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# From Ramanujan graphs to Ramanujan complexes

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Ramanujan graphs are graphs whose spectrum is bounded optimally. Such graphs have found numerous applications in combinatorics and computer science. In recent years, a high-dimensional theory has emerged. In this paper, these developments are surveyed. After explaining their connection to the Ramanujan conjecture, we will present some old and new results with an emphasis on random walks on these discrete objects and on the Euclidean spheres. The latter lead to 'golden gates' which are of importance in quantum computation.

This article is part of a discussion meeting issue 'Srinivasa Ramanujan: in celebration of the centenary of his election as FRS'.

## 1. Introduction

Let $X$ be a finite connected $k$-regular graph and $A$ its adjacency matrix. The graph $X$ is called Ramanujan graph if every eigenvalue $\lambda$ of $A$ satisfies either $|\lambda| = k$ or $|\lambda| \leq 2\sqrt{k-1}$. This term was coined in [1].

While Ramanujan had an interest in combinatorics (the partition function etc.), it does not seem as though he has had a special interest in graph theory. So why are these graphs named after him? This will be explained in §2. The explanation will suggest what should be the definition of a Ramanujan graph, for a general graph, not necessarily $k$-regular. Moreover, it will suggest the definition for directed graphs (digraphs) and even high-dimensional simplicial complexes (the so-called Ramanujan complexes), as will be explained in §3 and in §4.

Ramanujan graphs have found plenty of applications in computer science and pure mathematics. Most of them have to do with the fact that they provide optimal expanders (see [2–4] and the references therein). Lately, Ramanujan complexes and high-dimensional expanders have also started to be a popular subject of research (cf. [5,6] and references therein).



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Here, we concentrate on describing their aspects which truly use the full power of being Ramanujan, and not merely expansion. In §5, we will describe random walks on Ramanujan graphs and complexes and in §6, 'golden gates', which is a new and fascinating application of them to quantum computation.

## 2. Why Ramanujan?

Let $X$ be a finite connected $k$-regular graph, $k \geq 3$, with $n$ vertices, and $A$ its adjacency $n \times n$ matrix. Being symmetric, all its eigenvalues $\lambda$ are real and it is easy to see that $|\lambda| \leq k$, $k$ is always an eigenvalue, and $-k$ is an eigenvalue if and only if $X$ is bi-partite. The graph $X$ is called a *Ramanujan graph* if every eigenvalue $\lambda$ satisfies either $|\lambda| = k$ or $|\lambda| \leq 2\sqrt{k-1}$. The bound $2\sqrt{k-1}$ is significant: by the Alon–Boppana theorem, (cf. [1, prop. 4.2]) this is the best possible bound one can hope for, for an infinite family of $k$-regular graphs. The real reason behind it is as follows: the universal cover of $X$ (in the sense of algebraic topology) is $\tilde{X} = T_k$—the infinite $k$-regular tree. An old result of Kesten asserts that the spectrum of the adjacency operator acting on $L^2(T_k)$ is the interval $\left[-2\sqrt{k-1}, 2\sqrt{k-1}\right]$. So, being Ramanujan means for $X$, that all its non-trivial eigenvalues are in the spectrum of its universal cover $\tilde{X}$.

Ramanujan graphs are optimal expanders from the spectral point of view. Recall that a finite $k$-regular graph $X$ is called $\varepsilon$-expander if $h(X) \geq \varepsilon$ when $h(X)$ is the Cheeger constant of $X$, namely

$$h(X) = \min\left\{ \frac{|E(Y, \bar{Y})|}{|Y|} \;\middle|\; \begin{array}{c} Y \subseteq X, \\ 0 < |Y| \leq \frac{|X|}{2} \end{array} \right\},$$

when $E(Y, \bar{Y})$ is the set of edges between $Y$ and its complement.

