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Reconciling at-a-Station and at-Many-Stations Hydraulic Geometry Through River-Wide Geomorphology

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Key Points:
- We derive AMHG, explain why it exists in some rivers and not others, and reconcile AMHG and AHG, validated across 155 rivers
- We suggest that AMHG is a necessary consequence of any well-fit river-wide relationship of slope, roughness, and geometry imposed on AHG physics
- We show that AMHG presence/absence is a function of the explanatory power of both AHG and a river-wide slope roughness model

Supporting Information:
- Supporting Information S1

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Abstract At-many-stations hydraulic geometry (AMHG), while useful for estimating river discharge from satellite data, remains empirical and has yet to be reconciled with the at-a-station hydraulic geometry (AHG) from which it was originally derived. Here we present evidence, using United States Geological Survey field measurements of channel hydraulics for 155 rivers, that AMHG can be hydraulically and geomorphically reconciled with AHG. Our results indicate that AMHG is rightly understood as an expression of a river-wide model of hydraulics driven by changes in slope imposed upon AHG physics. The explanatory power of AHG and this river-wide model combine to determine whether AMHG exists: if both AHG and the river-wide model adequately describe hydraulics, then we show that AMHG is a necessary mathematical consequence of these two phenomena. We also orient these findings in the context of river discharge estimation and other applications.

Plain Language Summary Hydraulic geometry (HG) is an empirical phenomenon that predicts river width, depth, and velocity given river discharge and is fundamental to our ability to predict floods, river habitats, and water availability for human and ecosystem use. Recently, a new form of HG was discovered (at-many-stations HG, or AMHG), and it has been successfully deployed in a range of applications and proven to exist in a wide variety of rivers. However, the novel phenomenon of AMHG remains empirical with puzzling and seemingly contradictory links to other, traditional forms of HG, which are well understood and have been studied since the 1950s. AMHG is also not manifested in all rivers, adding to confusion as to its origin. Here, we show for the first time that we can reconcile AMHG with all other traditional variants of HG and gain a complete understanding of how and when AMHG occurs in rivers. This puts the most puzzling aspects of AMHG—why is it observed in some rivers but not others, and what causes the phenomenon—to rest. We have fundamentally changed the conception of AMHG and suggest this work as a basis for all future AMHG research.

1. Introduction

Fluvial geomorphology, the study of the form and process of rivers, plays a uniquely forward role in Earth surface processes and sediment manipulation as facilitated by the hydraulics of a river channel. Hydraulic geometry (HG), conceptualized in 1953, describes simple power law relationships between discharge (Q) and channel width (W), depth (d) and velocity (V; Leopold & Maddock, 1953). These are defined as

$$w = aQ^b \tag{1}$$

$$d = cQ^f \tag{2}$$

$$v = kQ^m \tag{3}$$

Via continuity, $ack = 1$ and $b+f+m = 1$. The same principles have been expanded to drainage area and slope, but equations (1)–(3) are the original HG equations. Table A1 in Appendix A gives nomenclature for all variables used in this study.

HG is divided into at-a-station HG (AHG), where equations (1)–(3) apply at a single cross section to describe changing flows over time, and downstream HG (DHG), where equations (1)–(3) apply across multiple stations for a given characteristic flow (i.e., bankfull discharge) through space (Leopold & Maddock, 1953). The widely adopted Leopold and Maddock publication used the same nomenclature for both AHG and DHG, which can lead to confusion. AHG and DHG serve important roles for monitoring streamflow via rating curves, habitat risk assessments, minimum flow requirements for fish passage, remote sensing of surface water, and many more applications (e.g., Ferguson, 1986; Gleason, 2015; Gleason & Smith, 2014; Rosenfeld et al., 2007; Singh & Broeren, 1989; Smith & Pavelsky, 2008; Wiele & Smith, 1996). It has long been debated whether the "discovery" of HG was a foundational development in assessing river form or not; HG's impact on geomorphology, however, is irrefutably significant (Ferguson, 1986).

In a seminal paper on HG, Ferguson (1986) proved that channel cross-sectional shape and velocity-depth relationships (e.g., Manning's or Chezy's relations) govern AHG below-bankfull discharges. Ferguson verified that AHG is a result of a flow resistance law for depth and velocity imposed on a given channel cross-sectional geometry: reducing AHG to "hydraulics and geometry." Based upon this conceptual model, Dingman (2007) introduced physically based expressions for AHG coefficients and exponents by constraining channel geometry at bankfull flow (w_b, d_b, v_b, Q_b) imposed on a channel shape parameter r (where r defines the exponent in a width-depth channel form as approximated by $\frac{f}{b}$. $r = 1$ is a triangle, r = 2 is a parabola, and r represents an increasingly rectangular channel as it approaches infinity) to implicitly scale at-a-station hydraulics within these boundary conditions, yielding equations (4)–(6).

$$w = \left[W_b^{\frac{r+rp}{\delta}} d_b^{-\frac{1+p}{\delta}} \left(\frac{r}{r+1} \right)^{-\frac{1+p}{\delta}} (KS^q)^{-\frac{1}{\delta}} \right] Q^{\frac{1}{\delta}} \tag{4}$$

$$d = \left[W_b^{\frac{-r}{\delta}} d_b^{\frac{1}{\delta}} \left(\frac{r}{r+1} \right)^{\frac{1}{\delta}} (KS^q)^{-\frac{r}{\delta}} \right] Q^{\frac{r}{\delta}} \tag{5}$$

$$v = \left[W_b^{\frac{-rp}{\delta}} d_b^{\frac{p}{\delta}} \left[\frac{r}{r+1} \right]^{\frac{p}{\delta}} [KS^q]^{\frac{1+r}{\delta}} \right] Q^{\frac{rp}{\delta}} \tag{6}$$

