THE COMPLEX RELATIONSHIP BETWEEN CONCEPTUAL UNDERSTANDING AND PROCEDURAL FLUENCY IN DEVELOPMENTAL ALGEBRA IN COLLEGE

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In this study we use latent class and distractor analysis, and qualitative analysis of cognitive interviews, to investigate how student responses to conceptual items may reflect different patterns of algebraic conceptual understanding and procedural fluency. Our analysis reveals three groups of students, which we label "mostly random guessing", "some procedural fluency with key misconceptions", and "procedural fluency with emergent conceptual understanding". Student responses revealed high rates of misconceptions that stem from misuse or misunderstanding of procedures, whose prevalence correlates with higher levels of procedural fluency.

Keywords: algebra, developmental algebra, conceptual understanding, concept inventory

Elementary algebra and other developmental courses have consistently been identified as barriers to college persistence and degree progress (see e.g., Bailey, Jeong, & Cho, 2010). There is evidence that students struggle in these courses because they do not understand fundamental algebraic concepts (see e.g., Givvin, Stigler, & Thompson, 2011; Stigler, Givvin, & Thompson, 2010), and many research studies have documented the negative consequences of learning algebraic procedures without any connection to the underlying concepts (see e.g., Hiebert & Grouws, 2007). However, developmental mathematics classes currently focus heavily on recall and procedural skills without integrating reasoning and sense-making (Goldrick-Rab, 2007; Hammerman & Goldberg, 2003). This focus on procedural skills in isolation may increase the probability that students use procedures inappropriately because they lack understanding of when and why the procedures work (e.g., Givvin et al., 2011; Stigler et al., 2010). In this paper we explore student responses to conceptual questions at the end of an elementary algebra course in college. We combine quantitative analysis of responses (using latent class analysis and distractor analysis) with qualitative analysis of cognitive interviews to better understand different typologies of student reasoning around some basic concepts in algebra, and to better understand how conceptual understanding and procedural fluency may relate to one another in this context.

Theoretical Framework

In this paper we use Fishbein's (1994) typology of mathematics as a human activity as a framework for analyzing student responses. Fishbein outlines three basic components of mathematics as a human activity: 1) *the formal component* (which we call *conceptual understanding*), which consists of axioms, definitions, theorems and proofs, which need to be “invented or learned, organized, checked and used actively” by students; 2) *the algorithmic component* (which we call *procedural fluency*), which consists of skills used to solve mathematical problems in specific contexts and stems from algorithmic practice; and 3) *the intuition component*, which is an “apparently” self-evident mathematical statement that is accepted directly with the feeling that no justification is necessary.

In this study we use the term conceptual understanding to denote both a formal understanding of abstract concepts (e.g. axioms), but also of how, when, and why procedures can be used. This is in contrast to procedural fluency in standard problem contexts, in which a student may be able to quickly solve particular types of standard problems correctly but may not understand of how, why, or when these methods work. Using these definitions, no question is wholly conceptual or procedural, but falls on a spectrum. In this paper we explore how procedural skills and conceptual understanding may relate to one another and how student justifications of answer choices may exhibit intuition components (either correct or incorrect), as well as how these intuitions may relate to both the processes of developing procedural fluency as well as conceptual understanding.

Methods

This study focuses on student responses to the multiple choice questions on the Elementary Algebra Concept Inventory (EACI). For details on the development and validation of the EACI, see (Wladis, Offenholley, Licwinko, Dawes, & Lee, 2018). Here we focus on 698 students who took the inventory at the end of their elementary algebra class in 2016-2017 as well as 10 cognitive interviews that were conducted towards the end of the semester with these students; these were analyzed using grounded theory (Glaser & Strauss, 1967), although a full qualitative analysis is not presented here due to space constraints. The distribution of interviewees among the three classes was not significantly different from the whole quantitative sample. In this paper we used latent class analysis (LCA) of the nine binary scored (right/wrong) multiple-choice items on the inventory (e.g., Collins & Lanza, 2010).

