	24.8%	3600	0	275,594	**15.5%**	25.0%	25.7%	3600	0	130,787	40.1%	63.7%	61.1%
200,20	LF	3600	0	118,471	87.8%	88.6%	88.6%	3600	0	107,640	77.4%	78.1%	77.9%	3600	0	184,061	97.5%	99.2%	98.5%
	LEF	3600	0	47,486	**30.0%**	36.2%	35.4%	3600	0	58,945	31.1%	36.0%	35.6%	3600	0	63,610	**70.6%**	83.0%	78.8%
	CF	3600	0	27,323	99.5%	80.4%	99.5%	3600	0	17,547	89.6%	72.0%	88.7%	3600	0	24,168	99.7%	81.0%	100.0%
	CEF	3600	0	20,677	30.9%	30.9%	33.3%	3600	0	22,387	**30.0%**	23.0%	32.7%	3600	0	4559	76.4%	76.0%	75.8%
500,50	LF	3600	0	1232	90.2%	89.4%	89.4%	3600	0	369	82.3%	78.3%	78.3%	3600	0	6619	99.5%	99.8%	99.5%
	LEF	3600	0	636	**42.8%**	43.0%	42.7%	3600	0	1324	**41.1%**	38.6%	38.5%	3600	0	113	**90.3%**	91.4%	89.3%
	CF	3600	0	9997	100.0%	86.1%	100.0%	3602	0	3292	100.0%	75.0%	100.0%	3600	0	13,213	100.0%	89.8%	100.0%
	CEF	3603	0	17	**42.8%**	20.4%	41.3%	3604	0	26	41.8%	5.7%	37.3%	3603	0	1	93.4%	80.5%	90.9%

Table 8 continued

n, m	κ	$10\% \cdot n$						$20\% \cdot n$						Unconstrained					
	Ref.	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap	Time	#	Nodes	End-gap	Rlx-gap	Ron-gap
1000,100	LF	3601	0	†	†	†	†	3600	0	21	**81.6%**	79.9%	79.9%	3601	0	4	**99.9%**	99.9%	99.8%
	LEF	3601	0	†	†	†	†	3601	0	†	†	†	†	3601	0	†	†	†	†
	CF	3600	0	911	**100.0%**	87.3%	100.0%	3600	0	1210	100.0%	77.8%	100.0%	3600	0	663	100.0%	92.8%	100.0%
	CEF	3600	0	†	†	†	†	3600	0	†	†	†	†	3600	0	†	†	†	†

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (Time) in seconds, number (#) of instances solved to optimality, number of nodes (Nodes) processed, end gap (End-gap), continuous relaxation gap (Rlx-gap), root-node gap (Ron-gap), best bound gap (Bbn-gap) and optimality gap (Opt-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

Table 9 Computational results to compare binary-expansion formulations with their basic counterparts for the assortment data set [33]

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
25,2	LF	**0**	30	0.0%	14.1%	4.7%	2	1395	0.0%	42.6%	24.5%	2	1538	0.0%	35.0%	12.1%
	LF_{log}	**0**	5	0.0%	15.3%	1.1%	**0**	27	0.0%	28.1%	4.3%	1	386	0.0%	56.1%	18.1%
	CEF	**0**	0	0.0%	0.0%	0.0%	**0**	1	0.0%	0.1%	0.1%	**0**	0	0.0%	0.0%	0.0%
	CEF_{log}	1	1	0.0%	0.1%	0.1%	1	3	0.0%	0.7%	0.3%	1	316	0.0%	5.8%	5.8%
50,5	LF	55	40,100	0.0%	51.7%	41.3%	3600	935,667	29.4%	60.1%	49.8%	3600	1,495,669	19.3%	52.9%	41.4%
	LF_{log}	**1**	233	0.0%	30.0%	5.7%	2	2109	0.0%	44.4%	14.5%	18	35,496	0.0%	65.6%	28.2%
	CEF	**1**	1	0.0%	0.0%	0.0%	**1**	7	0.0%	0.2%	0.2%	**1**	2	0.0%	0.0%	0.0%
	CEF_{log}	**1**	6	0.0%	0.6%	0.3%	2	371	0.0%	2.1%	2.0%	18	13,917	0.0%	12.4%	12.3%
100,10	LF	3600	170,481	70.1%	78.3%	77.7%	3600	473,158	61.7%	72.1%	70.2%	3600	730,892	51.5%	67.5%	64.2%
	LF_{log}	979	364,141	0.0%	42.7%	14.5%	3155	1,732,777	0.4%	55.3%	23.7%	3600	1,543,428	1.6%	75.9%	38.5%
	CEF	**6**	14	0.0%	0.7%	0.2%	**4**	23	0.0%	0.6%	0.2%	**6**	17	0.0%	0.3%	0.0%
	CEF_{log}	10	1457	0.0%	1.3%	1.2%	212	27,571	0.0%	4.0%	3.9%	3465	292,906	1.0%	20.2%	20.2%
200,20	LF	3600	28,528	82.3%	84.4%	84.4%	3600	47,569	77.0%	79.4%	79.3%	3600	49,188	73.4%	78.3%	77.9%
	LF_{log}	3600	549,079	6.7%	52.9%	24.7%	3600	383,827	8.7%	64.7%	31.9%	3600	300,111	24.1%	82.7%	49.9%
	CEF	**73**	190	0.0%	1.5%	0.4%	**40**	31	0.0%	1.5%	0.1%	**59**	332	0.0%	1.1%	0.1%
	CEF_{log}	3600	137,672	0.9%	2.6%	2.6%	3600	121,652	3.8%	8.2%	8.2%	3600	96,988	22.0%	28.9%	29.0%
500,50	LF	3600	1620	90.3%	89.0%	89.0%	3600	2097	86.7%	86.2%	86.2%	3601	5755	86.2%	86.4%	86.4%
	LF_{log}	3600	102,004	39.8%	53.0%	35.4%	3600	84,392	54.0%	68.6%	35.0%	3600	92,414	55.7%	91.8%	74.7%
	CEF	3611	779	**0.2%**	32.6%	0.3%	**2620**	842	0.0%	38.4%	0.5%	3604	272	**0.5%**	23.9%	0.7%
	CEF_{log}	3600	49,090	5.1%	5.4%	5.4%	3600	55,757	13.3%	15.8%	15.7%	3600	53,280	36.9%	37.3%	38.5%

Table 9 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
1000,100	LF	3601	3	99.2%	93.1%	93.1%	3600	10	99.0%	90.4%	90.4%	3601	33	99.0%	90.5%	90.5%
	$\mathrm{LF}_{\mathrm{log}}$	3600	55,776	55.9%	60.6%	51.0%	3600	58,847	62.7%	77.4%	61.2%	3600	55,641	76.5%	94.3%	79.1%
	CEF	3605	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	$\mathrm{CEF}_{\mathrm{log}}$	3600	36,151	**25.3%**	9.8%	9.9%	3600	36,647	**23.7%**	23.6%	23.9%	3600	35,213	**48.7%**	44.6%	45.7%
2000,100	$\mathrm{LF}_{\mathrm{log}}$	3600	58,217	57.8%	68.0%	62.7%	3600	56,546	70.5%	84.0%	79.1%	3600	39,585	78.3%	96.0%	81.6%
	$\mathrm{CEF}_{\mathrm{log}}$	3600	26,386	**30.3%**	15.5%	16.1%	3600	22,548	**60.0%**	31.9%	38.2%	3600	26,785	**71.0%**	50.7%	52.7%
5000,100	$\mathrm{LF}_{\mathrm{log}}$	3600	23,558	78.1%	86.8%	84.7%	3600	37,298	80.6%	93.4%	93.4%	3601	12,870	**83.5%**	96.8%	96.8%
	$\mathrm{CEF}_{\mathrm{log}}$	3600	15,535	**48.0%**	26.7%	57.7%	3600	10,662	**77.7%**	39.3%	60.0%	3600	11,067	86.5%	57.2%	86.5%
10000,100	$\mathrm{LF}_{\mathrm{log}}$	3600	13,230	88.4%	90.0%	90.0%	3600	8857	**83.1%**	94.7%	94.7%	3602	5082	**93.0%**	97.6%	97.6%
	$\mathrm{CEF}_{\mathrm{log}}$	3600	7551	**53.8%**	29.5%	52.2%	3600	3781	84.6%	45.0%	85.1%	3600	2786	95.4%	70.3%	95.0%

