# Edge Expansion and Spectral Gap of Nonnegative Matrices 

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#### Abstract

The classic graphical Cheeger inequalities state that if $M$ is an $n \times n$ symmetric doubly stochastic matrix, then $$
\frac{1-\lambda_{2}(M)}{2} \leq \phi(M) \leq \sqrt{2 \cdot\left(1-\lambda_{2}(M)\right)}
$$


where $\phi(M)=\min _{S \subseteq[n],|S| \leq n / 2}\left(\frac{1}{|S|} \sum_{i \in S, j \notin S} M_{i, j}\right)$ is the edge expansion of $M$, and $\lambda_{2}(M)$ is the second largest eigenvalue of $M$. We study the relationship between $\phi(A)$ and the spectral gap $1-\operatorname{Re} \lambda_{2}(A)$ for any doubly stochastic matrix $A$ (not necessarily symmetric), where $\lambda_{2}(A)$ is a nontrivial eigenvalue of $A$ with maximum real part. Fiedler showed that the upper bound on $\phi(A)$ is unaffected, i.e., $\phi(A) \leq \sqrt{2 \cdot\left(1-\operatorname{Re} \lambda_{2}(A)\right)}$. With regards to the lower bound on $\phi(A)$, there are known constructions with

$$
\phi(A) \in \Theta\left(\frac{1-\operatorname{Re} \lambda_{2}(A)}{\log n}\right)
$$

indicating that at least a mild dependence on $n$ is necessary to lower bound $\phi(A)$.

In our first result, we provide an exponentially better construction of $n \times n$ doubly stochastic matrices $A_{n}$, for which

$$
\phi\left(A_{n}\right) \leq \frac{1-\operatorname{Re} \lambda_{2}\left(A_{n}\right)}{\sqrt{n}}
$$

In fact, all nontrivial eigenvalues of our matrices are 0 , even though the matrices are highly nonexpanding. We further show that this bound is in the correct range (up to the exponent of $n$ ), by showing that for any doubly stochastic matrix $A$,

$$
\phi(A) \geq \frac{1-\operatorname{Re} \lambda_{2}(A)}{35 \cdot n}
$$

As a consequence, unlike the symmetric case, there is a (necessary) loss of a factor of $n^{\alpha}$ for $\frac{1}{2} \leq \alpha \leq 1$ in lower bounding $\phi$ by the spectral gap in the nonsymmetric setting.

Our second result extends these bounds to general matrices $R$ with nonnegative entries, to obtain a twosided gapped refinement of the Perron-Frobenius theorem.

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Recall from the Perron-Frobenius theorem that for such $R$, there is a nonnegative eigenvalue $r$ such that all eigenvalues of $R$ lie within the closed disk of radius $r$ about 0 . Further, if $R$ is irreducible, which means $\phi(R)>0$ (for suitably defined $\phi$ ), then $r$ is positive and all other eigenvalues lie within the open disk, so (with eigenvalues sorted by real part), $\operatorname{Re} \lambda_{2}(R)<r$. An extension of Fiedler's result provides an upper bound and our result provides the corresponding lower bound on $\phi(R)$ in terms of $r-\operatorname{Re} \lambda_{2}(R)$, obtaining a two-sided quantitative version of the Perron-Frobenius theorem.

## 1 Introduction.

1.1 Motivation and main result. We study the relationship between edge expansion and second eigenvalue of nonnegative matrices. We restrict to doubly stochastic matrices for exposition in the introduction, since the definitions are simpler and it captures most of the key ideas. The extension to general nonnegative matrices is treated in Section 1.2. Let $A$ be an $n \times n$ doubly stochastic matrix, equivalently interpreted as a bi-regular weighted digraph. The edge expansion of $A$, denoted as $\phi(A)$, is defined as

$$
\phi(A)=\min _{S \subseteq[n],|S| \leq n / 2} \frac{\sum_{i \in S, j \notin S} A_{i, j}}{|S|}
$$

The fact that $A$ is doubly stochastic implies that $\phi(A)=$ $\phi\left(A^{T}\right) . \phi$ is a measure of how much the graph would have to be modified for it to lose strong-connectedness; it also lower bounds how frequently, in steady state, the associated Markov chain switches between the blocks of any bi-partition; thus, it fundamentally expresses the extent to which the graph is connected.

The connection between edge expansion and the second eigenvalue has been of central importance in the case of symmetric doubly stochastic matrices $M$ (equivalently, reversible Markov chains with uniform stationary distribution). For such $M$, let the eigenvalues be $1=\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{n} \geq-1$. The Cheeger inequalities give two-sided bounds between the edge expansion $\phi(M)$ and the spectral gap $1-\lambda_{2}(M)$.

Theorem 1.1. [Dod84, AM85, Alo86] (Cheeger's inequalities for symmetric doubly stochastic matrices) Let
$M$ be a symmetric doubly stochastic matrix, then

$$
\frac{1-\lambda_{2}(M)}{2} \leq \phi(M) \leq \sqrt{2 \cdot\left(1-\lambda_{2}(M)\right)}
$$

This is closely related to earlier versions for Riemannian manifolds [Che70, Bus82]. Notably, the inequalities in Theorem 1.1 do not depend on $n$. Further, they are tight up to constants - the upper bound on $\phi$ is achieved by the cycle and the lower bound by the hypercube.

The key question we address in this work is whether or to what extent the Cheeger inequalities survive for nonsymmetric doubly stochastic matrices (the question was already asked, for e.g., in [MT06]). Let $A$ be a doubly stochastic matrix, not necessarily symmetric. The eigenvalues of $A$ lie in the unit disk around the origin in the complex plane, with an eigenvalue 1 called the trivial or stochastic eigenvalue corresponding to the eigenvector 1 (the all 1's vector), and all other eigenvalues considered nontrivial. Let the eigenvalues of $A$ be ordered so that $1=\lambda_{1} \geq \operatorname{Re} \lambda_{2} \geq \ldots \geq \operatorname{Re} \lambda_{n} \geq-1$. The spectral gap will be defined as $1-\operatorname{Re} \lambda_{2}(A)$. There are three motivations for this definition: first, the continuous-time Markov chain based on $A$ is $\exp (t \cdot(A-I))$, and this spectral gap specifies the largest norm of any of the latter's nontrivial eigenvalues; second, $\operatorname{Re} \lambda_{2}(A)=1$ if and only if $\phi=0$ (i.e., if the matrix is reducible); finally, this gap lower bounds the distance (in the complex plane) between the trivial and any nontrivial eigenvalue.

It was noted by Fiedler [Fie95] that the upper bound on $\phi$ in Cheeger's inequality (Theorem 1.1) carries over easily to general doubly stochastic matrices, because for $M=\left(A+A^{T}\right) / 2, \phi(A)=\phi(M)$ and $\operatorname{Re} \lambda_{2}(A) \leq \lambda_{2}(M)$. (In fact, Fiedler made a slightly different conclusion with these observations, but they immediately give the upper bound on $\phi$, see the extended version of the paper [MS19] for an extension and proof.

Lemma 1.1. (Fiedler [Fie95]) Let $A$ be any $n \times n$ doubly stochastic matrix, then

$$
\phi(A) \leq \sqrt{2 \cdot\left(1-\operatorname{Re} \lambda_{2}(A)\right)}
$$

It remained to investigate the other direction, i.e., the lower bound on $\phi(A)$ in terms of the spectral gap $1-\operatorname{Re} \lambda_{2}(A)$. Towards this end, we define the function
Definition 1.1. $\Gamma(n)=\min _{A} \frac{\phi(A)}{1-\operatorname{Re} \lambda_{2}(A)}$ where $A$ is an $n \times n$ doubly stochastic matrix.

For symmetric doubly stochastic matrices this minimum is no less than $1 / 2$. However, for doubly stochastic matrices $A$ that are not necessarily symmetric, there are known (but perhaps not widely known) examples demonstrating that it is impossible to have a function $\Gamma$
entirely independent of $n$. These examples, discussed in Section 3.1, show that

$$
\begin{equation*}
\Gamma(n) \leq \frac{1}{\log n} \tag{1.1}
\end{equation*}
$$

One reason that $\phi$ and $1-\operatorname{Re} \lambda_{2}$ are important is their connection to mixing time $\tau$ - the number of steps after which a random walk starting at any vertex converges to the uniform distribution over the vertices. For the case of symmetric doubly stochastic matrices - or in general reversible Markov chains - it is simple to show that $\tau \in O\left(\frac{\log n}{1-\lambda_{2}}\right)$, and by Cheeger's inequality (Theorem 1.1), it further gives $\tau \in O\left(\frac{\log n}{\phi^{2}}\right)$ where $n$ is the number of vertices. For the case of general doubly stochastic matrices - or in general not-necessarily-reversible Markov chains - it still holds that $\tau \in O\left(\frac{\log n}{\phi^{2}}\right)$ by a result of Mihail ([Mih89], and see [Fil91]). Depending on the situation, either $\operatorname{Re} \lambda_{2}$ or $\phi$ may be easier to estimate. Most often, one is interested in concisely-specified chains on exponential-size sets, i.e., the number of vertices is $n$ but the complexity parameter is $\log n$. In this case, either $\phi$ or $\lambda_{2}$ (for the reversible case) can be used to estimate $\tau$. However, if the matrix or chain is given explicitly, then one reason the Cheeger inequalites are useful is because it is simpler to estimate $\tau$ using $\lambda_{2}$ which is computable in P while computing $\phi$ is NP-hard [GJS74].

From the point of view of applications to Markov Chain Monte Carlo (MCMC) algorithms, a $\log n$ loss in the relation between $\phi$ and $1-\operatorname{Re} \lambda_{2}$ as implied by (1.1), is not a large factor. For MCMC algorithms, since " $n$ " is the size of the state space being sampled or counted and the underlying complexity parameter is $\log n$, if it were true that $\Gamma(n) \geq \log ^{-c} n$ for some constant $c$, then the loss in mixing time estimates would be polynomial in the complexity parameter, and thus, the quantity $1-\operatorname{Re} \lambda_{2}$ could still be used to obtain a reasonable estimate of the mixing time even in the case of nonsymmetric doubly stochastic matrices.

