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Optimal Incentives for Teams: A Multiscale Decision Theory Approach

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Abstract We present a novel modeling approach for supervised teams, which can determine optimal incentives when individual team member contributions are unknown. Our approach is based on multiscale decision theory (MSDT), which models the agents' decision processes and their mutual influence. To estimate the initially unknown influence of team members on their supervisor's success, we develop a linear approximation method that estimates model parameters from historic team performance data. In our analysis, we derive the optimal incentives the supervisor should offer to team members accounting for their varying skill levels. In addition, we identify the information and communication requirements between all agents such that the supervisor can calculate the optimal incentives, and such that team members can calculate their optimal effort responses. We illustrate our methods and the results through a systems engineering example.

Keywords Teams · Incentives · Multiscale decision theory · Principal-agent problem · Systems engineering

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1 Introduction

In organizations, incentives are used to motivate employees to do their best and make decisions in the interest of their supervisors. A well-designed incentive scheme aligns the preferences of all employees such that their actions contribute to the organizational goals [1, 10, 48]. Organizational incentive alignment is challenging due the multiscale effect of incentives across organizational levels and time scales [23, 24, 38, 49]. Organizational incentive design becomes even more challenging when team dynamics need to be considered. In teams, the performance of each team member is interdependent and cannot be easily quantified or isolated [2, 23, 37].

Multiscale decision theory (MSDT), developed by Wernz [42], is a modeling approach that analyzes multi-level incentives in organizations and other hierarchical systems with many decision makers. MSDT has proven to be an effective and efficient approach for incentive analysis and design in complex socio-technical systems, ranging from manufacturing enterprises [44] to the US healthcare system [51]. Current MSDT models, however, have not yet accounted for teams and team incentivization.

Motivated by the prevalence and importance of teams in organizations, and the challenges associated with incentivizing teams, the goal of this paper is to develop a novel mathematical model of teams using the MSDT approach. While the team incentivization problem is a well-known challenge in agency theory [13, 15, 16, 20], current mathematical models are not scalable, i.e., they cannot account for the multi-level interactions and incentive effects throughout a large organization [5, 6, 37, 40, 41]. Further, data-driven methods to isolate the contributions of each team member are not sufficiently developed, but are needed to quantify how each team member contributes to the success of the team, the supervisor and the organization overall [3, 4].

In response, this paper's objective is to develop a team model that can derive effective team incentives, and that lays the foundation for a multiscale team model. The specific contributions of the paper are: (1) extension of the MSDT framework to account for teams, (2) introduction of a continuous decision variable to MSDT (prior models only considered binary decisions), and (3) development of a data-driven approach to isolate the contribution of each team member to overall team performance.

We will present a two-level model with one supervisor overseeing a team with multiple members. Each team member chooses and exerts a level of work effort, which is that agent's continuous decision variable. Incentives are used to align the team members' interests with that of the supervisor, i.e., the team member's individually optimal effort becomes also optimal for the supervisor. Using the MSDT extension approach [49], this two-level, single-period model can be extended in future research to a multiscale model with many teams across different organizational levels operating on different time scales. The data-driven method to quantify and isolate the contributions of team members is based on linear regression. It derives the model parameter values needed for the team model.

The paper is organized as follows: Section 2 discusses related work on teams and team incentivization, and provides a brief overview of MSDT. Section 3 introduces the model, followed by the parameter estimation approach. The analysis and results are presented in Section 4, followed by an illustrative examples in Section 5. Section 6 provides the conclusions.

2 Related Work

2.1 Teams and Incentives

To determine optimal incentives for teams, a variety of quantitative, economic models have been developed [7, 26, 29, 33]. The majority of these works builds upon the principal-agent model, which in its basic form studies the interaction between one supervisor (principal) and one subordinate (agent) [35]. A key problem in the principal-agent relationship is that of moral hazard [8, 32], which refers to the agent's tendency to exert less work effort, since effort is not observable or enforceable. The supervisor's inability to obverse the agent's effort results in information asymmetry [22, 25], and thus the supervisor has to use an indirect signal to determine the incentive, such as the stochastic outcome of effort [2, 19, 21].

Information asymmetries not only exist between a supervisor and the team, but also between team members. The role of information in teams was comprehensively studied by Marschak and Radner [30]. Groves [16] built upon their work and explored how incentives can motivate supervisors to communicate accurate information to the team, thereby enabling team members to make optimal decisions. Information also plays a central role in our paper. We determine the minimal information set that must be exchanged so that supervisor and team members can make optimal decisions.

Incentives for teams can take on various forms. One can distinguish between two classes of incentive mechanisms: relative performance incentives and independent performance incentives [9, 28, 29, 34]. For relative performance incentives, the incentive for the team member depends on the performance of their team mates, while for independent performance incentives, only the team members' absolute performance counts. Relative performance incentives are more effective when stochastic, external events have a strong effect on team performance, while independent performance incentives are best when only team-internal effects are relevant [14, 27]. For our model assumptions, results show that the independent performance mechanism is optimal, even though stochastic, external events have a strong performance influence.

2.2 Multiscale Decision Theory (MSDT)

MSDT is a normative decision-theoretic framework that combines game theory [12], Markov decision processes [36], and hierarchical and graphical modeling [11, 31]. It allows for the analysis of stochastic interactions in complex systems, where decision makers affect each other across multiple system levels and time scales. In particular, MSDT can be used to determine optimal incentives and organizational design parameters that align and coordinate decisions throughout an organization.

The current MSDT framework incorporates two interconnected scales: the organizational scale and the time scale. Decisions at higher organizational levels are more strategic and long term in nature, while lower level decisions are more operational and short term. Prior operations research methods have either modeled decision-making over time, e.g., via Markov decision processes, or decision-making across system levels, e.g., via game theory or principal-agent models. Comprehensive and scalable methods for multi-level and dynamic systems, where each level operates at a different time scale, had been missing.

The foundational paper on MSDT, by Wernz and Deshmukh [48], introduced the MSDT term and concept and focused on the organizational scale of large hierarchical organizations. The paper considered a cascading incentive mechanism where each supervisor offers performance-based rewards to their immediate subordinates. Even for large systems, closed-form analytic solutions could be derived, which described the optimal multi-level incentives, and each agent's optimal decision response.

Building upon this multi-organizational scale model, the temporal scale was then integrated into the MSDT framework [49]. Using multi-time-scale Markov decision processes, closed-form solutions for a 3-level, 3-time-scale and 3-period decision problem were determined. In the next step, Wernz [43] derived a sequence of recursive solutions for the many-period problem with two levels operating at different time scales.

In parallel to the methodological development of MSDT, various applications were explored, which show the benefit this method can bring to complex decision-making and system design problems. MSDT has been applied to a 2-level manufacturing enterprise problem [44], general management problems [45–47], a 3-level service operations problem [17, 50], a 3-level supply chain problem [18], and a 3-level, 4-agent healthcare problem [51, 52].

In our paper, we consider the effect of the organizational scale on the interaction between team members and the supervisor. The supervisor's performance is influenced by the aggregate team performance, and thus each team member affects the supervisor's probability of success. To quantify this influence, we use MSDT's influence function concept. In addition to developing a team model, our work also extends MSDT by incorporating a continuous decision variable, which in this model is the work effort of team members. Prior MSDT models used discrete action sets. This extension presents an important methodological advancement for MSDT, which required a new solution approach.

3 Model

The team consists of a group of subordinates working for a supervisor. The supervisor receives her objectives from the organization and then proceeds to allocate jobs to each subordinate to achieve these objectives. We do not consider the job allocation or the job completion sequence in our model. Instead, we focus on the overall team and team member performance and the resulting effect on the supervisor's success.

The interactions between the supervisor and the team members is modeled as a one-period game. Team members are characterized by their skill level, their effort decision, the outcome (performance) they achieve, and the rewards they receive. A team member's skill is a function that describes the relationship between their effort and their performance. A team member's reward consists of two components: a base reward and an incentive. The base reward is provided by the organization, while the incentive is provided and paid for by the supervisor. Both rewards are performance/outcome dependent. A team member's effort is a continuous decision variable, and combined with a team member's skill characteristics, results in the probability of achieving a satisfactory outcome. A team member's effort is costly, and thus a team member's decision problem is to choose the optimal effort. We assume that team members are risk-neutral and aim to maximize their expected rewards.

The supervisor is characterized similarly to team members. This structural similarity supports the multiscale extension of the model, where a supervisor not only oversees a team, but is also part of a team working under another supervisor, who in turn reports to and gets incentivized by a higher level authority, and so on. In our two-level case, we take the supervisor's effort as given, i.e., it is a model parameter, not a decision variable.

The supervisor's outcome is probabilistic and is either satisfactory or unsatisfactory. Her outcome-based reward is reduced by the incentive she pays to her team. The incentive that she offers to each team member is a percentage of her reward. Thus, the supervisor's decision problem is to chose the percentage of her reward each team member is offered.

The team's performance affects the performance of the supervisor. We use MSDT's influence function concept to model this bottom-up influence. Since the supervisor offers a percentage of her final reward, team members need to consider the effort responses and incentives of their team mates while computing their own effort decision. Thus, incentives and efforts are interdependent, and determining an optimal level for each requires a game-theoretic analysis by all team members and the supervisor. To perform this analysis, agents may need private information from the other agents, and we will analyze the communication requirements.