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (Time) in seconds, number of nodes (Nodes) processed, end gap (End-gap), continuous relaxation gap (Rlx-gap), and root-node gap (Ron-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

Table 10 Computational results to compare binary-expansion formulations with their basic counterparts for the uniformly generated data set [9]

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
25,2	LF	**0**	38	0.0%	81.5%	48.1%	1	433	0.0%	70.4%	41.5%	**1**	497	0.0%	89.7%	46.9%
	$\mathrm{LF}_{\log}$	**0**	23	0.0%	49.8%	28.0%	1	95	0.0%	50.1%	32.3%	**1**	199	0.0%	93.0%	59.8%
	CEF	**0**	2	0.0%	4.0%	3.8%	**0**	14	0.0%	7.9%	7.0%	**1**	16	0.0%	18.6%	13.2%
	$\mathrm{CEF}_{\log}$	**0**	6	0.0%	10.3%	7.9%	1	89	0.0%	19.9%	19.3%	3	353	0.0%	45.7%	44.2%
50,5	LF	1554	589,909	0.0%	85.7%	82.8%	3600	2,006,223	46.9%	75.2%	69.9%	3600	2,791,336	44.9%	96.0%	91.5%
	$\mathrm{LF}_{\log}$	**3**	3364	0.0%	50.7%	43.7%	20	21,061	0.0%	54.2%	45.0%	**52**	55,437	0.0%	96.9%	77.1%
	CEF	**3**	172	0.0%	14.6%	13.7%	**18**	6093	0.0%	15.6%	15.3%	100	35,410	0.0%	43.6%	40.9%
	$\mathrm{CEF}_{\log}$	11	4746	0.0%	24.4%	24.0%	22	8046	0.0%	30.0%	29.6%	521	78,437	0.0%	64.4%	62.1%
100,10	LF	3600	1,100,713	82.1%	87.5%	87.1%	3600	1,252,405	71.1%	77.1%	76.0%	3600	1,308,606	90.2%	98.3%	96.7%
	$\mathrm{LF}_{\log}$	3600	2,079,337	**5.0%**	54.5%	48.7%	3600	2,153,102	**5.0%**	56.4%	49.8%	3600	2,487,103	**11.2%**	98.6%	84.6%
	CEF	3600	221,990	10.7%	25.6%	24.8%	3600	275,594	15.5%	25.0%	25.7%	3600	130,787	40.1%	63.7%	61.1%
	$\mathrm{CEF}_{\log}$	3601	433,421	8.1%	34.8%	34.7%	3600	394,433	7.9%	36.6%	36.5%	3600	368,512	20.1%	76.0%	74.6%
200,20	LF	3600	118,471	87.8%	88.6%	88.6%	3600	107,640	77.4%	78.1%	77.9%	3600	184,061	97.5%	99.2%	98.5%
	$\mathrm{LF}_{\log}$	3600	612,063	41.7%	56.8%	54.5%	3600	490,278	37.7%	58.0%	54.7%	3600	519,981	**58.2%**	99.3%	89.9%
	CEF	3600	20,677	**30.9%**	30.9%	33.3%	3600	22,387	**30.0%**	23.0%	32.7%	3600	4559	76.4%	76.0%	75.8%
	$\mathrm{CEF}_{\log}$	3600	131,182	39.6%	40.1%	40.1%	3600	88,037	36.6%	40.0%	40.0%	3600	285,525	64.6%	83.6%	83.3%
500,50	LF	3600	1232	90.2%	89.4%	89.4%	3600	369	82.3%	78.3%	78.3%	3600	6619	99.5%	99.8%	99.5%
	$\mathrm{LF}_{\log}$	3600	81,055	48.7%	49.0%	48.9%	3600	60,815	48.7%	47.2%	47.1%	3600	139,697	**87.0%**	99.9%	96.1%
	CEF	3603	17	**42.8%**	20.4%	41.3%	3604	26	**41.8%**	5.7%	37.3%	3603	1	93.4%	80.5%	90.9%
	$\mathrm{CEF}_{\log}$	3600	34,703	53.2%	45.2%	45.1%	3600	26,390	42.8%	40.7%	40.7%	3600	82,878	91.0%	90.8%	90.8%

Table 10 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
1000,100	LF	3601	†	91.0%	89.5%	89.5%	3600	21	81.6%	79.9%	79.9%	3601	4	99.9%	99.9%	99.8%
	$\mathrm{LF}_{\mathrm{log}}$	3600	52,994	50.3%	48.7%	48.7%	3600	41,825	50.1%	50.9%	50.8%	3600	48,644	96.6%	99.9%	97.3%
	CEF	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	$\mathrm{CEF}_{\mathrm{log}}$	3600	12,062	**46.0%**	45.3%	45.3%	3601	8843	**47.9%**	44.7%	45.3%	3600	37,767	**93.7%**	93.3%	93.3%
2000,100	$\mathrm{LF}_{\mathrm{log}}$	3600	41,092	50.7%	51.2%	51.2%	3600	30,062	50.6%	50.8%	50.7%	3600	35,408	97.8%	100.0%	98.2%
	$\mathrm{CEF}_{\mathrm{log}}$	3601	5139	**48.8%**	47.9%	48.4%	3600	4909	**48.5%**	44.4%	45.2%	3600	25,840	**97.0%**	95.5%	95.6%
5000,100	$\mathrm{LF}_{\mathrm{log}}$	3600	18,499	67.9%	68.6%	68.6%	3600	34,661	65.0%	69.5%	69.5%	3601	13,907	98.8%	100.0%	98.8%
	$\mathrm{CEF}_{\mathrm{log}}$	3600	5092	**51.4%**	46.4%	47.1%	3600	3305	**48.0%**	44.6%	45.6%	3600	11,678	**97.0%**	96.7%	96.7%
10000,100	$\mathrm{LF}_{\mathrm{log}}$	3600	15,052	68.6%	69.0%	69.0%	3600	11,855	68.2%	69.2%	69.2%	3601	2471	**99.4%**	100.0%	99.3%
	$\mathrm{CEF}_{\mathrm{log}}$	3600	1873	**50.5%**	47.2%	47.7%	3600	1010	**48.2%**	44.3%	45.1%	3601	475	**99.4%**	98.0%	99.0%

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (Time) in seconds, number of nodes (Nodes) processed, end gap (End-gap), continuous relaxation gap (Rlx-gap), the root-node gap (Ron-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

