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Nonlinear Optimal Velocity Car Following Dynamics (I): Approximation in Presence of Deterministic and Stochastic Perturbations

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Abstract— **The behavior of the optimal velocity model is investigated in this paper. Both deterministic and stochastic perturbations are considered in the Optimal velocity model and the behavior of the dynamical systems and their convergence to their associated averaged problems is studied in detail.**

I. INTRODUCTION AND RELATED WORKS

Microscopic dynamical models are one of the most important traffic flow models which consider the behavior of each individual vehicle-driver in response to its interaction with other vehicles. These dynamical models are suitable frameworks for Intelligent Transportation Systems (ITS) including the Adaptive Cruise Control (ACC) and Advanced Driver Assistance Systems (ADS) (for details about the applications, different microscopic models and their assumptions, we refer the interested readers to [1]). From theoretical point of view, these problems are very complex and have many interesting properties that make them theoretically rich and valuable. The literature is extensive and diverse in the sense of proposed mathematical models and the realistic traffic factors that are included.

One main type of microscopic traffic dynamics is so called *car-following models*. Different categorization, models and analysis in this area can be found in [1]. A generic form of the these model is in the form of

$$\ddot{X}^{(n)}(t) = F(\Delta X^{(n)}(t), V^{(n-1)}(t), V^{(n)}(t)), \tag{1}$$

where $\Delta X^{(n)}(t) = X^{(n-1)}(t) - X^{(n)}(t)$, *i.e.* the distance between the n-th vehicle and its leading one. $V^{(n)}$ denotes the velocity of the n-th vehicle. Function F defines the acceleration and in fact can be interpreted as response to the interactions with other vehicles.

Reuschel [2] and Pipe [3] introduced the first car following models. Many other dynamical models have been proposed to improve realistic assumptions on these original models. We refer the readers to some of the fundamental works in this area [3], [4], [5], [6], [7], [8], [9].

We are mainly interested in a particular type of car-following models in this paper. Bando et al. [10], [11] introduce *Optimal velocity* (OV) model in which each vehicle has an optimal velocity function (see equation (2) below). The comparison of the current and a so called optimal speed decides the acceleration or deceleration of vehicle under consideration. This legal or optimal speed is defined based on the distance between a vehicle and its preceding car.

Most of the work in this area are related to study of stability of the corresponding dynamical systems. In this work, however, we investigate this problem from a different point of view. We are interested in interpreting this OV model as a basic model which governs the behavior of an autonomous system. We believe that in order to improve this dynamical model from this point of view, we need to first investigate the system locally by isolating some of the component (vehicles) and investigate the behavior of this system under limited source of interactions. Then we can extend the source of interactions to contain other vehicles and drivers. To do so, We consider the optimal velocity model proposed by Bando et al. [11] as our primary dynamical model. Some details about this dynamical model is explained in the next section and more details on defining the model, its stability, nonlinearity, and its performance in the numerical analysis can be found in [11]. We localize this model to contain the first two vehicles only.

One important aspect in study of any dynamical model is investigating the behavior of such system in presence of deterministic or stochastic perturbations. In this regards, element of stochasticity has been discussed in several papers, including the drivers uncertain behavior [1, Section 12], stochasticity which causes the traffic breakdown in [12] and references therein, and more recently [13], [14] which discuss the stochastic stability of OV models. In this work we consider a particular deterministic and stochastic perturbations in OV dynamical model and discuss the convergence of these systems to an unperturbed model known as averaged problem. This is in particular important since any real dynamical model encounters perturbation due to different reasons including the uncertainty in parameters and uncertain environmental conditions among others. In addition, the analysis and some other proven properties of the solution are insightful on the their own right. Separately, In [15] the order of convergence and the quality of such approximation is discussed by authors.

The organization of the paper is as follows. We first review the construction of the OV dynamical model and its modifications in section II. Then in section III we discuss the bound on the solution of the system, which is essential to prove our main results. In the next step we show the asymptotic behavior of the system in the presence of

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deterministic perturbation. We further extend our result by discussing the stochastic oscillations in the system and the averaging behavior of the dynamical model in section IV.

