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Nonlinear Optimal Velocity Car Following Dynamics (II): Rate of Convergence In the Presence of Fast Perturbation

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Abstract— **Traffic flow models have been the subject of extensive studies for decades. The interest in these models is both as the result of their important applications as well as their complex behavior which makes them theoretically challenging. In this paper, an optimal velocity dynamical model is considered and analyzed. We consider a dynamical model in the presence of perturbation and show that not only such a perturbed system converges to an averaged problem, but also we can show its order of convergence. Such understanding is important from different aspects, and in particular, it shows how well we can approximate a perturbed system with its associated averaged problem.**

I. INTRODUCTION AND RELATED WORKS

As the result of their huge social and economic importance as well as their applications, traffic flow models have attracted researchers for many years. Various models have been proposed to represent the nature of this complex problem. A review of classifications of traffic flow models can be found in [1].

In microscopic models including car-following and optimal velocity dynamics, the individual vehicles and their interaction with other vehicles construct the traffic flow. These models have different important applications, e.g., Advanced Driver Assistance Systems (ADS) and Adaptive Cruise Control (ACC) can be presented by the microscopic models [1]. In addition, these models are well descriptive when the effect of the drivers' response in the traffic flow dynamics is of interest. Different works in the literature, model and analyze the interaction between vehicles in traffic flow, [2], [3], [4], [5], [6]. Variety of extensions have been proposed by researchers to improve the original models in order to make them more representative of the situations that may arise in real world applications. For example, car-following dynamics are extended to include the delay in the response of each vehicle to sudden changes in other vehicles' behavior, [7], [8], [9], [10], [11], [12] and the references therein.

Bando et al. in [13] and [14] introduce *Optimal velocity* (OV) dynamical model in which the difference between the optimal and the current velocity determines the acceleration or deceleration of the vehicle. In fact, OV model aims to capture the very complex relation between the vehicles by defining an optimal velocity of the following cars based on the distance from the leading cars. In [15] we considered the OV model proposed by [14] in the form of

$$\ddot{X}^{(n)}(t) = \alpha \left\{ V\left(\frac{X^{(n-1)}(t) - X^{(n)}(t)}{d} \right) - \dot{X}^{(n)}(t) \right\}, \quad (1)$$

where $X^{(n)}(t)$ is the location of the n-th vehicle, $\alpha > 0$ is a constant predefined by manufacturer, d is a scaling parameter and function $V : \mathbb{R} \to \mathbb{R}$ was considered to be

$$V(x) \stackrel{\text{def}}{=} \tanh(x-2) + \tanh(2), \quad (2)$$

which is a monotone increasing and bounded function (see [14] for more details on the choice of this function). In this model in fact function V defines a kind of velocity based on the distance between the leading and following vehicles. Such velocity which is the optimal velocity mentioned before, is then compared with the current velocity of the following vehicle and determines the response of the following vehicle. OV models can naturally be considered as a dynamical model governing the autonomous systems as well.

In order to simplify the analysis of this model in [15] we discussed how to reduce the dynamics by considering

$$z^{(2n-1)}(t) \stackrel{\text{def}}{=} X^{(1)}(t) - X^{(n)}(t) = v_0 t - X^{(n)}(t)$$
$$z^{(2n)}(t) \stackrel{\text{def}}{=} \dot{X}^{(1)}(t) - \dot{X}^{(n)}(t) = v_0 - \dot{X}^{(n)}(t)$$

for $n \in \{1, \cdots, N\}$, and where v_0 is the constant velocity of the first car. This implies that equation (1) can be written in the form of

$$\begin{aligned} &\dot{z}^{(2n-1)}(t) = z^{(2n)}(t) \\ &\dot{z}^{(2n)}(t) = -\alpha \left\{ V\left(\frac{z^{(2n-1)}(t) - z^{(2n-3)}(t)}{d} \right) - v_0 + z^{(2n)}(t) \right\} \end{aligned} \quad (3)$$

Different analytical and experimental results have been discussed in the literature for OV models. Stability analysis of linearized model of (1) and regions of stability is discussed in [14]. OV models are simple models which are successful in explaining traffic congestion. Many different works extend the OV models from this point of view. Peng et al. [16] add the information from multiple preceding vehicles to the original OV model. In fact, they modify the optimal velocity term to $V(\Delta X^{(1)}, \cdots, \Delta X^{(m)})$, where $\Delta X^{(k)} = X^{(k)} - X^{(k-1)}$. They perform stability analysis for this extended model and

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discuss the stability regions. Nakayama et al. [17] introduce another type of extension for OV models by incorporating the effect of looking back at the following car in addition to the leading car and they study such effect on the stability of traffic flow congestion.