However, the truth is much different. Our main result is that $\Gamma(n)$ does not scale as $\log ^{-c} n$, but is exponentially smaller.

Theorem 1.2. (Bounds on $\Gamma$ )

$$
\frac{1}{35 \cdot n} \leq \Gamma(n) \leq \frac{1}{\sqrt{n}}
$$

We give an explicit construction of doubly stochastic matrices $A_{n}$ for the upper bound on $\Gamma$. This construction of highly nonexpanding doubly stochastic matrices has, in addition, the surprising property that every nontrivial eigenvalue is 0 . Thus, for non-reversible

Markov chains, the connection between $\phi$ and $\operatorname{Re} \lambda_{2}$ breaks down substantially, in that the upper bound on mixing time obtained by lower bounding $\phi$ by the spectral gap $1-\operatorname{Re} \lambda_{2}$ can be be exponentially weaker (when the complexity parameter is $\log n$ ) than the actual mixing time, whereas for reversible chains there is at worst a quadratic loss.

This theorem has a very natural extension to general nonnegative matrices, as we next describe.
1.2 General nonnegative matrices and a twosided quantitative refinement of the PerronFrobenius theorem We extend our results to general nonnegative matrices $R$. By the Perron-Frobenius theorem (see Theorem 2.1), since $R$ is nonnegative, it has a nonnegative eigenvalue $r$ (called the PF eigenvalue) that is also largest in magnitude amongst all eigenvalues, and the corresponding left and right "PF eigenvectors" $u$ and $v$ have all nonnegative entries. Further, if $R$ is irreducible, i.e., the underlying weighted digraph on edges with positive weight is strongly connected (for every $(i, j)$ there is a $k$ such that $\left.R^{k}(i, j)>0\right)$, then $r$ is a simple eigenvalue, and $u$ and $v$ have all positive entries. We henceforth assume that nonzero nonnegative $R$ has been scaled ( to $\frac{1}{r} R$ ) so that $r=1$. Thus we can again write the eigenvalues of $R$ as $1=\lambda_{1}(R) \geq \operatorname{Re} \lambda_{2}(R) \geq \cdots \geq \operatorname{Re} \lambda_{n}(R) \geq-1$. Scaling changes the spectrum but not the edge expansion (which we define next), so this canonical scaling is necessary before the two quantities can be compared.

Defining the edge expansion of $R$ is slightly delicate so it will be done in Definition 2.3, with explanation in Lemma 2.1; here we excerpt here only the final definition. Let $R$ be a nonnegative matrix with PF eigenvalue 1. If every pair of left and right PF eigenvectors $(u, v)$ for $R$ have some $i$ such that $u(i)=0$ or $v(i)=0$, then define $\phi(R)=0$. Otherwise, fix any left and right PF eigenvectors $u$ and $v$, normalized so that $\langle u, v\rangle=1$, and define the edge expansion of $R$ as

$$
\phi(R)=\min _{S \subseteq[n], \sum_{i \in S} u_{i} \cdot v_{i} \leq \frac{1}{2}} \frac{\sum_{i \in S, j \in \bar{S}} R_{i, j} \cdot u_{i} \cdot v_{j}}{\sum_{i \in S} u_{i} \cdot v_{i}}
$$

Given this definition of $\phi(R)$, we show the following.
Theorem 1.3. Let $R$ be a nonnegative matrix with PF eigenvalue 1, and let $u$ and $v$ be any corresponding left and right eigenvectors. Let $\kappa=\min _{i} u_{i} \cdot v_{i}$, and if $\kappa>0$, let $u$ and $v$ be normalized so that $\langle u, v\rangle=1$. Then

$$
\frac{1}{30} \cdot \frac{1-\operatorname{Re} \lambda_{2}(R)}{n+\ln \left(\frac{1}{\kappa}\right)} \leq \phi(R) \leq \sqrt{2 \cdot\left(1-\operatorname{Re} \lambda_{2}(R)\right)}
$$

The upper bound in Theorem 1.2 is a straightforward extension of Fiedler's bound (Lemma 1.1) based on
the above mentioned definition of $\phi$ (Definition 2.3). Also note that the lower bound in Theorem 1.2 can be obtained by setting $\kappa=\frac{1}{n}$ in Theorem 1.3.

Since Theorem 1.3 gives a two-sided relation between the second eigenvalue of nonnegative matrices and their edge expansion, it gives a two-sided quantitative refinement of the Perron-Frobenius theorem. Although the Perron-Frobenius theorem implies that the nontrivial eigenvalues of an irreducible nonnegative matrix $R$ with PF eigenvalue 1 have real part strictly less than 1 , it does not give any concrete separation. Further, it also does not provide a qualitative (or quantitative) implication in the other direction - whether a nonnegative matrix $R$ with all nontrivial eigenvalues having real part strictly less than 1 implies that $R$ is irreducible. Theorem 1.3 comes to remedy this by giving a lower and upper bound on the spectral gap in terms of $\phi$, a quantitative measure of the irreducibility of $R$. We are not aware of any previous result of this form.
1.3 Mixing time The third quantity we study is the mixing time of general nonnegative matrices, and we relate it to their singular values, edge expansion, and spectral gap. This helps us obtain new bounds on mixing time, and also obtain elementary proofs for known results. These results are treated in detail in the second part of the paper, in Section 5 .

### 1.4 Perspectives

1.4.1 Matrix perturbations Let $A$ be a nonnegative matrix with PF eigenvalue 1 and corresponding positive left and right eigenvector $w$ with $\langle w, w\rangle=1$, and let $\kappa=\min _{i} w_{i}^{2}$. Given such $A$, it is certainly easier to calculate its spectrum than its edge expansion. However, in other cases, e.g., if the matrix is implicit in a nicely structured randomized algorithm (as in the canonical paths method [SJ89]), the edge expansion may actually be easier to bound. From this point of view, a lower bound on $\phi(A)$ in terms of the spectral gap is an eigenvalue perturbation bound. Specifically, one might write a nonnegative matrix $A$ with small edge expansion $\phi(A)$ as a perturbation of another nonnegative matrix $A_{0}$, i.e.,

$$
A=A_{0}+\delta \cdot B
$$

where $A_{0}$ has disconnected components $S, S^{c}$ (for $S$ achieving $\phi(A)$ ), and $B$ is a matrix such that $\|B\|_{2} \leq 1$, $B w=0$ and $B^{T} w=0$. Due to the conditions on $A$ and $B, A_{0}$ has PF eigenvalue 1 with left and right eigenvector
$w$, and since $B D_{w} \mathbf{1}=0$, writing $\mathbf{1}_{\bar{S}}=\mathbf{1}-\mathbf{1}_{S}$, we have, eigenvector $w$, let $\kappa=\min _{i} w_{i}^{2}$ and $M=\frac{1}{2}\left(A+A^{T}\right)$. Then

$$
\begin{aligned}
\left|\frac{\left\langle\mathbf{1}_{S}, D_{w} B D_{w} \mathbf{1}_{\bar{S}}\right\rangle}{\left\langle\mathbf{1}_{S}, D_{w} D_{w} \mathbf{1}_{S}\right\rangle}\right| & =\left|\frac{\left\langle\mathbf{1}_{S}, D_{w} B D_{w} \mathbf{1}_{S}\right\rangle}{\left\langle\mathbf{1}_{S}, D_{w} D_{w} \mathbf{1}_{S}\right\rangle}\right| \\
& =\left|\frac{\left\langle D_{w} \mathbf{1}_{S}, B D_{w} \mathbf{1}_{S}\right\rangle}{\left\langle D_{w} \mathbf{1}_{S}, D_{w} \mathbf{1}_{S}\right\rangle}\right| \\
& \leq\|B\|_{2} \\
& \leq 1,
\end{aligned}
$$

and it follows then that $\phi\left(R_{0}\right)-\delta \leq \phi(R) \leq \phi\left(R_{0}\right)+\delta$, and since in this case $\phi\left(R_{0}\right)=0$, so $\phi(A) \leq \delta$. Edge expansion is therefore stable with respect to perturbation by $B$. What about the spectral gap?
$A_{0}$ has (at least) a double eigenvalue at 1 , and $A$ retains a simple eigenvalue at 1 , so it is natural to try to apply eigenvalue stability results, specifically BauerFike ([BF60], [Bha97] §VIII), to obtain an upper bound on $\left|1-\lambda_{2}(A)\right|$ (and therefore also on $1-\operatorname{Re} \lambda_{2}(A)$ ). However, Bauer-Fike requires $A_{0}$ to be diagonalizable, and the quality of the bound depends upon the condition number of the diagonalizing change of basis ${ }^{1}$. There are extensions of Bauer-Fike which do not require diagonalizability, but deteriorate exponentially in the size of the Jordan block, and the bound still depends on the condition number of the (now Jordan-izing) change of basis ([Saa11] Corollary 3.2). Since there is no a priori (i.e., function of $n$ ) bound on these condition numbers, these tools unfortunately do not imply any, even weak, result analogous to Theorem 1.2.

In summary, the lower bound in Theorem 1.2 should be viewed as a new eigenvalue perturbation bound:

$$
1-\operatorname{Re} \lambda_{2}(A) \leq 30 \cdot \delta \cdot\left(n+\ln \left(\frac{1}{\kappa}\right)\right)
$$

A novel aspect of this bound, in comparison with the prior literature on eigenvalue perturbations, is that it does not depend on the condition number (or even the diagonalizability) of $A$ or of $A_{0}$.
1.4.2 Relating eigenvalues of a nonnegative matrix and its additive symmetrization Let $A$ be any nonnegative matrix with PF eigenvalue 1, and corresponding positive left and right eigenvector $w$. Our result helps to give the following bounds between the second eigenvalue of a nonnegative matrix $A$ and that of its additive symmetrization $\frac{1}{2}\left(A+A^{T}\right)$.