K is a roughness coefficient, S is bed slope, and κ, p, and q are fitted model parameters in the equation (7) generalized flow relation, implemented in a later validation of equations (4)–(6) using river cross sections from New Zealand (Dingman & Afshari, 2018):

$$v = \kappa d^p, \text{where } \kappa \cong KS^q \tag{7}$$

It has been repeatedly shown that q and p vary by station and take different values than assumed in either Manning's or Chezy's relations and that AHG should change over long timescales with geologic changes in S, despite the common use of a constant bed slope S in equations (4)–(6) as consistent with uniform flow (e.g., Bjerklie et al., 2005; Dingman, 2007; Dingman & Sharma, 1997; Ferguson, 2010; Knighton, 1975). It should be further noted that AHG is also not always strongly present at all stations of a river. This is because its construction is too simple to adequately describe hydraulics at all stations. Richards (1973) first raised this idea, showing that compound functions fit hydraulic data better than the simple power laws of equations (1)–(3), and Ferguson (1986) later confirmed that the power law form of AHG is coincident with specific channel geometries. Furthering this confirmation, Dingman (2007) added explicit controls on geometry (equations (4)–(6)), and others have addressed a physical basis for AHG (e.g. Parker et al., 2007; Singh, 2003).

A new form of HG was recently discovered, termed AMHG or at-many-stations HG (Gleason & Smith, 2014). AMHG asserts that calibrated AHG coefficients and exponents (equations (1)–(3)) can exhibit a strong log-linear relationship through space along the same river, thus spatially associating cross-sectional hydraulic variation at-a-station with other stations downstream. This would seem to suggest that the site-specific AHG is not in fact site specific as we know it to be. AMHG is defined by equations (8)–(10):

$$\mathrm{w_c} = \mathrm{a_i Q_{cw}^{b_i}} \tag{8}$$

$$\mathrm{d_c} = \mathrm{c_i Q_{cd}^{f_i}} \tag{9}$$

$$\mathrm{v_c} = \mathrm{k_i Q_{cv}^{m_i}} \tag{10}$$

where subscript i refers to different stations in space along a river and subscript c refers to what are termed

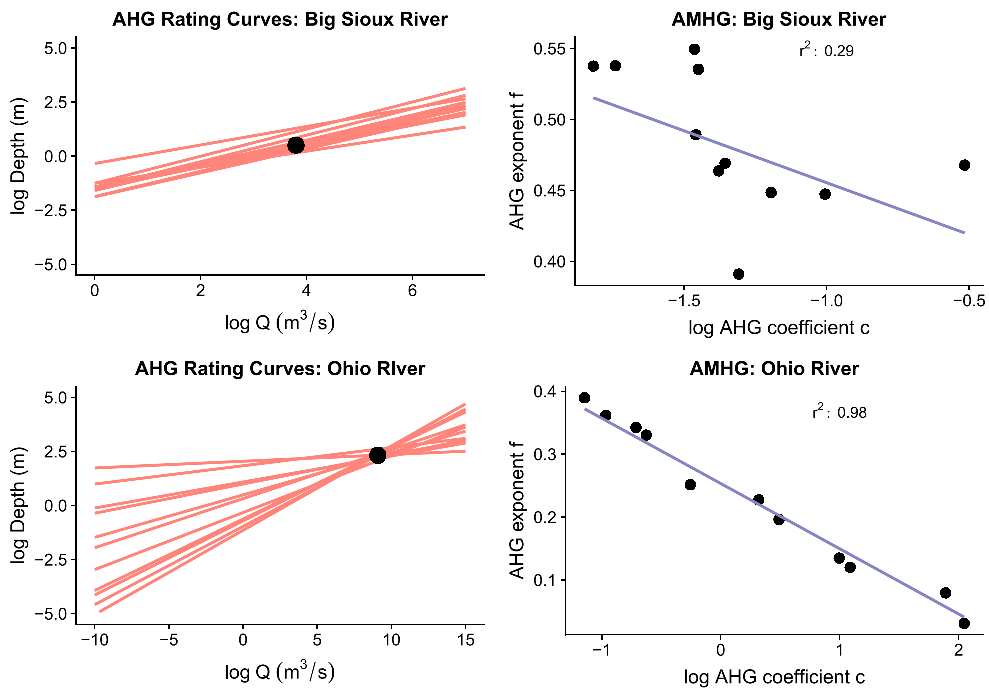

Figure 1. Individual AHG curves are shown in the left panels (with black dots highlighting d_c, Q_{cd}) and resulting AMHG shown on the right panels. Gleason and Wang (2015) showed that the stronger an AMHG, the more that rivers formed "bow tie" shapes in hydraulic space. Data are taken from United States Geological Survey databases of field measurements and are filtered consistent with the rest of the data in this paper. AHG = at-a-station hydraulic geometry; AMHG = at-many-stations hydraulic geometry.

"congruent hydraulics," the empirically fit parameters that define AMHG, and a, *b, c, f, k,* and *m* are identical to equations (1)–(3) (Gleason & Wang, 2015). Barber and Gleason (2018) used United States Geological Survey (USGS) measurements across the continental United States to show that 118/181 rivers in their database exhibited strong AMHG (conceptually defined as the ability of AMHG to satisfactorily predict AHG parameters with an r^2 of equations (8)–(10) greater than 0.60: AMHG can explain 60% of the variance in AHG parameters when it is strong by their definition).