While most of the studies in OV literature are related to stability analysis, we consider a different approach to investigate this model's properties. In fact we are more interested in considering the OV model as a representative of autonomous vehicles' dynamics and to gradually improve this dynamics to construct a model which captures more real assumptions from this point of view. We believe in order to be able to extend the model properly, we need to first consider the entire system locally by isolating some of the components (particles) of the system. We need to ensure that the local interaction between these components is well understood before we study their interactions with many other particles. To do so, in [15] we considered the first two vehicles (particles) isolated from other platoon of vehicles and with the representative dynamical system of the form

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -\alpha\left\{V\left(\frac{x(t)}{d}\right) + y(t) - v_0\right\} \\ (x(t_0), y(t_0)) &= (x_0, y_0), \end{aligned} \tag{4}$$

which is obtained from (3) by only considering the first two vehicles. We then studied both deterministic and stochastic perturbation in this system and showed that such perturbed system converges uniformly to it associated averaged problem.

In this work we show that while studying of convergence of perturbed system is essential in order to have a better understanding of asymptotic behavior of the system, we also need to learn how fast such convergence can be obtained. In other words, although [15] suggests that we can approximate the system in presence of oscillations, we still need to understand how accurate is such approximation. From a practical point of view even if the convergence of the system happens but in a slow rate, the limiting system cannot be an accurate representative of the behavior of the system. This later inquiry is the subject of our rigorous investigations in this paper.

The organization of the paper is as follows. We first introduce some the commonly used notations in this text in section II. Then in section III we recall the perturbed model which we are interested in and conclude the section with a basic lemma which is essential for the rest of the results. Section IV contains the main result of this paper including a crucial bound and the main theorem. We conclude our work in section V with discussing some possible future directions.

II. Notations

Let us fix some notations before we proceed to the next sections. We denote by $C([0,T],\mathbb{R})$ the space of continuous functions from $[0,T]$ to $\mathbb{R}$. On this function space, we define a norm which is used frequently to prove some of the main results. For any function $f \in C([0,T],\mathbb{R})$ we define the supremum norm to be $\|f\|_{C_{0,T}} \stackrel{\text{def}}{=} \sup_{t\in[0,T]} |f(t)|$ and $\mathscr{T}_{\|\cdot\|_{C_{0,T}}}$ denotes the topology generated by this norm on this space and it is known that they introduce a Banach space. We need this property to complete the proof of theorem 2.

Similarly, for a bounded and continuous function f in $\mathbb{R}^n$, we define $\|f\|_{C(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |f(x)|$. This norm will be used in couple of places when we discuss the boundedness of a particular function in theorem 1.

In proving theorem 1 we found it easier to employ the multi-index notation as we need to use Taylor series to represent a function of interest in this theorem. consider a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $|\alpha| = k$ and a function $f \in C^k(\mathbb{R}^n)$. Then

$$\mathscr{D}^\alpha f(x) \stackrel{\text{def}}{=} \frac{\partial^\alpha f(x)}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n}.$$

We refer the readers to any standard book in analysis for more details on the multi-index notations (e.g. see [18]). The second derivative with respect to x_1-coordinate of function f is denoted by $\mathscr{D}^{(2,0,\cdots,0)} f$. If there is no chance of confusion we equivalently use $\partial_{x_1} f \equiv \mathscr{D}^{(1,0,\cdots,0)} f$ (and similarly with respect to other coordinates) for simplicity of notations.