Now if we denote $\lambda_1(X) = \max\{\lambda \mid \lambda \neq k, \lambda \text{ e.v. of } A\}$, then

$$\frac{h(X)^2}{2k} \leq k - \lambda_1(X) \leq 2h(X) \text{ (cf. [3, §4.2])}.$$

So, Ramanujan graphs are expanders. Expander graphs are of great importance in combinatorics and computer science (cf. [2] and the references therein) and also in pure mathematics (cf. [4]). Expander graphs serve as basic building blocks in various network constructions, in many algorithms and so on. The bound on their eigenvalues ensures that the random walk on such graphs converges quickly to the uniform distribution and on Ramanujan graphs this happens in the fastest possible way (see §5). This is one more reason that makes them so useful.

The existence of Ramanujan graphs is by no means a trivial issue: while it is known that random $k$-regular graphs are expanders, it is not known if they are Ramanujan. First examples of infinite families of such graphs were given by explicit construction in [1,7] for $k = q + 1$, $q$ prime. In [8], it is shown, by a non-constructive method, that for every $k \geq 3$ there exist infinitely many $k$-regular bi-partite Ramanujan graphs.

Why are Ramanujan graphs named after Ramanujan? As far as we know, Ramanujan had no special interest in graph theory. Let us explain the reason for this name which was coined in [1].

Observe the following power series:

$$\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24} = \Sigma \tau(n) q^n = q - 24q^2 + 252q^3 + \cdots.$$

The coefficients $\tau(n)$ define the so-called Ramanujan tau function. Ramanujan conjectured that $|\tau(p)| \leq 2p^{(11/2)}$ for every prime $p$. The importance of $\Delta$ comes from the fact that if we write $q = e^{2\pi i z}$ then $\Delta(z)$ is a cusp form of weight 12 on the upper half plane $\mathbb{H} = \{z = x + iy \mid x, y \in \mathbb{R},\ y > 0\}$ with respect to the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}$ by Möbius transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = (az + b)/(cz + d)$. Now if $\Gamma_0(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid c \equiv 0 \mod N\right\}$ we denote $S_k(N)$ (or more generally $S_k(N, w)$ for a Dirichlet character $w$ of $\mathbb{Z}/N\mathbb{Z}$) the space of cusp forms on $\mathbb{H}$ w.r.t. $\Gamma_0(N)$ (and $w$). The Hecke operators $T_p$ ($p$ prime, $(p, N) = 1$), act, and commute, on each $S_k(N, w)$, and their common eigenfunctions are the Hecke eigenforms. Now, $S_{12}(\Gamma = \Gamma_0(1))$ is one dimensional

and so $\Delta(z)$ above is such a Hecke eigenform. Moreover, $\tau(p)$ above is equal to the eigenvalue of $T_p$ acting on $S_{12}(\Gamma)$. A natural and far reaching generalization of the Ramanujan conjecture mentioned above on the size of $\tau(p)$ is the so-called Ramanujan–Peterson (RP) conjecture: for every Hecke eigenform $f$ in $S_k(N,w)$, the eigenvalues $\lambda_p$ of $T_p$, $(p,N)=1$, satisfy $|\lambda_p| \le 2p^{k-1/2}$. The reader is referred to [9] for a concise and clear explanation of all these notions.

The modern approach to automorphic functions via representation theory brought in another point of view on the RP conjecture. Satake [10] showed that the RP conjecture is equivalent to the assertion: let $\mathbb{A}=\mathbb{A}_{\mathbb{Q}}$ be the ring of adeles of $\mathbb{Q}$, and $\pi$ an irreducible cuspidal $\mathrm{GL}_2$-representation in $L^2(\mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathbb{Q}))$, such that its component at infinity $\pi_\infty$ is square-integrable, then for every prime $p$ the local factor at the $p$-component $\pi_p$ is a tempered representation. See [9] for exact definitions. Here, we only mention that a representation of a (simple) $p$-adic or real Lie group $G$ is tempered if it is weakly contained in $L^2(G)$. The RP conjecture was proved by Deligne (for the special representations that are relevant to the Ramanujan graphs, the RP conjecture was actually proven earlier by Eichler). The representation-theoretic formulation suggests vast generalizations to other algebraic groups.