The three pairs of congruent hydraulics (w_c Q_{cw}, d_c Q_{cd}, and v_c, Q_{cv}) in equations (8)–(10) are mathematically defined by Gleason and Wang (2015) as single discharge-width, discharge-depth, and discharge-velocity tuples that exist in all AHG curves for that river. That is, if all AHG curves for a river are plotted on the same axes, all will intersect at exactly one point, forming a "bow tie" in log-log hydraulic coordinate space. These intersection points are the congruent hydraulic pairs. Gleason and Wang (2015) showed mathematically that the degree of empirical clustering around these theoretical intersection points thus defines the strength of AMHG as a geomorphic phenomenon (see Figure 1). The mathematical definition of AMHG and congruent hydraulics is thus secure, yet the physical definition of these phenomena is an open research question. Shen et al. (2016) proposed that width AMHG (equation (8)) is largely mathematical while depth and velocity AMHG (equations (9) and (10)) represent distinct geomorphic phenomena with a physical basis that conveniently coincides with Gleason and Wang's (2015) mathematical definition. However, this was tested statistically and yielded no further illumination on what this physical basis is or how AMHG is physically associated to AHG.

In theory, AMHG and AHG should be reconcilable as AMHG fundamentally describes the intersection point and degree of intersection of AHG curves in the same river. However, we do not know (1) why AHG curves intersect in this manner, (2) whether AMHG is purely mathematical or is in fact a geomorphic phenomenon, or (3) why some rivers exhibit AMHG and some do not. Here, we address these uncertainties by first analytically reconciling AHG and AMHG, using the fullest expression of AHG possible (Dingman, 2007) as a basis for our reconciliation. With this established, we then test the hypothesis that AMHG is a necessary consequence of a systematic downstream variation in channel shape, roughness, and slope, as suggested by the results of our first analysis. We make this test by introducing a series of well-established models of river-wide geomorphology that control these parameters into our newly derived analytical expressions for AMHG's congruent hydraulics. What results is a new understanding of AMHG.

2. Data

To develop our theory, we required independent field measurements of river hydraulics, which we derived from the USGS's collection of field-measured quantities. We joined the Barber and Gleason (2018) database of field-measured w, v, and Q with the USGS's National Hydrography Data set (Geological Survey, U. S, 2019), to assign an S and distance along the river l to each station in the data set for later model calibration. V is the cross-sectional average velocity in the data set measured using a variety of field techniques, and d was derived as $d = Q/wv$. More details on our data set are in Text S1 in the supporting information but ultimately, we have field-measured data (filtered for measurement quality) for almost 1,600 cross sections across 155 rivers, covering an extensive array of AHG and AMHG strengths (again defined by r^2).

Next, we calculated station-specific parameters for Dingman's (2007) expressions for AHG (equations (4)–(6)). These are necessarily constrained by bankfull depth and width. Acknowledging that field studies have shown Q_b to be associated with a range of return periods given local geomorphology (Petit & Pauquet, 1997; Williams, 1978), we used a range of return periods to define bankfull conditions. The results were not sensitive to these changes, so we chose the commonly accepted definition of a 2-year return period for Q_b. Defining these bankfull flows allows us to obtain r for each cross section in conjunction with other measurements of d and w, following Dingman (2007). Finally, to obtain K for each cross section, we followed Dingman and Afshari (2018), solving explicitly for K by assuming q always approximates one half, as in Manning's equation. More sophisticated treatment using variable exponent depth flow resistance (q) introduces another free parameter that prevents our analysis here, despite our previous assertion that q should vary. While a station (and time) varying q is desirable, application of the Manning's equation is standard practice in most HG research and applications, and we lack data for more sophisticated expressions of flow. More details on this process are outlined in Text S1.

3. Reconciling AMHG With Dingman's AHG Formulations

We first orient congruent hydraulics (i.e., AMHG) within physical expressions for AHG. Congruent hydraulic pairs from AMHG (equations (8)–(10)) are substituted for the generalized hydraulic pairs in equations (4)–(6) and algebraic manipulation allows equations (4)–(6) to take the forms highlighted in equations (11)–(13):

$$w_c^{\delta} Q_{cw}^{-1} = W_b^{r+rp} d_b^{-(1+p)} \left(\frac{r}{r+1}\right)^{-(1+p)} \left(KS^{\frac{1}{2}}\right)^{-1} \tag{11}$$

$$d_c^{\frac{\delta}{r}} Q_{cd}^{-1} = W_b^{-1} d_b^{\frac{1}{r}} \left(\frac{r}{r+1}\right)^{\frac{1}{r}} \left(KS^{\frac{1}{2}}\right)^{-1} \tag{12}$$

$$V_c^{\frac{\delta}{rp}} Q_{cv}^{-1} = W_b^{-1} d_b^{\frac{1}{r}} \left(\frac{r}{r+1}\right)^{\frac{1}{r}} \left(KS^{\frac{1}{2}}\right)^{\frac{1+r}{rp}} \tag{13}$$

AMHG's congruent hydraulics are now defined explicitly as some function of K, S, and r, with bankfull geometry as boundary conditions that vary downstream. These equations have been created via substitution: we have placed AMHG into an AHG context assuming they are reconcilable. To test whether these equations are valid, we turn to our 155-river data set. For each cross section, the left-hand sides of equations (11)–(13) predict a station varying quantity (out of algebraic necessity) built from river-wide congruent hydraulics, whereas the right-hand sides have only site-specific AHG parameters. If AMHG's congruent hydraulics physically coexist with AHG, then the left- and right-hand sides should be equivalent. We test this at our approximately 1,600 cross sections in Figure 2, using empirically defined congruent hydraulics following Gleason and Wang (2015). We also calculated two One-Sided t tests to determine if the equivalence between the two sides of equations (11)–(13) are statistically significant across our data set ("control" in Table S2).

