COUNTEREXAMPLES TO QUASICONCAVITY FOR THE HEAT EQUATION

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Abstract. We construct solutions to the heat equation on convex rings showing that quasiconcavity may not be preserved along the flow, even for smooth and subharmonic initial data.

1. Introduction

Let Ω_0 and Ω_1 be convex open sets with smooth boundary in $\mathbb{R}^n$ with $\overline{\Omega}_1 \subset \Omega_0$. Assume that Ω_1 contains the origin. Denote by $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ the open convex ring. We say that a function $u(x)$ on $\overline{\Omega}$ is *quasiconcave* if the sets

$$\{x \in \overline{\Omega} \mid u(x) \geq c\} \cup \Omega_1$$

are convex subsets of $\mathbb{R}^n$ for every $c \in \mathbb{R}$. Fix T with $0 < T \leq \infty$. A function $u = u(x,t)$ on $\overline{\Omega} \times [0,T)$ is called *space-time quasiconcave* if the sets

$$\{(x,t) \in \overline{\Omega} \times [0,T) \mid u(x,t) \geq c\} \cup (\Omega_1 \times [0,T))$$

are convex subsets of $\mathbb{R}^{n+1}$ for every $c \in \mathbb{R}$. Note that space-time quasiconcavity of $u(x,t)$ implies that $x \mapsto u(x,t)$ is quasiconcave on $\overline{\Omega}$ for each t.

It is a classical result that if u is a harmonic function on Ω satisfying the Dirichlet boundary conditions

$$u|_{\partial\Omega_0} = 0, \quad u|_{\partial\Omega_1} = 1 \tag{1.1}$$

then u is quasiconcave [1, 17, 29]. This result has been extended to solutions u of more general elliptic PDEs, where there is a general principle that convexity properties of Ω_0 and Ω_1 imply convexity of the superlevel sets of u. These results are proved via "macroscopic" approaches involving functions of two points which could be far apart, or "microscopic" approaches using functions of the principal curvatures of the level sets together with constant rank theorems. See for example [2, 3, 4, 5, 7, 8, 9, 10, 11, 24, 25, 26, 30, 31, 32, 33] and the references therein. On the other hand, convexity properties fail for solutions to some elliptic PDEs [19, 34].

There has been considerable interest in *parabolic* versions of these classical results. Parabolic constant rank theorems in rather general contexts have been established by Hu-Ma [20], Chen-Hu [12] and Chen-Shi [13]. An older result of Borell [6] assumes that the initial data is identically zero and shows that the solution $u(x,t)$ to the heat equation with boundary conditions (1.1) is space-time quasiconcave. This result has been extended to more general parabolic equations by Ishige-Salani [22, 23]. However,

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the assumption of identically vanishing initial data is rather restrictive. This begs the question: what assumption on the initial data is necessary to ensure space-time quasiconcavity of the solution to the heat equation? Ishige-Salani [21] gave examples to show that quasiconcavity of the initial data is not sufficient. A natural condition considered in [14, 15] is that u_0 in addition be subharmonic (with sufficient regularity), namely $\Delta u_0 \geq 0$. In this paper we provide a counterexample to show that this is still not sufficient to ensure quasiconcavity of the solution to the heat equation.

More precisely, we consider a classical solution u of the following problem:

$$\begin{cases} \partial u/\partial t = \Delta u, & \text{on } \Omega \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega \\ u(x,t) = 0, & (x,t) \in \partial\Omega_0 \times [0,T) \\ u(x,t) = 1, & (x,t) \in \partial\Omega_1 \times [0,T), \end{cases} \tag{1.2}$$

for $0 < T \leq \infty$. Here u_0 is a smooth function on $\overline{\Omega}$ which satisfies the conditions

$$\begin{aligned} &u_0 = 1 \text{ on } \partial\Omega_1, \quad u_0 = 0 \text{ on } \partial\Omega_0, \quad x \cdot \nabla u_0(x) \leq 0 \text{ on } \Omega \\ &\Delta u_0 \geq 0 \text{ on } \Omega \quad \text{but not identically zero.} \end{aligned} \tag{1.3}$$

We call such a function u_0 *admissible.*

We consider the following question: if an admissible u_0 is quasiconcave on $\overline{\Omega}$, does it follow that the solution $u(x,t)$ to (1.2) is space-time quasiconcave on $\overline{\Omega} \times [0,T)$? If not, is $x \mapsto u(x,t)$ quasiconcave on $\overline{\Omega}$ for each $t > 0$?