III. Preliminaries

We consider the following dynamical system:

$$\begin{aligned} \dot{x}_\varepsilon(t) &= y_\varepsilon(t), \\ \dot{y}_\varepsilon(t) &= -\alpha\left\{V\left(\frac{x_\varepsilon(t)}{d}\right) + y_\varepsilon(t) - v_0\right\} + g(t/\varepsilon) \\ (x_\varepsilon(0), y_\varepsilon(0)) &= (x_0, y_0), \end{aligned} \tag{5}$$

where function V and other parameters are defined in (2) and (1) respectively. Properties of function g will be discussed later in this section. In [15] authors show that such a perturbed system converges to the averaged dynamical system of the form:

$$\begin{aligned} \dot{x}^\circ(t) &= y^\circ(t), \\ \dot{y}^\circ(t) &= -\alpha\left\{V\left(\frac{x^\circ(t)}{d}\right) + y^\circ(t) - v\right\} + \bar{g} \\ (x^\circ(0), y^\circ(0)) &= (x_0, y_0), \end{aligned} \tag{6}$$

where $\bar{g} = \lim_{T\nearrow\infty} \frac{1}{T}\int_0^T g(t)dt$ (we will redefine $\bar{g}$ for the purpose of this paper below). In fact, we showed that

$$(x_\varepsilon, y_\varepsilon) \to (x^\circ, y^\circ), \quad \text{in } \left(C([0,T],\mathbb{R}), \mathscr{T}_{\|\cdot\|_{C_{0,T}}}\right),$$

as $\varepsilon \to 0$ (see section II for the notational conventions).

In this section we study the rate of such convergence. We fix the interval $[0,T]$ for some $T > 0$ throughout this section and introduce the following definitions before stating the main results of this section.

In this paper we consider function $g : \mathbb{R} \to \mathbb{R}$ to be a continuously differentiable, bounded, and 1-periodic function (see Remark 1 below). We define:

$$\begin{aligned}\bar{g} &\stackrel{\text{def}}{=} \int_0^1 g(s)ds,\\ A(t) &\stackrel{\text{def}}{=} \int_0^t (g(s)-\bar{g})ds\\ \bar{A} &\stackrel{\text{def}}{=} \int_0^1 A(s)ds,\\ A^\dagger(t) &\stackrel{\text{def}}{=} \int_0^t \left(A(s)-\bar{A}\right)ds.\end{aligned} \tag{7}$$

Next, we prove some properties of the functions defined above and are crucial in proving the next theorems.

Lemma 1 Suppose function g satisfies the above mentioned conditions. Then, the following properties hold:

(a) Function A is bounded.
(b) Function A is 1-periodic.
(c) Function $A^\dagger$ is bounded.

Proof: It is clear from the definition that function A is a continuous function on $\mathbb{R}$. Considering any integer n and periodicity of function g, we have

$$\begin{aligned}A(n) &= \sum_{n'=0}^{n-1}\int_{n'}^{n'+1}(g(s)-\bar{g})ds\\ &= n\int_0^1 (g(s)-\bar{g}) = 0.\end{aligned}$$

Thus, continuity of function A implies that in fact A is bounded. To prove the second statement, we consider that

$$\begin{aligned}A(t+1) &= \int_0^{t+1}(g(s)-\bar{g})ds\\ &= \int_0^1 (g(s)-\bar{g})ds + \int_1^{t+1}(g(s)-\bar{g})ds\\ &= \int_0^t (g(s)-\bar{g})ds = A(t).\end{aligned}$$

The proof of the last statement is the same as the first part by appropriately replacing function g by A in the proof. ■

IV. CONVERGENCE RATE OF PERTURBED DYNAMICS TO THE AVERAGED PROBLEM

In this section we state the main results on the order of convergence of the dynamical system under study. The next theorem is one of the main results of this paper and provides a critical bound which will be necessary to prove the rate of convergence of the solution.