II. OPTIMAL VELOCITY CAR FOLLOWING DYNAMICS

In this section we first elaborate the dynamics governing the motion of a platoon of vehicles and then we discuss some properties of the solution.
We start by considering N number of vehicles and let $X^{(n)}(t)$ be the position of the n-th car at time t. Assuming that the platoon is moving rightward and the first car is in the right most position. In addition, for simplicity we assume that the first car is moving at a constant speed v_0 and the starting position is the origin. *i.e.*

$$X^{(1)}(t) = v_0 t.$$

In this paper we consider the *optimal velocity dynamics* proposed by [11] and can be presented as follows. For the n-th car we have

$$\ddot{X}^{(n)}(t) = \alpha\left\{V\left(\frac{X^{(n-1)}(t) - X^{(n)}(t)}{d}\right) - \dot{X}^{(n)}(t)\right\}, \quad (2)$$

where $\alpha > 0$ is a constant associated with the driver's sensitivity and $d > 0$ is an adjusting constant and function V which will be defined later. If we define $\Delta X \stackrel{\text{def}}{=} X^{(n-1)} - X^{(n)}$, in fact, function $V(\frac{\Delta X}{d})$ captures a so called *optimal* velocity of the vehicle n which depends on the distance between the n-th car and its preceding $(n-1)$-th car. This dynamic explains the acceleration and deceleration of the following car n. That is, considering (2) if

$$\Delta X^{(n)} > dV^{-1}\left(\dot{X}^{(n)}\right),$$

where V^{-1} is the inverse function, then a positive force (acceleration) needs be imposed. Similarly, if this identity is negative, then a negative force (deceleration) needs to be applied in order to keep the vehicle in a safe distance.

Based on the role that function V plays in defining the above dynamics, we expect this function to be *(a)* monotonically increasing and *(b)* bounded above (maximum permitted speed). In addition, for our analytic investigation we like function V to satisfy Lipschitz condition and to be sufficiently smooth. In this paper, we consider function V to be:

$$V(x) \stackrel{\text{def}}{=} \tanh(x-2) + \tanh(2);$$

which satisfies the required properties (figure 1).

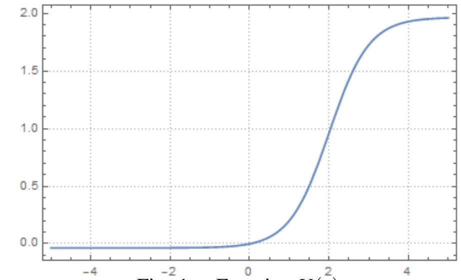

Fig. 1. Function $V(x)$.

A. Reducing the Dynamics

To investigate the dynamics as well as the interaction between the vehicles, we first take a closer look at the first two cars. To do so, let us first define:

$$z^{(2n-1)}(t) \stackrel{\text{def}}{=} X^{(1)}(t) - X^{(n)}(t) = v_0 t - X^{(n)}(t)$$
$$z^{(2n)}(t) \stackrel{\text{def}}{=} \dot{X}^{(1)}(t) - \dot{X}^{(n)}(t) = v_0 - \dot{X}^{(n)}(t),$$

for $n \in \{1, \cdots, N\}$. This implies that $z^{(1)}(t) = z^{(2)}(t) = 0$ and also

$$v_0 - \dot{z}^{(2n-1)}(t) = v_0 - z^{(2n)}(t)$$
$$\dot{z}^{(2n)}(t) = -\ddot{X}_t^{(n)}.$$

Therefore, (2) can be written in the form of

$$\dot{z}^{(2n-1)}(t) = z^{(2n)}(t)$$
$$\dot{z}^{(2n)}(t) = -\alpha\left\{V\left(\frac{z^{(2n-1)}(t) - z^{(2n-3)}(t)}{d}\right) - v_0 + z^{(2n)}(t)\right\}.$$

We start by considering the first and the second cars only. In this case, the dynamics of the second car is

$$\dot{z}^{(3)}(t) = z^{(4)}(t)$$
$$\dot{z}^{(4)}(t) = -\alpha\left\{V\left(\frac{z^{(3)}(t)}{d}\right) + z^{(4)}(t) - v_0\right\}.$$

Let us denote $x(t) = z^{(3)}(t)$ and $y(t) = z^{(4)}(t)$ and then we have the following optimal velocity dynamic for the first two vehicles:

$$\begin{aligned} &\dot{x}(t) = y(t) \\ &\dot{y}(t) = -\alpha\left\{V\left(\frac{x(t)}{d}\right) + y(t) - v_0\right\} \\ &(x(t_0), y(t_0)) = (x_0, y_0). \end{aligned} \quad (3)$$

As mentioned before, we believe that understanding the interaction between two vehicles is crucial in theoretical investigation of more interacting vehicles.