Table 11 The size of selected formulations versus their binary-expansion versions for the assortment data set [33]

n, m	κ	$10\% \cdot n$				$20\% \cdot n$				Unconstrained			
	Ref.	C-var	B-var	L-const	C-const	C-var	B-var	L-const	C-const	C-var	B-var	L-const	C-const
25,2	LF	52	25	203	–	52	25	203	–	52	25	202	–
	LF_{log}	12	35	35	–	14	37	41	–	17	40	50	–
	CEF	56	25	207	52	56	25	207	52	56	25	206	52
	CEF_{log}	16	45	49	12	18	49	57	14	21	55	65	17
50,5	LF	255	50	1006	–	255	50	1006	–	255	50	1005	–
	LF_{log}	35	80	101	–	40	85	116	–	47	92	135	–
	CEF	265	50	1016	255	265	50	1016	255	265	50	1015	255
	CEF_{log}	49	114	152	39	53	123	168	43	57	133	183	47
100,10	LF	1010	100	4011	–	1010	100	4011	–	1010	100	4010	–
	LF_{log}	80	170	231	–	90	180	261	–	104	194	302	–
	CEF	1030	100	4031	1010	1030	100	4031	1010	1030	100	4030	1010
	CEF_{log}	110	250	351	90	114	264	367	94	124	286	406	104
200,20	LF	4020	200	16,021	–	4020	200	16,021	–	4020	200	16,020	–
	LF_{log}	180	360	521	–	200	380	581	–	226	406	657	–
	CEF	4060	200	16,061	4020	4060	200	16,061	4020	4060	200	16,060	4020
	CEF_{log}	240	540	781	200	244	564	797	204	264	608	876	224
500,50	LF	25,050	500	100,051	–	25,050	500	100,051	–	25,050	500	100,050	–
	LF_{log}	500	950	1451	–	550	1000	1601	–	650	1100	1900	–
	CEF	25,150	500	100,151	25,050	25,150	500	100,151	25,050	25,150	500	100,150	25,050
	CEF_{log}	650	1450	2151	550	700	1550	2351	600	750	1700	2550	650

Table 11 continued

n, m	κ	$10\% \cdot n$				$20\% \cdot n$				Unconstrained			
	Ref.	C-var	B-var	L-const	C-const	C-var	B-var	L-const	C-const	C-var	B-var	L-const	C-const
1000,100	LF	100,100	1000	400,101	–	100,100	1000	400,101	–	100,100	1000	400,100	–
	$\mathtt{LF}_{\mathtt{log}}$	1100	2000	3201	–	1200	2100	3501	–	1400	2300	4100	–
	CEF	100,300	1000	400,301	100,100	100,300	1000	400,301	100,100	100,300	1000	400,300	100,100
	$\mathtt{CEF}_{\mathtt{log}}$	1400	3100	4701	1200	1500	3300	5101	1300	1600	3600	5500	1400
2000,100	$\mathtt{LF}_{\mathtt{log}}$	1200	3100	3501	–	1300	3200	3801	–	1500	3400	4400	–
	$\mathtt{CEF}_{\mathtt{log}}$	1500	4300	5101	1300	1600	4500	5501	1400	1700	4800	5900	1500
5000,100	$\mathtt{LF}_{\mathtt{log}}$	1400	6300	4101	–	1500	6400	4401	–	1600	6500	4700	–
	$\mathtt{CEF}_{\mathtt{log}}$	1600	7600	5501	1400	1700	7800	5901	1500	1800	8000	6300	1600
10000,100	$\mathtt{LF}_{\mathtt{log}}$	1500	11,400	4401	–	1600	11,500	4701	–	1700	11,600	5000	–
	$\mathtt{CEF}_{\mathtt{log}}$	1700	12,800	5901	1500	1800	13,000	6301	1600	1900	13,200	6700	1700

In each row, we present averages over five instances for: the number of continuous (C-var) and binary (B-var) variables as well as the number of linear (L-const) and rotated cone (C-const) constraints

Table 12 The size of selected formulations versus their binary-expansion versions for the uniformly generated data set [9]

n, m	κ	$10\% \cdot n$				$20\% \cdot n$				Unconstrained			
	Ref.	C-var	B-var	L-const	C-const	C-var	B-var	L-const	C-const	C-var	B-var	L-const	C-const
25,2	LF	52	25	203	–	52	25	203	–	52	25	202	–
	$\mathrm{LF}_{\mathrm{log}}$	14	37	41	–	16	39	47	–	19	42	54	–
	CEF	56	25	207	52	56	25	207	52	56	25	206	52
	$\mathrm{CEF}_{\mathrm{log}}$	18	49	55	14	20	53	63	16	23	59	73	19
50,5	LF	255	50	1006	–	255	50	1006	–	255	50	1005	–
	$\mathrm{LF}_{\mathrm{log}}$	40	85	116	–	45	90	131	–	51	96	149	–
	CEF	265	50	1016	255	265	50	1016	255	265	50	1015	255
	$\mathrm{CEF}_{\mathrm{log}}$	50	120	156	40	55	130	176	45	63	144	207	53
100,10	LF	1010	100	4011	–	1010	100	4011	–	1010	100	4010	–
	$\mathrm{LF}_{\mathrm{log}}$	90	180	261	–	100	190	291	–	113	203	329	–
	CEF	1030	100	4031	1010	1030	100	4031	1010	1030	100	4030	1010
	$\mathrm{CEF}_{\mathrm{log}}$	110	260	351	90	120	280	391	100	132	306	438	112
200,20	LF	4020	200	16,021	–	4020	200	16,021	–	4020	200	16,020	–
	$\mathrm{LF}_{\mathrm{log}}$	200	380	581	–	220	400	641	–	244	424	713	–
	CEF	4060	200	16,061	4020	4060	200	16,061	4020	4060	200	16,060	4020
	$\mathrm{CEF}_{\mathrm{log}}$	240	560	781	200	260	600	861	220	288	656	972	248
500,50	LF	25,050	500	100,051	–	25,050	500	100,051	–	25,050	500	100,050	–
	$\mathrm{LF}_{\mathrm{log}}$	550	1000	1601	–	600	1050	1751	–	700	1150	2050	–
	CEF	25,150	500	100,151	25,050	25,150	500	100,151	25,050	25,150	500	100,150	25,050
	$\mathrm{CEF}_{\mathrm{log}}$	650	1500	2151	550	700	1600	2351	600	800	1800	2750	700

Table 12 continued

n, m	κ	$10\% \cdot n$				$20\% \cdot n$				Unconstrained			
	Ref.	C-var	B-var	L-const	C-const	C-var	B-var	L-const	C-const	C-var	B-var	L-const	C-const
1000,100	LF	100,100	1000	400,101	–	100,100	1000	400,101	–	100,100	1000	400,100	–
	$\mathrm{LF}_{\mathrm{log}}$	1200	2100	3501	–	1300	2200	3801	–	1500	2400	4400	–
	CEF	100,300	1000	400,301	100,100	100,300	1000	400,301	100,100	100,300	1000	400,300	100,100
	$\mathrm{CEF}_{\mathrm{log}}$	1400	3200	4701	1200	1500	3400	5101	1300	1700	3800	5900	1500
2000,100	$\mathrm{LF}_{\mathrm{log}}$	1300	3200	3801	–	1400	3300	4101	–	1600	3500	4700	–
	$\mathrm{CEF}_{\mathrm{log}}$	1500	4400	5101	1300	1600	4600	5501	1400	1800	5000	6300	1600
5000,100	$\mathrm{LF}_{\mathrm{log}}$	1500	6400	4401	–	1600	6500	4701	–	1700	6600	5000	–
	$\mathrm{CEF}_{\mathrm{log}}$	1700	7800	5901	1500	1800	8000	6301	1600	1900	8200	6700	1700
10000,100	$\mathrm{LF}_{\mathrm{log}}$	1600	11,500	4701	–	1700	11,600	5001	–	1800	11,700	5300	–
	$\mathrm{CEF}_{\mathrm{log}}$	1800	13,000	6301	1600	1900	13,200	6701	1700	2000	13,400	7100	1800

In each row, we present averages over five instances for: the number of continuous (C-var) and binary (B-var) variables as well as the number of linear (L-const) and rotated cone (C-const) constraints

Table 13 Computational results to evaluate the impact of the polymatroid cuts on the basic formulations for the assortment data set [33]