Lemma 1.2. Let $A$ be any nonnegative matrix with $P F$ eigenvalue 1 and corresponding positive left and right

[^1]$$
\frac{1}{1800}\left(\frac{1-\operatorname{Re} \lambda_{2}(A)}{n+\ln \left(\frac{1}{\kappa}\right)}\right)^{2} \leq 1-\lambda_{2}(M) \leq 1-\operatorname{Re} \lambda_{2}(A) .
$$

The bounds immediately follows from Theorem 1.3 and the fact that $\phi$ is unchanged by additive symmetrization. We remark that any improved lower bound on $1-\lambda_{2}(M)$ in terms of $1-\operatorname{Re} \lambda_{2}(A)$ will help to improve the lower bound in Theorem 1.3. As a consequence of Lemma 1.2, any bound based on the second eigenvalue of symmetric nonnegative matrices can be applied, with dimension-dependent loss, to nonnegative matrices that have identical left and right eigenvector for the PF eigenvalue.

An example application of Lemma 1.2 is the following. For some doubly stochastic $A$ (not necessarily symmetric), consider the continuous time Markov Chain associated with it, $\exp (t \cdot(A-I)$ ). It is well-known (for instance, [DSC96]) that for any standard basis vector $x_{i}$,

$$
\begin{aligned}
&\left\|\exp (t \cdot(A-I)) x_{i}-\frac{1}{n} \mathbf{1}\right\|_{1}^{2} \\
& \leq n \cdot \exp \left(-2 \cdot\left(1-\lambda_{2}(M)\right) \cdot t\right) .
\end{aligned}
$$

Thus, using Lemma 1.2, we immediately get the bound in terms of the second eigenvalue of $A$ itself (instead of its additive symmetrization),

$$
\begin{aligned}
& \left\|\exp (t \cdot(A-I)) x_{i}-\frac{1}{n} 1\right\|_{1}^{2} \\
& \quad \leq n \cdot \exp \left(-\frac{1}{1800}\left(\frac{1-\operatorname{Re} \lambda_{2}(A)}{n+\ln \left(\frac{1}{\kappa}\right)}\right)^{2} \cdot t\right)
\end{aligned}
$$

1.4.3 The role of singular values Although in the symmetric case singular values are simply the absolute values of the eigenvalues, the two sets can be much less related in the nonsymmetric case. It is not difficult to show the following (see the extended version of the paper [MS19]).

Lemma 1.3. Let $A$ be a nonnegative matrix with $P F$ eigenvalue 1, and let $w$ be the positive vector such that $A w=w$ and $A^{T} w=w$. Then

$$
\frac{1-\sigma_{2}(A)}{2} \leq \phi(A)
$$

where $\sigma_{2}(A)$ is the second largest singular value of $A$.
Despite appearances, this tells us little about edge expansion. To see this, consider the directed cycle on $n$
vertices, for which every singular value (and in particular $\sigma_{2}$ ) is 1 , so Lemma 1.3 gives a lower bound of 0 for $\phi$, although $\phi=2 / n$. A meaningful lower bound should be 0 if and only if the graph is disconnected, i.e. it should be continuous in $\phi$. An even more striking example is that of de Bruijn graphs (described in Section 3.1), for which half the singular values are 1 , although $\phi=\Theta(1 / \log n)$. Eigenvalues, on the other hand, are more informative, since a nonnegative matrix with PF eigenvalue 1 has a multiple eigenvalue at 1 if and only if, as a weighted graph, it is disconnected.

Despite these observations, singular values can be useful to infer information about mixing time, and can be used to recover all known upper bounds on mixing time using $\phi$, as discussed in Section 5 and Lemma 5.1.

Outline: We state the preliminary definitions, theorems, and notations in Section 2. We give the construction for the upper bound on $\Gamma$ in Theorem 1.2 in Section 3, and the lower bound on $\Gamma$ will follow from the general lower bound on $\phi$ in Theorem 1.3. We give details regarding the lower bound on $\phi$ in Theorem 1.3 in Section 4. We relate mixing time to singular values, edge expansion, and the spectral gap in Section 5 respectively. We defer all proofs to the extended version of the paper [MS19].

## 2 Preliminaries

We consider doubly stochastic matrices which are nonnegative matrices with entries in every row and column summing to 1 . We also consider general nonnegative matrices $R$, and say that $R$ is strongly connected or $i r$ reducible, if there is a path from $s$ to $t$ for every pair $(s, t)$ of vertices in the underlying digraph on edges with positive weight, i.e. for every $(s, t)$ there exists $k>0$ such that $R^{k}(s, t)>0$. We say $R$ is weakly connected, if there is a pair of vertices $(s, t)$ such that there is a path from $s$ to $t$ but no path from $t$ to $s$ in the underlying digraph (on edges with positive weight). We restate the Perron-Frobenius theorem for convenience.

Theorem 2.1. (Perron-Frobenius
theorem [Per07, Fro12]) Let $R \in \mathbb{R}^{n \times n}$ be a nonnegative matrix. Then the following hold for $R$.

1. $R$ has some nonnegative eigenvalue $r$, such that all other eigenvalues have magnitude at most $r$, and $R$ has nonnegative left and right eigenvectors $u$ and $v$ for $r$.
2. If $R$ has some positive left and right eigenvectors $u$ and $v$ for some eigenvalue $\lambda$, then $\lambda=r$.
3. If $R$ is irreducible, then $r$ is positive and simple (unique), $u$ and $v$ are positive and unique, and all
other eigenvalues have real part strictly less than $r$.
We denote the all 1 's vector by $\mathbf{1}$, and note that for a doubly stochastic matrix $A, \mathbf{1}$ is both the left and right eigenvector with eigenvalue 1 . We say that 1 is the trivial (or stochastic) eigenvalue, and $\mathbf{1}$ is the trivial (or stochastic) eigenvector, of $A$. All other eigenvalues of $A$ (corresponding to eigenvectors that are not trivial) will be called nontrivial (or nonstochastic) eigenvalues of $A$. Similarly, by Perron-Frobenius (Theorem 2.1, part 1), a nonnegative matrix $R$ will have a simple nonnegative eigenvalue $r$ such that all eigenvalues have magnitude at most $r$, and it will be called the trivial or PF eigenvalue of $R$, and all other eigenvalues of $R$ will be called nontrivial. The left and right eigenvectors corresponding to $r$ will be called the trivial or PF left eigenvector and trivial or PF right eigenvector. This leads us to the following definition.

Definition 2.1. (Second eigenvalue) If $A$ is a doubly stochastic matrix, then $\lambda_{2}(A)$ is the nontrivial eigenvalue of $A$ with the maximum real part, and $\lambda_{m}(A)$ is the nontrivial eigenvalue that is largest in magnitude. Similarly, if $R$ is any general nonnegative matrix with PF eigenvalue 1, then $\lambda_{2}(R)$ is the nontrivial eigenvalue with the maximum real part, i.e., $1=\lambda_{1}(R) \geq \operatorname{Re} \lambda_{2}(R) \geq \cdots \geq$ $\operatorname{Re} \lambda_{n}(R) \geq-1$, and $\lambda_{m}(R)$ is the nontrivial eigenvalue that is largest in magnitude.

We will also consider singular values of nonnegative matrices $A$ with identical positive left and right eigenvector $w$ for PF eigenvalue 1, and denote them as $1=\sigma_{1}(A) \geq \sigma_{2}(A) \geq \cdots \geq \sigma_{n}(A) \geq 0$ (see Lemma 4.4 for proof of $\sigma_{1}(A)=1$ ). We denote $(i, j)$ 'th entry of $M \in \mathbb{C}^{n \times n}$ by $M(i, j)$ or $M_{i, j}$ depending on the importance of indices in context, denote the conjugatetranspose of $M$ as $M^{*}$ and the transpose of $M$ as $M^{T}$. Any $M \in \mathbb{C}^{n \times n}$ has a Schur decomposition (see, e.g., [Lax07]) $M=U T U^{*}$ where $T$ is an upper triangular matrix whose diagonal entries are the eigenvalues of $M$, and $U$ is a unitary matrix. When we write "vector" we mean by default a column vector. For a vector $v$, we write $v(i)$ or $v_{i}$ to denote its $i$ 'th entry. For any two vectors $x, y \in \mathbb{C}^{n}$, we use the standard inner product $\langle x, y\rangle=\sum_{i=1}^{n} x_{i}^{*} \cdot y_{i}$ defining the norm $\|x\|_{2}=\sqrt{\langle x, x\rangle}$. We write $u \perp v$ to indicate that $\langle u, v\rangle=0$. Note that $\langle x, M y\rangle=\left\langle M^{*} x, y\right\rangle$. We denote the operator norm of $M$ by $\|M\|_{2}=\max _{u:\|u\|_{2}=1}\|M u\|_{2}$, and recall that the operator norm is at most the Frobenius norm, i.e., $\|M\|_{2} \leq \sqrt{\sum_{i, j}\left|M_{i, j}\right|^{2}}$. We write $D_{u}$ for the diagonal matrix whose diagonal contains the vector $u$. Recall the Courant-Fischer variational characterization of eigenvalues for symmetric real matrices, applied to the second
eigenvalue:

$$
\max _{u \perp v_{1}} \frac{\langle u, M u\rangle}{\langle u, u\rangle}=\lambda_{2}(M)
$$

where $v_{1}$ is the eigenvector for the largest eigenvalue of $M$. We will use the symbol $J$ for the all 1 's matrix divided by $n$, i.e., $J=\frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{T}$.

We say that any subset $S \subseteq[n]$ is a cut, denote its complement by $\bar{S}$, and denote the characteristic vector of a cut as $\mathbf{1}_{S}$, where $\mathbf{1}_{S}(i)=1$ if $i \in S$ and 0 otherwise.