3.1 Model Formulation

The team consists of n team members. As customary in MSDT, we refer to a team member as an infimal agent, or INF, and to the supervisor as the supremal agent, or SUP. Team members in general are referred to as INFs, and a specific team member is referred to as INFx, with $x \in \{1,\dots,n\}$. The performance of each agent is characterized by a state variable, with SUP's states being $S^{\text{SUP}} \in \{0,1\}$, and INF$x$'s states being $S^{\text{INF}x} \in \{0,1\}$. State 1 refers to the satisfactory outcomes, and state 0 to the unsatisfactory outcome for each agent. SUP's base reward is h^{SUP} for $S^{\text{SUP}} = 1$, and l^{SUP} for $S^{\text{SUP}} = 0$, with $h^{\text{SUP}} > l^{\text{SUP}}$. Similarly, the base rewards for INFx are $h^{\text{INF}x}$ for $S^{\text{INF}x} = 1$, and $l^{\text{INF}x}$ for $S^{\text{INF}x} = 0$, with $h^{\text{INF}x} > l^{\text{INF}x}$.

SUP has the option to offer INFx a share of her final reward as an incentive. INFx's reward share is denoted by $b^{\text{INF}x}$. Thus, INFx's incentive will either be $b^{\text{INF}x} \cdot h^{\text{SUP}}$ or $b^{\text{INF}x} \cdot l^{\text{SUP}}$ depending on SUP's outcome. We denote the vector of the rewards shares for the INFs by $b = (b^{\text{INF}1},\dots,b^{\text{INF}n})$.

The efforts chosen by the agents are e^{SUP} and $e^{\text{INF}x}$, with $e^{\text{INF}x}, e^{\text{SUP}} \geq 0$. For the two-level model, we assume SUP's effort e^{SUP} as given, i.e., a fixed model parameter, whereas the effort $e^{\mathrm{INF}x}$ is INFx's decision variable. The vector of effort chosen by the INFs is denoted by $e = (e^{\mathrm{INF1}}, \ldots, e^{\mathrm{INF}n})$.

INFx bears an effort-dependent cost $k^{\mathrm{INF}x}(e^{\mathrm{INF}x})$. A higher effort leads to higher effort cost, but also to a higher probability of achieving the satisfactory outcome. The extent to which a higher effort leads to a better outcome is captured by INFx's individual skill function $\alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x})$.

We define the skill function as the effort-dependent probability of achieving the satisfactory outcome, i.e., $\alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x}) := p(S^{\mathrm{INF}x} = 1|e^{\mathrm{INF}x})$. For SUP, whose effort is given, the skill is a fixed probability α^{SUP} with $\alpha^{\mathrm{SUP}} := p(S^{\mathrm{SUP}} = 1)$. SUP's effort cost is already accounted for in her rewards.

We assume that with increasing effort by INFx, its probability of success and effort-dependent costs strictly increase. However, with increasing effort for INFx, the marginal gains in the probability of success decrease, whereas the marginal losses from the effort dependent cost increase. Therefore, $\alpha^{\mathrm{INF}x}(\cdot)$ is a strictly increasing concave function of $e^{\mathrm{INF}x}$ for all x, and $k^{\mathrm{INF}x}(\cdot)$ is a strictly increasing convex function of $e^{\mathrm{INF}x}$ for all x. Lastly, we assume that $\alpha^{\mathrm{INF}x}(\cdot)$ and $k^{\mathrm{INF}x}(\cdot)$ are C^3 functions for all x.

The influence of the INFs on SUP's outcome is modeled through an influence function. We use the additive influence function approach of MSDT, where a so called change coefficient is either added or subtracted from the initial probability α^{SUP} of SUP's success. As shown by Wernz [42], an additive influence function is equivalent to a multiplicative one, as every multiplicative probability modification ultimately has additive characteristics, since probabilities need to add up to 1. In the case of teams, the change coefficient is a function of all of INFx's performances. Thus, SUP's probability function of achieving a satisfactory outcome is

$$p(S^{\mathrm{SUP}} = 1|e^{\mathrm{SUP}}, S^{\mathrm{INF1}}, \ldots, S^{\mathrm{INF}n}) = \alpha^{\mathrm{SUP}} + f_{\mathrm{team}}(S^{\mathrm{SUP}}, S^{\mathrm{INF1}}, \ldots, S^{\mathrm{INF}n}), \tag{1}$$

with $f_{\mathrm{team}}(\cdot)$ denoting the team influence function. Figure 1 graphically summarize the model described above.

For the ensuing analysis, we assume that the team influence function $f_{\mathrm{team}}(\cdot)$ can be approximated through a linear combination of the individual team member's influences. We define

$$f_{\mathrm{team}}(S^{\mathrm{SUP}}, S^{\mathrm{INF1}}, \ldots, S^{\mathrm{INF}n}) := \sum_{x=1}^{n} f_x(S^{\mathrm{SUP}}, S^{\mathrm{INF}x}), \tag{2}$$

where $f_x(\cdot)$ is the influence of each INFx's outcome on SUP's probability of success. For each of the influence functions $f_x(\cdot)$, a satisfactory outcome has an additive positive impact on SUP's probability of achieving her preferred outcome, and a negative influence otherwise. We define the influence functions $f_x(\cdot)$ by

$$f_x(S^{\mathrm{SUP}}, S^{\mathrm{INF}x}) := \begin{cases} c^{\mathrm{INF}x} & \text{if } S^{\mathrm{INF}x} = S^{\mathrm{SUP}} \\ -\bar{c}^{\mathrm{INF}x} & \text{if } S^{\mathrm{INF}x} \neq S^{\mathrm{SUP}} \end{cases} \tag{3}$$

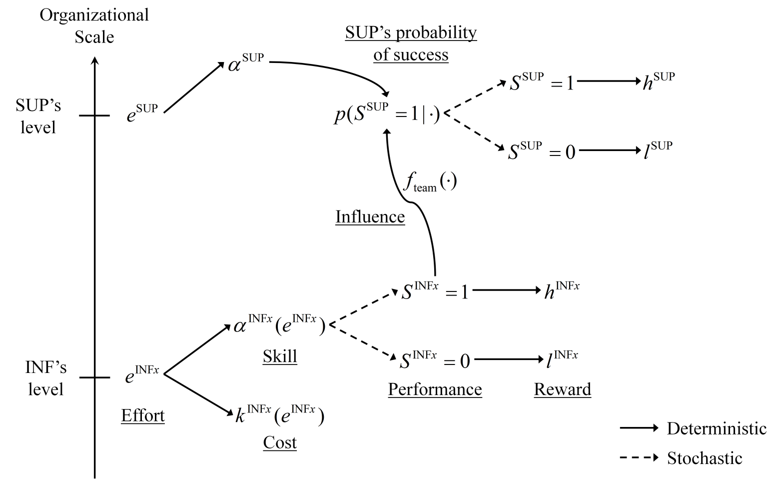

Fig. 1: Model outline

with change coefficients $c^{\mathrm{INF}x}, \tilde{c}^{\mathrm{INF}x} \geq 0$. SUP's conditional probability of success is thus

$$p(S^{\mathrm{SUP}} = 1 | S^{\mathrm{INF1}} = j^{\mathrm{INF1}}, .., S^{\mathrm{INF}n} = j^{\mathrm{INF}n}) = \alpha^{\mathrm{SUP}} + \sum_{x=1}^{n} j^{\mathrm{INF}x}\, c^{\mathrm{INF}x} - \sum_{x=1}^{n} (1 - j^{\mathrm{INF}x})\, \tilde{c}^{\mathrm{INF}x}, \quad (4)$$

with $j^{\mathrm{INF}x} \in \{0,1\}$ specifying the outcome of each INFx. Table 1 summarizes the model's notation.

In equation (2), we assumed that the team influence function $f_{\mathrm{team}}(\cdot)$ can be approximated through a linear combination of each INFx's influence on SUP. This assumption is only reasonable when the individual performances of the INFs are not significantly affected by the interactions between the INFs. The following section on data-driven team influence estimation builds upon this assumption, and a different team influence function would require a different estimation approach. For the model approach itself, the assumption is not necessary, and any other team influence function could be considered, though the results would differ.

3.2 Data-Driven Team Influence Estimation

To determine optimal incentives for each INFx, SUP must assess the contribution of the team's performance to her chance of success. In our model, this means estimating the change coefficients $c^{\mathrm{INF}x}, \tilde{c}^{\mathrm{INF}x}$. In this section, we present a data-driven approach for estimating the change coefficients and provide a numerical example. The approach can be divided into three steps:

1. Obtain data on the historic performance of SUP and INFs.

Table 1: Summary of notation

S^{SUP}, $S^{\text{INF}x}$	State variables denoting satisfactory or unsatisfactory outcomes
h^{SUP}, l^{SUP}	SUP's outcome-based rewards for satisfactory and unsatisfactory outcomes
$b^{\text{INF}x}$	Reward share offered by SUP to INFx
$h^{\text{INF}x}$, $l^{\text{INF}x}$	INFx's outcome-based rewards for satisfactory and unsatisfactory outcome
e^{SUP}, $e^{\text{INF}x}$	SUP's fixed effort and INFx's effort decision variable
$k^{\text{INF}x}(e^{\text{INF}x})$	Effort cost function
α^{SUP}, $\alpha^{\text{INF}x}(\cdot)$	SUP's base skill and INFx's skill function
$p(S^{\text{SUP}} = 1\|\cdot)$	SUP's probability of success
$f_{\text{Team}}(\cdot)$, $f_x(\cdot)$	Influence function of overall team and individual team members
$c^{\text{INF}x}$, $\tilde{c}^{\text{INF}x}$	Change coefficients

2. Qualitatively assess the type of influence each INFx has on SUP's chance of success.
3. Quantitatively assess each INFx's influence through linear regression.