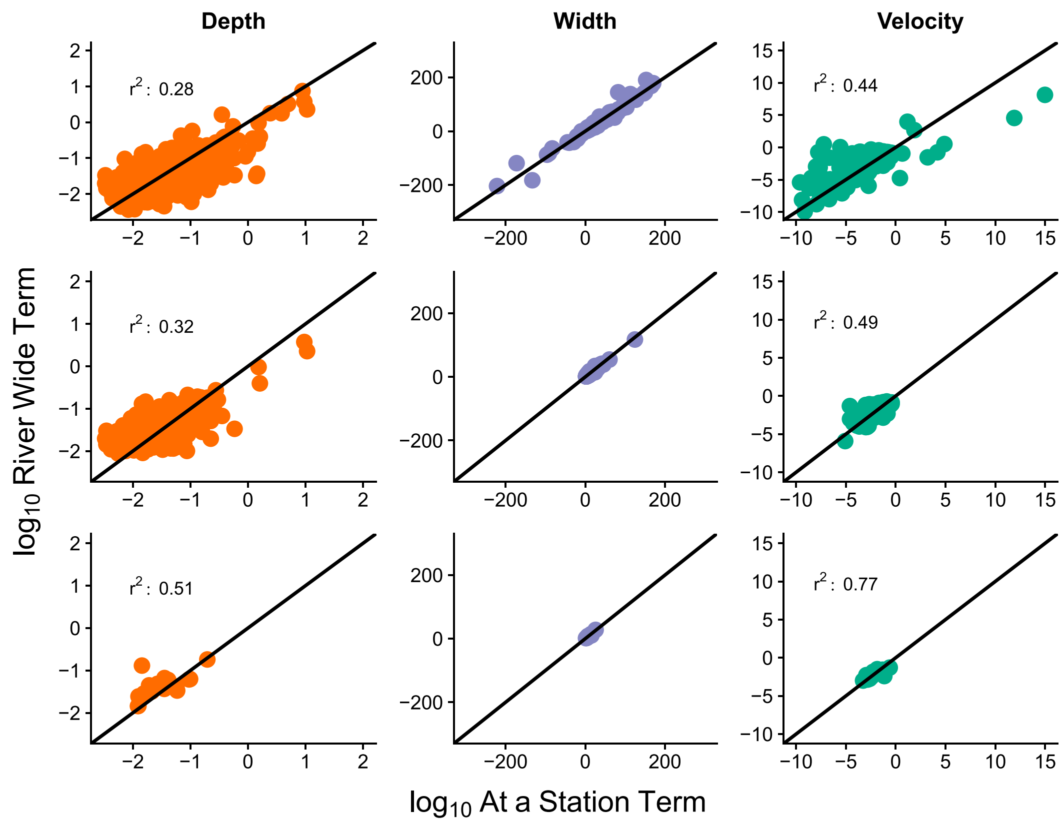

Figure 2. At-a-station term (RHS) plotted against the river-wide term (LHS) for equations (11)–(13). This progressive filtering indicates the better that AHG describes at-a-station hydraulics, the better agreement between AHG and AMHG. AHG strength is independently calculated on traditional AHG built from w, d, v, and Q without any river-wide hydraulics added. Axes were scaled to aid in visualization, and thus 19, 25, and 14 data points are not shown in the upper left, upper right, and middle left panels, respectively. Width r^2 is not shown, as width AHG and AMHG are derived by continuity and agreement is artificially high. AHG = at-a-station hydraulic geometry; AMHG = at-many-stations hydraulic geometry.

Figure 2 shows that AMHG's congruent hydraulics are satisfyingly reconciled with AHG expressions across our data set. This is corroborated by the equivalence tests, where all relations were statistically similar at $p <$ 0.003 ("control" in Table S2). The agreement between left-and-right hand sides of equations (11)–(13) increases with progressive filtering for AHG strength: the stronger AHG is at a cross section, the better AHG and AMHG agree. Width, conversely, yields nearly perfect recovery regardless of AHG strength as it is derived via continuity. This demonstrates that the congruent hydraulic pairs of w_c and Q_{cw}, v_c and Q_{cv}, and d_c and Q_{cd} are physically viable hydraulic geometry relations (i.e., these quantities are physically realistic), but this viability seems to be a function of how well AHG describes observed hydraulics (e.g., the narrowing ranges in Figure 2 as AHG strength increases). Thus, we conclude that AMHG is compatible with the physics that yield AHG. Further exploration of Figure 2 is given in Text S3, including a discussion on why filtering for stronger AHG yields values in Figure 2 that fall within physically realistic ranges of the values plotted on either axis of Figure 2 (i.e., the left- and right-hand sides of equations (11)–(13)).