Let us look at the $p$-adic group $G=\mathrm{PGL}_2(\mathbb{Q}_p)$. The Bruhat–Tits building associated with $G$ is, in this special case, the $(p+1)$-regular tree $T=T_{p+1}$ which can be identified as $T=G/K$ when $K$ is a maximal compact subgroup of $G$. If $\Gamma$ is a discrete cocompact subgroup of $G$, then $X=\Gamma\backslash T=\Gamma\backslash G/K$ is a finite $(p+1)$-regular graph. One can show (see [3]) that $X$ is a Ramanujan graph if and only if every infinite-dimensional $K$-spherical $G$-sub-representation of $L^2(\Gamma\backslash G)$ is tempered. Deligne theorem, combined with the so-called Jacquet–Langlands correspondence, enables the construction of such arithmetic subgroups $\Gamma$ for which the temperedness condition is satisfied and hence Ramanujan graphs are obtained. This was the method of [1,7]. Let us mention that for every $k$, if $G$ is the full automorphism group of $T_k$ and $\Gamma$ a discrete cocompact subgroup of $G$, then $X=\Gamma\backslash T_k$ is $k$-regular Ramanujan graph if and only if the same temperedness condition is satisfied: in other words, every non-trivial eigenvalue of $X=\Gamma\backslash T_k$ is coming from the spectrum of $T_k$ if and only if every non-trivial spherical subrepresentation of $L^2(\Gamma\backslash G)$ is coming from $L^2(G)$. This illustrates the connection between the notion of Ramanujan graph and the Ramanujan conjecture.

As mentioned above, the RP conjecture was generalized to other groups, and some of its generalizations to $\mathrm{GL}_d$ (instead of only $\mathrm{GL}_2$) led to higher dimensional versions of Ramanujan graphs, the so-called Ramanujan complexes. We will see more on it in §4.

Finally, another interesting hint to a connection with number theory: Ihara defined the notion of Zeta function of a $k$-regular graph $X$, and Sunada observed that $X$ is Ramanujan if and only if this Zeta function satisfies 'the Riemann hypothesis'. We refer the reader to the survey [11] for more details.

## 3. General graphs and digraphs

The first paragraph of §2 suggests what should be the general definition of Ramanujan graphs. This was carried out for the first time in the Greenberg thesis ([12], which is unfortunately not published and available only in Hebrew), and was vastly generalized in [13].

Here is the main point. Let $X$ be any finite connected graph and $\widetilde{X}$ its universal cover. Let $A_{\widetilde{X}}$ be the adjacency operator acting on $L^2(\widetilde{X})$ by $A_{\widetilde{X}}(f)(x)=\sum_{x'\sim x} f(x')$ where $x'$ runs over the neighbours of $x$ in $\widetilde{X}$. Now, it is shown in [12] that there exists a positive real number $\kappa$ depending only on $\widetilde{X}$, such that if $Y$ is a finite graph covered by $\widetilde{X}$, then $\kappa$ is the largest (Perron–Frobenius) eigenvalue of the adjacency matrix $A_Y$ of $Y$. When $X$ is $k$-regular $\kappa=k$, and when $X$ is bipartite $(k_1,k_2)$-biregular, $\kappa=\sqrt{k_1k_2}$.

**Definition 3.1 ([12]).** The graph $X$ is called Ramanujan if every eigenvalue $\lambda$ of $A_X$ satisfies either $|\lambda|=\kappa$ or $\lambda\in\mathrm{Spec}(A_{L^2(\widetilde{X})})$.

This recovers the classical definition of Ramanujan graphs for $k$-regular graphs since $\mathrm{Spec}(A|_{L^2(T_k)})=\left[-2\sqrt{k-1},2\sqrt{k-1}\right]$. For bipartite $(k_1,k_2)$-biregular graphs $X$ with $k_1\le k_2$, the

universal cover is the $(k_1, k_2)$-biregular tree $T_{k_1,k_2}$ and

$$\mathrm{Spec}(A|_{L^2(T_{k_1,k_2})}) = \left[-\sqrt{k_2-1}-\sqrt{k_1-1}, -\sqrt{k_2-1}+\sqrt{k_1-1}\right] \cup \{0\} \cup \left[\sqrt{k_2-1}-\sqrt{k_1-1}, \sqrt{k_2-1}+\sqrt{k_1-1}\right].$$