We construct a counterexample to show that the answer to both of these questions is negative.

Theorem 1.1. *For any $n \geq 2$, let Ω_1 and Ω_0 be balls in $\mathbb{R}^n$ centered at the origin, of radii 1 and 2 respectively, so that Ω is the annulus $1 < r < 2$. There is an admissible quasiconcave function u_0 with the following properties:*

- (i) *The solution $u(x,t)$ to (1.2) is smooth on $\Omega \times (0,\infty)$ and continuous on $\overline{\Omega} \times [0,\infty)$.*
- (ii) *There exists $t_0 > 0$ such that the function $x \mapsto u(x,t_0)$ fails to be quasiconcave on $\overline{\Omega}$.*

This example implies that the statement of [15, Theorem 3] (see the discussion in [22, Section 7]) requires additional hypotheses.

Our construction in Theorem 1.1 is based on the simple observation that the union of interiors of a sphere and a non-spherical ellipsoid is non-convex unless one is contained in the other. We use this observation as follows. We first find a radially symmetric admissible function V which is close to 1 near the boundary of Ω_1 and drops off rapidly to zero. For every positive time, the level sets of the heat flow solution starting from V will then give a foliation of Ω by spheres. We then construct an admissible function W whose level sets are spherical near the boundary of Ω_1 but non-spherical ellipsoids as one goes outwards. We choose $u_0 = (1-\varepsilon)V + \varepsilon W$, for $\varepsilon > 0$ small, as initial data. The relatively large radially symmetric heat distribution of $(1-\varepsilon)V$ quickly emanates out and interacts with the ellipsoidal level sets of εW to give a non-convex superlevel set after some positive time. The proof of Theorem 1.1, given in Section 2, makes this heuristic argument precise.

Note that by necessity our counterexample is not radially symmetric, and must have dimension $n > 1$. If radial symmetry is imposed for u_0, which implies quasiconcavity of $x \mapsto u(x,t)$ for each t (since the superlevel sets are balls in $\mathbb{R}^n$) it is natural to ask whether the stronger condition of space-time quasiconcavity follows. Our next counterexample shows that the answer to this is again negative for any $n \geq 1$.

Theorem 1.2. *For any $n \geq 1$, let Ω_1 and Ω_0 be balls in $\mathbb{R}^n$ of radii R and $R+1$ respectively, for a constant $R > 1$. For R sufficiently large, there is an admissible function u_0 on $\overline{\Omega} = \{R \leq r \leq R+1\} \subset \mathbb{R}^n$ with the following properties:*

(i) *u_0 is radially symmetric and hence if $u(x,t)$ solves (1.2) then $x \mapsto u(x,t)$ is quasiconcave on $\overline{\Omega}$ for every $t \geq 0$.*
(ii) *$u(x,t)$ is smooth on $\overline{\Omega} \times [0,\infty)$.*
(iii) *$u(x,t)$ is not space-time quasiconcave on $\overline{\Omega} \times [0,T)$ for any $T > 0$.*

In Theorem 1.2 any space-time level set $\partial\Omega_c := \{(x,t) \in \overline{\Omega} \times [0,\infty) \mid u(x,t) = c\}$ for $c \in (0,1)$ will be given by a graph $t = f(|x|)$ where $f(r)$ is a smooth strictly increasing function defined on some interval $[r_0, r_1)$ where $f(r_0) = 0$. In particular, f is defined implicitly by $u(r, f(r)) = c$ and differentiating this and using (1.2) gives

$$f''(r) = \frac{-1}{u_t}(u_{rr} + 2u_{rt}f' + u_{tt}(f')^2)$$

We show that by solving an ordinary differential equation, we may choose the function u_0 so that the right hand side above is negative at $(r_0, 0)$, implying $f''(r_0) < 0$ and thus $\partial\Omega_c$ is not convex. The details of this argument are given in Section 3 where we prove Theorem 1.2.

Finally, in Section 4 we give a different counterexample to space-time quasiconcavity using a "two-point function" as in [35] and inspired by the work of Rosay-Rudin [30]. It satisfies properties (i), (ii) of Theorem 1.2, but (iii) must be replaced by

(iii)* $u(x,t)$ is not space-time quasiconcave on $\overline{\Omega}_0 \times [0,T)$ for **some** $T > 0$.