4. AMHG as a Function of River-Wide Geomorphology

We have shown that AMHG can be reconciled with AHG, but we have not yet shown why we should expect AMHG to exist in the first place. Equations (11)–(13) have numerous station-varying terms on both sides of the equation. If we are to discover the source of the constant-in-space-and-time congruent hydraulics that define AMHG, equations (11)–(13) necessitate that some function of bed slope, bed roughness, and channel shape is needed to produce congruent hydraulics anywhere along a river. AMHG will only be fully explained if this function holds everywhere in a river: the station varying parameters in equations (11)–(13) coexist with river-wide parameters, and these thus must "trade off" explicitly to produce congruent hydraulics.

What phenomenon should guarantee this arrangement is not shown in Figure 2: we do not know *why* these parameters trade-off so precisely to allow AMHG-AHG reconciliation. We posit that any well-fit river-wide model of variation in cross-sectional geomorphology as a function of slope will yield reconciled congruent hydraulics. That is, AMHG and AHG can be reconciled only if (1) AHG adequately describes changes in hydraulics at a station in time (per section 3), and (2) there is an additional geomorphic relationship driven by changes in S that adequately describes at-a-station hydraulics through space. We introduce six well-known such additional relationships here (equations (14)–(19): "Graded Profile," "Regime Theory," "Einstein'," "Bjerklie," "Bjerklie Global," and "DHG". Again, consult Table A1 in Appendix A for variable nomenclature.

$$\textit{Graded Profile} : S = S_0 e^{-\alpha L} \tag{14}$$

$$\textit{Regime Theory} : S = 0.44 D_e^{1.15} Q_b^{-0.46} \tag{15}$$

$$\textit{Einstein} : S = \left(\frac{D}{u}\right)^{\frac{1}{t}} \tag{16}$$

$$\textit{Bjerklie} : S = \left(\frac{F_b}{x}\right)^{\frac{1}{y}} \tag{17}$$

$$\textit{Bjerklie Global} : S = \left(\frac{F_b}{2.85}\right)^{\frac{1}{0.31}} \tag{18}$$

$$\textit{DHG} : S = eQ_b^g \tag{19}$$

All these theories predict geomorphic relationships along a river, and all are driven by S. None are explicitly designed for AMHG or AHG, but all are well known and frequently observed. Some of these theories have globally constant parameters (Regime Theory, Bjerklie Global) and some require river or data set specific calibration (DHG, Einstein, Graded Profile, Bjerklie). Model definitions are outlined in Table S1, with additional grain size calibration methods necessary for the Einstein model in Text S1. The Bjerklie models are derived from Bjerklie et al. (2018), Graded Profile from Mackin (1948), and Regime Theory from Henderson (1966).

We substituted these six river-wide models (equations (14)–(19)) for the S term in equations (11)–(13), and rearranging yields, for example, equations (20)–(25) for depth. Text S2 gives similar expressions for width and velocity and also steps through all necessary algebra to arrive at equations (20)–(25).

$$\textit{Graded Profile} : d_c^{\frac{\delta}{r}} Q_{cd}^{-1} \left(S_0 e^{-\alpha L}\right)^{\frac{1}{2}} = W_b^{-1} d_b^{\frac{1}{r}} \left(\frac{r}{r+1}\right)^{\frac{1}{r}} K^{-1} \tag{20}$$

$$\textit{Regime Theory} : d_c^{\frac{\delta}{r}} Q_{cd}^{-1} \left(0.44 D_e^{1.15} Q_b^{-0.46}\right)^{\frac{1}{2}} = W_b^{-1} d_b^{\frac{1}{r}} \left(\frac{r}{r+1}\right)^{\frac{1}{r}} K^{-1} \tag{21}$$

$$\textit{Einstein} : d_c^{\frac{\delta}{r}} Q_{cd}^{-1} \left(\frac{D}{u}\right)^{\frac{1}{2t}} = W_b^{-1} d_b^{\frac{1}{r}} \left(\frac{r}{r+1}\right)^{\frac{1}{r}} K^{-1} \tag{22}$$

$$\textit{Bjerklie} : d_c^{\frac{\delta}{r}} Q_{cd}^{-1} \left(\frac{Fb}{x}\right)^{\frac{1}{2y}} = W_b^{-1} d_b^{\frac{1}{r}} \left(\frac{r}{r+1}\right)^{\frac{1}{r}} K^{-1} \tag{23}$$

$$\textit{Bjerklie Global} : d_c^{\frac{\delta}{r}} Q_{cd}^{-1} \left(\frac{Fb}{2.85}\right)^{\frac{1}{2*0.31}} = W_b^{-1} d_b^{\frac{1}{r}} \left(\frac{r}{r+1}\right)^{\frac{1}{r}} K^{-1} \tag{24}$$

$$\textit{DHG} : d_c^{\frac{\delta}{r}} Q_{cd}^{-1} \left(eQ_b^g\right)^{\frac{1}{2}} = W_b^{-1} d_b^{\frac{1}{r}} \left(\frac{r}{r+1}\right)^{\frac{1}{r}} K^{-1} \tag{25}$$

As before, we have isolated the congruent AMHG terms on the left-hand side. Equations (20)–(25) are still simply Dingman's AHG expressions at a station, now with both AMHG and a river-wide slope model imposed upon them.