It is known that for every $3 \le k \in \mathbb{N}$, there exist infinitely many $k$-regular Ramanujan graphs (explicit constructions for every $k = p^e + 1$, $p$ prime [14], and non explicit for every $k$ [8]). But for $(k_1, k_2)$-biregular, it is known only for special values:

**Theorem 3.2 ([15,16]).** *Let $p$ be a prime, $k_1 = p + 1$ and $k_2 = p^3 + 1$, then there exist infinitely many bipartite $(k_1, k_2)$-biregular Ramanujan graphs.*

In [15] existence was shown as the quotients of the bi-regular tree associated with a rank one simple $p$-adic Lie group. Explicit constructions (in the sense of computer science) are given for $p \equiv 3 \mod (4)$ in [16].

Let us mention that [8] gives existence of 'weak-Ramanujan' $(k_1, k_2)$-biregular graphs in the following sense: every eigenvalue $\lambda$ is either $|\lambda| = \kappa = \sqrt{k_1 k_2}$ or $|\lambda| \le \sqrt{k_2 - 1} + \sqrt{k_1 - 1}$.

In [17] it was shown that there exist finite graphs $X$ for which $\widetilde{X}$ does not cover any Ramanujan graph. This was put in a more general framework in [18].

Turning to digraphs (directed graph), denote by $A = A_{\mathcal{D}}$ the adjacency matrix of the digraph $\mathcal{D}$, namely $A_{v,w} = 1$ if $v \to w$ in $\mathcal{D}$ and $A_{v,w} = 0$ otherwise. We say that $\mathcal{D}$ is $k$-regular if every vertex has $k$ incoming edges, and $k$ outgoing ones. The notion of Ramanujan digraphs (directed graphs) was considered only quite recently [19–21]. A main reason for this is that the adjacency matrix of a digraph can be non-normal, in which case its spectrum reveals much less information on the graph.

**Definition 3.3.** A $k$-regular digraph is a Ramanujan digraph if every eigenvalue of $A_X$ satisfies either $|\lambda| = k$ or $|\lambda| \le \sqrt{k}$.

Here the trivial eigenvalues can be $e^{2\pi i/m}k$ for any $m \in \mathbb{N}$, indicating that the digraph is *$m$-periodic*: its vertices can be partitioned into $m$ sets $V_0, \ldots, V_{m-1}$, with every edge starting in $V_j$ terminating in $V_{(j+1 \mod m)}$. Once again, the non-trivial spectrum agrees with the 'directed universal cover' $T_k^{\rightleftarrows}$, which is the $2k$-regular tree, directed to have constant in-degree and out-degree $k$. Indeed, $\mathrm{Spec}(A|_{L^2(T_k^{\rightleftarrows})}) = \{z \in \mathbb{C} || z| \le \sqrt{k}\}$ by [22].

A general example of a Ramanujan digraph arises from Hashimoto's approach to Ihara's zeta function [23]. Given a $(k+1)$-regular (undirected) graph $X$, define the $k$-regular digraph $D_X$, whose vertices correspond to directed edges in $X$, and whose edges correspond to non-backtracking steps in $X$. Namely, $e \to e'$ in $D_X$ iff $e, e'$ form a non-backtracking path in $X$. Hashimoto's work shows that $D_X$ is a Ramanujan digraph if and only if $X$ is a Ramanujan graph.

It is interesting to note that the Alon–Boppana theorem fails for digraphs: the *De-Bruijn* digraphs (cf. [21, §3.4]) are $k$-regular digraphs, of arbitrarily large size, whose non-trivial spectrum consists entirely of zeros! However, these graphs have non-normal adjacency matrices. It turns out that normality, and even 'almost-normality' recovers an Alon–Boppana bound, for which Ramanujan digraphs are again optimal. We say that a family of digraphs is *almost-normal* if the adjacency matrices of its members are unitarily equivalent to block-diagonal matrices with blocks of globally bounded size.

**Theorem 3.4 ([21]).** *The smallest upper bound for the non-trivial spectrum of an infinite almost-normal family of $k$-regular, $m$-periodic digraphs, is $\sqrt{k}$.*

It turns out that almost-normality appears naturally in the context of digraphs which arise from Ramanujan graphs and complexes (see §4), and that it serves as a substitute for normality in the spectral analysis of these digraphs.