Remark 1 (Existence and Uniqueness of Solution) The existence, uniqueness and smoothness of the solution of (3) can be easily discussed based on properties of function V and in particular Lipschitz continuity.

III. FAST PERTURBATION

In this section we introduce a fast perturbation term into equation (3) and we discuss that asymptotically this system will behave like the averaged dynamical problem.

Fast perturbations can happen in different situations. For example variations in the velocity due to rotational fluctuations as the result of weather and/or road condition. Consider a car driving 60 m/h a bounded variation happens in the velocity as a result of road conditions (e.g. bumps) every 5 feet. Then the variation in velocity happens in the order of $\varepsilon = 0.05$ seconds. [16] includes detailed examples on

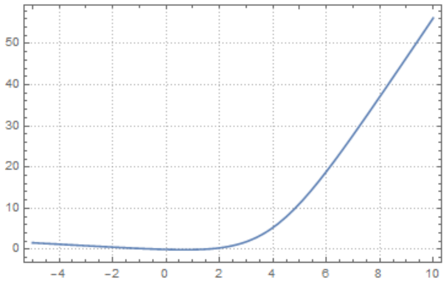

Fig. 2. Function $W(x)$, for $d=5, v_0=0.5, \alpha=10$.

oscillation theory of nonlinear and deterministic dynamical systems.

We consider

$$\begin{aligned} &\dot{x}_\varepsilon(t) = y_\varepsilon(t) \\ &\dot{y}_\varepsilon(t) = -\alpha\left\{V\left(\frac{x_\varepsilon(t)}{d}\right) + y_\varepsilon(t) - v_0\right\} + g(t/\varepsilon) \\ &(x_\varepsilon(0), y_\varepsilon(0)) = (x_0, y_0), \end{aligned} \tag{4}$$

for a bounded, and smooth function $g:\mathbb{R}\to\mathbb{R}$. We like to study the behavior of system (4) over the time interval $[0,T]$, for some $T>0$.

Theorem 1 Let $(x_\varepsilon(t), y_\varepsilon(t))$ be the solution of (4) on interval $[0,T]$. Furthermore, we suppose that function g is bounded by constant M_g. Then there exists a constant $M=M(T)>0$ such that:

$$\sup_{\varepsilon>0}\sup_{t\in[0,T]} |y_\varepsilon(t)| \le M.$$

Proof: Let us define function $H:\mathbb{R}^2\to\mathbb{R}$:

$$H(x,y) \stackrel{\text{def}}{=} \frac{1}{2}y^2 + W(x), \tag{5}$$

where,

$$\begin{aligned} W(x) &\stackrel{\text{def}}{=} \alpha\int_0^x V(x'/d)dx' - \alpha v_0 x \\ &= \frac{\alpha d}{2}\log\left(\frac{\cosh(x/d-2)}{\cosh(2)}\right) + \alpha x\tanh(2) - \alpha v_0 x. \end{aligned}$$

Using the definition of function W and assuming that $v_0 \in (\tanh(2)-\frac{1}{2}, \tanh(2)+\frac{1}{2})$, we have:

$$\begin{aligned} &W(x) \asymp kx, \quad \text{as } x\nearrow\infty, \quad \text{where} \\ &k \stackrel{\text{def}}{=} \alpha(\frac{1}{2} + \tanh(2) - v_0) > 0, \\ &W(x) \asymp k'x, \quad \text{as } x\searrow -\infty, \quad \text{where} \\ &k' \stackrel{\text{def}}{=} \alpha(-\frac{1}{2} + \tanh(2) - v_0) < 0. \end{aligned}$$

In addition, it is simple to observe that $W''(x)>0$. Hence,

$$\underline{W} \stackrel{\text{def}}{=} \inf_x W(x) > -\infty$$

Figure III shows the behavior of function W for given values of the parameters. Considering function H defined in (5), we have that:

$$\begin{aligned} \partial_x H(x,y) &= \alpha\left\{V\left(\frac{x}{d}\right) - v_0\right\} \\ \partial_y H(x,y) &= y. \end{aligned}$$