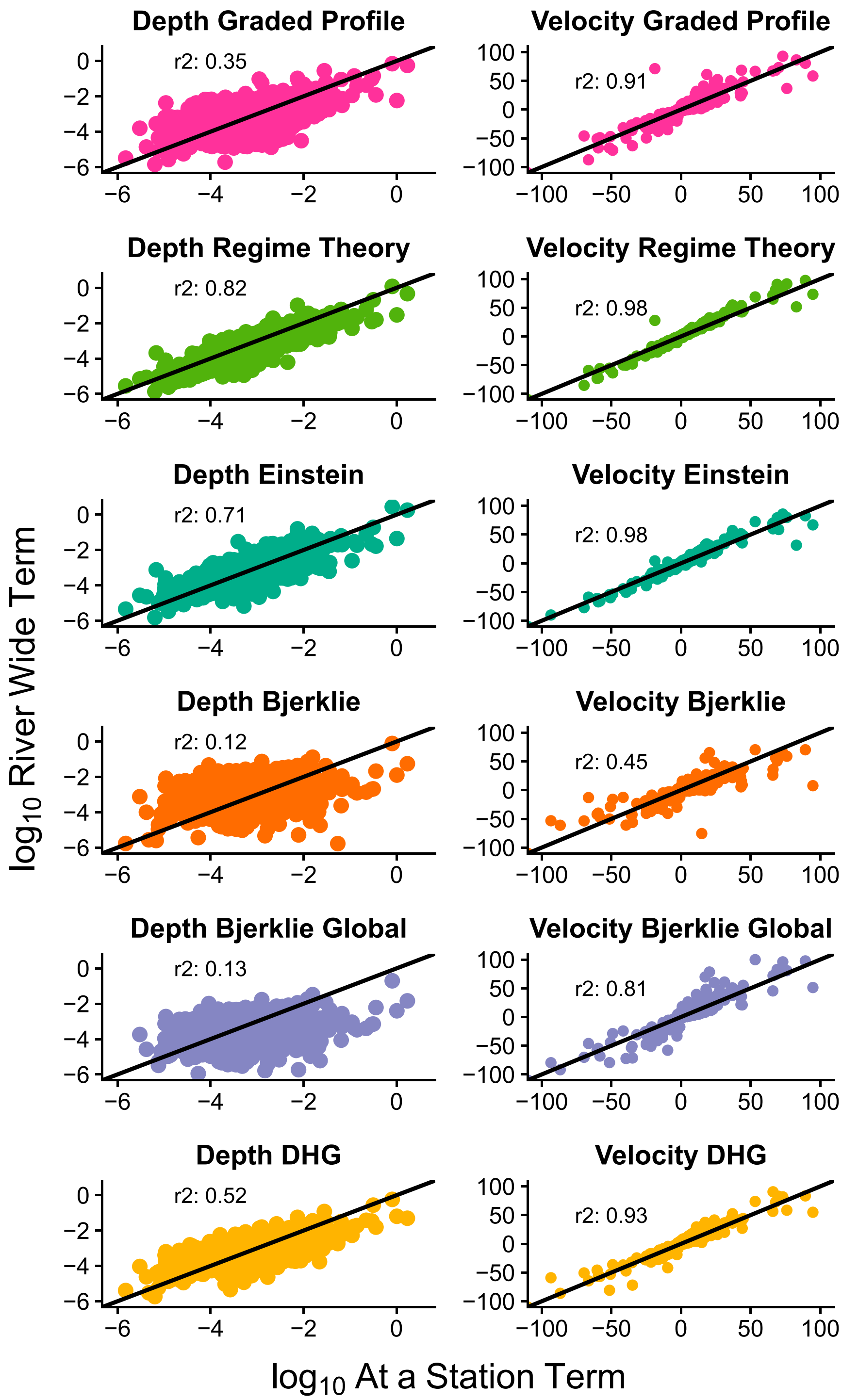

Figure 3. River-wide models introduced into equations (11)–(13) via equations (20)–(25) for depth (left column) and via equations (S6c)–(S11c) in supporting information Text S2 for velocity (right column) hold that agreement in these plots indicates that a given model is a possible origin of AMHG in that river. Multiple successful models suggest that more than one river-wide geomorphic relation can yield AMHG. Axes limits were again set to aid visualization, resulting in 9–16 points used in error assessment not appearing across all plots. Agreement is stronger for velocity than depth. AMHG = at-many-stations hydraulic geometry.

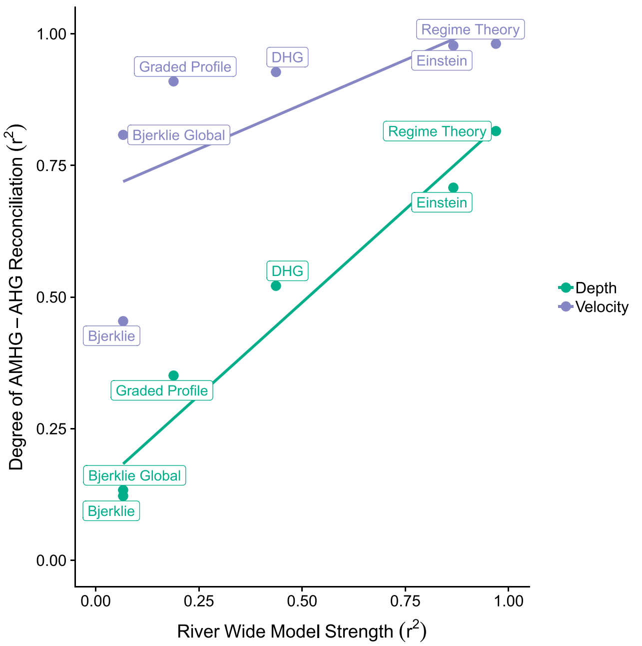

Figure 4. Strength of fit (r^2) between left- and right-hand sides of equations (20)–(25) for depth (equations (S6c)–(S11c) in supporting information Text S1 for velocity) from Figure 3 and the strength-of-fit of the river-wide models (r^2) across all cross sections. For both variables, agreement between the two sides of the equations (and thus AMHG strength) increases as models better represent a cross section. We have added a linear trend only for visualization purposes and do not propose that this relationship is linear. AMHG = at-many-stations hydraulic geometry.