## 4. Ramanujan complexes

Combinatorial graphs are one-dimensional simplicial complexes, and it is natural to ask for analogues of expanders and Ramanujan graphs in higher dimension. Here even the definition is not straightforward, as there is no clear counterpart to the $k$-regular tree $T_k$ in general dimension. The explicit construction of Ramanujan graphs suggests one answer: since for $k = p + 1$ the tree $T_k$ arose as the Bruhat–Tits building of $G = \mathrm{PGL}_2(\mathbb{Q}_p)$, one can replace it with the Bruhat–Tits building $\mathcal{B} = \mathcal{B}(G)$ of $G = \mathrm{PGL}_{d+1}(\mathbb{Q}_p)$, which is an infinite, contractible, $d$-dimensional complex. This is indeed the approach taken in [24,25], except for the replacement of $\mathbb{Q}_p$ by $\mathbb{F}_p((t))$—the reason being that the Ramanujan conjecture for $\mathrm{PGL}_d$ over $\mathbb{Q}$ is still open for $d \geq 3$, whereas for $\mathrm{PGL}_d$ over $\mathbb{F}_p(t)$ it was proved by Lafforgue in [26]. A more general approach is to look at any non-Archimedean local field $F$, and $G = \mathcal{G}(F)$, where $\mathcal{G}$ is a simple $F$-algebraic group. Bruhat–Tits theory associates with $G$ a building $\mathcal{B}$ (the so-called Bruhat–Tits building) which is a contractible simplicial complex of dimension $d$ equal to the $F$-rank of $\mathcal{G}$. The group $G$ acts on $\mathcal{B}$, transitively on the $d$-cells. Every torsion-free discrete cocompact subgroup $\Gamma$ of $G$ gives rise to a finite complex $X = \Gamma\backslash\mathcal{B}$, which can then be compared to its universal cover $\widetilde{X} = \mathcal{B}$.

For this comparison, one should decide which adjacency operator should one look at, as the standard adjacency relation between vertices depends only on the 1-skeleton of the complex, and does not capture the high-dimensional structure. One can ask, for example, about operators such as the discrete $j$-dimensional Laplacian, which acts on cells in dimension $j$ and detects the presence of real $j$-th cohomology. We take an inclusive approach: we call an operator $T$ on (a subset of) the cells of the building $\mathcal{B}(G)$ *geometric* if it commutes with the action of $G$. If $X$ is a finite quotient of $\mathcal{B}$, this implies that $T$ descends to a well-defined operator $T|_X$ on $X$, and we define:

**Definition 4.1.** Let $F$ be a non-archimedean local field, $\mathcal{B}$ the Bruhat–Tits building associated with $\mathrm{PGL}_{d+1}(F)$, and $X$ a quotient of $\mathcal{B}$.

(1) For a geometric operator $T$, an eigenvalue of $T|_X$ is *trivial* if the associated eigenfunction on $X$ lifts to a $PSL_{d+1}(F)$-invariant function on $\mathcal{B}$.
(2) The complex $X$ is a *Ramanujan complex* if for every geometric operator $T$ on $\mathcal{B}$, the non-trivial spectrum of $T|_X$ is contained in the $L^2$-spectrum of $T$ on $\mathcal{B}$.

The definition generalizes to other groups than $\mathrm{PGL}_d$, once we understand which are the trivial eigenfunctions—see [19] for the case of simple algebraic groups, and [27] for a more general one.

We remark that the original definition of Ramanujan complexes in [24,25] only requires (2) for geometric operators on the vertices of $\mathcal{B}$. However, all the known constructions of Ramanujan complexes [16,24,27–29] satisfy the stronger definition!

As in the case of graphs, the Ramanujan property can be related to representation theory: The Iwahori group of $G$ is the pointwise stabilizer of a cell of maximal dimension in $\mathcal{B}$, and the complex $X = \Gamma\backslash\mathcal{B}$ is Ramanujan if and only if every infinite dimensional, Iwahori-spherical, irreducible $G$-sub-representation of $L^2(\Gamma\backslash G)$ is tempered [19,27,30].