Definition 2.2. (Edge expansion of doubly stochastic matrices) For a doubly stochastic matrix $A$, the edge expansion of the cut $S$ is defined as

$$
\phi_{S}(A):=\frac{\left\langle\mathbf{1}_{S}, A \mathbf{1}_{\bar{S}}\right\rangle}{\min \left\{\left\langle\mathbf{1}_{S}, A \mathbf{1}\right\rangle,\left\langle\mathbf{1}_{\bar{S}}, A \mathbf{1}\right\rangle\right\}}
$$

and the edge expansion of $A$ is defined as

$$
\begin{aligned}
\phi(A) & =\min _{S \subseteq[n]} \phi_{S}(A) \\
& =\min _{S \subseteq[n]} \frac{\left\langle\mathbf{1}_{S}, A \mathbf{1}_{\bar{S}}\right\rangle}{\min \left\{\left\langle\mathbf{1}_{S}, A \mathbf{1}\right\rangle,\left\langle\mathbf{1}_{\bar{S}}, A \mathbf{1}\right\rangle\right\}} \\
& =\min _{S,|S| \leq n / 2} \frac{\sum_{i \in S, j \in \bar{S}} A_{i, j}}{|S|}
\end{aligned}
$$

We wish to extend these notions to general nonnegative matrices $R$. Since eigenvalues and singular values of real matrices remain unchanged whether we consider $R$ or $R^{T}$, the same should hold of a meaningful definition of edge expansion. However, note that Definition 2.2 has this independence only if the matrix is Eulerian, i.e., $R \mathbf{1}=R^{T} \mathbf{1}$. Thus, to define edge expansion for general matrices, we transform $R$ using its left and right eigenvectors $u$ and $v$ for eigenvalue 1 to obtain $D_{u} R D_{v}$, which is indeed Eulerian, since

$$
\begin{aligned}
D_{u} R D_{v} \mathbf{1}=D_{u} R v & =D_{u} v=D_{u} D_{v} \mathbf{1} \\
=D_{v} D_{u} \mathbf{1} & =D_{v} u=D_{v} R^{T} u=D_{v} R^{T} D_{u} \mathbf{1}
\end{aligned}
$$

Since $D_{u} R D_{v}$ is Eulerian, we can define the edge expansion of $R$ similar to that for doubly stochastic matrices:

Definition 2.3. (Edge expansion of nonnegative matrices) Let $R \in \mathbb{R}^{n \times n}$ be a nonnegative matrix with $P F$ eigenvalue 1. If there are no positive (i.e., everywhere positive) left and right eigenvectors $u$ and $v$ for eigenvalue 1 , then define the edge expansion $\phi(R)=0$. Else, let $u$ and $v$ be any (see Lemma 2.1 for justification) positive left and right eigenvectors for eigenvalue 1, normalized so that $\langle u, v\rangle=1$. The edge expansion of the cut $S$ is defined as

$$
\begin{equation*}
\phi_{S}(R):=\frac{\left\langle\mathbf{1}_{S}, D_{u} R D_{v} \mathbf{1}_{\bar{S}}\right\rangle}{\min \left\{\left\langle\mathbf{1}_{S}, D_{u} R D_{v} \mathbf{1}\right\rangle,\left\langle\mathbf{1}_{\bar{S}}, D_{u} R D_{v} \mathbf{1}\right\rangle\right\}} \tag{2.2}
\end{equation*}
$$

and the edge expansion of $R$ is defined as

$$
\begin{aligned}
\phi(R) & =\min _{S \subseteq[n]} \phi_{S}(R) \\
& =\min _{S \subseteq[n], \sum_{i \in S} u_{i} \cdot v_{i} \leq \frac{1}{2}} \frac{\left\langle 1_{S}, D_{u} R D_{v} 1 \bar{S}\right\rangle}{\left\langle 1_{S}, D_{u} D_{v} \mathbf{1}\right\rangle} \\
& =\min _{S \subseteq[n], \sum_{i \in S} u_{i} \cdot v_{i} \leq \frac{1}{2}} \frac{\sum_{i \in S, j \in \bar{S}} R_{i, j} \cdot u_{i} \cdot v_{j}}{\sum_{i \in S} u_{i} \cdot v_{i}}
\end{aligned}
$$

Lemma 2.1. Let $R \in \mathbb{R}^{n \times n}$ be a nonnegative matrix with PF eigenvalue 1. Then the value of $\phi(R)$ according to Definition 2.3 is independent of the choice of the specific (left and right) eigenvectors $u$ and $v$ for eigenvalue 1 of $R$.

Proof. Let $G$ be the underlying unweighted directed graph for $R$, where there is an edge $(u, v)$ in $G$ if and only if $R_{u, v}>0$. We prove the lemma based on the structure of $G$. Let $G$ be maximally partitioned into $k$ weakly connected components.

1. If $G$ has some weakly connected component which does not have a 1 eigenvalue, then for any pair of left and right eigenvectors $u$ and $v$ for eigenvalue 1 , there will be at least one entry $i$ such that $u_{i}=0$ or $v_{i}=0$. For such matrices $R$, from Definition 2.3, $\phi(R)=0$.
2. If all weakly connected components of $G$ have eigenvalue 1 , but there is some weakly connected component $S$ that is not strongly connected, then there is no positive pair of eigenvectors $u$ and $v$ for $R$, or even for $S$. Observe that $S=\left[\begin{array}{cc}A & B \\ 0 & C\end{array}\right]$ with $B \neq 0$, else $G$ is not maximally partitioned. For the sake of contradiction, let $x$ and $y$ be positive left and right eigenvectors for eigenvalue 1 of $S$. From $S y=y$ and $S^{T} x=x$, we get that $A y_{1}+B y_{2}=y_{1}$ and $A^{T} x_{1}=x_{1}$ with $x$ and $y$ partitioned into $x_{1}, x_{2}$ and $y_{1}, y_{2}$ based on the sizes of $A, B, C$. Thus,

$$
\begin{array}{r}
\left\langle x_{1}, y_{1}\right\rangle=\left\langle x_{1}, A y_{1}+B y_{2}\right\rangle=\left\langle A^{T} x_{1}, y_{1}\right\rangle+\left\langle x_{1}, B y_{2}\right\rangle \\
=\left\langle x_{1}, y_{1}\right\rangle+\left\langle x_{1}, B y_{2}\right\rangle
\end{array}
$$

implying $\left\langle x_{1}, B y_{2}\right\rangle=0$ which is a contradiction since $x_{1}$ and $y_{1}$ are positive and there is some entry in $B$ which is not 0 . Thus, since every pair of eigenvectors $x$ and $y$ for eigenvalue 1 of $S$ has some entry $i$ with $x_{i}=0$ or $y_{i}=0$, every pair of (left and right) eigenvectors $u$ and $v$ for $R$ (for eigenvalue 1) has some entry $i$ which is 0 , and so by Definition 2.3, $\phi(R)=0$.
3. If $G$ consists of one strongly connected component (i.e., $k=1$ ), then by Perron-Frobenius (Theorem 2.1, part 3), there is a unique (up to scaling) pair of positive eigenvectors $u$ and and $v$ for eigenvalue 1 .
4. The remaining case is that $G$ has $k \geq 2$ strongly connected components each with eigenvalue 1. By Perron-Frobenius (Theorem 2.1), there is some pair of positive left and right eigenvectors $u$ and $v$ for eigenvalue 1 (obtained by concatenating the positive left and right eigenvectors for each component individually). Choose the set $S$ to be one of the components (the one for which the denominator of equation (2.2) is at most $\frac{1}{2}$ ), then the numerator of equation (2.2) will be 0 , and thus $\phi(R)=0$ even in this case, corresponding to the existence of a strict subset of vertices with no expansion.

Thus, Definition 2.3 for $\phi(R)$ does not depend on the specific choice of $u$ and $v$.

The Perron-Frobenius theorem (Theorem 2.1, part 3) can now be restated in terms of $\phi(R)$ and $\operatorname{Re} \lambda_{2}(R)$ as follows.
Lemma 2.2. (Perron-Frobenius, part 3 of Theorem 2.1, restated) Let $R$ be a nonnegative matrix with PF eigenvalue 1. If $\phi(R)>0$, then $\operatorname{Re} \lambda_{2}(R)<1$.

Further, we obtain the following converse of Lemma 2.2.

Lemma 2.3. (Converse of Lemma 2.2) Let $R$ be a nonnegative matrix with PF eigenvalue 1. If $\operatorname{Re} \lambda_{2}(R)<$ 1, and there exists a pair of positive left and right eigenvectors $u$ and $v$ for eigenvalue 1 of $R$, then $\phi(R)>$ 0.

Proof. We show the contrapositive. Let $R$ be as stated and let $\phi(R)=0$. From Definition 2.3, if there are no positive eigenvectors $u$ and $v$ for eigenvalue 1, the lemma holds. So assume there are some positive $u$ and $v$ for eigenvalue 1 of $R$. Since $\phi(R)=0$, there is some set $S$ for which $\phi_{S}(R)=0$, or $\sum_{i \in S, j \in \bar{S}} R_{i, j} \cdot u_{i} \cdot v_{j}=0$. But since $u_{i}>0$ and $v_{i}>0$ for each $i$, and since $R$ is nonnegative, it implies that for each $i \in S, j \in \bar{S}, R_{i, j}=0$. Further, since $D_{u} R D_{v}$ is Eulerian, i.e. $D_{u} R D_{v} \mathbf{1}=D_{v} R^{T} D_{u} \mathbf{1}$, it implies that $\sum_{i \in \bar{S}, j \in S} R_{i, j} \cdot u_{i} \cdot v_{j}=0$, further implying that $R_{i, j}=0$ for each $i \in \bar{S}, j \in S$. As a consequence, $v$ can be rewritten as two vectors $v_{S}$ and $v_{\bar{S}}$, where $v_{S}(i)=v(i)$ if $i \in S$ and $v_{S}(i)=0$ otherwise, and similarly $v_{\bar{S}}$. Similarly, split $u$ into $u_{S}$ and $u_{\bar{S}}$. Note that $v_{S}$ and $v_{\bar{S}}$ are linearly independent (in fact, orthogonal), and both are right eigenvectors for eigenvalue 1 (similarly $u_{S}$ and $u_{\bar{S}}$ as left eigenvectors). Thus, this implies that eigenvalue 1 for $R$ has multiplicity at least 2 , and thus $\lambda_{2}(R)=1$, as required.