Theorem 1 Let $(x_\varepsilon(t), y_\varepsilon(t))$ be the solution of dynamical system (5) and $(x^\circ(t), y^\circ(t))$ be the solution of averaged system (6). We define:

$$\zeta_t^\varepsilon \stackrel{\text{def}}{=} \frac{y_\varepsilon(t)-y^\circ(t)}{\varepsilon} - A(t/\varepsilon). \tag{8}$$

Then there exists an $\varepsilon_0 > 0$ such that

$$\sup_{0<\varepsilon<\varepsilon_0} \|\zeta^\varepsilon\|_{C_{0,T}} < \infty.$$

Remark 1 The result of this theorem holds true for function g with any periodicity. Moreover, this result holds true for more general function for which

$$\bar{g} = \lim_{T\nearrow\infty}\frac{1}{T}\int_0^T g(s)ds,$$

exists. In this paper, however, we consider a 1-periodic function g since the analysis will be more straightforward. □

Proof: We introduce some notations for convenience in this section. For any $x, y \in \mathbb{R}$:

$$\begin{aligned}&L(x,y) \stackrel{\text{def}}{=} -\alpha\left\{V(x/d)+y-v_0\right\}\\ &z_\varepsilon(t) \stackrel{\text{def}}{=} (x_\varepsilon(t), y_\varepsilon(t)), \quad z^\circ(t) \stackrel{\text{def}}{=} (x^\circ(t), y^\circ(t)).\end{aligned}$$

Equations (5) and (6) then imply that:

$$\begin{aligned}y_\varepsilon(t)-y^\circ(t) &= \int_0^t \Big\{L(z_\varepsilon(s))-L(z^\circ(s))+(g(s/\varepsilon)-\bar{g})\Big\}ds\\ &= \int_0^t \Big\{L(z_\varepsilon(s))-L(z^\circ(s))\Big\}ds + \varepsilon A(t/\varepsilon);\end{aligned} \tag{9}$$

where the last term is by (7).

Function $L \in C^\infty(\mathbb{R}^2)$ and for any $s \in [0,T]$ we apply the Taylor series and denoting the partial derivatives by ∂_x and ∂_y, we have:

$$\begin{aligned}L(z_\varepsilon(s))-L(z^\circ(s)) &= \partial_x L(z^\circ(s))(x_\varepsilon(s)-x^\circ(s))\\ &+ \partial_y L(z^\circ(s))(y_\varepsilon(s)-y^\circ(s))\\ &+ \sum_{|\beta|=2}\mathscr{E}_\beta(z_\varepsilon(s);z^\circ(s))(z_\varepsilon(s)-z^\circ(s))^\beta,\end{aligned} \tag{10}$$

where for any $z, a \in \mathbb{R}^2$ the error (the remainder) term is defined by (see section II for the notations):

$$\begin{aligned}\mathscr{E}_\beta(z;a) &\stackrel{\text{def}}{=} \frac{|\beta|}{\beta!}\int_0^1 (1-\gamma)^{|\beta|-1}\mathscr{D}^\beta L(a+\gamma(z-a))d\gamma\\ &= \frac{2}{\beta!}\int_0^1 (1-\gamma)\mathscr{D}^\beta L(a+\gamma(z-a))d\gamma.\end{aligned}$$

From the the definition of function $L(x,y)$ and boundedness of function $V(x)$ and its second derivatives we have that:

$$\left|\mathscr{E}_\beta(z_\varepsilon(t);z^\circ(t))\right| \le \frac{1}{\beta!}\sup_{|\beta|=2}\sup_{z\in\mathbb{R}^2}\left|\mathscr{D}^\beta L(z)\right| < \infty.$$

Replacing the error term in the Taylor series and considering that some of the second order terms vanish, we can write:

$$\begin{aligned}L(z_\varepsilon(s))-L(z^\circ(s)) &= \partial_x L(z^\circ(s))(x_\varepsilon(s)-x^\circ(s))\\ &+ \partial_y L(z^\circ(s))(y_\varepsilon(s)-y^\circ(s))\\ &+ (x_\varepsilon(s)-x^\circ(s))^2\mathscr{E}_{(2,0)}(z_\varepsilon(s);z^\circ(s)).\end{aligned}$$