## 5. Random walks

A highly useful property of expanders is that random walks on them converge rapidly to the stationary distribution: let $X$ be a non-bipartite $k$-regular graph, $\{v_t\}_{t=0}^{\infty}$ a simple random walk (SRW) process on $X$, and $P_X^t$: $v \mapsto \mathrm{Prob}[v_t = v]$ the distribution of the walk at time $t$. It is a standard exercise to show that $\|P_X^t - \mathbf{u}\|_2$, the $L^2$-distance of $P_X^t$ from the uniform distribution, is bounded by $(\lambda/k)^t$, where $\lambda$ is the largest non-trivial eigenvalue of $A_X$ (in absolute value).

It turns out, however, that Ramanujan graphs are optimally mixing not only in $L^2$-norm but also in $L^p$ for all $1 \leq p \leq \infty$. Furthermore, they manifest a *cut-off* phenomena: the $L^p$-distance $\|P_X^t - \mathbf{u}\|_p$ drops abruptly from being near-maximal to being near zero, over a short interval of time called the *cut-off window*. We focus on $L^1$, the *total-variation* norm, which is hardest to bound, and the most useful for many purposes (see [31]).

**Theorem 5.1 ([32]).** *Let $X$ be a $k$-regular Ramanujan graph on $n$ vertices.*

(1) *The SRW on $X$ has $L^1$-cut-off at time $(k/k-2)\log_{k-1} n$.*
(2) *The non-backtracking random walk (NBRW) on $X$ has $L^1$-cut-off at time $\log_{k-1} n$.*

Note that the location of cut-off for NBRW is optimal: a non-backtracking walker on a $k$-regular graph sees at most $k-1$ new vertices at every step, with the exception of the first one. Thus, a walk of length $(1-\delta)\log_{k-1} n$ can reach only a small fraction of the graph for $\delta > 0$ (and even for $\delta = \log\log n/\log n$), resulting in $L^1$-distance $1-o(1)$ from equilibrium. In a similar manner one can show that the first bound is optimal when taking into account the hindrance caused by backtracking.

In [32], the authors first prove the bound for NBRW on $X$, and then show that it implies the bound for SRW. Let us give a glimpse of how the bound for NBRW is proved. Recall the digraph $D_X$ from §3: this is a **$(k-1)$**-regular Ramanujan digraph, and by its construction SRW on $D_X$ is equivalent to NBRW on $X$. If the adjacency matrix $A_{D_X}$ was symmetric, or even normal, then we would have $\|P^t_{D_X} - \mathbf{u}\|_2 \le (k-1)^{-t/2}$ as for undirected expanders, and a standard $L^2$ to $L^1$ bound would then give the desired result. However, $A_{D_X}$ is not normal when $k \ge 3$. The main step in [32] is to show that $D_X$ is 2-normal, namely, $A_{D_X}$ is unitarily equivalent to a block-diagonal matrix with blocks of size $2\times 2$. This is then shown to imply the bound $\|P^t_{D_X} - \mathbf{u}\|_2 \le (t+1)(k-1)^{-(t+1)/2}$, which only differs by a logarithmic factor, and suffices to prove cut-off.

Let us stress that the work of Lubetzky & Peres [32] uses the full strength of the Ramanujan property to deduce the cut-off phenomenon. It is still a widely open conjecture of Peres that such phenomena happens in all transitive expander graphs. It is known that it is not always the case for general expanders [33].

In Ramanujan complexes of higher dimension, it turns out that the digraph $D_X$ induced by NBRW is not a Ramanujan digraph anymore. However, it is shown in [19] that other operators on the cells of these complexes do induce Ramanujan digraphs. The crucial property is that these operators should describe *collision-free* walks on the building: this means that all the paths which descend from a fixed starting cell never meet one another (for example, non-backtracking walk on a tree has this property). It is shown in [19] that if a geometric operator induces a collision-free walk on $\mathcal{B}$, and $X$ is a Ramanujan quotient of $\mathcal{B}$, then the digraph which represents the walk by $T$ on $X$ is a Ramanujan digraph. Furthermore, it is shown that the digraphs which arise from quotients of a fixed building are almost-normal, which leading again to cut-off at the optimal time:

**Theorem 5.2 ([19]).** *Let $T$ be a geometric, $k$-regular, collision-free operator on $\mathcal{B}$, the Bruhat–Tits building of a simple $p$-adic group $G$. Then the walk induced by $T$ on a Ramanujan complex $X = \Gamma\backslash\mathcal{B}$ has $L^1$-cut-off at time $\log_k |X|$.*

In addition, it is shown in [19] that such walks do exist: for $G = \mathrm{PGL}_{d+1}(F)$, a collision-free walk on $j$-cells is exhibited for each $1 \le j \le d$, the so-called geodesic $j$-flow. For example, geodesic 1-flow goes from a (coloured) edge $v \to w$ to $w \to u$ if the cell $\{v, w, u\}$ does *not* belong to the complex. The situation when $j = 0$ is different: due to commutativity of the Hecke algebra, no geometric operator on vertices induces a Ramanujan digraph (see [21, Rem. 3.5(b)]). However, it is shown in [34] that by combining the optimal cut-off result for the $j$-flow operators in all dimensions, it is possible to recover cut-off for SRW on vertices.

**Theorem 5.3 ([34]).** *SRW on the vertices of Ramanujan complexes associated with $\mathrm{PGL}_d(F)$ exhibit $L^1$-cut-off.*

Once again, the proof requires the strength of the Ramanujan property, and not merely expansion. Moreover, it needs the full high dimensional structure of X, even when we study the SRW only on the vertices.

Finally, we mention that in [35] a different direction is taken: replacing $\mathrm{PGL}_2(\mathbb{Q}_p)$ with $\mathrm{PGL}_2(\mathbb{R})$, the authors suggest the notion of *Ramanujan surfaces*, which are hyperbolic Riemann surfaces

which spectrally behave like their universal cover, the hyperbolic plane. It is then shown that a discrete random walk with constant-length steps on these surfaces exhibits $L^1$-cut-off.

## 6. Golden gates

Recently, Ramanujan graphs and complexes have found a surprising application to the theory of quantum computation. In classical computation, one decomposes any function into basic logical gates such as XOR, AND, NOT. In quantum computation, the classical bits are replaced by qubits, which are vectors in projective Hilbert space $\mathbb{C}P^n$, and the logical gates are *all* the elements of the projective unitary group $G = PU(n)$. In the real world, one must implement some finite set of these gates, and use them to approximate the others. Denoting by $S^{(\ell)}$ the set of $\ell$-wise products of elements in $S \subset G$, we say that $S$ is *universal* if $\langle S\rangle = \cup_{\ell\geq 0} S^{(\ell)}$ is dense in $G$ (with respect to the standard bi-invariant metric $d^2(A,B) = 1 - (|\mathrm{trace}(A^*B)|/2)$). This means that any gate can be approximated with arbitrary precision as a product of elements of $S$. The notion of Golden Gates is a much stronger one, loosely requiring the following (see [16,20] for precise definitions):

(1) The covering rate of $G$ by $\langle S\rangle$ is (almost) optimal. Namely, for every $\ell$ the set $S^{(\ell)}$ distributes in $G$ as a perfect sphere packing (or randomly placed points) would, up to a negligible factor.
(2) Approximation: given $A \in PU(n)$ and $\varepsilon > 0$, there is an efficient algorithm to find some $A' \in B_\varepsilon(A)$ (the $\varepsilon$-ball around $A$) such that $A' \in S^{(\ell)}$ with $\ell$ (almost) minimal.
(3) Compiling: given $A \in \langle S\rangle$ as a matrix, there is an efficient algorithm to write $A$ as a word in $S$ of the smallest possible length.

These requirements ensure that any gate can be approximated and compiled as an efficient circuit using the gates in $S$.

To see the connection between covering and spectral expansion, denote by $T_S$ the $S$-adjacency operator on $L^2(G)$, namely, $(T_S f)(g) = \sum_{s\in S} f(sg)$. Clearly, $T_S(\mathbb{1}) = |S| \cdot \mathbb{1}$, and we denote $\lambda_S = \|T_S|_{\mathbb{1}^\perp}\|$, where $\mathbb{1}^\perp = \{f \mid \int_G f\, d\mu = 0\}$ and $\mu$ is the normalized Haar measure on $G$.