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
25,2	LF	**0**	30	0.0%	14.1%	4.7%	2	1395	0.0%	42.6%	24.5%	2	1538	0.0%	35.0%	12.1%
	LF^{P}	**0**	0	0.0%	0.4%	0.3%	**0**	1	0.0%	0.9%	0.7%	**0**	4	0.0%	1.8%	0.0%
	LEF	**0**	2	0.0%	0.7%	0.7%	**0**	9	0.0%	0.9%	0.8%	**0**	0	0.0%	0.0%	0.0%
	$\mathrm{LEF}^{\mathrm{P}}$	**0**	0	0.0%	0.2%	0.1%	**0**	1	0.0%	0.4%	0.2%	**0**	0	0.0%	0.0%	0.0%
	CF	1	13	0.0%	1.5%	1.5%	1	70	0.0%	2.8%	2.8%	2	1278	0.0%	8.2%	8.2%
	CF^{P}	**0**	0	0.0%	1.5%	0.8%	**0**	0	0.0%	2.8%	1.9%	1	0	0.0%	8.2%	1.2%
	CEF	**0**	0	0.0%	0.0%	0.0%	**0**	1	0.0%	0.1%	0.1%	**0**	0	0.0%	0.0%	0.0%
	$\mathrm{CEF}^{\mathrm{P}}$	**0**	0	0.0%	0.0%	0.0%	**0**	1	0.0%	0.1%	0.1%	**0**	0	0.0%	0.0%	0.0%
50,5	LF	55	40,100	0.0%	51.7%	41.3%	3600	935,667	29.4%	60.1%	49.8%	3600	1,495,669	19.3%	52.9%	41.4%
	LF^{P}	**0**	1	0.0%	0.9%	0.1%	**0**	17	0.0%	2.3%	0.1%	4	3492	0.0%	5.9%	0.1%
	LEF	**0**	123	0.0%	3.0%	3.0%	1	1111	0.0%	3.2%	3.2%	**0**	2	0.0%	0.0%	0.0%
	$\mathrm{LEF}^{\mathrm{P}}$	**0**	1	0.0%	0.7%	0.0%	**0**	10	0.0%	1.8%	0.2%	**0**	2	0.0%	0.0%	0.0%
	CF	2	157	0.0%	2.3%	2.2%	7	3016	0.0%	5.5%	5.5%	2916	802,844	1.0%	13.4%	13.4%
	CF^{P}	1	0	0.0%	2.3%	0.1%	2	0	0.0%	5.5%	0.0%	4	4	0.0%	13.4%	0.9%
	CEF	1	1	0.0%	0.0%	0.0%	1	7	0.0%	0.2%	0.2%	1	2	0.0%	0.0%	0.0%
	$\mathrm{CEF}^{\mathrm{P}}$	1	0	0.0%	0.0%	0.0%	1	8	0.0%	0.2%	0.2%	**0**	2	0.0%	0.0%	0.0%
100,10	LF	3600	170,481	70.1%	78.3%	77.7%	3600	473,158	61.7%	72.1%	70.2%	3600	730,892	51.5%	67.5%	64.2%
	LF^{P}	4	152	0.0%	1.9%	0.1%	10	1761	0.0%	4.4%	0.1%	2884	410,429	1.1%	12.0%	1.2%
	LEF	3357	345,641	1.6%	8.3%	8.3%	2190	361,599	0.2%	5.0%	5.0%	**1**	35	0.0%	0.1%	0.0%
	$\mathrm{LEF}^{\mathrm{P}}$	**2**	55	0.0%	1.8%	0.1%	6	792	0.0%	3.5%	1.9%	**1**	28	0.0%	0.1%	0.0%
	CF	67	7786	0.0%	3.8%	3.6%	3371	231,729	3.5%	7.8%	8.6%	3600	94,567	11.9%	17.8%	17.9%
	CF^{P}	10	0	0.0%	3.8%	0.0%	20	0	0.0%	7.8%	0.0%	25	370	0.0%	17.8%	0.1%
	CEF	6	14	0.0%	0.7%	0.2%	**4**	23	0.0%	0.6%	0.2%	6	17	0.0%	0.3%	0.0%
	$\mathrm{CEF}^{\mathrm{P}}$	6	14	0.0%	1.6%	0.2%	**4**	25	0.0%	0.4%	0.2%	5	16	0.0%	0.3%	0.0%

Table 13 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
200,20	LF	3600	28,528	82.3%	84.4%	84.4%	3600	47,569	77.0%	79.4%	79.3%	3600	49,188	73.4%	78.3%	77.9%
	LF^{P}	1755	27,150	0.0%	4.1%	0.2%	3600	87,785	0.3%	10.9%	0.4%	3600	37,866	4.5%	22.1%	4.4%
	LEF	3600	42,413	8.6%	10.6%	10.6%	3600	129,049	1.1%	3.0%	3.0%	**29**	1327	0.0%	0.3%	0.3%
	LEF^{P}	1236	22,636	0.2%	3.8%	1.1%	3524	152,434	1.0%	3.0%	2.9%	31	1309	0.0%	0.3%	0.3%
	CF	3600	55,709	32.9%	5.6%	36.0%	3600	29,821	62.3%	13.1%	63.7%	3600	14,939	65.2%	23.6%	73.1%
	CF^{P}	**27**	5	0.0%	5.6%	0.0%	64	131	0.0%	13.1%	0.0%	1562	26,768	0.2%	23.6%	0.5%
	CEF	73	190	0.0%	1.5%	0.4%	40	31	0.0%	1.5%	0.1%	59	332	0.0%	1.1%	0.1%
	CEF^{P}	61	230	0.0%	1.6%	0.4%	**38**	37	0.0%	1.3%	0.1%	69	407	0.0%	2.7%	0.1%
500,50	LF	3600	1620	90.3%	89.0%	89.0%	3600	2097	86.7%	86.2%	86.2%	3601	5755	86.2%	86.4%	86.4%
	LF^{P}	3600	†	†	†	†	3600	†	13.1%	24.8%	12.0%	3600	†	100.0%	33.4%	100.0%
	LEF	3600	2548	8.3%	8.7%	8.7%	**2520**	8118	0.2%	0.8%	0.2%	3501	12,727	**0.4%**	1.8%	0.4%
	LEF^{P}	3600	884	6.9%	7.2%	7.2%	2554	9193	0.2%	0.8%	0.2%	3552	13,582	**0.4%**	1.8%	0.4%
	CF	3600	11,986	96.4%	8.9%	100.0%	3600	11,367	100.0%	22.2%	100.0%	3600	4110	96.1%	26.7%	100.0%
	CF^{P}	**1194**	311	0.0%	8.9%	0.1%	3452	707	0.3%	22.2%	0.7%	3600	440	7.7%	26.7%	0.5%
	CEF	3611	779	0.2%	32.6%	0.3%	2620	842	0.0%	38.4%	0.5%	3604	272	0.5%	23.9%	0.7%
	CEF^{P}	3609	534	0.2%	29.0%	0.3%	2778	648	0.0%	42.9%	0.5%	3613	160	0.6%	30.9%	0.7%
1000,100	LF	3601	3	99.2%	93.1%	93.1%	3600	10	99.0%	90.4%	90.4%	3601	33	99.0%	90.5%	90.5%
	LF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	LEF	3600	215	**13.9%**	4.4%	4.4%	3722	488	**0.9%**	1.2%	0.0%	3600	351	**1.7%**	1.7%	1.1%
	LEF^{P}	3600	40	81.0%	4.4%	4.4%	3600	1	**0.9%**	1.2%	0.0%	3600	3	1.8%	1.7%	1.1%
	CF	3600	4962	100.0%	15.7%	100.0%	3605	4612	100.0%	29.7%	100.0%	3600	1882	100.0%	30.2%	100.0%
	CF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	CEF	3605	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	CEF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†