Therefore, (9) and the fact that $\partial_y L \equiv -\alpha$ imply that

$$\begin{aligned} y_\varepsilon(t) - y^\circ(t) &= \int_0^t \partial_x L(z^\circ(s))\,(x_\varepsilon(s) - x^\circ(s))\,ds \\ &\quad - \alpha \int_0^t (y_\varepsilon(s) - y^\circ(s))\,ds \\ &\quad + \int_0^t \mathscr{E}_{(2,0)}(z_\varepsilon(s); z^\circ(s))(x_\varepsilon(s) - x^\circ(s))^2 ds \\ &\quad + \varepsilon A(t/\varepsilon). \end{aligned} \tag{11}$$

Now, by the definition of ζ_t^ε in (8) it is immediate that

$$y_\varepsilon(t) - y^\circ(t) = \varepsilon\,(\zeta_t^\varepsilon + A(t/\varepsilon)).$$

Replacing in (11), we get that

$$\begin{aligned} \zeta_t^\varepsilon &= \frac{1}{\varepsilon} \int_0^t \partial_x L(z^\circ(s))\,(x_\varepsilon(s) - x^\circ(s))\,ds \\ &\quad - \alpha \int_0^t (\zeta_s^\varepsilon + A(s/\varepsilon))\,ds. \\ &\quad + \frac{1}{\varepsilon} \int_0^t \mathscr{E}_{(2,0)}(z_\varepsilon(s); z^\circ(s))(x_\varepsilon(s) - x^\circ(s))^2 ds. \end{aligned} \tag{12}$$

In addition, from the equation (5) we have that:

$$\begin{aligned} x_\varepsilon(s) - x^\circ(s) &= \int_0^s (y_\varepsilon(r) - y^\circ(r))dr \\ &= \int_0^s \varepsilon(\zeta_r^\varepsilon + A(r/\varepsilon))dr. \end{aligned} \tag{13}$$

Therefore, from equation (12) we can write:

$$\begin{aligned} \zeta_t^\varepsilon &= \int_0^t \partial_x L(z^\circ(s)) \int_0^s (\zeta_r^\varepsilon + A(r/\varepsilon))dr\,ds \\ &\quad - \alpha \int_0^t (\zeta_s^\varepsilon + A(s/\varepsilon))\,ds \\ &\quad + \varepsilon \int_0^t \mathscr{E}_{(2,0)}(z_\varepsilon(s); z^\circ(s)) \left(\int_0^s (\zeta_r^\varepsilon + A(r/\varepsilon))dr \right)^2 ds. \end{aligned}$$

By expanding the integral terms we have:

$$\begin{aligned} \zeta_t^\varepsilon &= \int_0^t \partial_x L(z^\circ(s)) \int_0^s \zeta_r^\varepsilon dr\,ds \\ &\quad + \int_0^t \partial_x L(z^\circ(s)) \int_0^s A(r/\varepsilon)dr\,ds \\ &\quad - \alpha \int_0^t \zeta_s^\varepsilon ds - \alpha \int_0^t A(s/\varepsilon)ds \\ &\quad + \varepsilon \int_0^t \mathscr{E}_{(2,0)}(z_\varepsilon(s); z^\circ(s)) \left(\int_0^s (\zeta_r^\varepsilon + A(r/\varepsilon))dr \right)^2 ds. \end{aligned} \tag{14}$$

Considering the definition of $A^\dagger$ in (7) and a simple change of variables in the integral, we have:

$$\begin{aligned} \int_0^s A(r/\varepsilon)dr &= \int_0^s \left(A(r/\varepsilon) - \bar{A}\right) dr + \int_0^s \bar{A} dr \\ &= \varepsilon A^\dagger(s/\varepsilon) + \bar{A}s, \end{aligned} \tag{15}$$

Thus, (14) and (15) imply that:

$$\begin{aligned} \zeta_t^\varepsilon &= \int_0^t \partial_x L(z^\circ(s)) \int_0^s \zeta_r^\varepsilon dr\,ds \\ &\quad + \varepsilon \int_0^t \partial_x L(z^\circ(s)) A^\dagger(s/\varepsilon)ds + \int_0^t \partial_x L(z^\circ(s))\bar{A}\,s\,ds \\ &\quad - \alpha \int_0^t \zeta_s^\varepsilon ds - \alpha\varepsilon A^\dagger(t/\varepsilon) - \alpha\bar{A}t \\ &\quad + \varepsilon \int_0^t \mathscr{E}_{(2,0)}(z_\varepsilon(s); z^\circ(s)) \left(\int_0^s (\zeta_r^\varepsilon + A(r/\varepsilon))dr \right)^2 ds. \end{aligned} \tag{16}$$