**Theorem 6.1 ([20, §3]).** *Denoting by $\mu_\varepsilon = \mu(B_\varepsilon(1))$ the volume of an $\varepsilon$-ball in $G$, the $\varepsilon$-neighbourhood of $S$ satisfies*

$$\mu\left(\bigcup\nolimits_{s\in S} B_\varepsilon(s)\right) \geq 1 - \frac{\lambda_S^2}{|S|^2\mu_\varepsilon}.$$

Thus, as in the case of expander graphs, one aims to minimize the non-trivial eigenvalues of an adjacency operator. It turns out that the spectral bounds for Ramanujan graphs reappear in these settings.

**Theorem 6.2 ([36,37]).**

(1) *If $S \subset PU(2)$ is a symmetric set of size $k$, then $\lambda_S \geq 2\sqrt{k-1}$.*
(2) *For prime $p \equiv 1 \pmod 4$, there is an explicit symmetric set $S_p \subset PU(2)$ of size $k = p+1$ such that $\lambda_{S_p} = 2\sqrt{k-1}$.*

In fact, the connection to Ramanujan graphs runs deeper than the spectral bound. The construction of $S_p$, and of the $(p+1)$-regular Ramanujan graphs in [1], can be described using a single subgroup of $PU_2(\mathbb{Q})$, which acts simply-transitively on the Bruhat–Tits tree of $PU_2(\mathbb{Q}_p) \cong \mathrm{PGL}_2(\mathbb{Q}_p)$ (this isomorphism follows from $p \equiv 1 \pmod 4$). This also solves the compiling problem: by writing any $A \in \langle S_p\rangle$ in $p$-adic coordinates, one recovers its decomposition in $\langle S_p\rangle$ by following the (unique) path leading from $A$ to the root of the tree (cf. [20]).

The proof of the spectral bound $\lambda_{S_p} = 2\sqrt{k-1}$ uses again the RP conjecture (Deligne's theorem), but while [1] uses the RP conjecture for automorphic representations of weight two and arbitrary level, [36,37] use the conjecture for representations of level two and arbitrary weight. To

see that the gates of [36,37] are optimally covering (compared with random ones), one needs to bound $\lambda_{S_p^{(\ell)}}$ for general $\ell$; we refer the reader to [20] for a full account, which addresses also the approximation problem for these gates by the Ross–Selinger algorithm [38].

As Ramanujan graphs appear when studying $PU(2)$, one expects Ramanujan complexes to appear when moving to general $PU(n)$. This is indeed so, but the direction taken in §4, of replacing $\mathbb{Q}$ by $\mathbb{F}_p(t)$, cannot be used anymore, since the latter does not embed in $\mathbb{R}$. The task also becomes more complicated due to the fact that the naive generalization of RP conjecture to $\mathrm{PGL}_d$ fails, due to the appearance of functorial lifts (cf. [39]). For general $n$, this is still work in progress, but for $PU(3)$ (which corresponds to quantum computation on a single *qutrit*), a complete solution exists:

**Theorem 6.3 ([16]).** *For $p \equiv 1 \pmod 4$, there is an explicit Golden Gate set $S_p \subset PU_3(\mathbb{Q})$, such that $\langle S_p \rangle$ acts simply transitively on the Bruhat–Tits building of* $\mathrm{PGL}_3(\mathbb{Q}_p)$.

The compiling problem for these gates is solved by studying their action on the two-dimensional building of $\mathrm{PGL}_3(\mathbb{Q}_p)$. The optimal covering rate is obtained by showing that the spectral bound $\lambda_{S_p}$ is the same as the maximal non-trivial adjacency eigenvalue of a two-dimensional Ramanujan complex! Let us mention that the proof of this bound uses Rogawski's work [40], as well as some state-of-the-art results of the Langlands program, in particular, Ngô's proof of the Fundamental Lemma, which enabled Shin to prove the RP conjecture for cuspidal self-dual representations of $\mathrm{PGL}_d$ over CM fields [41].

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