The argument using the two-point function is perhaps slightly more intuitive than that of Theorem 1.2 and the counterexample is defined on the annulus $\{1 < r < 2\}$.

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2. A counterexample to quasiconcavity

Let Ω_0, Ω_1 and Ω be as in the introduction. We first gather some well-known facts about solutions to (1.2).

Proposition 2.1. *Let u_0 be an admissible function on $\overline{\Omega}$. Then there exists a unique continuous solution $u(x,t)$ to (1.2) on $\overline{\Omega} \times [0,\infty)$ which is smooth on $\Omega \times (0,\infty)$ and satisfies the following conditions for $(x,t) \in \Omega \times (0,\infty)$*

(i) $0 < u(x,t) < 1$.
(ii) $u_t(x,t) = \Delta u(x,t) > 0$.
(iii) $x \cdot \nabla u(x,t) < 0$.

Moreover, as $t \to \infty$, $u(x,t)$ converges smoothly on Ω to the harmonic function u_∞ with boundary conditions $u_\infty|_{\partial\Omega_0} = 0$ and $u_\infty|_{\partial\Omega_1} = 1$.

Proof. The existence of a unique solution $u(x,t)$ to (1.2) with the stated regularity, and the convergence as $t \to \infty$ are classical, see for example [16]. From (1.3) we have $0 \leq u_0 \leq 1$ and then (i) follows from the strong maximum principle for parabolic equations. Parts (ii) and (iii) are proved in [15, Lemma 1] and are also consequences of the maximum principle. $\square$

We now start the proof of Theorem 1.1. Let $\Omega = \{1 < r < 2\}$ be as in the statement of the theorem and we assume for the rest of this section that $n \geq 2$. Part (i) of Theorem 1.1 is a consequence of the above proposition. For part (ii), we begin by defining two auxiliary functions V_ρ and W.

Lemma 2.1. *For any $\rho \in (1, 3/2]$, define a radially symmetric function $V_\rho = V_\rho(r)$ on $\overline{\Omega}$ by*

$$V_\rho(r) = \begin{cases} \exp\left(\frac{n}{r-\rho} - \frac{n}{1-\rho}\right), & 1 \leq r < \rho \\ 0, & \rho \leq r \leq 2. \end{cases}$$

Then V_ρ is an admissible function.

Proof. We drop the ρ subscript. Observe that $V(r) \geq 0$ is smooth, decreasing on $[1,2]$, satisfies $V(1) = 1$ and

$$\begin{aligned} (\Delta V)(r) &= V''(r) + \frac{(n-1)V'(r)}{r} \\ &= nV(r)\left(\frac{2r(r-\rho) + nr - (n-1)(r-\rho)^2}{r(r-\rho)^4}\right) \\ &\geq \frac{n}{4r(r-\rho)^4} V(r) \geq 0 \end{aligned} \tag{2.1}$$

where the second-to-last inequality follows from the inequalities $|r - \rho| \leq 1/2$ and $1 \leq r \leq 3/2$ when $r < \rho$. $\square$

We now use V_ρ to define a non-radially symmetric function W.

Lemma 2.2. *Fix r_0, r_1 with $1 < r_0 < r_1 \leq 3/2$. There exists an admissible function W on $\overline{\Omega}$ with the following properties.*

(i) *W is radially symmetric on $1 \leq r \leq r_0$.*

(ii) *There exists a smooth strictly decreasing function b on $[r_0, r_1]$ with $b(r_0) = 1$ and $b(r_1) \in (0,1)$ such that if we define E_R to be the ellipsoid with equation*

$$b(R)^2 x_1^2 + x_2^2 + \cdots + x_n^2 = R^2, \quad \textit{for } R \in [r_0, r_1], \tag{2.2}$$

then the level sets $\{W = c\}$ for $0 < c < W(r_0)$ are the non-spherical ellipsoids E_R for $R \in (r_0, r_1)$.