Therefore,

$$\begin{aligned} &H(x_\varepsilon(t), y_\varepsilon(t)) - H(x_0,y_0) \\ &= \int_0^t \Big(\dot{x}_\varepsilon(s)\partial_x H(x_\varepsilon(s), y_\varepsilon(s)) \\ &\qquad + \dot{y}_\varepsilon(s)\partial_y H(x_\varepsilon(s), y_\varepsilon(s))\Big) ds \\ &= \int_0^t \Big[\alpha y_\varepsilon(s)\left(V(x_\varepsilon(s)/d) - v_0\right) + y_\varepsilon(s)\times \\ &\qquad \{-\alpha\{V(x_\varepsilon(s)/d) - v_0 + y_\varepsilon(s)\} + g(s/\varepsilon)\}\Big] ds \\ &= \int_0^t \{-\alpha(y_\varepsilon(s))^2 + y_\varepsilon(s)g(s/\varepsilon)\} ds \\ &\le \int_0^t y_\varepsilon(s)g(s/\varepsilon)ds \end{aligned}$$

Therefore, using Young inequality and definition of M_g over $[0,T]$, we have that:

$$\begin{aligned} &H(x_\varepsilon(t), y_\varepsilon(t)) - H(x_0,y_0) \\ &\le \frac{1}{2}\int_0^t (y_\varepsilon(s))^2 ds + \frac{1}{2}M_g^2 T. \end{aligned}$$

On the other hand, by the definition of H and $\underline{W}$:

$$H(x,y) \ge \frac{1}{2}y^2 + \underline{W}.$$

This implies that on $[0,T]$:

$$\begin{aligned} \frac{1}{2}(y_\varepsilon(t))^2 \le\; &H(x_0,y_0) - \underline{W} + \frac{1}{2}\int_0^t (y_\varepsilon(s))^2 ds \\ &+ \frac{1}{2}M_g^2 T. \end{aligned}$$

Therefore by applying *Gronwall* inequality on $y_\varepsilon^2(t)$ and taking square root, we get:

$$\sup_{\varepsilon>0}\sup_{t\in[0,T]} |y_\varepsilon(t)| \le \left\{\sqrt{2H(x_0,y_0) - 2\underline{W} + M_g^2 T}\right\} e^{T/2} \tag{6}$$

which completes the proof. ■

We define

$$\bar{g} \stackrel{\text{def}}{=} \lim_{T\nearrow\infty}\frac{1}{T}\int_0^T g(t)dt, \tag{7}$$

and assuming that such limit exists. Next theorem shows that the dynamical system (4) as a perturbed system behaves in a certain way like and averaged problem. It should be noted that while convergence to the averaged problem may look natural for the fast oscillating dynamics but the proof requires detailed and subtle discussions in general.

Theorem 2 Let g be bounded and smooth function. Then the perturbed dynamic in (4) can be approximated by the solution to the following dynamical system

$$\begin{aligned} &\dot{x}^\circ(t) = y^\circ(t), \\ &\dot{y}^\circ(t) = -\alpha\left\{V\left(\frac{x^\circ(t)}{d}\right) + y^\circ(t) - v\right\} + \bar{g}, \\ &(x^\circ(0), y^\circ(0)) = (x_0, y_0), \end{aligned} \tag{8}$$

over time interval $[0,T]$. In other words,

$$\sup_{t\in[0,T]} |X_\varepsilon(t) - X^\circ(t)| \to 0, \quad \text{as } \varepsilon \to 0,$$

where,

$$X_\varepsilon(t) = \begin{pmatrix} x_\varepsilon(t) \\ y_\varepsilon(t) \end{pmatrix}, \quad X^\circ(t) = \begin{pmatrix} x^\circ(t) \\ y^\circ(t) \end{pmatrix},$$

and $X^\circ(t)$ is the solution of (8).

Proof: Let us define

$$A(t) \stackrel{\text{def}}{=} \int_0^t \{g(s) - \bar{g}\}\, ds.$$

The following observations will be used through the entire discussion. First, we recall that:

$$M_g = \sup_{t\in\mathbb{R}} |g(t)|.$$

Thus,

$$\sup_{t\in[0,T]} |A(t)| \le 2M_g T.$$

This means function A is only locally bounded.