Table 13 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
2000,100	LEF	3601	†	†	†	†	3600	†	†	†	†	3601	†	†	†	†
	LEF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
5000,100	LEF	7807	†	†	†	†	8155	†	†	†	†	7241	†	†	†	†
	LEF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
10000,100	LEF	4225	†	†	†	†	4026	†	†	†	†	3603	†	†	†	†
	LEF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (Time) in seconds, number of nodes (Nodes) processed, end gap (End-gap), continuous relaxation gap (Rlx-gap), and root-node gap (Ron-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

Table 14 Computational results to evaluate the impact of the polymatroid cuts on the basic formulations for the uniformly generated data set [9]

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
25,2	LF	**0**	38	0.0%	81.5%	48.1%	1	433	0.0%	70.4%	41.5%	1	497	0.0%	89.7%	46.9%
	LF^P	1	22	0.0%	33.6%	10.2%	2	340	0.0%	46.9%	12.6%	1	275	0.0%	44.7%	12.0%
	LEF	**0**	1	0.0%	6.8%	5.4%	**0**	24	0.0%	10.8%	9.2%	**0**	21	0.0%	28.1%	22.7%
	LEF^P	**0**	2	0.0%	6.7%	4.7%	**0**	20	0.0%	10.8%	9.0%	1	17	0.0%	26.6%	15.5%
	CF	1	114	0.0%	31.4%	24.8%	1	1198	0.0%	40.5%	40.5%	1	1657	0.0%	37.1%	37.1%
	CF^P	3	2	0.0%	31.4%	4.6%	4	179	0.0%	40.5%	6.3%	4	24	0.0%	37.1%	3.4%
	CEF	**0**	2	0.0%	4.0%	3.8%	**0**	14	0.0%	7.9%	7.0%	1	16	0.0%	18.6%	13.2%
	CEF^P	**0**	2	0.0%	4.0%	2.8%	**0**	14	0.0%	7.9%	7.0%	2	16	0.0%	18.6%	5.9%
50,5	LF	1554	589,909	0.0%	85.7%	82.8%	3600	2,006,223	46.9%	75.2%	69.9%	3600	2,791,336	44.9%	96.0%	91.5%
	LF^P	437	126,715	0.0%	55.9%	27.2%	3600	1,542,663	20.2%	61.2%	35.5%	2530	826,742	4.2%	59.6%	26.5%
	LEF	2	381	0.0%	18.5%	16.4%	13	9831	0.0%	20.9%	19.9%	**43**	35,334	0.0%	56.3%	49.6%
	LEF^P	**1**	327	0.0%	18.5%	16.2%	**10**	10,679	0.0%	20.9%	19.9%	65	34,778	0.0%	50.1%	28.3%
	CF	75	31,692	0.0%	56.2%	56.2%	3606	1,112,962	6.0%	56.3%	56.3%	2594	677,850	7.3%	56.6%	56.6%
	CF^P	78	22,068	0.0%	56.2%	23.9%	3601	1,058,360	6.5%	56.3%	25.4%	2903	1,043,778	3.0%	56.6%	23.9%
	CEF	3	172	0.0%	14.6%	13.7%	18	6,093	0.0%	15.6%	15.3%	100	35,410	0.0%	43.6%	40.9%
	CEF^P	2	162	0.0%	14.6%	13.7%	17	6266	0.0%	15.6%	15.3%	311	32,574	0.0%	40.6%	24.1%
100,10	LF	3600	1,100,713	82.1%	87.5%	87.1%	3600	1,252,405	71.1%	77.1%	76.0%	3600	1,308,606	90.2%	98.3%	96.7%
	LF^P	3600	303,009	51.7%	74.3%	56.5%	3600	337,167	51.4%	73.5%	54.7%	3600	153,764	42.5%	73.9%	49.7%
	LEF	3600	480,988	12.3%	29.4%	27.5%	3600	616,551	17.1%	30.4%	29.7%	3600	654,126	**38.5%**	72.5%	66.6%
	LEF^P	3600	526,364	12.3%	29.4%	27.4%	3600	687,839	17.0%	30.4%	29.7%	3600	438,734	37.2%	66.9%	51.5%
	CF	3600	223,083	52.3%	72.8%	72.8%	3600	154,439	54.9%	66.9%	66.9%	3600	220,110	53.2%	71.5%	71.4%
	CF^P	3600	462,737	43.5%	72.8%	50.8%	3600	166,635	44.3%	66.9%	47.3%	3600	330,256	42.0%	71.5%	50.4%
	CEF	3600	221,990	**10.7%**	25.6%	24.8%	3600	275,594	15.5%	25.0%	25.7%	3600	130,787	40.1%	63.7%	61.1%
	CEF^P	3601	204,084	**10.7%**	25.7%	24.8%	3600	260,464	**15.4%**	25.2%	25.7%	3600	132,248	40.3%	60.5%	50.8%

Table 14 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
200,20	LF	3600	118,471	87.8%	88.6%	88.6%	3600	107,640	77.4%	78.1%	77.9%	3600	184,061	97.5%	99.2%	98.5%
	LF^P	3600	17,965	72.1%	82.0%	71.2%	3600	18,122	65.5%	77.0%	65.0%	3600	9401	73.1%	81.8%	70.4%
	LEF	3600	47,486	**30.0%**	36.2%	35.4%	3600	58,945	31.1%	36.0%	35.6%	3600	63,610	70.6%	83.0%	78.8%
	LEF^P	3600	74,821	29.3%	36.2%	35.3%	3600	100,973	30.6%	36.0%	35.6%	3600	37,575	65.4%	77.8%	67.3%
	CF	3600	27,323	99.5%	80.4%	99.5%	3600	17,547	89.6%	72.0%	88.7%	3600	24,168	99.7%	81.0%	100.0%
	CF^P	3600	25,375	65.8%	80.4%	66.0%	3600	1113	61.6%	72.0%	58.9%	3600	7872	**70.9%**	81.0%	66.3%
	CEF	3600	20,677	30.9%	30.9%	33.3%	3600	22,387	**30.0%**	23.0%	32.7%	3600	4559	76.4%	76.0%	75.8%
	CEF^P	3600	17,843	30.8%	32.3%	33.3%	3600	19,082	30.1%	25.5%	32.7%	3601	5142	71.5%	72.7%	67.8%
500,50	LF	3600	1232	90.2%	89.4%	89.4%	3600	369	82.3%	78.3%	78.3%	3600	6619	99.5%	99.8%	99.5%
	LF^P	3600	606	88.4%	87.1%	87.1%	3600	395	81.5%	78.0%	77.4%	3600	†	†	†	†
	LEF	3600	636	42.8%	43.0%	42.7%	3600	1324	41.1%	38.6%	38.5%	3600	113	90.3%	91.4%	89.3%
	LEF^P	3600	850	**42.5%**	43.0%	42.7%	3600	1616	**41.0%**	38.6%	38.5%	3600	39	**83.1%**	88.0%	82.8%
	CF	3600	9997	100.0%	86.1%	100.0%	3602	3292	100.0%	75.0%	100.0%	3600	13,213	100.0%	89.8%	100.0%
	CF^P	3600	†	†	†	†	3600	†	†	†	†	3600	1208	84.9%	89.8%	83.1%
	CEF	3603	17	42.8%	20.4%	41.3%	3604	26	41.8%	5.7%	37.3%	3603	1	93.4%	80.5%	90.9%
	CEF^P	3603	7	42.9%	23.3%	41.3%	3604	11	53.6%	7.2%	37.3%	3600	†	†	†	†
1000,100	LF	3601	†	†	†	†	3600	21	**81.6%**	79.9%	79.9%	3601	4	99.9%	99.9%	99.8%
	LF^P	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	LEF	3601	†	†	†	†	3601	†	†	†	†	3601	†	†	†	†
	LEF^P	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	CF	3600	911	**100.0%**	87.3%	100.0%	3600	1210	100.0%	77.8%	100.0%	3600	663	100.0%	92.8%	100.0%
	CF^P	3600	†	†	†	†	3600	†	†	†	†	3600	406	**95.6%**	83.4%	95.5%
	CEF	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	CEF^P	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†