The upper and lower bounds on $\phi(R)$ in Theorem 1.3 are quantitative versions of Lemmas 2.2 and 2.3 respectively.

We note that Cheeger's inequalities hold not only for any symmetric doubly stochastic matrix, but also for any nonnegative matrix $R$ which satisfies detailed balance. We say that a nonnegative matrix $R$ with positive left and right eigenvectors $u$ and $v$ for PF eigenvalue 1 satisfies detailed balance if $D_{u} R D_{v}$ is symmetric, which generalizes the usual definition of detailed balance (or reversibility) for stochastic matrices. We first note that if $R$ satisfies the condition of detailed balance, then $R$ has all real eigenvalues. To see this, let $W=D_{u}^{\frac{1}{2}} D_{v}^{-\frac{1}{2}}$ where the inverses are well-defined since we assume $u$ and $v$ are positive (else $\phi=0$ by definition), and $A=W R W^{-1}$ where $A$ has same eigenvalues as $R$. For $w=D_{u}^{\frac{1}{2}} D_{v}^{\frac{1}{2}} \mathbf{1}$ which is both the left and right eigenvector of $A$ for eigenvalue 1, the detailed balance condition translates to $D_{w} A D_{w}=D_{w} A^{T} D_{w}$, which implies $A=A^{T}$, which further implies that all eigenvalues of $A$ (and $R$ ) are real. We can thus state Cheeger inequalities for nonnegative matrices satisfying detailed balance.

Theorem 2.2. (Cheeger's inequalities for nonnegative matrices satisfying detailed balance) Let $R$ be a nonnegative matrix with PF eigenvalue 1 and positive left and right eigenvectors $u$ and $v$ (else $\phi(R)=0$ by definition), and let $R$ satisfy the condition of detailed balance, i.e. $D_{u} R D_{v}=D_{v} R^{T} D_{u}$. Then

$$
\frac{1-\lambda_{2}(R)}{2} \leq \phi(R) \leq \sqrt{2 \cdot\left(1-\lambda_{2}(R)\right)}
$$

3 Construction of doubly stochastic matrices with small edge expansion and large spectral gap
As discussed, it might have been tempting to think that $\Gamma(n)$ (see Definition 1.1) should be independent of the matrix size $n$, since this holds for the symmetric case, and also for the upper bound on $\phi$ for the general nonsymmetric doubly stochastic case (Lemma 1.1). However, two known examples showed that a mild dependence on $n$ cannot be avoided.

### 3.1 Known Constructions

Klawe-Vazirani construction: The first is a construction of Klawe [Kla84] - these families of $d$-regular undirected graphs have edge expansion $(\log n)^{-\gamma}$ for various $0<\gamma<1$. However, there is a natural way in which to direct the edges of these graphs and obtain a $(d / 2)$ in, $(d / 2)$-out -regular graph, and it was noted by $U$. Vazirani [Vaz17] in the 1980s that for this digraph $A$, which shares the same edge expansion as Klawe's, all eigenvalues (except the stochastic eigenvalue) have norm
$\leq 1 / 2$. Specifically, one construction is as follows: let $n$ be an odd prime, and create the graph on $n$ vertices with in-degree and out-degree 2 by connecting every vertex $v \in \mathbb{Z} / n$ to two vertices, $1+v$ and $2 v$. Dividing the adjacency matrix of the graph by 2 gives a doubly stochastic matrix $A_{K V}$. It is simple to see (by transforming to the Fourier basis over $\mathbb{Z} / n)$ that the characteristic polynomial of $A_{K V}$ is $x(x-1)\left((2 x)^{n-1}-1\right) /(2 x-1)$, so apart from the trivial eigenvalue $1, A_{K V}$ has $n-2$ nontrivial eigenvalues $\lambda$ such that $|\lambda|=\frac{1}{2}$ and one eigenvalue 0 , and thus, $\operatorname{Re} \lambda_{2}\left(A_{K V}\right) \leq \frac{1}{2}$. Further, upper bounding the edge expansion $\phi$ by the vertex expansion bound (Theorem 2.1 in [Kla84]), it follows that for some constant $c$,

$$
\Gamma(n) \leq \frac{\phi\left(A_{K V}\right)}{1-\operatorname{Re} \lambda_{2}\left(A_{K V}\right)} \leq c \cdot\left(\frac{\log \log n}{\log n}\right)^{1 / 5}
$$

de Bruijn construction: A second example is the de Bruijn digraph [Bru46]. This is if anything even more striking: the doubly stochastic matrix (again representing random walk along an Eulerian digraph) has edge expansion $\Theta(1 / \log n)$ [DT98], yet all the nontrivial eigenvalues are 0 . More specifically, define a special case of de Bruijn [Bru46] graphs as follows: Let $n=2^{k}$ for some integer $k$, and create the graph of degree 2 on $n$ vertices by directing edges from each vertex $v=\left(v_{1}, v_{2}, \ldots, v_{k}\right) \in\{0,1\}^{k}$ to two vertices, $\left(v_{2}, v_{3}, \ldots, v_{k}, 0\right)$ and $\left(v_{2}, v_{3}, \ldots, v_{k}, 1\right)$. Dividing the adjacency matrix of the graph by 2 gives a doubly stochastic matrix $A_{d B}$. Since this random walk completely forgets its starting point after $k$ steps, every nontrivial eigenvalue of $A_{d B}$ is 0 (and each of its Jordan blocks is of dimension at most $k$ ). Further, it was shown in [DT98] that $\phi\left(A_{d B}\right) \in \Theta(1 / k)$, and thus,

$$
\Gamma(n) \leq \frac{\phi\left(A_{d B}\right)}{1-\operatorname{Re} \lambda_{2}\left(A_{d B}\right)} \leq \frac{1}{k}=\frac{1}{\log n}
$$

Other literature - Feng-Li construction: We round out this discussion by recalling that Alon and Boppana [Alo86, Nil91] showed that for any infinite family of $d$-regular undirected graphs, the adjacency matrices, normalized to be doubly stochastic and with eigenvalues $1=\lambda_{1} \geq \lambda_{2} \geq \ldots \geq-1$, have $\lambda_{2} \geq \frac{2 \sqrt{d-1}}{d}-o(1)$. Feng and $\mathrm{Li}[\mathrm{Li} 92]$ showed that undirectedness is essential to this bound: they provide a construction of cyclicallydirected $r$-partite $(r \geq 2) d$-regular digraphs (with $n=k r$ vertices for $k>d, \operatorname{gcd}(k, d)=1$ ), whose normalized adjacency matrices have (apart from $r$ "trivial" eigenvalues), only eigenvalues of norm $\leq 1 / d$. The construction is of an affine-linear nature quite similar to the preceding two, and to our knowledge does not give an upper bound on $\Gamma$ any stronger than those.
3.2 A new construction To achieve the upper bound in Theorem 1.2, we need a construction that is exponentially better than known examples. We give an explicit construction of $n \times n$ doubly stochastic matrices $A_{n}$ (for every $n$ ) that are highly nonexpanding, since they contain sets with edge expansion less than $1 / \sqrt{n}$, even though every nontrivial eigenvalue is 0 . The construction might seem nonintuitive, but in the extended version of the paper [MS19], we give some explanation of how to arrive at it.

ThEOREM 3.1. Let $m=\sqrt{n}$,

$$
\begin{aligned}
a_{n} & =\frac{m^{2}+m-1}{m \cdot(m+2)}, & b_{n} & =\frac{m+1}{m \cdot(m+2)}, \\
c_{n} & =\frac{1}{m \cdot(m+1)}, & d_{n} & =\frac{m^{3}+2 m^{2}+m+1}{m \cdot(m+1) \cdot(m+2)}, \\
e_{n} & =\frac{1}{m \cdot(m+1) \cdot(m+2)}, & f_{n} & =\frac{2 m+3}{m \cdot(m+1) \cdot(m+2)},
\end{aligned}
$$

and define the $n \times n$ matrix

$$
A_{n}=\left[\begin{array}{ccccccccc}
a_{n} & b_{n} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & c_{n} & d_{n} & e_{n} & e_{n} & e_{n} & e_{n} & \cdots & e_{n} \\
0 & c_{n} & e_{n} & d_{n} & e_{n} & e_{n} & e_{n} & \cdots & e_{n} \\
0 & c_{n} & e_{n} & e_{n} & d_{n} & e_{n} & e_{n} & \cdots & e_{n} \\
0 & c_{n} & e_{n} & e_{n} & e_{n} & d_{n} & e_{n} & \cdots & e_{n} \\
0 & c_{n} & e_{n} & e_{n} & e_{n} & e_{n} & d_{n} & \cdots & e_{n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & c_{n} & e_{n} & e_{n} & e_{n} & e_{n} & e_{n} & \cdots & d_{n} \\
b_{n} & f_{n} & c_{n} & c_{n} & c_{n} & c_{n} & c_{n} & \cdots & c_{n}
\end{array}\right]
$$

Then the following hold for $A_{n}$ :

1. $A_{n}$ is doubly stochastic.
2. Every nontrivial eigenvalue of $A_{n}$ is 0 .
3. The edge expansion is bounded as

$$
\frac{1}{6 \sqrt{n}} \leq \phi\left(A_{n}\right) \leq \frac{1}{\sqrt{n}}
$$

4. As a consequence of $1,2,3$,

$$
\phi\left(A_{n}\right) \leq \frac{1-\operatorname{Re} \lambda_{2}\left(A_{n}\right)}{\sqrt{n}}
$$

and thus

$$
\Gamma(n) \leq \frac{1}{\sqrt{n}}
$$

Proof. The proof is given in the extended version of the paper [MS19].