(iii) *W vanishes outside the ellipsoid E_{r_1}.*

Proof. Let $V = V_{r_1}$ be as in Lemma 2.1 defined with $\rho = r_1$. Regarding V as a function of $x_1, \ldots, x_n$ we compute for any $i, j = 1, ..., n$:

$$V_{x_i} = -V(r)\frac{nx_i}{r(r-r_1)^2}$$

and

$$V_{x_i x_j} = V(r)\left(\frac{n^2 x_i x_j}{r^2(r-r_1)^4} + \frac{2n x_i x_j}{r^2(r-r_1)^3} - \frac{n(r^2\delta_{ij} - x_i x_j)}{r^3(r-r_1)^2}\right).$$

It follows from this and (2.1) that for some constant $\beta = \beta(n)$,

$$\Delta V + \sum_{i,j} c_{i,j} V_{x_i x_j} + \sum_i c_i V_{x_i} \geq 0 \tag{2.3}$$

as long as $|c_{i,j}|, |c_i| \leq \beta$.

Fix a smooth non-decreasing function $a(r) : [1,2] \to [0,1]$ with $a(1) = 0$ and $a(2) = 1$ and with $\{r \mid a'(r) > 0\} = (r_0, r_1)$. Let $b(r) = 1 - \kappa a(r)$ for some constant $\kappa \in (0,1)$ to be determined. Treating b as a rotationally symmetric function of $x_1, \ldots, x_n$ consider the map $(y_1, \ldots, y_n) = (x_1/b, x_2, \ldots, x_n)$, which we will write as $y = \Psi(x)$. Note that Ψ is invertible and $\Psi(x) \to \mathrm{Id}$ as $\kappa \to 0$ where the convergence is uniform in any C^k norm on $\overline{\Omega}$. It follows from the inverse function theorem that we likewise have $\Psi^{-1}(y)|_{\overline{\Omega}} \to \mathrm{Id}$ uniformly in any C^k norm on $\overline{\Omega}$ as $\kappa \to 0$.

The map $\Psi(x)$ is the identity on $\{1 \leq r \leq r_0\}$ and takes the sphere $x_1^2 + \cdots + x_n^2 = R^2$ to the ellipsoid $b(R)^2 y_1^2 + y_2^2 + \cdots + y_n^2 = R^2$ which is non-spherical exactly when $R \in (r_0, 2]$. We choose κ sufficiently small so that the ellipsoid $(1-\kappa)^2 y_1^2 + y_2^2 + \cdots + y_n^2 = (r_1)^2$ is contained inside the sphere of radius 2.

Define $W : \overline{\Omega} \to \mathbb{R}$ by $W(y) = V(x(y))$ for $x(y) = \Psi^{-1}(y)$. Note that $W = 0$ outside the ellipsoid $(1-\kappa)^2 y_1^2 + y_2^2 + \cdots + y_n^2 = (r_1)^2$. Thus $W(y)$ satisfies (i), (ii) and (iii) of the Lemma.

To see that $y \cdot (\nabla W)(y) \leq 0$ we compute,

$$y \cdot (\nabla W)(y) = \sum_{i,j} y_i V_{x_j}(x(y)) \frac{\partial x_j}{\partial y_i} = -(1+E) \frac{nr}{(r - r_1)^2}(x(y)) V(x(y))$$

where E is an "error" term which converges uniformly to zero as κ tends to zero. Hence $y \cdot \nabla W \leq 0$ for κ sufficiently small.

All that remains is to show now is that $\Delta W \geq 0$ in Ω. We compute

$$\begin{aligned}(\Delta W)(y) &= \sum_{i,j,k} V_{x_i x_j}(x(y)) \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_k} + \sum_{i,k} V_{x_i}(x(y)) \frac{\partial^2 x_i}{\partial y_k^2} \\ &= (\Delta V)(x(y)) + \sum_{i,j} c_{i,j} V_{x_i x_j}(x(y)) + \sum_i c_i V_{x_i}(x(y)),\end{aligned}$$

for $c_{i,j}$ and c_i which converge uniformly to zero as κ tends to zero. From (2.3) it follows that $\Delta W \geq 0$ for κ sufficiently small. □

Next we have an elementary lemma about radially symmetric subharmonic functions.

Lemma 2.3. *Let $f = f(r)$ be a smooth radially symmetric function on Ω with $f(1) = 1$, $f(2) = 0$, $f_r \leq 0$ and $\Delta f \geq 0$. Fix r_0, r_1 with $1 < r_0 < r_1 < 2$. Then*

$$f(r_1) \leq (1 - \sigma) f(r_0),$$

for $\sigma = (r_1 - r_0)/(2 + r_1 - 2r_0) > 0$.