1. *Obtain data.* SUP needs to obtain historic team performance data. The data must contain sufficient information to determine SUP's past probability of success for all possible permutations of INF-level outcomes. With two possible outcomes for each INFx, there are 2^n permutations for $(S^{\text{INF1}}, ..., S^{\text{INF}n})$. For each permutation, SUP's frequency of success needs to be assessed. Based on SUP's historic success rate, her future probability of success $p(S^{\text{SUP}} = 1|S^{\text{INF1}} = j^{\text{INF1}}, .., S^{\text{INF}n} = j^{\text{INF}n})$ can be estimated. Henceforth, we use $p^{\text{SUP}}_{j^{\text{INF1}},...,j^{\text{INF}n}}$ to denote this probability.

The following example will help to illustrate our method. Consider a three-member team and a supervisor, who have worked together on 40 projects. With three team members, there are $2^3 = 8$ possible team-level outcome permutations, or scenarios. Table 2 summarizes the data and analysis results. Consider scenario 8, for example, which refers to the case where every INFx had a satisfactory outcome. In this scenario, SUP was successful 9 out of 10 times, and thus $p^{\text{SUP}}_{1,1,1} = 0.9$. In addition to the data of Table 2, we assume that SUP's skill is $\alpha^{\text{SUP}} = 0.5$.

2. *Qualitatively assess the type of influence.* After SUP has determined $p^{\text{SUP}}_{j^{\text{INF1}},...,j^{\text{INF}n}}$ for all permutations of INFs' outcomes, SUP is tasked with qualitatively assessing the type of influence each INFx has on her chance of success. A type of influence can be categorized and characterized by whether the change coefficients $c^{\text{INF}x}, \tilde{c}^{\text{INF}x}$ are positive or zero. This qualitative assessment helps reduce the search space for the

Table 2: Example data

Scenario	S^{INF1}	S^{INF2}	S^{INF3}	# projects	# $(S^{\text{SUP}}=1)$	$p^{\text{SUP}}_{j^{\text{INF1}},j^{\text{INF2}},j^{\text{INF3}}}$
1	0	0	0	1	0	0
2	0	0	1	4	1	0.25
3	0	1	0	3	1	0.33
4	0	1	1	7	4	0.57
5	1	0	0	4	1	0.25
6	1	0	1	5	3	0.6
7	1	1	0	6	5	0.83
8	1	1	1	10	9	0.9

quantitative assessment via linear regression in the next step. We distinguish between four types of team member influences:

(i) Bi-directional effect: A satisfactory outcome by INFx has a positive effect on SUP's chance of success, and an unsatisfactory outcome has a negative effect SUP's chance of success, i.e., $c^{\text{INF}x} > 0$ and $\tilde{c}^{\text{INF}x} > 0$.
(ii) Positive effect: A satisfactory outcome by INFx improves SUP's chance of success, but an unsatisfactory outcome has no effect on SUP's chance of success, i.e., $c^{\text{INF}x} > 0$ and $\tilde{c}^{\text{INF}x} = 0$.
(iii) Negative effect: A satisfactory outcome by INFx has no effect on SUP's chance of success, but an unsatisfactory outcome has a negative effect SUP's chance of success, i.e., $c^{\text{INF}x} = 0$ and $\tilde{c}^{\text{INF}x} > 0$.
(iv) No effect: INFx's outcome has no effect on SUP's chance of success, i.e., $c^{\text{INF}x} = 0$ and $\tilde{c}^{\text{INF}x} = 0$.

If SUP cannot make such a qualitative assessment for one or more team members, the quantitative assessment can still be performed, but all four cases need to considered and need to be evaluated and compared. Continuing with our example, SUP makes the assessment that INF1 has a negative effect on her chance of success, and that INF2 has a positive effect. For INF3, SUP is unable to make the assessment.

3. *Quantitative assessment.* The final step is the numerical estimation of the change coefficients. We use a multiple linear regression model of the form

$$Y = \mathbf{X} \cdot C + \varepsilon \tag{5}$$

The dependent variable Y is a vector defined as $Y := (p^{\text{SUP}}_{0,\dots,0,1}, \dots, p^{\text{SUP}}_{1,1,\dots,1}) - \alpha^{\text{SUP}}$. Vector Y has 2^n elements covering all possible permutations of INF-level outcomes. Vector Y corresponds to the last column of Table 2, with the values being reduced by α^{SUP}. Vector $C := (c^{\text{INF1}}, \dots, c^{\text{INF}n}, \tilde{c}^{\text{INF1}}, \dots, \tilde{c}^{\text{INF}n})$ contains the $2n$ change coefficients. The matrix of independent variables is defined by $\mathbf{X} = (\hat{X}_1, \dots, \hat{X}_{2^n})^T$, where each $\hat{X}_i$ is a row vector with $\hat{X}_i := (j^{\text{INF1}}, \dots, j^{\text{INF}n}, j^{\text{INF1}} - 1, \dots, j^{\text{INF}n} - 1)$. The row vector $\hat{X}_i$ corresponds to the Y vector element $y_i = p^{\text{SUP}}_{j^{\text{INF1}},\dots,j^{\text{INF}n}}$, and captures the INFs with satisfactory outcomes via the values $j^{\text{INF1}}, \dots, j^{\text{INF}n}$, and the INFs with unsatisfactory outcomes via the values $j^{\text{INF1}} - 1, \dots, j^{\text{INF}n} - 1$. Lastly, ε is the vector of error variables.

In step 2, SUP had assessed that INF1 is of negative effect type, and INF2 is of positive effect type. For INF3, SUP could not make an assessment. Since SUP does not know INF3's influence, she needs to consider all four possible influence types. For each of these types, SUP must fit the linear model, equation (5), on the data given in Table 2. In each of the four variations of equation (5) that SUP considers, INF1 has a negative influence and INF2's has a positive influence. The results of linear regression analysis for all four influence types is shown in Table 3.

Table 3: Results of linear regression

	Bi-directional		***Positive***	
	R^2	0.96	R^2	0.88
	F-test p-value	0.01	F-test p-value	0.02
	Estimate	t-test p-value	Estimate	t-test p-value
$\tilde{c}^{\text{INF1}}$	0.35	< 0.01	0.44	< 0.01
c^{INF2}	0.38	< 0.01	0.30	0.01
c^{INF3}	-0.09	0.41	0.15	0.11
$\tilde{c}^{\text{INF3}}$	0.16	0.05	0	-NA-
	Negative		***No-effect***	
	R^2	0.95	R^2	0.69
	F-test p-value	< 0.01	F-test p-value	0.04
	Estimate	t-test p-value	Estimate	t-test p-value
$\tilde{c}^{\text{INF1}}$	0.39	< 0.01	0.21	0.07
c^{INF2}	0.35	< 0.01	0.19	0.03
c^{INF3}	0	-NA-	0	-NA-
$\tilde{c}^{\text{INF3}}$	0.11	0.01	0	-NA-

SUP must now determine which of the four influence types best fits the data. To make this decision, SUP considers the squared correlation coefficient R^2, the p-value of the F-test, and the p-value associated with the t-test for each change coefficient.

R^2 is the proportion of the variance explained by the linear model, and is a measure of how well a linear model fits the data. The closer R^2 is to 1, the better.

The F-test uses the null hypothesis that the current set of independent variables does not sufficiently describe the dependent variable. The p-value of the F-test is an indicator of whether the current set of independent variables needs to be modified to achieve a better model fit on the underlying data. The t-test for each change coefficient uses the null hypothesis that the change coefficient must be 0, and the p-value of a t-test on the estimate of a change coefficient is an indicator of how likely it is the estimate should actually be 0 rather than a non-zero value. Since hypothesis testing is based on rejecting or not rejecting the null hypothesis, the p-value of a hypothesis test is preferred to be close to 0.

The results of the linear regression in Table 3 show that all four models have a low p-value on the F-test, and that there is not sufficient evidence for SUP to reject a model based on those results alone. However, there are differences in the R^2 and the p-values associated with the t-tests, which SUP can utilize to make a choice. The no-effect model has the lowest R^2 value, and is thus inferior to the other three. Further, the bi-directional model shows a negative c^{INF3} value, which is not feasible. Between the remaining positive and negative model, the negative model has better p-values, and has a higher R^2. Thus, the negative model best describes the data. The quantitative assessment finds that INF3 is of negative effect type with $\tilde{c}^{\text{INF3}} = 0.11$, and that $\tilde{c}^{\text{INF1}} = 0.39$ and $c^{\text{INF2}} = 0.35$.

4 Analysis

In this section, we use a game-theoretic analysis to derive the optimal effort for each INFx as a function of the incentives, and then determine SUP's optimal incentive offer based on INFs' responses.

