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Phase retrieval of real-valued signals in a shift-invariant space ☆

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A B S T R A C T

In this paper, we consider an infinite-dimensional phase retrieval problem to reconstruct real-valued signals living in a shift-invariant space from their phaseless samples taken either on the whole line or on a discrete set with finite sampling density. We characterize all phase retrievable signals in a real-valued shift-invariant space using their nonseparability. For nonseparable signals generated by some function with support length L, we show that they can be well approximated, up to a sign, from their noisy phaseless samples taken on a discrete set with sampling density $2L-1$. In this paper, we also propose an algorithm with linear computational complexity to reconstruct nonseparable signals in a shift-invariant space from their phaseless samples corrupted by bounded noises.

1. Introduction

A function/signal f on the real line is defined by its values $f(t), t \in \mathbb{R}$. In this paper, we consider the problem whether a real-valued signal f on the real line is determined, up to a sign, from its magnitude measurements $|f(t)|, t \in \Omega$, where Ω is either the whole real line $\mathbb{R}$ or its discrete subset. The above problem to determine a signal of interest from its magnitude measurements is an infinite-dimensional phase retrieval problem and it is possible to solve only if we have additional information about the signal.

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Phase retrieval plays important roles in signal/image/speech processing [25–27,31–33,36,39,46,47]. The phase retrieval problem of finite-dimensional signals has received considerable attention in recent years, however there still are lots of open mathematical and engineering questions unanswered, see [9,10,17,15,57] and references therein. In the finite-dimensional setting, a fundamental problem is whether and how a (sparse) vector $x \in \mathbb{R}^d$ (or $\mathbb{C}^d$) can be reconstructed from its magnitude measurements $y = |Ax|$, where A is a measurement matrix. The phase retrievability has been characterized via the measurement matrix A [10,13,57], and many algorithms have been proposed to reconstruct the vector x from its magnitude measurements y [15–17,25,28,34,40–42,45,47].

The phase retrieval problem in an infinite-dimensional space is fundamentally different from a finite-dimensional setting. There are several papers devoted to that research field [5–7,14,19,38,43,44,48,53,58]. Thakur proved in [53] that real-valued bandlimited signals could be reconstructed from their phaseless samples taken at more than twice the Nyquist rate. Shenoy, Mulleti and Seelamantula studied the phase retrieval of signals living in a principal shift-invariant space when the magnitude measurements of their frequency are available [48]. In this paper, we study whether and how to determine a real-valued signal residing in a shift-invariant space

$$V(\phi) := \left\{ \sum_{k\in\mathbb{Z}} c(k)\phi(t-k) : \ c(k) \in \mathbb{R} \right\}, \tag{1.1}$$

up to a sign, from its magnitude measurements, where the generator ϕ is a real-valued continuous function with compact support. A representative generator ϕ is the B-spline B_N of order $N \geq 1$ [55,56], which is obtained by convolving the indicator function $\chi_{[0,1)}$ on the unit interval N times,

$$B_N = \underbrace{\chi_{[0,1)} * \cdots * \chi_{[0,1)}}_{N}. \tag{1.2}$$

The concept of shift-invariant spaces arose in sampling theory, wavelet theory, approximation theory and signal processing, see [1,4,20,21,35,37,54] and references therein.

The Paley–Wiener space for bandlimited signals to live in is a shift-invariant space generated by the function $\frac{\sin \pi t}{\pi t}$ with infinite support. We notice that not all real-valued signals in a shift-invariant space $V(\phi)$ generated by a compactly supported function ϕ are phase retrievable, which is a different phenomenon from bandlimited signals. In Theorem 2.1 of Section 2, we show that a real-valued signal $f \in V(\phi)$ can be determined, up to a sign, from its magnitude measurements $|f(t)|, t \in \mathbb{R}$, if and only if f is nonseparable, i.e., the signal f is not the sum of two nonzero signals in $V(\phi)$ with their supports being essentially disjoint. The above notion for a signal can be considered as a weak version of complement property for ideal sampling functionals, see Corollary 6.6 and cf. complement property for frames in Hilbert/Banach spaces [7,10,13,14]. Given an arbitrary signal $f \in V(\phi)$, we also find all functions/signals $g \in V(\phi)$ in Section 2 such that g and f have the same magnitude measurements on the real line.