Proof. We begin by showing

$$-f_r \geq \frac{1}{2 - r_0} f, \quad \text{on } [r_0, r_1]. \tag{2.4}$$

Indeed the condition $\Delta f \geq 0$ implies that $r^{n-1}(-f_r(r))$ is nonincreasing in r. Hence for $s \in [r_0, r_1]$ we have

$$(2-s)s^{n-1}(-f_r(s)) \geq \int_s^2 r^{n-1}(-f_r(r))dr \geq s^{n-1} \int_s^2 (-f_r(r))dr = s^{n-1} f(s),$$

where for the final equality we used $f(2) = 0$, and (2.4) follows.

Next compute

$$f(r_0) - f(r_1) = \int_{r_0}^{r_1} (-f_r(r))dr \geq \frac{1}{2-r_0} \int_{r_0}^{r_1} f(r)dr \geq \frac{(r_1 - r_0)}{2 - r_0} f(r_1),$$

where we recall for the last inequality that f is decreasing in r. The result follows. □

We can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $V = V_{5/4}$ be as in Lemma 2.1. Let W be as in Lemma 2.2 with $r_0 = 5/4$ and $r_1 = 3/2$. We will write $v(t)$ and $w(t)$ for the solutions to (1.2) with initial conditions V and W respectively. For $\varepsilon \in (0,1)$ to be determined, define

$$u_0 = (1-\varepsilon)V + \varepsilon W,$$

so that $u(t) = (1-\varepsilon)v(t) + \varepsilon w(t)$ is the solution of (1.2) starting at u_0. Clearly u_0 is an admissible function. In addition, note that for $\{1 < r \leq 5/4\}$ the level sets of u_0 are spheres and, since V vanishes for $r > 5/4$, the level sets for u_0 on $\{r > 5/4\}$ are the same as those of W. In particular, the level sets of u_0 are convex and so u_0 is quasiconcave.

Let $\eta_1, \eta_2 \in (0, 1/4)$ be small constants to be determined and write $R^- = 3/2 - \eta_1$. Then on the non-spherical ellipsoid E_{R^-} with equation given by (2.2), W takes a constant value, $W = s > 0$, say, which depends on η_1. We have $W = 0$ on and outside the ellipsoid $E_{3/2}$. Define $R^+ = 3/2 + \eta_2$ and write S_{R^+} for the sphere of radius R^+. Observe that for η_2 sufficiently small we can find $X \in S_{R^+}$ and $Y' \in E_{3/2}$ such that $(X + Y')/2$ lies outside both S_{R^+} and $E_{3/2}$. Next by choosing η_1 sufficiently small we can find a point $Y \in E_{R^-}$ close to Y' so that $Z = (X+Y)/2$ lies outside both S_{R^+} and $E_{3/2}$. See Figure 1. Note that now η_1 is chosen, s is a fixed positive number.

Since $V(X) = W(Z) = 0$, and using Proposition 2.1, there exist continuous functions $\alpha(t), \beta(t)$ which vanish at $t = 0$ and are positive for $t > 0$ such that

$$v(X,t) = s\alpha(t), \quad w(Z,t) = \beta(t).$$

Next we use Lemma 2.3 to see that there exists a constant $\sigma > 0$ independent of t such that

$$v(Z,t) \leq s(1-\sigma)\alpha(t).$$

Indeed, the radially symmetric function $v(\cdot, t)$ satisfies the conditions of Lemma 2.3 and Z lies at a fixed distance outside the sphere S_{R^+} which contains X.

Next note that since $W(Y) = s$ and $w_t \geq 0$, we have

$$w(Y,t) \geq s$$

for all $t \geq 0$. Choose a small time $t_0 > 0$ such that

$$\varepsilon := \alpha(t_0) < 1/2, \quad \text{and } \beta(t_0) < \frac{\sigma s}{2}.$$