4.1 INFx's optimal effort

Each INFx seeks to maximize its expected reward. To determine INFx's expected reward, we must first determine SUP's expected reward, which is a function of INFs' effort vector e and reward share vector b. SUP's expected reward can be calculated as follows:

$$
\begin{aligned}
R^{\text{SUP}}(e,b) = \Big(1 - \sum_{x=1}^{n} b^{\text{INF}x}\Big)\Big(& l^{\text{SUP}} \\
&+ (h^{\text{SUP}} - l^{\text{SUP}}) \sum_{j_1=0}^{1} \cdots \sum_{j_n=0}^{1} \Big(p(S^{\text{SUP}} = 1 | S^{\text{INF1}} = j_1, .., S^{\text{INF}n} = j_n) \\
&\times p(S^{\text{INF1}} = j_1) \times \ldots \times p(S^{\text{INF}n} = j_n)\Big)\Big)
\end{aligned}
$$

$$= \Big(1 - \sum_{x=1}^{n} b^{\mathrm{INF}x}\Big)\bigg(l^{\mathrm{SUP}} + (h^{\mathrm{SUP}} - l^{\mathrm{SUP}})\Big(\alpha^{\mathrm{SUP}} + \sum_{x=1}^{n} c^{\mathrm{INF}x}\alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x}) - \sum_{x=1}^{n} \tilde{c}^{\mathrm{INF}x}\big(1 - \alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x})\big)\Big)\bigg). \quad (6)$$

Based on SUP's expected reward, the expected reward for INFx can be calculated as

$$R^{\mathrm{INF}x}(e,b) = (h^{\mathrm{INF}x} - l^{\mathrm{INF}x})\alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x}) + l^{\mathrm{INF}x} - k^{\mathrm{INF}x}(e^{\mathrm{INF}x}) + b^{\mathrm{INF}x}\Bigg(l^{\mathrm{SUP}} + (h^{\mathrm{SUP}} - l^{\mathrm{SUP}})\bigg(\alpha^{\mathrm{SUP}} + \sum_{x=1}^{n} c^{\mathrm{INF}x}\alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x}) - \sum_{x=1}^{n} \tilde{c}^{\mathrm{INF}x}\big(1 - \alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x})\big)\bigg)\Bigg). \quad (7)$$

To determine INFx's optimal effort response for a given incentive level $b^{\mathrm{INF}x}$, denoted by $e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})$, we solve the first order condition of optimality and verify the solution with the second order condition. For the analysis, we define e_{-x} as the effort vector of all INFs but INFx; the vector has $n-1$ elements. For given efforts e_{-x} and given reward share vector b, the optimal effort $e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})$ is the solution to the first order condition of optimality, which is

$$\frac{\partial}{\partial e^{\mathrm{INF}x}} R^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x}), e_{-x}, b\big) = 0$$
$$\Longrightarrow \Big((h^{\mathrm{INF}x} - l^{\mathrm{INF}x}) + b^{\mathrm{INF}x}(c^{\mathrm{INF}x} + \tilde{c}^{\mathrm{INF}x})(h^{\mathrm{SUP}} - l^{\mathrm{SUP}})\Big)\frac{\mathrm{d}}{\mathrm{d}e^{\mathrm{INF}x}}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big) - \frac{\mathrm{d}}{\mathrm{d}e^{\mathrm{INF}x}} k^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big) = 0. \quad (8)$$

To verify that $e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})$ *maximizes* INFx's expected reward, we evaluate the second order condition

$$\frac{\partial^2}{\partial (e^{\mathrm{INF}x})^2} R^{\mathrm{INF}x}(e^{\mathrm{INF}x}, e_{-x}, b)$$
$$= \Big((h^{\mathrm{INF}x} - l^{\mathrm{INF}x}) + b^{\mathrm{INF}x}(c^{\mathrm{INF}x} + \tilde{c}^{\mathrm{INF}x})(h^{\mathrm{SUP}} - l^{\mathrm{SUP}})\Big)\frac{\mathrm{d}^2}{\mathrm{d}(e^{\mathrm{INF}x})^2}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big) - \frac{\mathrm{d}^2}{\mathrm{d}(e^{\mathrm{INF}x})^2} k^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big), \quad (9)$$

which is strictly negative given the convexity of effort cost $k^{\mathrm{INF}x}$, and the concavity of skill function $\alpha^{\mathrm{INF}x}$.

Since no element of e_{-x} is present in the first order condition, equation (8), we know that INFx's optimal effort is independent of the effort of the other INFs. SUP accounting for the influence of each INFx's outcome on her outcome, allows each INFx to ignore the efforts and outcomes of the other INFs, even though SUP's expected reward, and thus each INFx's expected reward, is dependent on the performance of all

INFs. A significant implication of this result is that INFs can make optimal effort decisions without knowing other INFs' private information, in particular their rewards, skills of effort costs. Each INFx merely needs to know its own parameters, its incentive share offered by SUP, and SUP's reward difference between a satisfactory and unsatisfactory outcome . Communication and sharing of private information between INFs is not necessary.

4.2 Optimal Incentives

Unlike the INFs, SUP needs to know all the parameters of all INFs to determine the optimal reward shares. However, and as we will show next, SUP can compute each INFx's optimal effort response by herself, and does not need to observe or request that information from the INFs. This result even applies, if a closed-form analytic expression for INFs' optimal efforts cannot be derived.

Theorem 1 *SUP can numerically compute each INFx's optimal effort e_*^{INFx} for any reward share b^{INFx} to any level of accuracy.*

Proof. The proof consists of two parts. First we show that each INFx's optimal effort is a continuous function of that agent's incentive, i.e., $e_*^{\text{INF}x}$ is a continuous function of $b^{\text{INF}x}$. Then we derive the general form of the m^{th} order derivative of $e_*^{\text{INF}x}(b^{\text{INF}x})$, where $m \in \mathbb{N}^+$, and show that it exists for all values of $b^{\text{INF}x}$ and m. The existence of the m^{th} order derivative of $e_*^{\text{INF}x}(b^{\text{INF}x})$ implies that even if a closed-form expression for $e_*^{\text{INF}x}(b^{\text{INF}x})$ is not available, SUP can approximate $e_*^{\text{INF}x}(b^{\text{INF}x})$ via the Taylor series expansion [39]. To reduce the error in prediction, SUP can increase the order of the derivatives used in the Taylor series expansion.

For the first part of the proof, let

$$
\begin{aligned}
M_x(e^{\text{INF}x}, b^{\text{INF}x}) &= \frac{\partial}{\partial e^{\text{INF}x}} R^{\text{INF}x}\left(e^{\text{INF}x}, e_{-x}, b\right) \\
&= \Big((h^{\text{INF}x} - l^{\text{INF}x}) + b^{\text{INF}x}(\tilde{c}^{\text{INF}x} + c^{\text{INF}x})(h^{\text{SUP}} - l^{\text{SUP}})\Big)\frac{\mathrm{d}}{\mathrm{d}e^{\text{INF}x}}\alpha^{\text{INF}x}(e^{\text{INF}x}) \\
&\qquad - \frac{\mathrm{d}}{\mathrm{d}e^{\text{INF}x}} k^{\text{INF}x}(e^{\text{INF}x}). \quad (10)
\end{aligned}
$$

Given $b^{\text{INF}x}$, we know from equation (8) that $M_x(e_*^{\text{INF}x}(b^{\text{INF}x}), b^{\text{INF}x}) = 0$, and thus

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}b^{\text{INF}x}} M_x(e_*^{\text{INF}x}(b^{\text{INF}x}), b^{\text{INF}x}) = 0 \\
&\implies \frac{\partial}{\partial b^{\text{INF}x}} M_x(e_*^{\text{INF}x}(b^{\text{INF}x}), b^{\text{INF}x}) \\
&\qquad\qquad + \frac{\partial}{\partial e^{\text{INF}x}} M_x(e_*^{\text{INF}x}(b^{\text{INF}x}), b^{\text{INF}x}) \times \frac{\mathrm{d}e_*^{\text{INF}x}(b^{\text{INF}x})}{\mathrm{d}b^{\text{INF}x}} = 0 \\
&\implies \frac{\mathrm{d}e_*^{\text{INF}x}(b^{\text{INF}x})}{\mathrm{d}b^{\text{INF}x}} = -\left((c^{\text{INF}x} + \tilde{c}^{\text{INF}x})(h^{\text{SUP}} - l^{\text{SUP}})\frac{\mathrm{d}}{\mathrm{d}e^{\text{INF}x}}\alpha^{\text{INF}x}\big(e_*^{\text{INF}x}(b^{\text{INF}x})\big)\right)\Big/
\end{aligned}
$$

$$\Bigg(\Big((h^{\text{INF}x}-l^{\text{INF}x})+b^{\text{INF}x}(\tilde{c}^{\text{INF}x}+c^{\text{INF}x})(h^{\text{SUP}}-l^{\text{SUP}})\Big)\frac{\mathrm{d}^2}{\mathrm{d}(e^{\text{INF}x})^2}\alpha^{\text{INF}x}\big(e_*^{\text{INF}x}(b^{\text{INF}x})\big)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}(e^{\text{INF}x})^2}k^{\text{INF}x}\big(e_*^{\text{INF}x}(b^{\text{INF}x})\big)\Bigg). \quad (11)$$

Since the denominator of $\mathrm{d}e_*^{\text{INF}x}(b^{\text{INF}x})\big/\mathrm{d}b^{\text{INF}x}$ is strictly negative for any effort of INFx, we know that $\mathrm{d}e_*^{\text{INF}x}(b^{\text{INF}x})\big/\mathrm{d}b^{\text{INF}x}$ exists for all efforts of INFx, and this in turn implies that INFx's optimal effort is a continuous function of its incentive share.