We can test agreement between the two sides of equations (20)–(25) for depth in the same manner as in section 4 when we compared the two sides of equations (11)–(13). We also calculated two One-Sided *t* tests and confirmed that the two sides of equations (20)–(25) are statistically significant across our data set. These results are for depth and velocity as provided in Figure 3, and width is not shown in Figure 3 as it exhibits near-perfect recovery across all models tested because we follow Dingman (2007) to derive width AHG via continuity, as also anticipated by Shen et al. (2016). Consult Text S2 for a velocity derivation and Text S3 and Figure S1 for a more in-depth explanation of the width phenomenon.

Figure 3 and the results of the equivalence tests (Table S2) confirm that there are statistically significant similarities between the left- and right-hand sides of equations (20)–(25), and all tests are significant at $p < 0.003$. As noted previously, the width test is not appropriate and is artificially perfectly recovered, regardless of model. Our six models show a range in strengths of recovery of congruent hydraulics, defined by their r^2. It logically follows that the better the river-wide model describes at-a-station hydraulics, the better it can explain the existence of AMHG. To address this, in Figure 4 we consider the strength of fit of each river-wide model in the context of the agreement between the left- and right-hand sides of equations (20)–(25).

Figure 4 shows that when a river-wide model better predicts cross-sectional hydraulics, AMHG is better recovered. Thus, AMHG appears not to be an outcome of a specific river-wide phenomenon, but rather is a function of any model that accurately reflects the river's unique physics along its longitudinal profile as predicted by changes in *S*. That is, if a river is physically well-represented by a river-wide model that constrains AHG hydraulics, then AMHG is more strongly reconciled than if the model was poorly fit at that cross section. Thus, we conclude that AMHG could be a mathematical consequence of any of these six river-wide phenomena imposed on AHG physics provided the river-wide model is strong. Regime Theory is the strongest predictor, but this finding should hold for *any* river-wide model so long as it physically represents the river's downstream changes in slope, roughness, and/or channel shape. AMHG is therefore driven by these larger geomorphic forces and manifests only when these conditions are met. This explains both the origins of AMHG and also why it exists only in certain rivers.

5. Implications and Application

Our results suggest that AMHG arises from systematic downstream changes in hydraulics driven by geological changes in *S*, which allows us to interrogate past AMHG research. Shen et al. (2016) suggested that depth and velocity AMHG represented physical phenomena and that width AMHG was largely mathematical by forcing random perturbations through empirical AMHG and finding that width did not vary in response. We find that the left- and right-hand sides of equations (S5a)–(S10a) in supporting information Text S2 agree perfectly for width (Figure S1), regardless of the goodness of fit of the river-wide model or AHG. We have thus corroborated Shen et al.'s (2016) findings using actual river physics across 155 rivers. This highlights the role of continuity in HG research—it is difficult to solve for all three HG relations independently. Text S3 also notes how our results bear on previous definitions of "sufficient variation" in AHG to define AMHG.

Our new definition of AMHG and congruent hydraulics should be used in applications moving forward. The above analysis suggests that width AMHG can continue to be reliably exploited to estimate discharge from remote sensing on rivers that fit systematic models of downstream river geomorphology well, as proposed by Gleason and Smith (2014), Gleason, et al. (2014), Hagemann et al. (2017), and others. Thus, an immediate implication of this work is to shift applications research toward predicting which rivers exhibit strong AMHG, now physically defined via equations (11)–(13) and equations (20)–(25), a priori. This in turn provides a basis for when an AMHG-based discharge estimation algorithm will prove useful: the better a river-wide model fits a river's observed hydraulics, the stronger AMHG-AHG reconciliation will be and the more likely discharge inversion is to be successful. As a start, researchers should begin by assuming Regime Theory, but researchers seeking to estimate discharge via an AMHG-based algorithm should consider how to identify the degree of fit of river-wide models to rivers in the context of remote sensing. Further, we now have an explicit at-a-station expression (right-hand side of equation (11)) which should yield a physically viable (e.g.,) Q_{cw}, w_c hydraulic pair, scaled for each cross-section. Because we now prove that these are physically viable hydraulics, prior distributions can be built from training data by substituting the distribution of $w_c^{\delta} Q_{cw}^{-1}$ for the resulting values of the right-hand side of equation (11). This crucially opens the door for developing physically based river classes for global rivers a priori, each with their own tailor-made $w_c^{\delta} Q_{cw}^{-1}$ distributions for Bayesian inversion algorithms (e.g., Hagemann et al., 2017). These advances should be incorporated into new discharge algorithms with new inversion physics.