Second, by the definition and boundedness of g, we can observe that $g(t) - \bar{g} \in L^1_{loc}(\mathbb{R})$, and therefore by Lebesgue differentiation theorem we have

$$A'(t) = g(t) - \bar{g}, \quad \text{for } t \in \mathbb{R},$$

and boundedness of g implies that:

$$\sup_{t\in\mathbb{R}} \left|A'(t)\right| \le 2M_g.$$

That is, A is continuously differentiable with bounded derivative.

The above fact in particular implies that $A : \mathbb{R} \to \mathbb{R}$ is *absolutely continuous* function and hence, for $t \in \mathbb{R}$ and for any $\varepsilon > 0$ we have

$$\int_0^t \{g(s/\varepsilon) - \bar{g}\}\, ds = \int_0^{t/\varepsilon} \varepsilon A'(\tau) d\tau = \varepsilon A(t/\varepsilon). \tag{9}$$

For simplicity, we denote by

$$L(x,y) \equiv -\alpha \left\{ V\left(\frac{x}{d}\right) + y - v \right\}.$$

Let us write the perturbed dynamics in the integral form as follows.

$$\begin{aligned} y_\varepsilon(t) &= y_0 + \int_0^t L(x_\varepsilon(s), y_\varepsilon(s)) ds + \int_0^t g(s/\varepsilon) ds \\ &= y_0 + \int_0^t L(x_\varepsilon(s), y_\varepsilon(s)) ds \\ &\quad + \int_0^t (g(s/\varepsilon) - \bar{g}) ds + \int_0^t \bar{g} ds \\ &= y_0 + \int_0^t L(x_\varepsilon(s), y_\varepsilon(s)) ds + \varepsilon A(t/\varepsilon) + \bar{g}t. \end{aligned} \tag{10}$$

First let us look at the following limit:

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon A(t/\varepsilon) &= \lim_{\varepsilon\to 0} \varepsilon \left(\int_0^{t/\varepsilon} g(s) ds - \bar{g}t/\varepsilon \right) \\ &= t \lim_{\varepsilon\to 0} \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} g(s) ds - \bar{g}t = 0, \end{aligned} \tag{11}$$

since by our assumption the limit exists and is equal to $\bar{g}$. Now, if we let $\varepsilon \to 0$ then (11) and (10) imply that:

$$\sup_{t\in[0,T]} \left| y_\varepsilon(t) - \left(y_0 + \int_0^t (L(x_\varepsilon(s), y_\varepsilon(s)) + \bar{g}) ds \right) \right| \to 0, \tag{12}$$

as $\varepsilon \to 0$. The next step is to show that x_ε and y_ε have limits when $\varepsilon \to 0$. Let us define the collection of functions of the form

$$\mathscr{F} = \{y_\varepsilon : \varepsilon > 0\}, \tag{13}$$

such that for each $\varepsilon > 0$, $y_\varepsilon \in C([0,T],\mathbb{R})$ and function $t \mapsto y_\varepsilon(t)$ solves (4). Then $\mathscr{F} \subset \left(C([0,T],\mathbb{R}), \mathscr{T}_{\|\cdot\|_u}\right)$ where $\|\cdot\|_u$ is the uniform norm on this space and $\mathscr{T}_{\|\cdot\|_u}$ denotes the associated topology with respect to this norm on $C([0,T],\mathbb{R})$. Equation (6) implies that $\mathscr{F}$ is bounded. Using dynamical system (4), theorem 1 and boundedness of $g(t)$, it is straightforward to see that:

$$\sup_{\varepsilon>0} |y_\varepsilon(t) - y_\varepsilon(s)| \le |t-s| K, \quad t,s \in [0,T],$$

for some constant $K > 0$. This proves the *equicontinuity* of the collection $\mathscr{F}$. Therefore by Arzela-Ascoli theorem we conclude that $\mathscr{F}$ is *totally bounded* (precompact). Hence, any sequence $\{y_{\varepsilon_m}\}_{m\in\mathbb{N}} \subset \mathscr{F}$ has a cauchy subsequence which converges in the Banach space $\left(C([0,T],\mathbb{R}), \mathscr{T}_{\|\cdot\|_u}\right)$. Let us consider any sequence $\{y_{\varepsilon_n}\}_{n\in\mathbb{N}}$ in which $\varepsilon_n \searrow 0$ as $n \nearrow \infty$. In addition, let $y^* \in C([0,T],\mathbb{R})$ be the limit point of this sequence. Similarly, we can write