For the second part of the proof, we know that

$$\begin{aligned}
&\frac{\mathrm{d}^m}{\mathrm{d}(b^{\text{INF}x})^m}M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})=0\\
\implies\ &\frac{\mathrm{d}^{m-1}}{\mathrm{d}(b^{\text{INF}x})^{m-1}}\Bigg(\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial b^{\text{INF}x}}\\
&\qquad+\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial e^{\text{INF}x}}\frac{\mathrm{d}e_*^{\text{INF}x}(b^{\text{INF}x})}{\mathrm{d}b^{\text{INF}x}}\Bigg)=0\\
\implies\ &\frac{\mathrm{d}^{m-1}}{\mathrm{d}(b^{\text{INF}x})^{m-1}}\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial b^{\text{INF}x}}\\
&\qquad+\frac{\mathrm{d}e_*^{\text{INF}x}(b^{\text{INF}x})}{\mathrm{d}b^{\text{INF}x}}\frac{\mathrm{d}^{m-1}}{\mathrm{d}(b^{\text{INF}x})^{m-1}}\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial e^{\text{INF}x}}\\
&\qquad+\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial e^{\text{INF}x}}\frac{\mathrm{d}^m e_*^{\text{INF}x}(b^{\text{INF}x})}{\mathrm{d}(b^{\text{INF}x})^m}=0\\
\implies\ &\frac{\mathrm{d}^m e_*^{\text{INF}x}(b^{\text{INF}x})}{\mathrm{d}(b^{\text{INF}x})^m}=-\Bigg(\frac{\mathrm{d}^{m-1}}{\mathrm{d}(b^{\text{INF}x})^{m-1}}\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial b^{\text{INF}x}}\\
&\qquad+\frac{\mathrm{d}e_*^{\text{INF}x}(b^{\text{INF}x})}{\mathrm{d}(b^{\text{INF}x})}\frac{\mathrm{d}^{m-1}}{\mathrm{d}(b^{\text{INF}x})^{m-1}}\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial e^{\text{INF}x}}\Bigg)\Bigg/\\
&\qquad\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial e^{\text{INF}x}}.
\end{aligned}$$

Since

$$\frac{\partial M_x(e_*^{\text{INF}x}(b^{\text{INF}x}),b^{\text{INF}x})}{\partial e^{\text{INF}x}}=$$
$$\Big((h^{\text{INF}x}-l^{\text{INF}x})+b^{\text{INF}x}(\tilde{c}^{\text{INF}x}+c^{\text{INF}x})(h^{\text{SUP}}-l^{\text{SUP}})\Big)\frac{\mathrm{d}^2}{\mathrm{d}(e^{\text{INF}x})^2}\alpha^{\text{INF}x}\big(e_*^{\text{INF}x}(b^{\text{INF}x})\big)$$
$$-\frac{\mathrm{d}^2}{\mathrm{d}(e^{\text{INF}x})^2}k^{\text{INF}x}\big(e_*^{\text{INF}x}(b^{\text{INF}x})\big)<0, \quad (12)$$

we know the m^{th} order derivative of $e_*^{\text{INF}x}(b^{\text{INF}x})$ exists, and thus SUP can numerically determine the optimal effort of INFx given $b^{\text{INF}x}$. We present the first and second order derivatives of $e_*^{\text{INF}x}(b^{\text{INF}x})$ in Appendix A. □

There are two significant implications of Theorem 1. The first is that SUP does not need to monitor, supervise or set the efforts of each INFx as long as SUP can assess their skill, effort cost function and influence (i.e., change coefficients). The second implication is that SUP only needs to consider the data of one INFx at a time to compute that agent's optimal effort response to an incentive.

Next, we determine the optimal incentives SUP will offer. SUP can determine the optimal reward share vector b by solving the optimization problem $\mathbb{O}_{\mathrm{S}}$:

$$\max_b R^{\mathrm{SUP}}(b) = \Big(1 - \sum_{x=1}^{n} b^{\mathrm{INF}x}\Big)\bigg((h^{\mathrm{SUP}} - l^{\mathrm{SUP}})\Big(\alpha^{\mathrm{SUP}} + \sum_{x=1}^{n} c^{\mathrm{INF}x}\alpha^{\mathrm{INF}x}(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})) - \sum_{x=1}^{n} \tilde{c}^{\mathrm{INF}x}(1 - \alpha^{\mathrm{INF}x}(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})))\Big) + l^{\mathrm{SUP}}\bigg)$$

s.t.

$$0 \leq b^{\mathrm{INF}q} \leq 1 \quad \text{for } q = 1, \ldots, n \tag{13}$$

$$\sum_{x=1}^{n} b^{\mathrm{INF}x} \leq 1. \tag{14}$$

In general, $\mathbb{O}_{\mathrm{S}}$ is a nonlinear optimization program. The first order partial derivative of $R^{\mathrm{SUP}}(b)$ with respect to INFx's reward share is

$$\begin{aligned}\frac{\partial}{\partial b^{\mathrm{INF}x}} R^{\mathrm{SUP}}(b) = \Big(1 - \sum_{k=1}^{n} b^{\mathrm{INF}k}\Big)\bigg((h^{\mathrm{SUP}} - l^{\mathrm{SUP}}) \times \\ (c^{\mathrm{INF}x} + \tilde{c}^{\mathrm{INF}x})\frac{\mathrm{d}}{\mathrm{d}e^{\mathrm{INF}x}}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big) \times \frac{\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}b^{\mathrm{INF}x}}\bigg) \\ - \bigg((h^{\mathrm{SUP}} - l^{\mathrm{SUP}})\Big(\alpha^{\mathrm{SUP}} + \sum_{k=1}^{n} c^{\mathrm{INF}k}\alpha^{\mathrm{INF}k}(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}k})) \\ - \sum_{k=1}^{n} \tilde{c}^{\mathrm{INF}k}(1 - \alpha^{\mathrm{INF}k}(e_*^{\mathrm{INF}k}(b^{\mathrm{INF}k})))\Big) + l^{\mathrm{SUP}}\bigg).\end{aligned} \tag{15}$$

With equation (15), we can determine the gradient of $R^{\mathrm{SUP}}(b)$ for a given reward share vector b, and thus SUP's optimization problem $\mathbb{O}_{\mathrm{S}}$ can be solved numerically. If closed-form expressions for any $\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})/\mathrm{d}b^{\mathrm{INF}x}$ are not available, then equation (24) in Appendix A can be used as a substitute. Together with suitable initial conditions, $\mathbb{O}_{\mathrm{S}}$ can be solved via numerical integration as discussed in the proof of Theorem 1.

The computational effort required to solve $\mathbb{O}_{\mathrm{S}}$ can be significantly reduced if $R^{\mathrm{SUP}}(b)$ is concave for all b. We know from equation (9) that INFx's marginal expected reward function, $\frac{\partial}{\partial e^{\mathrm{INF}x}} R^{\mathrm{INF}x}(e,b)$, is decreasing in INF$x$'s effort. In addition, we can show that if $\frac{\partial}{\partial e^{\mathrm{INF}x}} R^{\mathrm{INF}x}(e,b)$ is concave with respect to e, then $R^{\mathrm{SUP}}(b)$ is concave with respect to b. The concavity of SUP's expected reward function implies that a local maximum is the unique solution to $\mathbb{O}_{\mathrm{s}}$. The following theorem formally presents and proves this result.

Theorem 2 *A unique optimal incentivization strategy for SUP exists, if* $\frac{\partial}{\partial e^{INFx}} R^{\mathrm{INF}x}(e,b)$ *is concave with respect to* e^{INFx} *for all INFx.*

The proof of Theorem 2 is presented in Appendix B. The outline of the proof is as follows. A unique optimal incentivization strategy for SUP exists if $R^{\mathrm{SUP}}(b)$ is strictly concave with respect to the incentive vector b. As shown in Appendix B, if $\frac{\partial}{\partial e^{\mathrm{INF}x}} R^{\mathrm{INF}x}(e,b)$ is concave with respect to $e^{\mathrm{INF}x}$, then $e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})$ will always be strictly concave with respect to $b^{\mathrm{INF}x}$. This in turn will result in the Hessian matrix of $R^{\mathrm{SUP}}(b)$ being negative definite for all $b \in (0,1]^n$, which implies that $R^{\mathrm{SUP}}(b)$ is strictly concave with respect to b.

The sufficient condition for a unique optimal incentivization strategy for SUP to exist implies that the optimal effort for INFx, $e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})$, is a strictly increasing, concave function of $b^{\mathrm{INF}x}$. In other words, SUP can elicit a greater effort by INFx by offering a greater share of her reward. However, due to the concavity of $e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})$, the rate of increase in INFx's optimal effort is decreasing with increasing incentives. At the optimum, the costs and benefits of the incentives are in balance, which means that SUP's marginal reward gains from INFs' increase in efforts equal the marginal cost of providing the effort-inducing incentives.