Table 14 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
2000,100	LEF	3601	†	†	†	†	3602	†	†	†	†	3601	†	†	†	†
	LEF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
5000,100	LEF	4755	†	†	†	†	3938	†	†	†	†	3603	†	†	†	†
	LEF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
10000,100	LEF	9500	†	†	†	†	6022	†	†	†	†	5619	†	†	†	†
	LEF^{P}	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (`Time`) in seconds, number of nodes (`Nodes`) processed, end gap (`End-gap`), continuous relaxation gap (`Rlx-gap`), and root-node gap (`Ron-gap`). For each choice of n, m, and κ, among the solution methods, the best average time and the best average `End-gap` (if `Time`$\geqslant$ 3600 s) are in **bold**

Table 15 Computational results to evaluate the combined effect of binarization and polymatroid cuts on the performance of selected basic MILP and MICQP for the assortment data set [33]

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
25,2	LF	**0**	30	0.0%	14.1%	4.7%	2	1395	0.0%	42.6%	24.5%	2	1538	0.0%	35.0%	12.1%
	LF^{P}	**0**	0	0.0%	0.4%	0.3%	**0**	1	0.0%	0.9%	0.7%	**0**	4	0.0%	1.8%	0.0%
	$\mathrm{LF}_{\mathrm{log}}$	**0**	5	0.0%	15.3%	1.1%	**0**	27	0.0%	28.1%	4.3%	1	386	0.0%	56.1%	18.1%
	$\mathrm{LF}^{\mathrm{P}}_{\mathrm{log}}$	**0**	0	0.0%	0.5%	0.3%	**0**	1	0.0%	0.9%	0.7%	**0**	20	0.0%	1.8%	0.1%
	CEF	**0**	0	0.0%	0.0%	0.0%	**0**	1	0.0%	0.1%	0.1%	**0**	0	0.0%	0.0%	0.0%
	$\mathrm{CEF}^{\mathrm{P}}$	**0**	0	0.0%	0.0%	0.0%	**0**	1	0.0%	0.1%	0.1%	**0**	0	0.0%	0.0%	0.0%
	$\mathrm{CEF}_{\mathrm{log}}$	1	1	0.0%	0.1%	0.1%	1	3	0.0%	0.7%	0.3%	1	316	0.0%	5.8%	5.8%
	$\mathrm{CEF}^{\mathrm{P}}_{\mathrm{log}}$	1	1	0.0%	0.1%	0.1%	**0**	0	0.0%	0.6%	0.1%	2	379	0.0%	1.8%	1.5%
50,5	LF	55	40,100	0.0%	51.7%	41.3%	3600	935,667	29.4%	60.1%	49.8%	3600	1,495,669	19.3%	52.9%	41.4%
	LF^{P}	**0**	1	0.0%	0.9%	0.1%	**0**	17	0.0%	2.3%	0.1%	4	3492	0.0%	5.9%	0.1%
	$\mathrm{LF}_{\mathrm{log}}$	1	233	0.0%	30.0%	5.7%	2	2109	0.0%	44.4%	14.5%	18	35,496	0.0%	65.6%	28.2%
	$\mathrm{LF}^{\mathrm{P}}_{\mathrm{log}}$	**0**	0	0.0%	0.9%	0.3%	1	19	0.0%	2.4%	0.1%	6	6721	0.0%	5.9%	0.3%
	CEF	1	1	0.0%	0.0%	0.0%	1	7	0.0%	0.2%	0.2%	1	2	0.0%	0.0%	0.0%
	$\mathrm{CEF}^{\mathrm{P}}$	1	0	0.0%	0.0%	0.0%	1	8	0.0%	0.2%	0.2%	**0**	2	0.0%	0.0%	0.0%
	$\mathrm{CEF}_{\mathrm{log}}$	1	6	0.0%	0.6%	0.3%	2	371	0.0%	2.1%	2.0%	18	13,917	0.0%	12.4%	12.3%
	$\mathrm{CEF}^{\mathrm{P}}_{\mathrm{log}}$	**0**	4	0.0%	0.5%	0.3%	2	250	0.0%	1.9%	0.9%	21	11,132	0.0%	5.9%	5.2%

Table 15 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
100,10	LF	3600	170,481	70.1%	78.3%	77.7%	3600	473,158	61.7%	72.1%	70.2%	3600	730,892	51.5%	67.5%	64.2%
	LF^P	4	152	0.0%	1.9%	0.1%	10	1761	0.0%	4.4%	0.1%	2884	410,429	1.1%	12.0%	1.2%
	LF_{log}	979	364,141	0.0%	42.7%	14.5%	3155	1,732,777	0.4%	55.3%	23.7%	3600	1,543,428	1.6%	75.9%	38.5%
	LF^P_{log}	**1**	86	0.0%	1.9%	0.1%	6	2434	0.0%	4.4%	0.1%	3600	1,535,465	0.8%	12.1%	1.5%
	CEF	6	14	0.0%	0.7%	0.2%	**4**	23	0.0%	0.6%	0.2%	6	17	0.0%	0.3%	0.0%
	CEF^P	6	14	0.0%	1.6%	0.2%	**4**	25	0.0%	0.4%	0.2%	**5**	16	0.0%	0.3%	0.0%
	CEF_{log}	10	1457	0.0%	1.3%	1.2%	212	27,571	0.0%	4.0%	3.9%	3465	292,906	1.0%	20.2%	20.2%
	CEF^P_{log}	2	215	0.0%	1.2%	0.3%	22	3199	0.0%	3.7%	2.1%	3600	411,139	0.3%	12.1%	8.4%
200,20	LF	3600	28,528	82.3%	84.4%	84.4%	3600	47,569	77.0%	79.4%	79.3%	3600	49,188	73.4%	78.3%	77.9%
	LF^P	1755	27,150	0.0%	4.1%	0.2%	3600	87,785	0.3%	10.9%	0.4%	3600	37,866	4.5%	22.1%	4.4%
	LF_{log}	3600	549,079	6.7%	52.9%	24.7%	3600	383,827	8.7%	64.7%	31.9%	3600	300,111	24.1%	82.7%	49.9%
	LF^P_{log}	710	158,569	0.0%	4.1%	0.2%	3400	715,941	0.3%	10.9%	0.4%	3600	374,382	6.3%	22.3%	6.0%
	CEF	73	190	0.0%	1.5%	0.4%	40	31	0.0%	1.5%	0.1%	**59**	332	0.0%	1.1%	0.1%
	CEF^P	**61**	230	0.0%	1.6%	0.4%	**38**	37	0.0%	1.3%	0.1%	69	407	0.0%	2.7%	0.1%
	CEF_{log}	3600	137,672	0.9%	2.6%	2.6%	3600	121,652	3.8%	8.2%	8.2%	3600	96,988	22.0%	28.9%	29.0%
	CEF^P_{log}	2353	74,047	0.5%	2.6%	2.2%	3600	112,151	2.2%	8.2%	6.8%	3600	144,453	6.4%	22.2%	13.8%
500,50	LF	3600	1620	90.3%	89.0%	89.0%	3600	2097	86.7%	86.2%	86.2%	3601	5755	86.2%	86.4%	86.4%
	LF^P	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	LF_{log}	3600	102,004	39.8%	53.0%	35.4%	3600	84,392	54.0%	68.6%	35.0%	3600	92,414	55.7%	91.8%	74.7%
	LF^P_{log}	3600	110,452	0.8%	8.8%	0.3%	3600	57,797	3.3%	24.8%	1.3%	3600	65,850	15.2%	33.6%	13.4%
	CEF	3611	779	**0.2%**	32.6%	0.3%	**2620**	842	0.0%	38.4%	0.5%	3604	272	**0.5%**	23.9%	0.7%
	CEF^P	3609	534	**0.2%**	29.0%	0.3%	2778	648	0.0%	42.9%	0.5%	3613	160	0.6%	30.9%	0.7%
	CEF_{log}	3600	49,090	5.1%	5.4%	5.4%	3600	55,757	13.3%	15.8%	15.7%	3600	53,280	36.9%	37.3%	38.5%
	CEF^P_{log}	3600	55,687	4.7%	5.4%	5.0%	3600	63,450	12.2%	15.8%	14.0%	3601	129,520	26.1%	33.5%	20.6%