Looking at the integral equation in (16), it is reasonable to expect that $\lim_{\varepsilon \searrow 0} \zeta^\varepsilon = \zeta^\circ$ in $C([0,T], \mathbb{R})$ in corresponding topology where ζ° satisfies:

$$\begin{aligned} \zeta_t^\circ &= -\alpha \int_0^t \zeta_s^\circ ds + \int_0^t \partial_x L(z^\circ(s)) \int_0^s \zeta_r^\circ dr\,ds \\ &\quad + \int_0^t \partial_x L(z^\circ(s))\bar{A}\,s\,ds - \alpha\bar{A}t. \end{aligned} \tag{17}$$

This is in fact what will be discussed rigorously in the next theorem. Here having this convergence concept in mind, we try to find a bound on function ζ^ε. Let's first define some constants:

$$\begin{aligned} &K_1 \overset{\text{def}}{=} \|\partial_x L\|_{C(\mathbb{R}^2)}, \quad K_2 \overset{\text{def}}{=} \left\|A^\dagger\right\|_{C(\mathbb{R})} \\ &K_3 \overset{\text{def}}{=} \|\partial_x L\|_{C(\mathbb{R}^2)} |\bar{A}|, \quad K_4 \overset{\text{def}}{=} \alpha|\bar{A}| \\ &R \overset{\text{def}}{=} \frac{1}{2} K_3 T^2 + K_4 T, \quad M \overset{\text{def}}{=} \alpha + \frac{K_1}{\alpha}, \end{aligned} \tag{18}$$

where the norms are defined in section II. We note that by definition of $\partial_x L$ and boundedness of $A^\dagger$ from lemma 1 both norms are well-defined. Using the *Gronwall-Bellman* type of inequality (see Appendix VI-A for the details) we get the following bound on ζ° in (17):

$$\kappa \overset{\text{def}}{=} R\left(1 + \frac{\alpha^2}{\alpha^2 + K_1} e^{MT}\right)$$

We fix $\kappa' > \kappa$ and define

$$\tau^\varepsilon \overset{\text{def}}{=} \inf\{t > 0 : |\zeta_t^\varepsilon| \geq \kappa'\},$$

and for the moment assume that $\tau^\varepsilon \leq T$. In other words, we assume that ζ^ε exceeds κ' before time T and in fact τ^ε is the first such time.

For any $t \in [0, \tau^\varepsilon]$ for which $|\zeta_t^\varepsilon| \leq \kappa'$ and considering (16) and constants defined in (18):

$$\begin{aligned} |\zeta_t^\varepsilon| &\leq \alpha \int_0^t |\zeta_s^\varepsilon| ds + K_1 \int_0^t \int_0^s |\zeta_r^\varepsilon| dr\,ds + \frac{1}{2} K_3 T^2 + K_4 T \\ &\quad + \varepsilon \left\{ K_1 K_2 + \alpha K_2 + \left\|\mathscr{D}^{(2,0)} L\right\|_{C(\mathbb{R}^2)} \left(\kappa' + \|A\|_{C(\mathbb{R})}\right) T^3 \right\}. \end{aligned}$$

By applying the Gronwall-Bellman theorem in appendix VI-A we get:

$$|\zeta_t^\varepsilon| \leq \kappa + \varepsilon N \left(1 + \frac{\alpha^2}{\alpha^2 + K_1} e^{MT}\right)$$

where,

$$N \overset{\text{def}}{=} \left\{ K_1 K_2 + \alpha K_2 + \left\|\mathscr{D}^{(2,0)} L\right\|_{C(\mathbb{R}^2)} \left(\kappa' + \|A\|_{C(\mathbb{R})}\right) T^3 \right\}.$$