This completes the proof of the upper bound in Theorem 1.2. We remark that for the matrices $A_{n}$ constructed in Theorem 3.1, it holds that

$$
\phi\left(A_{n}\right) \leq \frac{1-\left|\lambda_{i}(A)\right|}{\sqrt{n}}
$$

for any $i \neq 1$, giving a stronger guarantee than that required for Theorem 1.2.

We also remark that it would be unlikely to arrive at such a construction by algorithmic simulation, since the eigenvalues of the matrices $A_{n}$ are extremely sensitive. Although $\lambda_{2}\left(A_{n}\right)=0$, if we shift only $O(1 / \sqrt{n})$ of the mass in the matrix $A_{n}$ to create a matrix $A_{n}^{\prime}$, by replacing $a_{n}$ with $a_{n}^{\prime}=a_{n}+b_{n}, b_{n}$ with $b_{n}^{\prime}=0, f_{n}$ with $f_{n}^{\prime}=f_{n}+b_{n}$ and keeping $c_{n}, d_{n}, e_{n}$ the same, then $\lambda_{2}\left(A_{n}^{\prime}\right)=1$. Thus, since perturbations of $O(1 / \sqrt{n})$ (which is tiny for large $n$ ) cause the second eigenvalue to jump from 0 to 1 (and the spectral gap from 1 to 0 ), it would not be possible to make tiny changes to random matrices to arrive at a construction satisfying the required properties in Theorem 3.1.

## 4 Lower bound on the edge expansion $\phi$ in terms of the spectral gap

In this section, we prove the lower bound on $\phi$ in Theorem 1.3, and the lower bound on $\phi$ in Theorem 1.2 will follow as a special case. The proof is a result of a sequence of lemmas that we state next. All proofs for the lemmas in this section can be found in the extended version of the paper [MS19]. The first lemma states that $\phi$ is sub-multiplicative in the following sense.

Lemma 4.1. Let $R \in \mathbb{R}^{n \times n}$ be a nonnegative matrix with left and right eigenvectors $u$ and $v$ for the PF eigenvalue 1. Then

$$
\phi\left(R^{k}\right) \leq k \cdot \phi(R)
$$

For the case of symmetric doubly stochastic matrices $R$, Lemma 4.1 follows from a theorem of Blakley and Roy [BR65]. (It does not fall into the framework of an extension of that result to the nonsymmetric case [Pat12]). Lemma 4.1 helps to lower bound $\phi(R)$ by taking powers of $R$, which is useful since we can take sufficient powers in order to make the matrix simple enough that its edge expansion is easily calculated. The next two lemmas follow by technical calculations.

Lemma 4.2. Let $T \in \mathbb{C}^{n \times n}$ be an upper triangular matrix with $\|T\|_{2}=\sigma$ and for every $i,\left|T_{i, i}\right| \leq \beta$. Then

$$
\left\|T^{k}\right\|_{2} \leq n \cdot \sigma^{n} \cdot\binom{k+n}{n} \cdot \beta^{k-n}
$$

Using Lemma 4.2, we can show the following lemma for the special case of upper triangular matrices with operator norm at most 1 .

Lemma 4.3. Let $T \in \mathbb{C}^{n \times n}$ be an upper triangular matrix with $\|T\|_{2} \leq 1$ and $\left|T_{i, i}\right| \leq \alpha<1$ for every i. Then $\left\|T^{k}\right\| \leq \epsilon$ for

$$
k \geq \frac{4 n+2 \ln \left(\frac{n}{\epsilon}\right)}{1-\alpha}
$$

Given lemmas 4.1 and 4.3, we can lower bound $\phi(R)$ in terms of $1-\left|\lambda_{m}(R)\right|$ (where $\lambda_{m}$ is the nontrivial eigenvalue that is maximum in magnitude). Our aim is to lower bound $\phi(R)$ by $\phi\left(R^{k}\right)$, but since the norm of $R^{k}$ increases by powering, we cannot use the lemmas directly, since we do not want a dependence on $\sigma(R)$ in the final bound. To handle this, we transform $R$ to $A$, such that $\phi(R)=\phi(A)$, the eigenvalues of $R$ and $A$ are the same, but $\sigma(A)=\|A\|_{2}=1$ irrespective of the norm of $R$.

Lemma 4.4. Let $R$ be a nonnegative matrix with positive (left and right) eigenvectors $u$ and $v$ for the PF eigenvalue 1, normalized so that $\langle u, v\rangle=1$. Define $A=D_{u}^{\frac{1}{2}} D_{v}^{-\frac{1}{2}} R D_{u}^{-\frac{1}{2}} D_{v}^{\frac{1}{2}}$. Then the following hold for $A$ :

1. $\phi(A)=\phi(R)$.
2. For every $i, \lambda_{i}(A)=\lambda_{i}(R)$.
3. $\|A\|_{2}=1$.

Given Lemma 4.4, we lower bound $\phi(A)$ using $\phi\left(A^{k}\right)$ in terms of $1-\left|\lambda_{m}(A)\right|$, to obtain the corresponding bounds for $R$.

Lemma 4.5. Let $R$ be a nonnegative matrix with positive (left and right) eigenvectors $u$ and $v$ for the PF eigenvalue 1, normalized so that $\langle u, v\rangle=1$. Let $\lambda_{m}$ be the nontrivial eigenvalue of $R$ that is maximum in magnitude and let $\kappa=\min _{i} u_{i} \cdot v_{i}$. Then

$$
\frac{1}{20} \cdot \frac{1-\left|\lambda_{m}\right|}{n+\ln \left(\frac{1}{\kappa}\right)} \leq \phi(R)
$$

Given Lemma 4.5, we use the trick of lazy random walks to get a bound on $1-\operatorname{Re} \lambda_{2}(R)$ from a bound on $1-\left|\lambda_{m}(R)\right|$.

Lemma 4.6. Let $R$ be a nonnegative matrix with positive (left and right) eigenvectors $u$ and $v$ for the PF eigenvalue 1, normalized so that $\langle u, v\rangle=1$. Let $\kappa=\min _{i} u_{i} \cdot v_{i}$. Then

$$
\frac{1}{30} \cdot \frac{1-\operatorname{Re} \lambda_{2}(R)}{n+\ln \left(\frac{1}{\kappa}\right)} \leq \phi(R)
$$

For any doubly stochastic matrix $A$,

$$
\frac{1-\operatorname{Re} \lambda_{2}(A)}{35 \cdot n} \leq \phi(A)
$$

and thus

$$
\frac{1}{35 \cdot n} \leq \Gamma(n)
$$

This completes the proof of the lower bound on $\phi$ in Theorem 1.3, and the upper bound on $\phi$ in Theorem 1.3 is shown in the extended version of the paper [MS19]. Combined with Theorem 3.1, this also completes the proof of Theorem 1.2.

## 5 Mixing time

We now study the mixing time of nonnegative matrices, and relate it to all the quantities we have studied so far. To motivate the definition of mixing time for general nonnegative matrices, we first consider the mixing time of doubly stochastic matrices. The mixing time of a doubly stochastic matrix $A$ (i.e., of the underlying Markov chain) is the worst-case number of steps required for a random walk starting at any vertex to reach a distribution approximately uniform over the vertices. To avoid complications of periodic chains, we assume that $A$ is $\frac{1}{2}$-lazy, meaning that for every $i, A_{i, i} \geq \frac{1}{2}$. Given any doubly stochastic matrix $A$, it can be easily converted to the lazy random walk $\frac{1}{2} I+\frac{1}{2} A$. This is still doubly stochastic and in the conversion both $\phi(A)$ and the spectral gap are halved. The mixing time will be finite provided only that the chain is connected. Consider the indicator vector $\mathbf{1}_{\{i\}}$ for any vertex $i$. We want to find the smallest $\tau$ such that $A^{\tau} \mathbf{1}_{\{i\}} \approx \frac{1}{n} \mathbf{1}$ or $A^{\tau} \mathbf{1}_{\{i\}}-\frac{1}{n} \mathbf{1} \approx 0$, which can further be written as $\left(A^{\tau}-\frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{T}\right) \mathbf{1}_{\{i\}} \approx 0$. Concretely, for any $\epsilon$, we want to find $\tau=\tau_{\epsilon}(A)$ such that for any $i$,

$$
\left\|\left(A^{\tau}-\frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{T}\right) \mathbf{1}_{\{i\}}\right\|_{1} \leq \epsilon
$$

Given such a value of $\tau$, for any vector $x$ such that $\|x\|_{1}=1$, we get

$$
\begin{aligned}
& \left\|\left(A^{\tau}-\frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{T}\right) x\right\|_{1} \\
& =\left\|\sum_{i}\left(A^{\tau}-\frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{T}\right) x_{i} \mathbf{1}_{\{i\}}\right\|_{1} \\
& \leq \sum_{i}\left|x_{i}\right|\left\|\left(A^{\tau}-\frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{T}\right) \mathbf{1}_{\{i\}}\right\|_{1} \\
& \leq \sum_{i}\left|x_{i}\right| \cdot \epsilon \\
& =\epsilon
\end{aligned}
$$

Thus, the mixing time $\tau_{\epsilon}(A)$ is the number $\tau$ for which $\left\|\left(A^{\tau}-J\right) \cdot x\right\|_{1} \leq \epsilon$ for any $x$ such that $\|x\|_{1}=1$.