5 Systems Engineering Example

We will consider the team of one SUP and three INFs introduced in Section 3. The team is part of a systems engineering firm that develops and produces drones for consumers. The firm seeks to launch a new model. The supervisor SUP and her team of three software engineers, the INFs, are tasked with developing the flight control software.

As discussed in Section 3, SUP and the three INFs have worked together in the past. The data in Table 2 summarize the team's historic performance. In addition and as in the prior example, SUP has assessed that INF1 is of negative influence type, INF2 is of positive influence type, and INF3's influence type is unknown. The result of the linear regression in Table 3 show that INF3 is of negative influence type and that the change coefficients $\tilde{c}^{\mathrm{INF1}} = 0.39$, $c^{\mathrm{INF2}} = 0.35$ and $\tilde{c}^{\mathrm{INF3}} = 0.11$ best fit the data.

For this example, we assume that the INFs are homogeneous, i.e., they have the same skill and effort cost functions. Effort $e^{\mathrm{INF}x}$ represents the hours spent by an INFx to complete the software tasks assigned by SUP. We assume that INFx's skill function is of the form $\alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x}) = 1 - \exp(-ve^{\mathrm{INF}x})$. Parameter v describes the skill level; greater v implies greater skill. The skill function is positive monotonic and concave with respect to effort, which means that additional hours lead to higher chances of success, but also that the marginal increase in the chance of success decreases as the number of hours increase.

We further assume INFx's cost function to be $k^{\mathrm{INF}x}(e^{\mathrm{INF}x}) = 20\exp(e^{\mathrm{INF}x}/w)$. The parameter w is a scaling factor, which represents the rate of increase in INFx's effort cost. It can be interpreted as INFx's endurance. The greater the value of w, the slower the growth in effort cost, i.e., the greater the agent's endurance. The effort cost function is positive monotonic and convex, which means that the cost of effort increases with greater effort, and that the marginal increase is also increasing with each extra hour worked.

Table 4: Parameter values for agents

Agent	Skill	Cost	Rewards	Influence
SUP	$\alpha^{\mathrm{SUP}} = 0.5$	-	$h^{\mathrm{SUP}} = 500$, $l^{\mathrm{SUP}} = 50$	-
INF1	$v = 10^{-3}$	$w = 2000$	$h^{\mathrm{INF1}} = 100$, $l^{\mathrm{INF1}} = 80$	$\tilde{c}^{\mathrm{INF1}} = 0.39$
INF2	$v = 10^{-3}$	$w = 2000$	$h^{\mathrm{INF2}} = 100$, $l^{\mathrm{INF2}} = 80$	$c^{\mathrm{INF2}} = 0.35$
INF3	$v = 10^{-3}$	$w = 2000$	$h^{\mathrm{INF3}} = 100$, $l^{\mathrm{INF3}} = 80$	$\tilde{c}^{\mathrm{INF3}} = 0.11$

Table 4 summarizes all model parameters for this example. We had assumed that except for their influence, all three INFs are identical. For SUP, the reward for achieving a satisfactory outcome is significantly larger than that for the INFs. The reason is that SUP has project level responsibility, which is greater than the individual responsibilities and rewards of the INFs. The following analysis will show if and what level of incentives SUP should offer to each INF such that her expected reward is maximized.

5.1 Optimal incentives for INFs

To determine the optimal incentives for the INFs, we formulate SUP's optimization problem $\mathbb{O}_{\mathrm{S}}$. Input to $\mathbb{O}_{\mathrm{S}}$ are INFs' optimal effort responses to the reward shares offered. The analytic expression of the optimal effort responses can be derived by solving equation (8). Together with the parameters of Table 4, the results are

$$e_*^{\mathrm{INF1}}(b^{\mathrm{INF1}}) = (2000/3)\log(2+11.7b^{\mathrm{INF1}}), \tag{16}$$

$$e_*^{\mathrm{INF2}}(b^{\mathrm{INF2}}) = (2000/3)\log(2+10.5b^{\mathrm{INF2}}) \text{ and} \tag{17}$$

$$e_*^{\mathrm{INF3}}(b^{\mathrm{INF3}}) = (2000/3)\log(2+3.3b^{\mathrm{INF3}}). \tag{18}$$

By substituting these equations in INFs' skill functions, we obtain the probability of them achieving a satisfactory performance given b^{INFx}:

$$\alpha_*^{\mathrm{INF1}}(b^{\mathrm{INF1}}) = 1-(20+117b^{\mathrm{INF1}})^{-\frac{2}{3}}, \tag{19}$$

$$\alpha_*^{\text{INF2}}(b^{\text{INF2}}) = 1 - (20 + 105b^{\text{INF2}})^{-\frac{2}{3}} \text{ and} \tag{20}$$

$$\alpha_*^{\text{INF3}}(b^{\text{INF3}}) = 1 - (20 + 33b^{\text{INF3}})^{-\frac{2}{3}}. \tag{21}$$

SUP's optimization problem $\mathbb{O}_{\text{S}}$ can now be specified as

$$\max_b R^{\text{SUP}}(b) = \Big(1 - \sum_{m=1}^{3} b^{\text{INF}m}\Big)\Big(205 - 117(20 + 117b^{\text{INF1}})^{-\frac{2}{3}} - 105 * (20 + 105b^{\text{INF2}})^{-\frac{2}{3}} - 33 * (20 + 33b^{\text{INF3}})^{-\frac{2}{3}})\Big)$$

s.t.

$$0 \leq b^{\text{INF}q} \leq 1 \quad \text{for } q \in \{1,2,3\} \tag{22}$$

$$\sum_{q=1}^{3} b^{\text{INF}q} \leq 1. \tag{23}$$

The optimal incentive vector $b_* = (b_*^{\text{INF1}}, b_*^{\text{INF2}}, b_*^{\text{INF3}})$ is determined by solving this nonlinear constrained optimization program $\mathbb{O}_{\text{S}}$. We used the *fmincon* function in MATLAB® to solve $\mathbb{O}_{\text{S}}$. Table 5 presents the solution of $\mathbb{O}_{\text{S}}$, and Table 6 shows the resulting increase of INFx's efforts and the improvements in the success probabilities of all agents.

Table 5: Optimal incentives and reward improvements

Agent	Initial reward	Reward share	Final reward	Reward increase
SUP	191.5	-0.131	212.6	11.0 %
INF1	62.5	0.074	79.3	27.5%
INF2	62.5	0.057	75.5	21.4%
INF3	62.5	0	62.2	0%

The results show that optimally incentivizing the INFs increases SUP's expected reward by 11.5%. This increase accounts for the incentive SUP pays to the INFs, which is 13.1% of her gross final reward. The increase in SUP's expected reward is due to an increase in SUP's probability of success, which went from 0.31 without incentives to 0.43 with incentives.

Though the INFs are equally skilled and have the same effort cost, SUP's optimal strategy is to incentivize only INF1 and INF2, but not INF3. This is due to the low influence that INF3 has on SUP, i.e., its low change coefficient (0.11 vs 0.39 and 0.35). SUP offers INF1 a 7.4% reward share and INF2 a 5.7% reward share, which result in effort increases of 72.1% and 53.3%, respectively.

Table 6: Improvements in effort and chances of success

Agent	Initial effort	Final effort	Effort increase	Initial success prob.	Final success prob.	Prob. increase
SUP	-	-	-	0.31	0.43	0.12
INF1	462.1	795.1	72.1%	0.37	0.55	0.18
INF2	462.1	708.6	53.3%	0.37	0.51	0.14
INF3	462.1	462.1	0%	0.37	0.37	0

5.2 Sensitivity analysis

To assess the effect of the model parameters in our example, we perform a sensitivity analysis. First, we study the effect of INFs' skill and endurance parameters on SUP's probability of success and expected reward. We consider two metrics: ΔR^{SUP} and $\Delta P(S^{\text{SUP}} = 1|\cdot)$. The metric ΔR^{SUP} is the percentage increase in SUP's expected reward with optimal incentivization, and $\Delta P(S^{\text{SUP}} = 1|\cdot)$ is the increase in SUP's final probability of success due to optimal incentivization.

We vary the skill parameter v for each INFx individually while keeping the skill parameters of the other INFs constant. Figure 2 shows ΔR^{SUP} (left) and $\Delta P(S^{\text{SUP}} = 1|\cdot)$ (right) as a function of $\Delta v^{\text{INF}x}(\%)$. The graphs show how SUP's optimal expected reward and SUP's optimal probability of success changes as the skill of each INF agent changes. With increasing skill, SUP's metrics increase, but the increase differs for each INF. SUP benefits the most from an increase in the skill of INF1, followed by INF2 and then INF3. The difference between the INFs is due to their different change coefficient values.