Our phase retrieval of signals in a shift-invariant space is a phaseless sampling and reconstruction problem. Sampling in shift-invariant spaces is well studied as it is a realistic model for signals with smooth spectrum, and suitable models for taking into account the real acquisition and reconstruction devices and the numerical implementation, see [1,2,4,23,49,50,54] and the extensive list of references therein. In Theorem 3.1 and Corollaries 3.3 and 3.4 of Section 3, we show that a nonseparable wavelet signal in $V(\phi)$ generated by some function ϕ with support length L is determined, up to a sign, from its phaseless samples taken on a shift-invariant set $X + \mathbb{Z}$, where $X \subset (0, 1)$ is a set containing $2L - 1$ distinct points.

Stability of phase retrieval is of paramount importance. The reader may refer to [11–13,24] for phase retrieval in the finite-dimensional setting and [51] for nonlinear frames. In this paper, we consider the scenario that phaseless samples

$$z_\epsilon(\gamma) = |f(\gamma)|^2 + \epsilon(\gamma),\ \gamma \in X + \mathbb{Z}, \tag{1.3}$$

of a signal $f \in V(\phi)$ taken on a shift-invariant set $X + \mathbb{Z}$ are corrupted, where $X \subset (0,1)$ is a finite set, and $\epsilon = (\epsilon(\gamma))_{\gamma \in X+\mathbb{Z}}$ are additive noises with bounded level

$$\|\epsilon\|_\infty = \sup\{|\epsilon(\gamma)| :\ \gamma \in X + \mathbb{Z}\}.$$

In Theorem 4.1 of Section 4, we establish the stability of phase retrieval in the above scenario when the original signal f is nonseparable. As an application of Theorem 4.1, any nonseparable signal in $V(\phi)$ generated by some function ϕ with support length L can be reconstructed, up to a sign, approximately from its noisy phaseless samples on a shift-invariant set $X + \mathbb{Z}$ with sampling density $2L - 1$.

Many algorithms have been proposed to solve the phase retrieval problem in the finite-dimensional setting [15–17,25,28,34,40–42,47]. A conventional approach to the scenario (1.3) is to solve the following min-max problem:

$$f_\epsilon := \mathrm{argmin}_{g \in V(\phi)} \max_{\gamma \in X+\mathbb{Z}} \big||g(\gamma)| - \sqrt{z_\epsilon(\gamma)}\big|, \tag{1.4}$$

which is infinite-dimensional and infeasible. In Section 4, we propose an MAPS algorithm to find an approximation $f_\epsilon \in V(\phi)$ to a nonseparable signal $f \in V(\phi)$, up to a sign, when the noisy phaseless samples $(z_\epsilon(\gamma))_{\gamma \in X+\mathbb{Z}}$ in (1.3) are available. The MAPS algorithm consists of three parts: minimizing, adjusting phases and sewing. It can be locally implemented, and has linear computational complexity $O(K_2 - K_1)$ to reconstruct a nonseparable signal $f = \sum_{k=K_1}^{K_2} c(k)\phi(\cdot - k) \in V(\phi)$ with finite duration approximately from its noisy phaseless sampling data. In Section 5, we present some simulations to demonstrate the stability of the proposed MAPS algorithm. Our numerical simulations indicate that the proposed MAPS algorithm is robust against bounded additive noises ϵ, and the error between the reconstructed signal f_ϵ and the original nonseparable signal f is $O(\sqrt{\|\epsilon\|_\infty})$.

Proofs of our conclusions are included in Section 6.

2. Phase retrievability and nonseparability

For a real-valued compactly supported generator ϕ of the shift-invariant space $V(\phi)$, let

$$L = \min_{L_1, L_2 \in \mathbb{Z}} \big\{L_2 - L_1 : \phi \text{ vanishes outside } [L_1, L_2]\big\}$$

be its support length. For the representative spline generator B_N, its support length is the same as the order $N \geq 1$. Without loss of generality, we assume that