Table 15 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
1000,100	LF	3601	3	99.2%	93.1%	93.1%	3600	10	99.0%	90.4%	90.4%	3601	33	99.0%	90.5%	90.5%
	LF^P	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	LF_{log}	3600	55,776	55.9%	60.6%	51.0%	3600	58,847	62.7%	77.4%	61.2%	3600	55,641	76.5%	94.3%	79.1%
	LF^P_{log}	3601	†	†	†	†	3601	6378	**20.9%**	39.4%	15.8%	3601	30,129	**26.1%**	43.6%	23.9%
	CEF	3605	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	CEF^P	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	CEF_{log}	3600	36,151	25.3%	9.8%	9.9%	3600	36,647	23.7%	23.6%	23.9%	3600	35,213	48.7%	44.6%	45.7%
	CEF^P_{log}	3601	32,326	**10.0%**	9.9%	9.5%	3600	26,843	22.6%	23.9%	22.5%	3600	4283	33.8%	42.9%	25.0%
2000,100	LF_{log}	3600	58,217	57.8%	68.0%	62.7%	3600	56,546	70.5%	84.0%	79.1%	3600	39,585	78.3%	96.0%	81.6%
	LF^P_{log}	3601	†	†	†	†	3600	32,898	41.4%	48.9%	41.2%	3601	8660	**33.1%**	52.2%	31.2%
	CEF_{log}	3600	26,386	30.3%	15.5%	16.1%	3600	22,548	60.0%	31.9%	38.2%	3600	26,785	71.0%	50.7%	52.7%
	CEF^P_{log}	3600	38,716	**16.1%**	15.8%	15.4%	3600	28,575	**30.7%**	32.5%	31.9%	3600	931	53.4%	48.2%	33.0%
5000,100	LF_{log}	3600	23,558	78.1%	86.8%	84.7%	3600	37,298	80.6%	93.4%	93.4%	3601	12,870	83.5%	96.8%	96.8%
	LF^P_{log}	3601	7220	**29.2%**	50.1%	25.1%	3601	15,186	49.0%	59.9%	47.6%	3601	6818	**50.7%**	61.1%	49.3%
	CEF_{log}	3600	15,535	48.0%	26.7%	57.7%	3600	10,662	77.7%	39.3%	60.0%	3600	11,067	86.5%	57.2%	86.5%
	CEF^P_{log}	3600	13,966	39.3%	27.5%	30.7%	3600	13,736	**40.6%**	33.9%	40.1%	3600	3257	58.4%	50.8%	47.5%
10000,100	LF_{log}	3600	13,230	88.4%	90.0%	90.0%	3600	8857	83.1%	94.7%	94.7%	3602	5082	93.0%	97.6%	97.6%
	LF^P_{log}	3601	5481	55.4%	58.6%	54.7%	3601	9440	53.2%	61.4%	49.2%	3601	5482	**54.7%**	65.0%	54.3%
	CEF_{log}	3600	7551	53.8%	29.5%	52.2%	3600	3781	84.6%	45.0%	85.1%	3600	2786	95.4%	70.3%	95.0%
	CEF^P_{log}	3600	9979	**33.4%**	5.0%	34.9%	3601	7247	**45.4%**	22.0%	37.6%	3601	†	†	†	†

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (Time) in seconds, number of nodes (Nodes) processed, end gap (End-gap), continuous relaxation gap (Rlx-gap), and root-node gap (Ron-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

Table 16 Computational results to evaluate the combined effect of binarization and polymatroid cuts on the performance of selected basic MILP and MICQP formulations for the uniformly generated data set [9]

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
25,2	LF	**0**	38	0.0%	81.5%	48.1%	1	433	0.0%	70.4%	41.5%	**1**	497	0.0%	89.7%	46.9%
	LF^P	1	22	0.0%	33.6%	10.2%	2	340	0.0%	46.9%	12.6%	**1**	275	0.0%	44.7%	12.0%
	LF_{log}	**0**	23	0.0%	49.8%	28.0%	1	95	0.0%	50.1%	32.3%	**1**	199	0.0%	93.0%	59.8%
	LF^P_{log}	**0**	13	0.0%	29.5%	11.5%	1	222	0.0%	42.3%	12.0%	**1**	142	0.0%	45.5%	14.6%
	CEF	**0**	2	0.0%	4.0%	3.8%	**0**	14	0.0%	7.9%	7.0%	**1**	16	0.0%	18.6%	13.2%
	CEF^P	**0**	2	0.0%	4.0%	2.8%	**0**	14	0.0%	7.9%	7.0%	2	16	0.0%	18.6%	5.9%
	CEF_{log}	**0**	6	0.0%	10.3%	7.9%	1	89	0.0%	19.9%	19.3%	3	353	0.0%	45.7%	44.2%
	CEF^P_{log}	1	4	0.0%	10.3%	6.9%	1	106	0.0%	19.9%	18.7%	6	133	0.0%	42.8%	27.1%
50,5	LF	1554	589,909	0.0%	85.7%	82.8%	3600	2,006,223	46.9%	75.2%	69.9%	3600	2,791,336	44.9%	96.0%	91.5%
	LF^P	437	126,715	0.0%	55.9%	27.2%	3600	1,542,663	20.2%	61.2%	35.5%	2530	826,742	4.2%	59.6%	26.5%
	LF_{log}	3	3364	0.0%	50.7%	43.7%	20	21,061	0.0%	54.2%	45.0%	**52**	55,437	0.0%	96.9%	77.1%
	LF^P_{log}	9	5014	0.0%	45.9%	21.3%	27	29,157	0.0%	51.4%	30.0%	85	81,310	0.0%	60.3%	25.0%
	CEF	3	172	0.0%	14.6%	13.7%	18	6093	0.0%	15.6%	15.3%	100	35,410	0.0%	43.6%	40.9%
	CEF^P	**2**	162	0.0%	14.6%	13.7%	**17**	6266	0.0%	15.6%	15.3%	311	32,574	0.0%	40.6%	24.1%
	CEF_{log}	11	4746	0.0%	24.4%	24.0%	22	8046	0.0%	30.0%	29.6%	521	78,437	0.0%	64.4%	62.1%
	CEF^P_{log}	6	2477	0.0%	24.4%	22.6%	26	8630	0.0%	29.9%	29.1%	86	25,435	0.0%	59.4%	46.4%