We want to extend this definition to any nonnegative matrix $R$ with PF eigenvalue 1 and corresponding positive left and right eigenvectors $u$ and $v$. Note that if $R$ is reducible (i.e., $\phi(R)=0$ ), then the mixing time is infinite. Further, if $R$ is periodic, then mixing time is again ill-defined. Thus, we again assume that $R$ is irreducible and $\frac{1}{2}$-lazy, i.e. $\quad R_{i, i} \geq \frac{1}{2}$ for every $i$. Let $x$ be any nonnegative vector for the sake of exposition, although our final definition will not require nonnegativity and will hold for any $x$. We want to find $\tau$ such that $R^{\tau} x$ about the same as the component of $x$ along the direction of $v$. Further, since we are right-multiplying and want convergence to the right eigenvector $v$, we will define the $\ell_{1}$-norm using the left eigenvector $u$. Thus, for the starting vector $x$, instead of requiring $\|x\|_{1}=1$ as in the doubly stochastic case, we will require $\left\|D_{u} x\right\|_{1}=1$. Since $x$ is nonnegative, $\left\|D_{u} x\right\|_{1}=\langle u, x\rangle=1$. Thus, we want to find $\tau$ such that $R^{\tau} x \approx v$, or $\left(R^{\tau}-v \cdot u^{T}\right) x \approx 0$. Since we measured the norm of the starting vector $x$ with respect to $u$, we will also measure the norm of the final vector $\left(R^{\tau}-v \cdot u^{T}\right) x$ with respect to $u$. Thus we arrive at the following definition.

Definition 5.1. (Mixing time of general nonnegative matrices $R$ ) Let $R$ be a $\frac{1}{2}$-lazy, irreducible nonnegative matrix with $P F$ eigenvalue 1 with $u$ and $v$ as the corresponding positive left and right eigenvectors, where $u$ and $v$ are normalized so that $\langle u, v\rangle=\left\|D_{u} v\right\|_{1}=1$. Then the mixing time $\tau_{\epsilon}(R)$ is the smallest number $\tau$ such that $\left\|D_{u}\left(R^{\tau}-v \cdot u^{T}\right) x\right\|_{1} \leq \epsilon$ for every vector $x$ with $\left\|D_{u} x\right\|_{1}=1$.

We remark that similar to the doubly stochastic case, using the triangle inequality, it is sufficient to find mixing time of standard basis vectors $\mathbf{1}_{\{i\}}$. Let $y_{i}=\frac{\mathbf{1}_{\{i\}}}{\left\|D_{u} \mathbf{1}_{\{i\}}\right\|_{1}}$, then $y_{i}$ is nonnegative, $\left\|D_{u} y_{i}\right\|_{1}=\left\langle u, y_{i}\right\rangle=1$, then for any $x$, such that $\left\|D_{u} x\right\|_{1}=1$, we can write

$$
x=\sum_{i} c_{i} \mathbf{1}_{\{i\}}=\sum_{i} c_{i}\left\|D_{u} \mathbf{1}_{\{i\}}\right\|_{1} y_{i}
$$

with

$$
\left\|D_{u} x\right\|_{1}=\left\|D_{u} \sum_{i} c_{i} \mathbf{1}_{\{i\}}\right\|_{1}=\sum_{i}\left|c_{i}\right|\left\|D_{u} \mathbf{1}_{\{i\}}\right\|_{1}=1
$$

Thus, if for every $i,\left\|D_{u}\left(R^{\tau}-v \cdot u^{T}\right) y_{i}\right\|_{1} \leq \epsilon$, then

$$
\begin{aligned}
& \left\|D_{u}\left(R^{\tau}-v \cdot u^{T}\right) x\right\|_{1} \\
& =\left\|D_{u}\left(R^{\tau}-v \cdot u^{T}\right) \sum_{i} c_{i}\right\| D_{u} \mathbf{1}_{\{i\}}\left\|_{1} y_{i}\right\|_{1} \\
& \leq \sum_{i} c_{i}\left\|D_{u} \mathbf{1}_{\{i\}}\right\|_{1}\left\|D_{u}\left(R^{\tau}-v \cdot u^{T}\right) y_{i}\right\|_{1} \\
& \leq \epsilon
\end{aligned}
$$

Thus, it is sufficient to find mixing time for every nonnegative $x$ with $\left\|D_{u} x\right\|_{1}=\langle u, x\rangle=1$, and it will hold for all $x$.

For the case of reversible nonnegative matrices $M$ with PF eigenvalue 1, the mixing time is well-understood, and it is easily shown that

$$
\begin{equation*}
\tau_{\epsilon}(M) \leq \frac{\ln \left(\frac{n}{\kappa \cdot \epsilon}\right)}{1-\lambda_{2}(M)} \stackrel{\text { Theorem 1.1) }}{\leq} \frac{2 \cdot \ln \left(\frac{n}{\kappa \cdot \epsilon}\right)}{\phi^{2}(M)} \tag{5.3}
\end{equation*}
$$

We will give corresponding bounds for the mixing time of general nonnegative matrices. All proofs for the lemmas in this Section can be found in the extended version of the paper [MS19].
5.1 Mixing time and singular values We first show a simple lemma relating the mixing time of nonnegative matrices to the second singular value. This lemma is powerful enough to recover the bounds obtained by Fill [Fil91] and Mihail [Mih89] in an elementary way. Since the largest singular value of any general nonnegative matrix $R$ with PF eigenvalue 1 could be much larger than 1 , the relation between mixing time and second singular value makes sense only for nonnegative matrices with the same left and right eigenvector for eigenvalue 1, which have largest singular value 1 by Lemma 4.4.

Lemma 5.1. (Mixing time and second singular value) Let A be a nonnegative matrix (not necessarily lazy) with $P F$ eigenvalue 1 , such that $A w=w$ and $A^{T} w=w$ for some $w$ with $\langle w, w\rangle=1$, and let $\kappa=\min _{i} w_{i}^{2}$. Then for every $c>0$,

$$
\tau_{\epsilon}(A) \leq \frac{c \cdot \ln \left(\frac{\sqrt{n}}{\sqrt{\kappa} \cdot \epsilon}\right)}{1-\sigma_{2}^{c}(A)} \leq \frac{c \cdot \ln \left(\frac{n}{\kappa \cdot \epsilon}\right)}{1-\sigma_{2}^{c}(A)}
$$

For the case of $c=2$, Lemma 5.1 was obtained by Fill [Fil91], but we find our proof simpler.
5.2 Mixing time and edge expansion We now relate the mixing time of general nonnegative matrices $R$ to its edge expansion $\phi(R)$. The upper bound for row stochastic matrices $R$ in terms of $\phi(R)$ were obtained
by Mihail [Mih89] and simplified by Fill [Fil91] using Lemma 5.1 for $c=2$. Thus, the following lemma is not new, but we state it for completeness, and also provide a simple proof in the extended version of the paper [MS19].

Lemma 5.2. (Mixing time and edge expansion) Let $\tau_{\epsilon}(R)$ be the mixing time of a $\frac{1}{2}$-lazy nonnegative matrix $R$ with PF eigenvalue 1 and corresponding positive left and right eigenvectors $u$ and $v$, and let $\kappa=\min _{i} u_{i} \cdot v_{i}$. Then

$$
\frac{\frac{1}{2}-\epsilon}{\phi(R)} \leq \tau_{\epsilon}(R) \leq \frac{4 \cdot \ln \left(\frac{n}{\kappa \cdot \epsilon}\right)}{\phi^{2}(R)}
$$

5.3 Mixing time and spectral gap We obtain bounds for the mixing time of nonnegative matrices in terms of the spectral gap, using methods similar to the ones used to obtain the upper bound on $\phi$ in Theorem 1.3.

Lemma 5.3. (Mixing time and spectral gap) Let $\tau_{\epsilon}(R)$ be the mixing time of a $\frac{1}{2}$-lazy nonnegative matrix $R$ with PF eigenvalue 1 and corresponding positive left and right eigenvectors $u$ and $v$, and let $\kappa=\min _{i} u_{i} \cdot v_{i}$. Then
$\frac{\frac{1}{2}-\epsilon}{\sqrt{2 \cdot\left(1-\operatorname{Re} \lambda_{2}(R)\right)}} \leq \tau_{\epsilon}(R) \leq 20 \cdot \frac{n+\ln \left(\frac{1}{\kappa \cdot \epsilon}\right)}{1-\operatorname{Re} \lambda_{2}(R)}$.
We remark that there is only additive and not multiplicative dependence on $\ln \left(\frac{n}{\kappa \cdot \epsilon}\right)$. Further, our construction for the upper bound in Theorem 1.2 also shows that the upper bound on $\tau$ using $\operatorname{Re} \lambda_{2}$ in Lemma 5.3 is also (almost) tight. For the construction of $A_{n}$ in Theorem 1.2, letting the columns of $U_{n}$ be $u_{1}, \ldots, u_{n}$, for $x=u_{2},\left(A_{n}^{k}-J\right) u_{2}=\left(1-(2+\sqrt{n})^{-1}\right)^{k} u_{3}$, and so for $k=O(\sqrt{n})$, the triangular block of $A^{O(\sqrt{n})}$ has norm about $1 / e$, which further becomes less than $\epsilon$ after about $\ln \left(\frac{n}{\epsilon}\right)$ powers. Thus for the matrices $A_{n}$, $\tau_{\epsilon}\left(A_{n}\right) \in O\left(\sqrt{n} \cdot \ln \left(\frac{n}{\epsilon}\right)\right)$. This shows Lemma 5.3 is also (almost) tight since $\lambda_{2}\left(A_{n}\right)=0$.
5.4 Mixing time of a nonnegative matrix and its additive symmetrization We can also bound the mixing time of a nonnegative matrix $A$ with the same left and right eigenvector $w$ for PF eigenvalue 1, with the mixing time of its additive symmetrization $M=\frac{1}{2}\left(A+A^{T}\right)$. Note that we obtained a similar bound on the spectral gaps of $A$ and $M$ in Lemma 1.2. Since $\phi(A)=\phi(M)$, we can bound $\tau_{\epsilon}(A)$ and $\tau_{\epsilon}(M)$ using the two sided bounds between edge expansion and mixing time in Lemma 5.2. For the lower bound, we get $\gamma_{1} \cdot \sqrt{\tau_{\epsilon}(M)} \leq \tau_{\epsilon}(A)$, and for the upper bound, we get