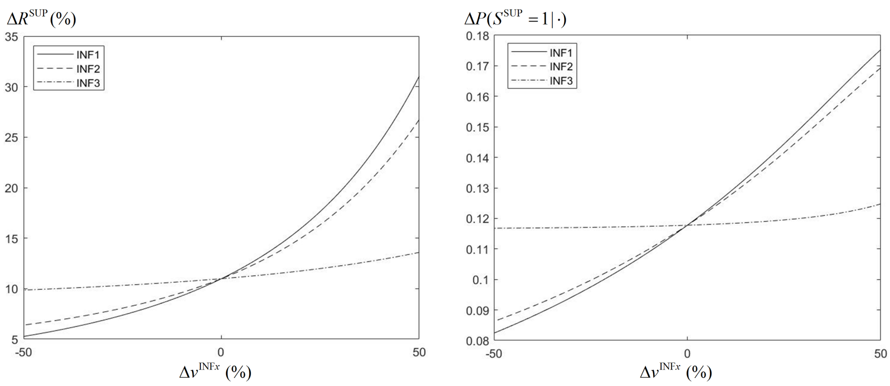

Fig. 2: Effect of INFs' skills on SUP. *Left*: Increase in optimal expected reward; *Right*: Increase in final probability of success

Next, we explore the changes in the reward shares offered to the INFs with respect to changes in their skill. Figure 3 shows the reward shares offered to the INFs when one of the INFx's skill parameter is varied, and the others INFs' skills are held constant. One can see that increasing the skills of any one of the INFs leads to higher reward shares for INF1 and INF2; for INF3 the incentive remains zero. A skill increase of INF1 benefits INF1 the most; the same applies to INF2. A skill increase of INF3 has no benefit for INF3, but benefits INF1 and INF2 — though only slightly. In all cases, the increase in reward share for INF1 and INF2 gets amplified by the increase in SUP's gross reward, of which the INFs get a percentage (the reward share).

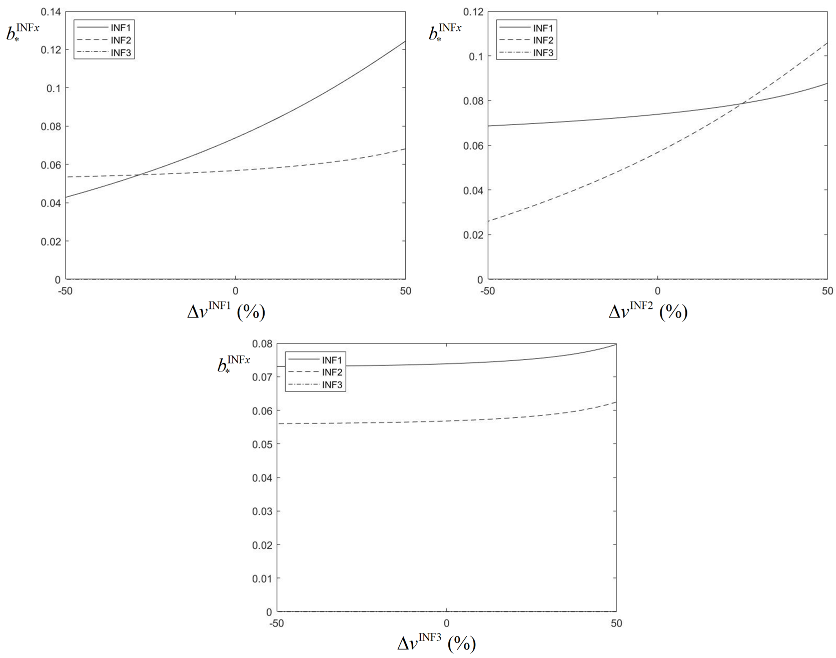

Fig. 3: Effect of INF's skill on reward shares. *Top Left*: Variation of INF1's skill; *Top Right*: Variation of INF2's skill; *Bottom Center*: Variation of INF3's skill

We observed similar results when we performed a sensitivity analysis on the endurance parameters of the INFs. SUP's optimal expected reward and probability of success increases when the endurance of an INF*x* increases. This is because an increase in the endurance of INF*x* lowers its effort cost, and thus INF*x* is able to choose a higher effort, improving its and thereby SUP's chance of success. As before, SUP's optimal strategy is to not offer an incentive to INF3.

Next, we analyze the impact of INFs' change coefficients on SUP. We varied the change coefficients such that the sum of the three $c^{\mathrm{INF}x}$ and the sum of the three $\tilde{c}^{\mathrm{INF}x}$ remained constant. We explored change coefficient changes for each of the three INFs. The three cases are summarized in Table 7. For case 1, we decreased $\tilde{c}^{\mathrm{INF1}}$ and redistributed the influence evenly across the other two change coefficients, $\tilde{c}^{\mathrm{INF2}}$ and $\tilde{c}^{\mathrm{INF3}}$. For case 2, we did the same for INF2's change coefficient c^{INF2}. For case 3, we increased $\tilde{c}^{\mathrm{INF3}}$ and c^{INF3} equally, and correspondingly decreased $\tilde{c}^{\mathrm{INF1}}$ and c^{INF2}.

Table 7: Change coefficient variation for sensitivity analysis

Agent	Variation for sensitivity analysis	Adjustment to satisfy unity
INF1	$\tilde{c}^{\mathrm{INF1}} - \Delta$	$\tilde{c}^{\mathrm{INF2}} + \Delta/2$, $\tilde{c}^{\mathrm{INF3}} + \Delta/2$
INF2	$c^{\mathrm{INF2}} - \Delta$	$c^{\mathrm{INF1}} + \Delta/2$, $c^{\mathrm{INF3}} + \Delta/2$
INF3	$\tilde{c}^{\mathrm{INF3}} + \Delta$, $c^{\mathrm{INF3}} + \Delta$	$\tilde{c}^{\mathrm{INF1}} - \Delta$, $c^{\mathrm{INF2}} - \Delta$

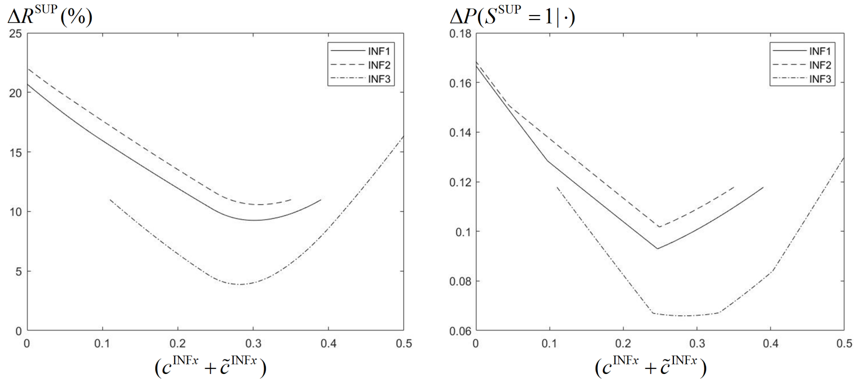

Fig. 4: Effect of influence on SUP. *Left*: Variation in optimal expected reward; *Right*: Variation in final probability of success

Figure 4 shows the results for ΔR^{SUP} (left) and $\Delta P(S^{\mathrm{SUP}} = 1|\cdot)$ (right). The sensitivity analysis for ΔR^{SUP} reveals a distinctive trough with local minima for all three cases. The reason for the minima and the convexity of ΔR^{SUP} is that SUP's expected reward decreases the more distributed the influence among the INFs. In addition, as the influence of one of the INFs drops below a certain threshold, SUP no longer incentivize that INFx, and thereby loses that agent's extra effort. The influence thresholds and incentivization transitions are the reasons for the kinks in the graph of $\Delta P(S^{\mathrm{SUP}} = 1|\cdot)$, which is a linear function of the change coefficients.

The incentivization transition points are shown explicitly in Figure 5. Comparing Figures 4 and 5, one can see that at the minima of ΔR^{SUP} and at the corresponding kinks in $\Delta P(S^{\mathrm{SUP}} = 1|\cdot)$, a transition in the incentivization of an INF occurs. Specifically, for case 1 (Figure 5, top left), moving from the right towards the minimum of ΔR^{SUP} in Figure 4, the incentivization of INF1 stops. INF2 is the only agent receiving an incentive, and thereby exerting an extra effort. The top right graph of Figure 5 shows that incentivization transition for INF 2, and the bottom center graph the one for INF 3.