Table 16 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
100,10	LF	3600	1,100,713	82.1%	87.5%	87.1%	3600	1,252,405	71.1%	77.1%	76.0%	3600	1,308,606	90.2%	98.3%	96.7%
	LF^P	3600	303,009	51.7%	74.3%	56.5%	3600	337,167	51.4%	73.5%	54.7%	3600	153,764	42.5%	73.9%	49.7%
	LF_{log}	3600	2,079,337	**5.0%**	54.5%	48.7%	3600	2,153,102	**5.0%**	56.4%	49.8%	3600	2,487,103	11.2%	98.6%	84.6%
	LF^P_{log}	3600	2,588,756	7.5%	54.1%	45.5%	3600	2,821,692	6.1%	56.3%	48.3%	3600	1,928,384	17.2%	74.5%	48.6%
	CEF	3600	221,990	10.7%	25.6%	24.8%	3600	275,594	15.5%	25.0%	25.7%	3600	130,787	40.1%	63.7%	61.1%
	CEF^P	3601	204,084	10.7%	25.7%	24.8%	3600	260,464	15.4%	25.2%	25.7%	3600	132,248	40.3%	60.5%	50.8%
	CEF_{log}	3601	433,421	8.1%	34.8%	34.7%	3600	394,433	7.9%	36.6%	36.5%	3600	368,512	20.1%	76.0%	74.6%
	CEF^P_{log}	3600	482,188	7.2%	34.8%	34.5%	3603	463,914	5.2%	36.6%	36.6%	3600	417,221	**10.9%**	73.6%	61.2%
200,20	LF	3600	118,471	87.8%	88.6%	88.6%	3600	107,640	77.4%	78.1%	77.9%	3600	184,061	97.5%	99.2%	98.5%
	LF^P	3600	17,965	72.1%	82.0%	71.2%	3600	18,122	65.5%	77.0%	65.0%	3600	9401	73.1%	81.8%	70.4%
	LF_{log}	3600	612,063	41.7%	56.8%	54.5%	3600	490,278	37.7%	58.0%	54.7%	3600	519,981	58.2%	99.3%	89.9%
	LF^P_{log}	3600	1,104,491	41.6%	56.7%	53.5%	3600	938,882	35.6%	57.9%	54.5%	3600	434,136	58.0%	82.1%	65.9%
	CEF	3600	20,677	30.9%	30.9%	33.3%	3600	22,387	**30.0%**	23.0%	32.7%	3600	4559	76.4%	76.0%	75.8%
	CEF^P	3600	17,843	**30.8%**	32.3%	33.3%	3600	19,082	30.1%	25.5%	32.7%	3601	5142	71.5%	72.7%	67.8%
	CEF_{log}	3600	131,182	39.6%	40.1%	40.1%	3600	88,037	36.6%	40.0%	40.0%	3600	285,525	64.6%	83.6%	83.3%
	CEF^P_{log}	3600	174,404	35.5%	40.1%	39.9%	3600	113,509	34.3%	40.0%	39.9%	3600	279,263	**54.4%**	81.4%	73.4%
500,50	LF	3600	1232	90.2%	89.4%	89.4%	3600	369	82.3%	78.3%	78.3%	3600	6619	99.5%	99.8%	99.5%
	LF^P	3600	606	88.4%	87.1%	87.1%	3600	395	81.5%	78.0%	77.4%	3600	†	†	†	†
	LF_{log}	3600	81,055	48.7%	49.0%	48.9%	3600	60,815	48.7%	47.2%	47.1%	3600	139,697	87.0%	99.9%	96.1%
	LF^P_{log}	3600	108,291	48.4%	49.0%	49.0%	3600	70,193	48.1%	47.2%	47.1%	3600	181,247	**82.9%**	90.5%	85.9%
	CEF	3603	17	**42.8%**	20.4%	41.3%	3604	26	**41.8%**	5.7%	37.3%	3603	1	93.4%	80.5%	90.9%
	CEF^P	3603	7	42.9%	23.3%	41.3%	3604	11	53.6%	7.2%	37.3%	3600	†	†	†	†
	CEF_{log}	3600	34,703	53.2%	45.2%	45.1%	3600	26,390	42.8%	40.7%	40.7%	3600	82,878	91.0%	90.8%	90.8%
	CEF^P_{log}	3600	29,818	46.3%	45.2%	44.9%	3600	24,696	43.1%	40.7%	40.7%	3600	23,870	86.7%	89.7%	86.6%

Table 16 continued

n, m	κ	$10\% \cdot n$					$20\% \cdot n$					Unconstrained				
	Ref.	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap	Time	Nodes	End-gap	Rlx-gap	Ron-gap
1000,100	LF	3601	†	†	†	†	3600	21	81.6%	79.9%	79.9%	3601	4	99.9%	99.9%	99.8%
	LF^P	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	LF_{log}	3600	52,994	50.3%	48.7%	48.7%	3600	41,825	50.1%	50.9%	50.8%	3600	48,644	96.6%	99.9%	97.3%
	LF^P_{log}	3600	48,719	50.2%	48.7%	48.7%	3600	24,225	50.2%	50.7%	50.8%	3600	108,734	**91.9%**	93.1%	92.0%
	CEF	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	CEF^P	3600	†	†	†	†	3600	†	†	†	†	3600	†	†	†	†
	CEF_{log}	3600	12,062	**46.0%**	45.3%	45.3%	3601	8843	47.9%	44.7%	45.3%	3600	37,767	93.7%	93.3%	93.3%
	CEF^P_{log}	3600	10,436	48.0%	45.3%	45.3%	3600	9445	**44.5%**	44.7%	45.0%	3600	476	92.2%	92.5%	90.2%
2000,100	LF_{log}	3600	41,092	50.7%	51.2%	51.2%	3600	30,062	50.6%	50.8%	50.7%	3600	35,408	97.8%	100.0%	98.2%
	LF^P_{log}	3600	15,925	50.8%	51.1%	51.2%	3600	14,228	50.7%	50.8%	50.8%	3600	69,565	**94.8%**	95.5%	95.1%
	CEF_{log}	3601	5,139	48.8%	47.9%	48.4%	3600	4909	48.5%	44.4%	45.2%	3600	25,840	97.0%	95.5%	95.6%
	CEF^P_{log}	3600	9576	**47.8%**	48.3%	48.2%	3600	7815	**44.6%**	45.1%	45.0%	3600	339	96.6%	95.1%	93.7%
5000,100	LF_{log}	3600	18,499	67.9%	68.6%	68.6%	3600	34,661	65.0%	69.5%	69.5%	3601	13,907	98.8%	100.0%	98.8%
	LF^P_{log}	3600	9434	68.8%	68.6%	68.6%	3600	12867	67.9%	69.5%	69.2%	3601	16,900	**96.9%**	96.7%	96.5%
	CEF_{log}	3600	5092	51.4%	46.4%	47.1%	3600	3305	48.0%	44.6%	45.6%	3600	11,678	97.0%	96.7%	96.7%
	CEF^P_{log}	3600	4295	**46.7%**	47.2%	47.0%	3601	3406	**45.2%**	45.7%	45.5%	3601	34	98.3%	96.4%	96.0%
10000,100	LF_{log}	3600	15,052	68.6%	69.0%	69.0%	3600	11,855	68.2%	69.2%	69.2%	3601	2471	99.4%	100.0%	99.3%
	LF^P_{log}	3601	5732	68.5%	69.0%	69.0%	3601	6058	68.4%	69.2%	68.8%	3601	6595	**97.8%**	98.0%	97.9%
	CEF_{log}	3600	1873	50.5%	47.2%	47.7%	3600	1010	48.2%	44.3%	45.1%	3601	475	99.4%	98.0%	99.0%
	CEF^P_{log}	3601	896	**47.5%**	47.9%	47.7%	3600	1165	**44.8%**	45.1%	45.0%	3600	†	†	†	†

For each combination of n, m, κ and each formulation, we present averages over five instances for: time (Time) in seconds, number of nodes (Nodes) processed, end gap (End-gap), continuous relaxation gap (Rlx-gap), and root-node gap (Ron-gap). For each choice of n, m, and κ, among the solution methods, the best average time and the best average End-gap (if Time$\geqslant$ 3600 s) are in **bold**

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