$$
\tau_{\epsilon}(A) \leq \gamma_{2} \cdot \tau_{\epsilon}^{2}(M)
$$

where $\gamma_{1}$ and $\gamma_{2}$ are some functions polylogarithmic in $n, \kappa, \frac{1}{\epsilon}$. However, by bounding the appropriate operator, we can show a tighter upper bound on $\tau_{\epsilon}(A)$, with only a linear instead of quadratic dependence on $\tau_{\epsilon}(M)$.
Lemma 5.4. Let $A$ be a $\frac{1}{2}$-lazy nonnegative matrix with positive left and right eigenvector $w$ for PF eigenvalue 1 , let $M=\frac{1}{2}\left(A+A^{T}\right)$, and $\kappa=\min _{i} w_{i}^{2}$. Then
$\frac{1-2 \epsilon}{4 \cdot \ln ^{\frac{1}{2}}\left(\frac{n}{\kappa \cdot \epsilon}\right)} \cdot \tau_{\epsilon}^{\frac{1}{2}}(M) \leq \tau_{\epsilon}(A) \leq \frac{2 \cdot \ln \left(\frac{n}{\kappa \cdot \epsilon}\right)}{\ln \left(\frac{1}{\epsilon}\right)} \cdot \tau_{\epsilon}(M)$.
One example application of Lemma 5.4 is the following: given any undirected graph $G$ such that each vertex has degree $d$, any manner of orienting the edges of $G$ to obtain a graph in which every vertex has in-degree and out-degree $d / 2$ cannot increase the mixing time of a random walk (up to a factor of $\ln \left(\frac{n}{\kappa \cdot \epsilon}\right)$ ).
5.5 Mixing time of the continuous operator Let $R$ be a nonnegative matrix with PF eigenvalue 1 and associated positive left and right eigenvectors $u$ and $v$. The continuous time operator associated with $R$ is defined as $\exp (t \cdot(R-I))$, where for any matrix $M$, we formally define $\exp (M)=\sum_{i=0} \frac{1}{i!} M^{i}$. The reason this operator is considered continuous, is that starting with any vector $x_{0}$, the vector $x_{t}$ at time $t \in \mathbb{R}_{\geq 0}$ is defined as $x_{t}=\exp (t \cdot(R-I)) x_{0}$. Since
$\exp (t \cdot(R-I))=\exp (t \cdot R) \cdot \exp (-t \cdot I)=e^{-t} \sum_{i=0}^{\infty} \frac{1}{i!} t^{i} R^{i}$
where we split the operator into two terms since $R$ and $I$ commute, it follows that $\exp (t \cdot(R-I))$ is nonnegative, and if $\lambda$ is any eigenvalue of $R$ for eigenvector $y$, then $e^{t(\lambda-1)}$ is an eigenvalue of $\exp (t \cdot(R-I))$ for the same eigenvector $y$. Thus, it further follows that $u$ and $v$ are the left and right eigenvectors for $\exp (t \cdot(R-I))$ with PF eigenvalue 1. The mixing time of $\exp (t \cdot(R-I))$, is the value of $t$ for which

$$
\left\|D_{u}\left(\exp (t \cdot(R-I))-v \cdot u^{T}\right) v_{0}\right\|_{1} \leq \epsilon
$$

for every $v_{0}$ such that $\left\|D_{u} v_{0}\right\|_{1}=1$, and thus, it is exactly same as considering the mixing time of $\exp (R-I)$ in the sense of Definition 5.1.

Lemma 5.5. Let $R$ be a nonnegative matrix (not necessarily lazy) with positive left and right eigenvectors $u$ and $v$ for PF eigenvalue 1, normalized so that $\langle u, v\rangle=1$ and let $\kappa=\min _{i} u_{i} \cdot v_{i}$. Then the mixing time of $\exp (t \cdot(R-I))$, or $\tau_{\epsilon}(\exp (R-I))$ is bounded as

$$
\frac{\frac{1}{2}-\epsilon}{\phi(R)} \leq \tau_{\epsilon}(\exp (R-I)) \leq \frac{100 \cdot \ln \left(\frac{n}{k \cdot \epsilon}\right)}{\phi^{2}(R)}
$$

5.6 Bounds using the canonical paths method For the case of symmetric nonnegative matrices $M$ with PF eigenvalue 1, as stated earlier in this section in equation 5.3 , since $\tau$ varies inversely with $1-\lambda_{2}$ (up to a loss of a factor of $\ln \left(\frac{n}{\kappa \cdot \epsilon}\right)$ ), it follows that any lower bound on the spectral gap can be used to upper bound $\tau_{\epsilon}(M)$. Further, since $1-\lambda_{2}$ can be written as a minimization problem for symmetric matrices (see Section 2), any relaxation of the optimization problem can be used to obtain a lower bound on $1-\lambda_{2}$, and inequalities obtained thus are referred to as Poincare inequalities. One such method is to use canonical paths [Sin92] in the underlying weighted graph, which helps to bound mixing time in certain cases in which computing $\lambda_{2}$ or $\phi$ is infeasible. However, since it is possible to define canonical paths in many different ways, it leads to multiple relaxations to bound $1-\lambda_{2}$, each useful in a different context. We remark one particular definition and lemma here, since it is relevant to our construction in Theorem 3.1, after suitably modifying it for the doubly stochastic case.

Lemma 5.6. [Sin92] Let $M$ represent a symmetric doubly stochastic matrix. Let $W$ be a set of paths in $M$, one between every pair of vertices. For any path $\gamma_{u, v} \in S$ between vertices $(u, v)$ where $\gamma_{u, v}$ is simply a set of edges between $u$ and $v$, let the number of edges or the (unweighted) length of the path be $\left|\gamma_{u, v}\right|$. Let

$$
\rho_{W}(M)=\max _{e=(x, y)} \frac{\sum_{(u, v): e \in \gamma_{u, v}}\left|\gamma_{u, v}\right|}{n \cdot M_{x, y}}
$$

Then for any $W$,

$$
1-\lambda_{2}(M) \geq \frac{1}{\rho_{W}(M)}
$$

and thus,

$$
\tau_{\epsilon}(M) \leq \rho_{W}(M) \cdot \ln \left(\frac{n}{\epsilon}\right)
$$

Corollary 5.1. Combining Lemma 1.3 and Lemma 5.6, it follows that for any doubly stochastic matrix A, and any set $W$ of paths in the underlying graph of $A A^{T}$,

$$
\begin{aligned}
\tau_{\epsilon}(A) \leq \frac{2 \cdot \ln \left(\frac{n}{\epsilon}\right)}{1-\sigma_{2}^{2}(A)}=\frac{2 \cdot \ln \left(\frac{n}{\epsilon}\right)}{1-} \begin{aligned}
& 1-\lambda_{2}\left(A A^{T}\right) \\
& \leq 2 \cdot \rho_{W}\left(A A^{T}\right) \cdot \ln \left(\frac{n}{\epsilon}\right)
\end{aligned} \text {. }
\end{aligned}
$$

Consider the example $A_{n}$ in Theorem 3.1. It is not difficult to see that

$$
\begin{equation*}
\tau_{\epsilon}\left(A_{n}\right) \in O\left(\sqrt{n} \cdot \ln \left(\frac{n}{\epsilon}\right)\right) \tag{5.4}
\end{equation*}
$$

This follows since the factor of $\sqrt{n}$ ensures that the only non zero entries in the triangular matrix $T_{n}$ (see the
extended version of the paper for details [MS19]) in the Schur form of $A^{\lceil\sqrt{n}\rceil}$ are about $e^{-1}$, and the factor of $\ln \left(\frac{n}{\epsilon}\right)$ further converts these entries to have magnitude at most $\frac{\epsilon}{n}$ in $A^{\tau}$. Thus, the operator norm becomes about $\frac{\epsilon}{n}$, and the $\ell_{1}$ norm gets upper bounded by $\epsilon$. However, from Theorem 3.1, since $\phi\left(A_{n}\right) \geq \frac{1}{6 \sqrt{n}}$, it follows from Lemma 5.2 that $\tau_{\epsilon}\left(A_{n}\right) \in O\left(n \cdot \ln \left(\frac{n}{\epsilon}\right)\right)$, about a quadratic factor off from the actual upper bound in equation 5.4. Further, from Theorem 3.1, the second eigenvalue of $A_{n}$ is 0 , and even employing Lemma 5.3 leads to a quadratic factor loss from the actual bound. However, Lemma 5.1 and Corollary 5.1 do give correct bounds. Since $\sigma_{2}\left(A_{n}\right)=1-\frac{1}{\sqrt{n}+2}$ from Theorem 3.1, it follows from Lemma 5.1 for $c=1$ that $\tau_{\epsilon}\left(A_{n}\right) \in O\left(\sqrt{n} \cdot \ln \left(\frac{n}{\epsilon}\right)\right)$, matching the bound in equation 5.4. Now to see the bound given by canonical paths and corollary 5.1, consider the matrix $M=A_{n} A_{n}^{T}$. Every entry of $M$ turns out to be positive, and the set $W$ is thus chosen so that the path between any pair of vertices is simply the edge between the vertices. Further, every entry of the matrix $M$ is at least $c \cdot n^{-\frac{3}{2}}$ for some constant $c$ (see the extended version of the paper [MS19] for details), and from Corollary 5.1, we get that $\tau_{\epsilon}\left(A_{n}\right) \in O\left(\sqrt{n} \cdot \ln \left(\frac{n}{\epsilon}\right)\right)$, matching the bound in equation 5.4.

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[^1]:    ${ }^{1}$ The condition number is $\inf \left(\|B\| \cdot\left\|B^{-1}\right\|\right)$ over $B$ such that $B A B^{-1}$ is diagonal, with $\|\cdot\|$ being operator norm.