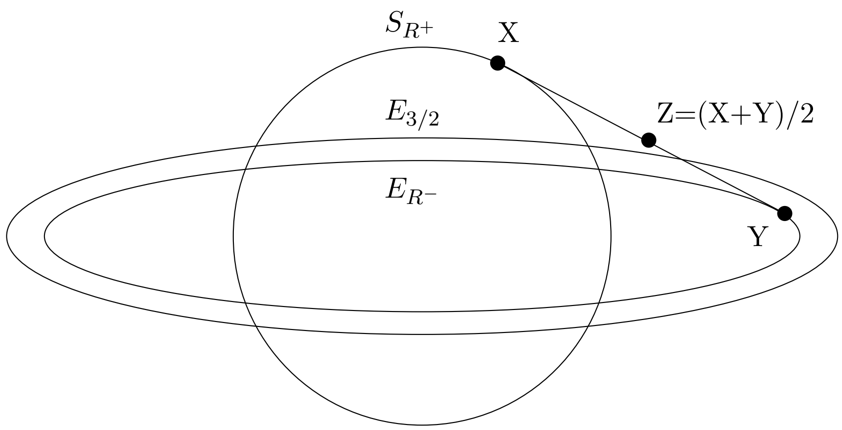

FIGURE 1. Schematic diagram showing the ellipsoids E_{R^-} and $E_{3/2}$, the sphere S_{R^+} and the points X, Y, Z.

Compute that

$$u(Z, t_0) \leq (1-\varepsilon)s(1-\sigma)\varepsilon + \frac{\varepsilon\sigma s}{2} < (1-\varepsilon)s\varepsilon.$$

But

$$u(X, t_0) \geq (1-\varepsilon)s\varepsilon, \quad u(Y, t_0) \geq \varepsilon s > (1-\varepsilon)s\varepsilon.$$

Hence X and Y lie in the superlevel set $\{P \mid u(P, t_0) \geq (1-\varepsilon)s\varepsilon\}$ but $Z = (X+Y)/2$ does not, showing that this superlevel set is not convex. $\square$

3. A radially symmetric example

In this section we give the proof of Theorem 1.2. First, define a smooth function $h : [0,1] \to \mathbb{R}$ to have the following properties

(a) $1/10 \geq h(r) > 0$ for $r \in (0,1)$.
(b) $h(r) = 1/10$ for $r \in [1/4, 3/4]$.
(c) $h^{(k)}(0) = 0 = h^{(k)}(1)$ for all $k = 0, 1, 2, \ldots$.

Next, for a constant $R > 1$, define $v_R : [0,1] \to \mathbb{R}$ to be the solution of the Dirichlet problem

$$v_R''(r) + (n-1)\frac{v_R'(r)}{r+R} = h(r), \ 0 < r < 1, \quad v_R(0) = 1, \ v_R(1) = 0. \tag{3.1}$$

We will determine the constant R later. By standard elliptic estimates [18] we have that $v_R(r)$ and its derivatives are bounded on $[0,1]$ uniformly with respect to $R > 1$.

Remark 3.1. In fact in what follows we only need that $|v_R'(r)| \leq C$ for a constant C independent of R. In this case we can actually write down the solution v_R explicitly and prove this directly. For example, when $n = 2$, $v_R(r)$ is given by

$$v_R(r) = \int_0^r \frac{1}{x+R}\left(\int_0^x (y+R)h(y)dy\right)dx + c_1 \log(r+R) + c_2$$

for constants c_1 and c_2 given by

$$c_1 = \frac{-1 - \int_0^1 \frac{1}{x+R}\left(\int_0^x (y+R)h(y)dy\right)dx}{\log((1+R)/R)}, \quad c_2 = 1 - c_1 \log R.$$

To see that $|v_R'(r)| \le C$ for a constant C independent of R, note that as $R \to \infty$ the term

$$\left|\frac{c_1}{r+R}\right| = \mathrm{O}\left(\frac{1/R}{\log(1+\frac{1}{R})}\right)$$

remains bounded.

We can now start the proof of Theorem 1.2.

Proof of Theorem 1.2. Define our radially symmetric function u_0 on $\overline{\Omega}$ by

$$u_0(r) = v_R(r-R), \quad \text{for } R \le r \le R+1,$$

which satisfies the boundary conditions $u_0(R) = 1$, $u_0(R+1) = 0$. Moreover, from (3.1) we have

$$\Delta u_0(r) = h(r-R), \quad \text{for } R \le r \le R+1.$$

Now notice that by the definition of h we have for $r = R$ or $r = R+1$,

$$\Delta^k u_0(r) = 0, \qquad \text{for every } k \ge 1.$$

Let $u(r,t)$ be the rotationally symmetric solution to (1.2) on Ω with initial condition u_0. Then from [28, Theorem 5.2] (or [27, Theorem 10.4.1]) it follows that the function $u(r,t)$ extends to a smooth function on $\overline{\Omega} \times [0,\infty)$.