$$\begin{aligned} \lim_{n\nearrow\infty} \|X_{\varepsilon_n}\|_u &= \lim_{n\nearrow\infty} \sup_{t\in[0,T]} |x_{\varepsilon_n}(t)| \\ &= \lim_{n\nearrow\infty} \sup_{t\in[0,T]} \left| x_0 + \int_0^t y_{\varepsilon_n}(s) ds \right| \\ &= \sup_{t\in[0,T]} \left| x_0 + \lim_{n\nearrow\infty} \int_0^t y_{\varepsilon_n}(s) ds \right|. \end{aligned}$$

Therefore (6) justifies the dominated convergence theorem and hence $x_{\varepsilon_n} \to x^* \in C([0,T],\mathbb{R})$ as $n \to \infty$ with respect to $\mathscr{T}_{\|\cdot\|_u}$ topology where,

$$x^*(t) = x_0 + \int_0^t y^*(s) ds.$$

Therefore, from (12) and using dominated convergence theorem, we have that

$$\begin{aligned} &\lim_{n\nearrow\infty} \sup_{t\in[0,T]} \left| y_{\varepsilon_n}(t) - \left(y_0 + \int_0^t (L(x_{\varepsilon_n}(s), y_{\varepsilon_n}(s)) + \bar{g} \right) ds \right| \\ &= \sup_{t\in[0,T]} \left| y^*(t) - (y_0 + \int_0^t (L(x^*(s), y^*(s)) + \bar{g}) ds \right| = 0. \end{aligned}$$

This result suggest that $(x^*(t), y^*(t))$ is the solution of the following dynamics over $[0,T]$

$$\begin{aligned} \dot{x}^*(t) &= y^*(t), \\ \dot{y}^*(t) &= L(x^*(t), y^*(t)) + \bar{g}, \end{aligned}$$

with the initial condition (x_0, y_0). On the other hand, the ODE has a unique solution; therefore, since any limit point satisfies the same ODE, $\lim_{\varepsilon\to 0} y_\varepsilon$ exists in $\left(C([0,T],\mathbb{R}), \mathscr{T}_{\|\cdot\|_u}\right)$. This completes the proof. ∎

IV. STOCHASTIC PERTURBATION

In this section we consider a small stochastic perturbation included in our dynamical model. As mentioned before, this kind of perturbation can be as a result of rotational noise in the velocity with small amplitude. Stochastic perturbation for general dynamical systems have been studied under different assumptions in various studies, e.g. [17] and references therein. In this paper we consider a stochastically perturbed OV model and show that this system can be approximated by the associated averaged dynamics. Let $\{W(t) : t \geq 0\}$ be a 1-dim Wiener process with $W_0 = 0$. Consider the following system:

$$\begin{aligned} &dx_\varepsilon(t) = y_\varepsilon(t)dt, \\ &dy_\varepsilon(t) = -\alpha\left\{V\left(\frac{x_\varepsilon(t)}{d}\right) + y_\varepsilon(t) - v_0\right\}dt + \varepsilon\, dW(t) \quad (14) \\ &(x_\varepsilon(0), y_\varepsilon(0)) = (x_0, y_0). \end{aligned}$$

In this paper we discuss a simple case of small Brownian perturbation. It should be noted that we could also discuss a more general case in which the perturbation is in the form of $\varepsilon\sigma(x_\varepsilon(t), y_\varepsilon(t))dW(t)$ in which $\sigma(.)$ satisfies Lipschitz and linear growth conditions and still similar result holds true.

Theorem 3 Let $(x_\varepsilon(s), y_\varepsilon(s))$ be the solution of (14) and $(x(t), y(t))$ be the solution of (3). Then,

$$\mathbb{P}\left\{\sup_{t\in[0,T]} |y_\varepsilon(t) - y(t)| > \delta\right\} \leq \varepsilon^2\delta^{-2}u(T),$$

for a bounded function u. A similar result holds true for solution $x_\varepsilon(t)$.