Therefore, there exists an $\varepsilon_0 < \varepsilon$ such that

$$\varepsilon_0 N\left(1+\frac{\alpha^2}{\alpha^2+K_1}e^{MT}\right) < \kappa' - \kappa. \qquad (19)$$

This implies that for $0 < \varepsilon < \varepsilon_0$:

$$|\zeta^{\varepsilon}_{\tau^{\varepsilon}}| < \kappa',$$

which is contradiction to definition of τ^{ε}. Therefore, we should have $\tau^{\varepsilon} > T$ for $0 < \varepsilon < \varepsilon_0$, which means

$$\sup_{0<\varepsilon<\varepsilon_0} \|\zeta^{\varepsilon}\|_{C_{0,T}} \leq \kappa',$$

and this completes the proof. ∎

In the next theorem, which is the main result of this paper, we will use the result of theorem 1 to show the rate of convergence of the perturbed problem to the averaged one.

Theorem 2 Let ζ^{ε} be defined in the statement of theorem 1. Then $\zeta^{\varepsilon} \to \zeta^{\circ}$ as $\varepsilon \to 0$ in $\left(C([0,T],\mathbb{R}), \mathscr{T}_{\|\cdot\|_{C_{0,T}}}\right)$ where

$$\begin{aligned}\zeta^{\circ}_t = &-\alpha \int_0^t \zeta^{\circ}_s ds \\ &+ \int_0^t \partial_x L(x^{\circ}(s), y^{\circ}(s)) \int_0^s \zeta^{\circ}_r dr ds \\ &+ \int_0^t \partial_x L(x^{\circ}(s), y^{\circ}(s))\bar{A} s ds - \alpha \bar{A} t.\end{aligned}$$

Proof: By the definition of ζ^{ε}_t in theorem 1 and the fact that $y_{\varepsilon}(t)$ and $y^{\circ}(t)$ are solutions of dynamical systems (5) and (6) respectively, function $t \mapsto \zeta^{\varepsilon}_t$ is continuous. We consider a collection of continuous functions of the form

$$\mathscr{F} \stackrel{\text{def}}{=} \{\zeta^{\varepsilon} : \varepsilon \in (0, \varepsilon_0)\},$$

such that ζ^{ε}_t satisfies (16) and ε_0 is defined in (19). First, we note that

$$\mathscr{F} \subset \left(C([0,T], \mathscr{T}_{\|.\|_{C_{0,T}}}\right).$$

Considering boundedness of ζ^{ε} by theorem 1, using the definition of $A^{\dagger}$ from (7) and finally employing equation (16), we notice that for any $t, t' \in [0,T]$

$$\sup_{\varepsilon\in(0,\varepsilon_0)} |\zeta^{\varepsilon}_t - \zeta^{\varepsilon}_s| \leq C|t-s|,$$

for some constant $C > 0$. This result proves that $\mathscr{F}$ is *equicontinuous*. Therefore, considering the fact that $\mathscr{F}$ is bounded as well then Arzela-Ascoli theorem implies that $\mathscr{F}$ is *totally bounded* (precompact). Hence, any sequence in $\mathscr{F}$ has a Cauchy subsequence. Let $\{\zeta^{\varepsilon_n}\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$ such that $\varepsilon_n \to 0$ as $n \to \infty$. Therefore there exists a subsequence of this sequence which is Cauchy in $\mathscr{F}$ and hence convergent in Banach space $\left(C([0,T], \mathscr{T}_{\|.\|_{C_{0,T}}}\right)$. With slight abuse of notation we show such subsequence by $\{\zeta^{\varepsilon_n}\}_{n\in\mathbb{N}}$ as well. Suppose now that ζ^* is the limit point of this subsequence in the space $C([0,T],\mathbb{R})$. Considering dominated convergence theorem and the fact that $(x_{\varepsilon}, y_{\varepsilon}) \to (x^{\circ}, y^{\circ})$ as $\varepsilon \to 0$ in $C([0,T],\mathbb{R})$, equation (16) implies that ζ^* should satisfy:

$$\begin{aligned}\zeta^*_t = &-\alpha \int_0^t \zeta^*_s ds \\ &+ \int_0^t \partial_x L(x^{\circ}(s), y^{\circ}(s)) \int_0^s \zeta^*_r dr ds \\ &+ \int_0^t \partial_x L(x^{\circ}(s), y^{\circ}(s))\bar{A} s ds - \alpha \bar{A} t.\end{aligned}$$