Description of the classes

LCA revealed three distinct classes of students. Item response patterns, distractor analysis, and qualitative coding of cognitive interviews were then used to interpret the classes, and evidence was found among these different complementary approaches for these characterizations:

- C1 (27%): Answers to most items are indistinguishable from random guessing, likely due to low procedural/conceptual knowledge, low self-efficacy, and/or low motivation.
- C2 (28%): Some well-developed procedural skills but limited conceptual understanding.
- C3 (45%): Some well-developed procedural skills and emergent conceptual understanding.

Firstly, we consider the response patterns of students from each of the three classes. Student responses in C1 do not vary much from what would be expected for random guessing on four-option multiple choice items. C2 answers significantly worse than chance on questions 2 and 6 because of the presence of attractive distractors that likely tap into misconceptions related to the misuse of procedures. C2 and C3 are distinguished by improved performance on the items overall but different proportions of key misconceptions. Students who passed the class were most likely to be in class 3, then class 2, and least likely to be in class 1. An end-of-course standardized procedural test showed a similar outcome. To illustrate how different response patterns distinguish these three classes, we performed a distractor analysis and analyzed cognitive interviews for two exemplars: items 2 and 6, using the Bayes modal assignment to determine class membership.

Two example questions: illustrating different class response patterns

First we consider item 6, which shows an interesting pattern of responses:

6. A student is trying to simplify two different expressions:

i. $(x^2y^3)^2$

ii. $(x^2 + y^3)^2$

Which one of the following steps could the student perform to correctly simplify each expression?

a. For both expressions, the student can distribute the exponent.
b. The student can distribute the exponent in the first expression, but not in the second.
c. The student can distribute the exponent in the second expression, but not in the first.
d. The student cannot distribute the exponent in either expression.

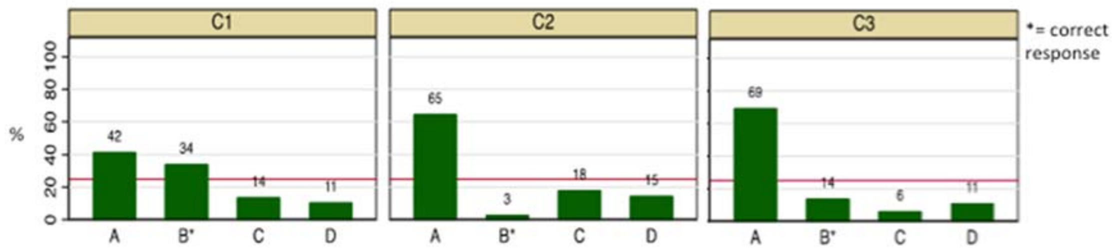

Figure 1. Item 6 distractor analysis

The correct answer is b. C2 and C3 were strongly attracted to option a (see Figure 1), likely because they have intuitions stemming from their experiences with procedures that use distributive properties, but they do not recognize the critical differences between distributing multiplication versus exponents—likely because they have no deeper conceptual understanding of how the distributive properties work. Selecting the correct answer is negatively correlated with scores on the procedural exam—students who selected the incorrect option a scored on average 7.1 percentage points higher on the procedural exam ($p < 0.000$) than others. Looking at student interview responses reinforces our interpretation of the three classes, and sheds light on how intuitions developed from procedural practice may impede conceptual understanding.

> **C1 (chose B):** [The difference between the first and second equation] is that there's a plus right there [pointing to the second equation]. I think for this one [pointing to the second equation], you have to add and for this one [pointing to the first equation] you don't….

> **C3 (chose A):** That's how you kind of get rid of the parenthesis and get rid of the outer exponents by distributing it in the inside. Whether it's with another exponent or with a number… You want to add or multiply that exponent [outside the parentheses] to the ones inside the parentheses but I can't remember whether you add or multiply…

Here the C1 student notices that there is a difference between the two equations and has an intuition that it is important, but doesn't actually know how to perform the distribution correctly. In contrast, none of the C2 or C3 students interviewed was able to describe when or why it is possible to distribute—they all cited different incorrect intuitions related to procedural methods.