Proof: By using the definition of the dynamical systems (3) and (14), we have that

$$\begin{aligned} \sup_{t\in[0,T]} |y_\varepsilon(t) - y(t)| \leq{}& \alpha\int_0^T \left|V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right| ds \\ &+ \alpha\int_0^T |y_\varepsilon(s) - y(s)|\, ds \\ &+ \varepsilon \sup_{t\in[0,T]} |W(t)|. \end{aligned}$$

Therefore we can write:

$$\begin{aligned} &\mathbb{P}\left\{\sup_{t\in[0,T]} |y_\varepsilon(t) - y(t)| > \delta\right\} \\ &\leq \mathbb{P}\left\{\alpha\int_0^T \left|V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right| ds > \delta/4\right\} \\ &+ \mathbb{P}\left\{\alpha\int_0^T |y_\varepsilon(s) - y(s)|\, ds > \delta/4\right\} \\ &+ \mathbb{P}\left\{\varepsilon \sup_{t\in[0,T]} |W(t)| > \delta/2\right\} \end{aligned}$$

Now we calculate each of these terms separately.

$$\begin{aligned} &\mathbb{P}\left\{\alpha\int_0^T |y_\varepsilon(s) - y(s)|\, ds > \delta/4\right\} \\ &\leq 16\alpha^2\delta^{-2}\mathbb{E}\left(\int_0^T |y_\varepsilon(s) - y(s)|\, ds\right)^2 \\ &\leq 16\alpha^2\delta^{-2}T\mathbb{E}\int_0^T (y_\varepsilon(s) - y(s))^2\, ds \\ &= 16\alpha^2\delta^{-2}T\int_0^T \mathbb{E}\,(y_\varepsilon(s) - y(s))^2\, ds, \end{aligned} \tag{15}$$

where the first inequality is by Chebyshev and the second one is by aplying the Holder's inequality and the last equality is by Fubini theorem. To calculate (15) we apply the Ito formula on $F(y_\varepsilon(t), y(t))$, where $F(v,w) \stackrel{\text{def}}{=} (v-w)^2$:

$$\begin{aligned} &(y_\varepsilon(t) - y(t))^2 \\ &= -2\alpha\int_0^t \left\{(y_\varepsilon(s) - y(s))\left\{V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right\}\right\}ds \\ &\quad - 2\alpha\int_0^t (y_\varepsilon(s) - y(s))^2\, ds \\ &\quad + 2\varepsilon\int_0^t (y_\varepsilon(s) - y(s))\, dW(s) + \varepsilon^2 t. \end{aligned}$$

Dropping the negative term, taking expectation from both sides and the fact that expectation of the martingale term is zero, we obtain:

$$\begin{aligned} &\mathbb{E}\,(y_\varepsilon(t) - y(t))^2 \\ &\leq -2\alpha\int_0^t \mathbb{E}\left\{(y_\varepsilon(s) - y(s))\left(V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right)\right\}ds \\ &\quad + \varepsilon^2 T. \end{aligned}$$

Applying Young inequality, we have

$$\begin{aligned} \mathbb{E}\,(y_\varepsilon(t) - y(t))^2 \leq{}& \alpha\int_0^t \mathbb{E}\,(y_\varepsilon(s) - y(s))^2\, ds \\ &+ \alpha\int_0^t \mathbb{E}\left(V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right)^2 ds \\ &+ \varepsilon^2 T \end{aligned} \tag{16}$$

Function V satisfies Lipschitz continuity by its definition and suppose K denotes the Lipschitz constant. Thus we can write:

$$\begin{aligned} \left|V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right| &\leq K\,|x_\varepsilon(s) - x(s)| \\ &\leq K\int_0^s |y_\varepsilon(r) - y(r)|\, dr. \end{aligned}$$

Therefore squaring both sides and applying Holder's inequality on th right hand side, we obtain:

$$\left(V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right)^2 \leq K^2T\int_0^s (y_\varepsilon(r) - y(r))^2\, dr. \tag{17}$$