Equivalently, ζ^* is the solution of an ODE of the form

$$\begin{aligned}&\dot{\zeta}(t) = \eta(t) \\ &\dot{\eta}(t) = -\alpha\eta(t) + \partial_x L(x^{\circ}(t), y^{\circ}(t))\zeta(t) \\ &\qquad + \partial_x L(x^{\circ}(t), y^{\circ}(t))\bar{A}t - \alpha\bar{A} \\ &(\zeta(0), \eta(0)) = (0,0).\end{aligned}$$

Existence and uniqueness of solution for this ODE and the fact that the limit point of any convergent subsequence satisfies this ODE implies that ζ° is the unique limit point of ζ^{ε} as $\varepsilon \to 0$ in $C([0,T],\mathbb{R})$ and this proves the desired result. ∎

Remark 2 In fact, theorem 2 and the definition of ζ^{ε} in (8) provide useful information about rate of convergence of y^{ε} to y°. We notice that by the convergence proved in theorem 1, for any constant $\delta > 0$ there exist an $\hat{\varepsilon}$ such that for any $\varepsilon < \hat{\varepsilon}$ we have

$$\begin{aligned}|y_{\varepsilon}(t) - y^{\circ}(t)| &\leq \varepsilon \left\{ |A(t/\varepsilon)| + \sup_{t\in[0,T]} |\zeta^{\circ}_t| + \delta \right\} \\ &\leq \varepsilon \left\{ \|A\|_{C(\mathbb{R})} + \sup_{t\in[0,T]} |\zeta^{\circ}_t| + \delta \right\} \leq \varepsilon C\end{aligned}$$

for some constant $C > 0$. This implies that the rate of convergence is of order of ε.

Remark 3 Considering (13) we notice that the same rate of convergence applies to x^{ε} as well.

V. CONCLUSION AND FUTURE WORKS

In this paper we extended our previous work in [15] by studying the rate in which the convergence of a perturbed problem happens. This is particularly important as it explains the accuracy of approximation of the perturbed model with non-perturbed model.

Much more work needs to be done in this area in order to be able to understand the complex behavior of the system theoretically from this point of view. In this paper we considered a particular class of perturbations, however, more complicated models may be required to explain the real problems in car following models and in particular in autonomous systems. To be able to present and analyze more descriptive models, our assumptions and analysis should be extended to cover more general cases. For instance, the results should be extended to include stochastic elements.

In addition to the improvement of the model, one important consideration is to extend the results such that they can explain the complex relation between more than two vehicles. In fact, authors are working on such generalization for future works.

VI. Appendix

A. Gronwall- Bellman Inequality

In this section we bring a Gronwall-Bellman inequality type without the proof which is used in the proof of theorem 1. The detailed proof and more similar inequalities can be found in [19].

Theorem 3 Let $u(t), f(t)$ and $g(t)$ be real-valued non-negative continuous functions defined on set $I = [0, \infty)$, such that

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s) \int_0^s g(r)u(r)dr\,ds, \quad t \in I,$$

holds. Then,

$$u(t) \leq u_0 \left(1 + \int_0^t f(s) \exp\left\{\int_0^s (f(r) + g(r))dr\right\} ds\right).$$

In the case of our problem, and considering the constants defined in (18) we have:

$$|\zeta_t^\circ| \leq \alpha \int_0^t |\zeta_s^\circ| ds + K_1 \int_0^t \int_0^s |\zeta_r^\circ| dr\,ds + R.$$

Therefore if we define

$$f(s) = \alpha, \quad g(s) = \frac{K_1}{\alpha}, \quad u_0 = R.$$

then in the view of theorem 3 we have

$$|\zeta_t^\circ| \leq R\left(1 + \frac{\alpha^2}{\alpha^2 + K_1} \exp\left\{\left(\alpha + \frac{K_1}{\alpha}\right)t\right\}\right).$$

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