Next we show that for sufficiently large R, the function u_0 satisfies the hypotheses of Proposition 2.1.

Lemma 3.1. *For sufficiently large R we have*

(i) *$u_0(r)$ is strictly decreasing in r.*
(ii) *$u_0''(R+1/2) \ge 1/10$.*
(iii) *$\Delta u_0 > 0$ for $R < r < R+1$.*

Proof. For (i), note that for all $r \in [R, R+1]$ we have $|u_0'(r)| \le C$ for some C independent of R as observed above. Thus for all $r \in [R, R+1]$ we have

$$\begin{aligned} |u_0''(r)| &\le |\Delta u_0(r)| + (n-1)|u_0'(r)/r| \\ &= |h(r-R)| + (n-1)|u_0'(r)/r| \\ &\le 1/10 + (n-1)C/R \le 1/5 \end{aligned}$$

as long as R is larger than $10C(n-1)$. On the other hand, by the boundary conditions on u_0 we have by the Mean Value Theorem that $u_0'(r_0) = -1$ at some point r_0 in $[R, R+1]$. Hence $u_0'(r) \le -1 + 1/5$ for all $r \in [R, R+1]$, giving (i).

For (ii), we have

$$\begin{aligned} u_0''(R+1/2) &= \Delta u_0(R+1/2) - (n-1)u_0'(R+1/2)/(R+1/2) \\ &\ge \Delta u_0(R+1/2) \\ &= h(1/2) = 1/10 \end{aligned}$$

where in the second inequality we have used part (i) and in the third inequality we have used the definition of u_0 and the property (b) of h.

Finally (iii) follows from the definition of h. $\square$

We now fix R as in the lemma above. Define $c = u_0(R + 1/2)$ and let $u(x,t)$ be the solution to (1.2). We consider the space-time superlevel set

$$\Omega_c := \{(x,t) \in \overline{\Omega} \times [0,\infty) : u(x,t) \geq c\} \cup (\Omega_1 \times [0,\infty)).$$

As noted above, $u(x,t)$ is smooth on $\overline{\Omega} \times [0,\infty)$ while from Lemma 3.1 and Proposition 2.1, $u_t(x,t)$ is smooth and strictly positive on $\Omega \times [0,\infty)$. By the Implicit Function Theorem, we may write $\partial\Omega_c = \{(x,t) \in \overline{\Omega} \times [0,\infty) : u(x,t) = c\}$ as the radial graph of the equation $t = f(r)$ where $u(r, f(r)) = c$ and $f(r)$ is a smooth increasing function on $[R+1/2, R+1/2+\varepsilon)$ for some $\varepsilon > 0$. Note that $f(R+1/2) = 0$. Differentiating the defining equation for f we obtain

$$f'(r) = -\frac{u_r}{u_t} \tag{3.2}$$

where the functions above are evaluated at $(r, f(r))$, and

$$f''(r) = \frac{-1}{u_t}(u_{rr} + 2u_{rt}f' + u_{tt}(f')^2) \tag{3.3}$$

where again these are all evaluated at $(r, f(r))$. Now evaluating the above at $(r, f(r)) = (R+1/2, 0)$, and noting that $u_t(r, 0) = \Delta u_0(r) = h(r-R) = 1/10$ in some neighborhood of $r = R + 1/2$, hence $u_{tr}(R + 1/2, 0) = u_{tt}(R + 1/2, 0) = 0$, we get

$$f''(R + 1/2) = -10u_0''(R + 1/2) \leq -1 < 0 \tag{3.4}$$

by Lemma 3.1. This contradicts that $\partial\Omega_c$, which is defined by $t = f(r)$, is convex in $\mathbb{R}^{n+1}$. $\square$

4. Two-point functions

In this section we discuss a different way to find a counterexample to space-time quasiconcavity using a two-point function as in [35] (see also [30]). We work in dimension $n \geq 1$ with the domain $\Omega = \{1 < r < 2\}$. Let $u(x,t)$ be a smooth function on $\overline{\Omega} \times [0,\infty)$. Consider the two-point function

$$\mathcal{H}((x,s),(y,t)) = (Du(y,t) - Du(x,s)) \cdot (y-x) + (u_t(y,t) - u_t(x,s))(t-s),$$