We are unable to give more explicit definitions for congruent hydraulics beyond equations (20)–(25) and equations (S5a)–(S10c) in Text S2. Ideally, we would build independent expressions where, for example, d_c and Q_{cd} would appear as separate terms rather than the way they are formulated in equations (20)–(25). However, this is not algebraically possible, which means we were unable to assess what physical quantities, if any, d_c and Q_{cd} specifically represent. However, our combined term still has merit as it places the river-wide and station-varying terms in balance with one another, and future work should seek to further this line of inquiry and perhaps develop explicit formulae for congruent hydraulics. Future AMHG research should also address the reliance on Manning's relation (here necessitated by a lack of required data) in the derivations of AMHG and AHG. That is, it is possible that our results are limited by Manning's physics and measurement error and AMHG and AHG are more universal than our data allows us to show.

Finally, we also note that Gleason and Smith (2014), Gleason and Wang (2015), Shen et al. (2016), and Barber and Gleason (2018) all observed that the three Q_c values were not equivalent. However, until now, no satisfactory explanation as to why has been provided due to a lack of physical expressions for congruent hydraulics. It is shown in Text S3 and Figure S2 that each pair of congruent hydraulics represents distinct geometric conditions; that is, for a given cross-sectional depth to equal d_c, the associated width need not be w_c and so accompanying congruent discharges necessarily vary. To put this another way, we can confirm that continuity does not hold on congruent discharges: $Q_c \neq w_c\, d_c\, v_c$.

6. Conclusions

We have shown that AMHG's congruent hydraulics can be reconciled with AHG using measurements from almost 1,600 cross sections across 155 rivers. We have redefined AMHG in the process: We suggest AMHG to be a mathematical expression of *any* systematic downstream variation in roughness, slope, and channel shape, expressed through AHG physics. This fundamentally alters previous definitions of AMHG and proves that this empirical phenomenon is not at odds with classical AHG, which has played a pivotal role in river geomorphology for decades. What results instead is a new understanding of AMHG, and why AMHG manifests in some rivers but not others has been shown here as well. This has important implications for the use of AMHG in estimating river discharge, and perhaps opens the door to its application in other hydrologic arenas.

We began this paper by asking (1) why AHG curves intersect to yield AMHG, (2) whether AMHG is purely mathematical or is in fact a geomorphic phenomenon, and (3) why some rivers exhibit AMHG and some do not. We have put forth answers to each of these questions here. We propose that (1) AHG curves will only intersect to yield AMHG if there is an underlying river-wide slope function that well describes hydraulics everywhere in a river, (2) we show that AMHG is a mathematical consequence of this larger geomorphic phenomenon imposed on AHG physics, and (3) we now know that the conditions for AMHG's presence in a river are strong AHG and a strong river-wide slope function. AMHG research is thus brought into line with more mainstream fluvial geomorphology concerned with these systematic changes in slope and flow resistance along rivers, and AMHG applications are thus now soundly grounded and can proceed with this new geomorphic basis at their core.

Table A1
Table of Nomenclature Used in Study

Variable Definition			
a	Coefficient in width-discharge relation [d]	Q_{cw}	Congruent width discharge [l^3/t]
b	Exponent in width-depth relation [d]	Q_{cd}	Congruent depth discharge [l^3/t]
c	Coefficient in depth-discharge relation [d]	Q_{cv}	Congruent velocity discharge [l^3/t]
d	Cross-sectional depth [l]	r	Channel shape factor [d]
D	Grain size, per Einstein and Barbarossa (1952) [l]	r^2	Coefficient of Determination [d]
D_b	Bankfull depth [l]	S	Bed slope [d]
d_c	Congruent depth [l]	S_0	Most upstream bed slope [d]
D_e	Minimum grain size not entrained, per Henderson (1966) [l]	t	Exponent in F_b-S relation [d]
e	Coefficient in slope-bankfull discharge relation [d]	u	Coefficient in F_b-S relation [d]
F	Exponent in depth-discharge relation [d]	v	Cross-sectional velocity [l/t]
F_b	Froude number [d]	v_c	Congruent velocity [l/t]
g	Exponent in slope-bankfull discharge relation [d]	w	Cross-sectional width [l]
k	Coefficient in velocity-discharge relation [d]	W_b	Bankfull width [l]
K	Generalized roughness coefficient [d]	w_c	Congruent width [l]
l	Distance Downstream [l]	x	Coefficient in D-S relation [d]
m	Exponent in velocity-discharge relation [d]	y	Exponent in D-S relation [d]
p	Exponent in velocity-depth relation [d]	α	Diminution coefficient, per Graded Profile (Lane, 1955; Mackin, 1948) [d]
q	Exponent in depth-velocity relation [d]	κ	Coefficient in velocity-depth relation [d]
Q	Instantaneous discharge [l^3/t]	δ	$1 + r + rp$
Q_b	Bankfull discharge [l^3/t]		

Note. Units in brackets. [d] denotes dimensionless values, [l] denotes length values, and [t] denotes time values.

Appendix A

Table A1 contains the variable nomenclature used throughout the study.

Acknowledgments
All data used in this study are freely available from the USGS and easily accessed using their dataRetrieval package for R (https://github.com/USGS-R/dataRetrieval) or their hydrography program (https://www.usgs.gov/core-science-systems/ngp/national-hydrography). The script and data sets necessary to repeat this analysis are available online (https://github.com/craigbrinkerhoff/2019_AMHG). Input from Mark Fonstad greatly improved initial drafts of the manuscript, as did thoughtful and useful review comments from Paul Bates. C. B. Brinkerhoff was supported by a grant to C. J. Gleason through the NSF RoL FELS RAISE program (PI Peter Raymond, Grant 1840243).

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Erratum

In the originally published version of this article, equations (12) and (13) were published incorrectly. These errors have since been corrected, and the present version may be considered the authoritative version of record.