Replacing in (16) then for any $t \in [0,T]$ we have:

$$\begin{aligned} \mathbb{E}\,(y_\varepsilon(t) - y(t))^2 \leq{}& \alpha\int_0^t \mathbb{E}\,(y_\varepsilon(s) - y(s))^2\, ds \\ &+ \alpha K^2 T\int_0^t\int_0^s \mathbb{E}\,(y_\varepsilon(r) - y(r))^2\, dr\, ds + \varepsilon^2 T \end{aligned}$$

Gronwall-Bellman type inequality (see [15, Appendix] for details about this inequality) suggests that for any $t \in [0,T]$:

$$\mathbb{E}\left(y_\varepsilon(t) - y(t)\right)^2 \le \varepsilon^2 T \left(1 + \frac{\alpha}{\alpha + K^2 T} e^{(\alpha + K^2 T)t}\right) \qquad (18)$$

Therefore, from (15) we may write:

$$\mathbb{P}\left\{\alpha \int_0^T |y_\varepsilon(s) - y(s)|\, ds > \delta/4\right\} \le \varepsilon^2 \delta^{-2} u_1(T). \qquad (19)$$

where,

$$u_1(T) \stackrel{\text{def}}{=} 16\alpha^2 T^2 \int_0^T \left\{1 + \frac{\alpha}{\alpha + K^2 T} e^{(\alpha + K^2 T)t}\right\} dt.$$

Now, we consider the next probability term:

$$\begin{aligned}
&\mathbb{P}\left\{\alpha \int_0^T \left| V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right| ds > \delta/4\right\} \\
&\le 16\alpha^2 \delta^{-2} \mathbb{E}\left\{\int_0^T \left| V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right| ds\right\}^2 \\
&\le 16\alpha^2 \delta^{-2} T \mathbb{E}\left\{\int_0^T \left(V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right)^2 ds\right\},
\end{aligned}$$

where, the first inequality is by Chebyshev and the second one is by applying the Holder's inequality. Using (17) and (18) we may write

$$\mathbb{P}\left\{\alpha \int_0^T \left| V\left(\frac{x_\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right)\right| ds > \delta/4\right\} \le \varepsilon^2 \delta^{-2} u_2(T)$$

where

$$u_2(T) \stackrel{\text{def}}{=} 16\alpha^2 K^2 T^3 \int_0^T \int_0^s \left\{1 + \frac{\alpha}{\alpha + K^2 T} e^{(\alpha + K^2 T)r}\right\} dr\, ds.$$

Finally applying Doob's maximal inequality, we have that

$$\begin{aligned}
\mathbb{P}\left\{\varepsilon \sup_{t \in [0,T]} |W(t)| > \delta/2\right\} &\le 4\varepsilon^2 \delta^{-2} \mathbb{E}\,|W(T)|^2 \\
&= 4\varepsilon^2 \delta^{-2} T = \varepsilon^2 \delta^{-2} u_3(T)
\end{aligned}$$

Therefore letting $u(T) \stackrel{\text{def}}{=} u_1(T) + u_2(T) + u_3(T)$ completes the proof. ■

We note that a similar proof can be used to prove the same result for solution x_ε. In fact, by considering the definition of x_ε in our dynamical system, the proof is trivial. This shows that the solution of stochastically perturbed problem can be uniformly approximated by the solution of unperturbed problem in probability.

V. CONCLUSION AND FUTURE WORKS

In this paper we considered the optimal velocity model with some perturbation included in the model. We first studied a simple fast deterministic perturbation in the system and showed that such model can be approximated with its associated averaged problem. Then, we investigated a stochastic perturbed model with simple small Brownian noise and we showed that similar to the deterministic case, this model also can be approximated by the averaged model.

We analyzed the system locally for only the first two vehicles. Many behaviors of the system is not still known and hence we believe it is essential to understand the behavior of the system when the interaction of the particles (vehicles in this case) is limited.

Study of OV models can be extended in some important directions. For instance, different sources of perturbation and in particular stochastic perturbation can bring the model closer to the real situations, e.g. the uncertainty in the behavior of the drivers in response to different interaction with other vehicles. Such models are already considered and studied in the sense of stability and a similar analysis as in this paper can be informative.

As mentioned before, we are considering the OV model in this paper mainly as a basic model governing an autonomous system and we considered a local system in which the interaction of following vehicle is in response to the leading car with constant velocity. There are several directions that the result of this paper can be extended, for example, a non constant leading velocity which definitely requires more involving analysis.

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