restricted to $(x,s),(y,t) \in \overline{\Omega} \times [0,\infty)$ with $u(x,s) = u(y,t)$. If $u(x,t)$ is space-time quasiconcave then $\mathcal{H} \leq 0$. Indeed, note that for a smooth function w in $\mathbb{R}^{n+1}$, if the superlevel set $\{w > c\}$ is convex with a smooth boundary that contains two points X and Y then $(Dw(Y) - Dw(X)) \cdot (Y - X) \leq 0$ since the vectors $Dw(X), Dw(Y)$ point in the inward normal direction. Applying this to $X = (x,s)$ and $Y = (y,t)$ in $\overline{\Omega} \times (0,\infty)$ with $u(x,s) = u(y,t)$, and using the continuity of $\mathcal{H}$, we have $\mathcal{H} \leq 0$ on its domain of definition.

For our counterexample, we construct a radially symmetric admissible function u_0 such that the corresponding solution $u(x,t)$ of the heat equation (1.2) is smooth on $\overline{\Omega} \times [0,\infty)$ and has $\mathcal{H}$ strictly positive somewhere.

Let $\varepsilon > 0$ be a small constant to be determined and let $g : [1,2] \to [0,\infty)$ be a smooth function with the following properties:

(a) $g(r) = 1/(2\varepsilon)$ for $1 \le r \le 1 + \varepsilon$, and $g(r) = 0$ for $2 - \varepsilon \le r \le 2$.
(b) g is decreasing on $[1, 2]$.
(c) $\displaystyle\int_1^2 r^{1-n} g(r) dr = 1$.

We then define a smooth function u_0 on $[1, 2]$ by

$$u_0(r) = -\int_1^r s^{1-n} g(s) ds + 1, \quad r \in [1, 2].$$

It is straightforward to check that u_0 is an admissible function. Moreover, Δu_0 vanishes identically in a neighborhood of $r = 1$ and $r = 2$. Let $u(x, t)$ be the corresponding solution of (1.2), which by [28, Theorem 5.2] is smooth on $\overline{\Omega} \times [0, \infty)$. From Proposition 2.1 we know that $u(x, t)$ converges smoothly as $t \to \infty$ to the harmonic function u_∞ given by

$$u_\infty(r) = \begin{cases} 2 - r, & n = 1 \\ 1 - \frac{\log r}{\log 2}, & n = 2 \\ \frac{(2/r)^{n-2} - 1}{2^{n-2} - 1}, & n > 2. \end{cases}$$

We now choose our points (x, s) and (y, t). We choose x and y to lie in the line $x_2 = \cdots = x_n = 0$. Pick $x = (1 + \varepsilon/2, 0, \ldots, 0)$ and $s = 0$. By the definition of u_0 we have $u(x, 0) = u_0(1 + \varepsilon/2) \approx 3/4$. Let γ solve $u_\infty(1 + \gamma) = u(x, 0)$, which satisfies $\gamma > c(n)$ for a constant $c(n) \in (0, 1)$ depending only on n. Writing $y(t) = (y_1(t), 0, \ldots, 0)$ solving $u(y(t), t) = u(x, 0)$ we have $y_1(t) \to 1 + \gamma$ and $u_r(y(t), t) \to (u_\infty)_r(1 + \gamma)$ as $t \to \infty$. Then for t sufficiently large and $\varepsilon > 0$ sufficiently small we have $y_1 > 1 + \varepsilon/2$ and $|u_r(y(t), t)| \le C$ for a uniform C. Writing y for $y(t)$ we have

$$\mathcal{H}((x, s), (y, t)) = (u_r(y, t) - u_r(x, 0))(y_1 - (1 + \varepsilon/2)) + (u_t(y, t) - u_t(x, 0))(t - 0).$$

From the definition of u_0 we have $u_r(x, 0) \approx -1/(2\varepsilon)$ and $u_t(x, 0) = \Delta u(x, 0) = 0$. On the other hand, from Proposition 2.1, $u_t(y, t) > 0$. Hence for ε sufficiently small, $\mathcal{H} > 0$.

Remark 4.1. As in [35] one can show that a maximum principle holds for the quantity $\mathcal{H}$ using a parabolic version of a Lemma of Rosay-Rudin [30]. This rules out $\mathcal{H}$ obtaining a positive interior maximum. However, it does not rule out a positive maximum occuring at a point $((x, s), (y, t))$ with $s = 0$ which would be needed to prove that quasiconcavity is preserved for the heat equation.

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