Next we consider item 2, which reveals another interesting pattern of responses:

2. Consider the equation $x + y = 10$. Which of the following statements must be true?

a. There is only one possible solution to this equation, a single point on the line $x + y = 10$.
b. There are an infinite number of possible solutions, all points on the line $x + y = 10$
c. This equation has no solution.
d. There are exactly two possible solutions to this equation: one for x and one for y.

For this question, the correct answer is b, (the most popular choice for students in classes 1 and 3) but no examinee in class 2 chose it (see Figure 2). They were strongly attracted to option d, which was also the second most popular choice for students in other classes, although at a much lower rate. Option d is a common response from students asked to solve a system of linear equations for *x* and *y*, which may explain its popularity. Looking at student interview responses reinforces our interpretation of the three classes, and sheds light on students' reasoning.

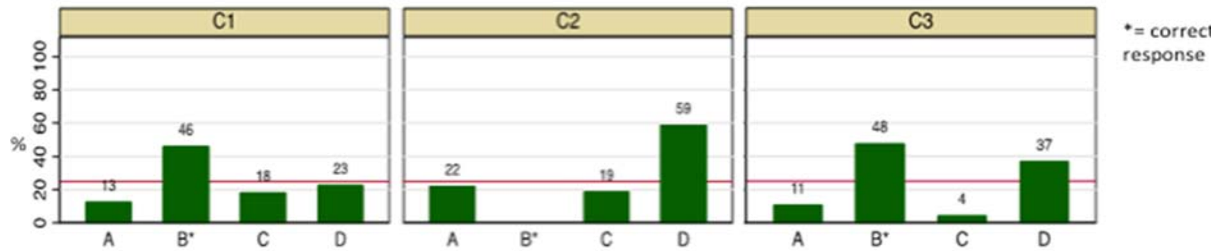

Figure 2. Item 2 distractor analysis

C1 (originally chose C, but drifted towards B in the interview): $x + y$ equals nothing so it can't be 10. Right?... It could be possible like it equals 10. [Option D isn't correct] maybe because *x* and *y* could be equal to anything?

C2 (chose D): I know there are certain numbers that will add up to ten, so there could be two solutions, since there's only a *x* term and a *y* term…

C3 (chose B): Ten could equal to many things. Like five plus five could equal ten. Nine plus one could equal ten. Seven plus three…it could be any number that will equal to ten.

The C1 student initially chose "no solution" because they didn't know what *x* and *y* could be, but then they started to relate this to the idea that *x* and *y* could be "anything". While their reasoning is not strictly correct, they are beginning to explore the idea that *x* and *y* may have many possible values, and they show no evidence of faulty intuitions stemming from procedural practice. The C2 student exhibits an intuition about what the equation means to find a single solution, but they do not explore whether there might be others, and they confuse the number of solutions with the number of variables in the solution set, suggesting that their intuitions about the definition of a solution set are incorrect. The student from C3 describes how this equation could have multiple solutions, demonstrating some conceptual understanding of solution sets, including the fact that they describe the relationship between the two variables.

Discussion and Limitations

This study revealed that roughly one quarter of students at the end of the course appeared to guess somewhat randomly on conceptual questions; however, cognitive interviews suggest that these students are able to make some progress towards conceptual understanding by relying initially on more naïve reasoning and that they are not typically hindered by incorrect intuitions stemming from misuse of procedures. About one quarter of students demonstrated some mastery of procedures in standard problem contexts, but demonstrated many misconceptions related to misuse of procedures on conceptual questions. In contrast, roughly half the class showed evidence of emergent conceptual understanding, with lower frequency of misconceptions related to misuse of procedures. For a number conceptual questions, particularly those that were more abstract or non-standard, conceptual understanding and procedural fluency were significantly strongly inversely related. Cognitive interviews revealed that this may happen when students develop incorrect intuitions stemming from the use of procedures.

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