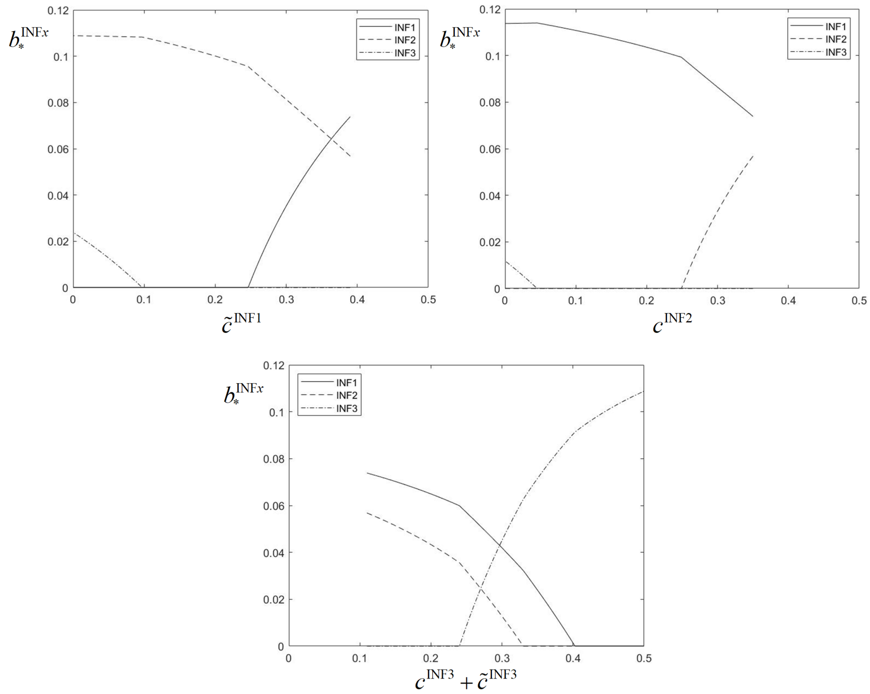

Fig. 5: Effect of influence on optimal reward share. *Top left*: Case 1–variation in INF1's influence; *Top right*: Case 2–variation in INF2's influence; *Bottom center*: Case 3–variation in INF3's influence

6 Conclusions

In this paper, we developed a model that can capture the interactions and interdependencies in supervised teams. The objective was to identify an effective incentivization scheme, and to determine the optimal decision responses of team members and the supervisor. In particular, we determined the optimal incentives, as a share of the supervisor's reward, and the optimal effort response by the team members. The model formulation is based on multiscale decision theory (MSDT). It uses MSDT's influence function concept, which describes how efforts and outcomes at lower levels affect the chances of success of the supervisor.

Our results show how optimal incentives and effort responses can be calculated. By only assuming general functional properties, including continuity, monotonicity and concavity/convexity, we proved that the supervisor can always determine an optimal and unique incentive for each team member through a Taylor series approximation.

We further showed that each team member's incentive can be calculated in isolation. Even though all team members jointly affect the supervisor, the supervisor can analyze one team member at a time, which reduces the computational complexity of her incentivization challenge.

For the team members, we showed that given the incentive offer by the supervisor, they can calculate their optimal effort response, and do so with little information and reduced communication requirements. Specifically, team members only need to know their personal information and aggregate information from and about the supervisor, but not any information from other team members. This result is non-intuitive, since the efforts and outcomes of the other team members affect the incentives they receive in the end.

In addition to the model analysis, we presented a data-driven estimation approach based on linear regression that the supervisor can use to assess and isolate the influence of each team member based on overall team performance data. We presented a three step approach that describes (1) what data needs to be obtained, (2) how qualitative information that the supervisor has on the team can be integrated, and (3) how the necessary model parameters can be computed through linear regression.

We illustrated the data-driven estimation approach through an example with a three-person team. We then built upon this example to develop a comprehensive example for the entire method presented in the paper, and showed how specific incentive and effort responses can be calculated. For the example, we assumed certain functional forms of effort, skill and effort cost. For these functions, we were able to derive closed-formed analytic solutions for optimal incentives and effort responses.

We performed a sensitivity analysis for the example and showed how changes in agent's skill and influence affect the supervisor and team members. We observed that when a team member's influence on the supervisor's chance of success is below a certain threshold, the supervisor no longer incentivizes that team member.

In future research, this two-level model can be extended to a multi-level model. Using the MSDT extension concepts [42], we expect that computationally scalable solutions can be found, and that under certain conditions, closed-form solutions can be derived.

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Appendix A

Using equation (12), the first order derivative of $e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})$ is given by

$$\frac{\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}b^{\mathrm{INF}x}} = -\left((c^{\mathrm{INF}x}+\tilde{c}^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})\frac{\mathrm{d}}{\mathrm{d}e^{\mathrm{INF}x}}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\right)\Bigg/$$
$$\left(\Big((h^{\mathrm{INF}x}-l^{\mathrm{INF}x})+b^{\mathrm{INF}x}(\tilde{c}^{\mathrm{INF}x}+c^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})\Big)\frac{\mathrm{d}^2}{\mathrm{d}(e^{\mathrm{INF}x})^2}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\right.$$
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}(e^{\mathrm{INF}x})^2}k^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\right). \quad (24)$$

Using equations (12) and (24), the second order derivative of $e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})$ is given by

$$\frac{\mathrm{d}^2e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}(b^{\mathrm{INF}x})^2} =$$
$$-\left(2(c^{\mathrm{INF}x}+\tilde{c}^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})\frac{\mathrm{d}^2}{\mathrm{d}(e^{\mathrm{INF}x})^2}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\times\frac{\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}b^{\mathrm{INF}x}}\right.$$
$$+\Big(b^{\mathrm{INF}x}(\tilde{c}^{\mathrm{INF}x}+c^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})+(h^{\mathrm{INF}x}-l^{\mathrm{INF}x})\Big)\frac{\mathrm{d}^3}{\mathrm{d}(e^{\mathrm{INF}x})^3}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)$$
$$\left.-\frac{\mathrm{d}^3}{\mathrm{d}(e^{\mathrm{INF}x})^3}k^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\Big)\times\left(\frac{\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}b^{\mathrm{INF}x}}\right)^2\right)\Bigg/$$

$$\left(\Big((h^{\mathrm{INF}x}-l^{\mathrm{INF}x})+b^{\mathrm{INF}x}(\tilde{c}^{\mathrm{INF}x}+c^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})\Big)\frac{\mathrm{d}^2}{\mathrm{d}(e^{\mathrm{INF}x})^2}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\right.$$
$$\left.\frac{\mathrm{d}^2}{\mathrm{d}(e^{\mathrm{INF}x})^2}k^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\right). \quad (25)$$

Appendix B

Proof of Theorem 2
From equation (24), we know that $\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big/\mathrm{d}b^{\mathrm{INF}x} > 0$ for all x and for all $b \in (0,1]^n$. In addition, if $\frac{\partial^2}{\partial(e^{\mathrm{INF}x})^2}\Big(\frac{\partial}{\partial e^{\mathrm{INF}x}}R^{\mathrm{INF}x}(e,b)\Big) \leq 0$, then

$$\Big(b^{\mathrm{INF}x}(c^{\mathrm{INF}x}+\tilde{c}^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})+(h^{\mathrm{INF}x}-l^{\mathrm{INF}x})\Big)\frac{\mathrm{d}^3}{\mathrm{d}(e^{\mathrm{INF}x})^3}\alpha^{\mathrm{INF}x}(e^{\mathrm{INF}x})$$
$$-\frac{\mathrm{d}^3}{\mathrm{d}(e^{\mathrm{INF}x})^3}k^{\mathrm{INF}x}(e^{\mathrm{INF}x}) \leq 0$$

for all x and for all e. From equation (25) it follows that $\mathrm{d}^2e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big/\mathrm{d}(b^{\mathrm{INF}x})^2 < 0$ for all x and for all $b \in (0,1]^n$. The second order partial derivatives of $R^{\mathrm{SUP}}(b)$ are

$$\frac{\partial^2 R^{\mathrm{SUP}}(b)}{\partial(b^{\mathrm{INF}x})^2} =$$
$$-2(c^{\mathrm{INF}x}+\tilde{c}^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})\frac{\mathrm{d}}{\mathrm{d}e^{\mathrm{INF}x}}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\times\frac{\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}b^{\mathrm{INF}x}}$$
$$+\Big(1-\sum_{k=1}^{n}b^{\mathrm{INF}k}\Big)(c^{\mathrm{INF}x}+\tilde{c}^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})\times\dots$$
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}(e^{\mathrm{INF}x})^2}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\times\Big(\frac{\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}b^{\mathrm{INF}x}}\Big)^2\right.$$
$$\left.+\frac{\mathrm{d}}{\mathrm{d}e^{\mathrm{INF}x}}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\times\frac{\mathrm{d}^2e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}(b^{\mathrm{INF}x})^2}\right) \quad (26)$$

and

$$\frac{\partial^2 R^{\mathrm{SUP}}(b)}{\partial b^{\mathrm{INF}x}\partial b^{\mathrm{INF}w}} = -(c^{\mathrm{INF}x}+\tilde{c}^{\mathrm{INF}x})(h^{\mathrm{SUP}}-l^{\mathrm{SUP}})\times\dots$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}e^{\mathrm{INF}x}}\alpha^{\mathrm{INF}x}\big(e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})\big)\times\frac{\mathrm{d}e_*^{\mathrm{INF}x}(b^{\mathrm{INF}x})}{\mathrm{d}b^{\mathrm{INF}x}}\right.$$
$$\left.+\frac{\mathrm{d}}{\mathrm{d}e^{\mathrm{INF}w}}\alpha^{\mathrm{INF}w}\big(e_*^{\mathrm{INF}w}(b^{\mathrm{INF}w})\big)\frac{\mathrm{d}e_*^{\mathrm{INF}w}(b^{\mathrm{INF}w})}{\mathrm{d}b^{\mathrm{INF}w}}\right) \quad (27)$$

for $w \in \{1,\dots,n\}$ and $w \neq x$. Since all the second order partial derivatives of $R^{\mathrm{SUP}}(b)$ are negative, we know that the Hessian matrix of $R^{\mathrm{SUP}}(b)$ is negative definite for all $b \in (0,1]^n$. This implies that $R^{\mathrm{SUP}}(b)$ is strictly concave for all $b \in (0,1]^n$. □