$$\phi(t) = 0 \text{ for all } t \notin [0, L], \tag{2.1}$$

otherwise replacing ϕ by $\phi(\cdot - L_0)$ for some $L_0 \in \mathbb{Z}$. Clearly, not all signals in $V(\phi)$ are determined, up to a sign, from their magnitude measurements on $\mathbb{R}$. For instance, signals $\phi(t) \pm \phi(t-L)$ have the same magnitude measurements $|\phi(t)| + |\phi(t-L)|$ on the real line, but they are not the same even up to a sign. Then it is natural to ask whether a signal f in $V(\phi)$ is determined, up to a sign, from its magnitude measurements, or equivalently,

$$\mathcal{M}_f = \{\pm f\},$$

where the set

$$\mathcal{M}_f := \{g \in V(\phi) : \ |g(x)| = |f(x)| \text{ for all } x \in \mathbb{R}\} \tag{2.2}$$

contains all signals g in $V(\phi)$ that have the same magnitude measurements as the signal f has.

Theorem 2.1. *Let ϕ be a real-valued continuous function with compact support and $V(\phi)$ be the shift-invariant space in (1.1) generated by ϕ. Then a signal $f \in V(\phi)$ is determined, up to a sign, by its magnitude measurements, i.e., $\mathcal{M}_f = \{\pm f\}$, if and only if there do not exist nonzero signals f_1 and f_2 in $V(\phi)$ such that*

$$f = f_1 + f_2 \quad \text{and} \quad f_1 f_2 = 0. \tag{2.3}$$

We call a signal $f \in V(\phi)$ to be *nonseparable* if there do not exist nonzero signals $f_1, f_2 \in V(\phi)$ such that (2.3) holds, see Definition 6.1 for the definition of nonseparable signals in a linear space. Then a separable signal $f \in V(\phi)$ can be written as the sum of two nonzero signals $f_1, f_2 \in V(\phi)$ satisfying $f_1 f_2 = 0$. The proof of Theorem 2.1 and connection between nonseparability of signals and complement property for ideal sampling functionals will be discussed in Subsection 6.1.

Remark 2.2. Let $\mathcal{S}(\phi)$ be the set of all nonseparable signals in a real-valued shift-invariant space $V(\phi)$. The set $\mathcal{S}(\phi)$ is a cone of $V(\phi)$ containing the zero signal, however it is neither a convex subset of $V(\phi)$ nor its closed subset. This phenomenon for phase retrievability is different from the bandlimited case, for which all bandlimited functions can be reconstructed, up to a sign, from their magnitude measurements on $\mathbb{R}$ [48,53].

Given a signal $f \in V(\phi)$, the next question to be addressed in this section is to find all signals $g \in V(\phi)$ such that g and f have the same magnitude measurements on the whole real line, cf. [6]. Let us start from the simplest case that $L = 1$ (i.e., the generator ϕ is supported on $[0, 1]$). In this case, one can verify that a signal $f \in V(\phi)$ is nonseparable if and only if there exists an integer k_0 such that

$$f(t) = c(k_0)\phi(t - k_0) \quad \text{for some} \quad c(k_0) \in \mathbb{R}. \tag{2.4}$$

Therefore any signal in $V(\phi)$ is a linear combination of nonseparable signals with mutually disjoint supports. For the case that the generator ϕ has its support length

$$L \geq 2, \tag{2.5}$$

as shown in Lemma 6.9, such a linear combination exists for any signal $f \in V(\phi)$, i.e., there exist nonseparable signals $f_i \in V(\phi), i \in I$, such that

$$f = \sum_{i \in I} f_i \tag{2.6}$$

and their support intervals $[a_i, a'_i]$ are essentially mutually disjoint in the sense that

$$[a_i, a'_i) \cap [a_j, a'_j) = \emptyset \text{ for all distinct } i, j \in I. \tag{2.7}$$

Clearly signals $g = \sum_{i \in I} \xi_i f_i$ with $\xi_i \in \{-1, 1\}, i \in I$, have the same magnitude measurements as the signal f has. In the following theorem, we show that the converse is true under some proper assumptions on the generator ϕ.

Theorem 2.3. *Let ϕ be a real-valued continuous function satisfying (2.1) and (2.5), and $X := \{x_m, 1 \leq m \leq 2L - 1\} \subset (0, 1)$ be so chosen that all $L \times L$ submatrices of*

$$\Phi = \big(\phi(x_m+n)\big)_{1\le m\le 2L-1, 0\le n\le L-1} \tag{2.8}$$

are nonsingular. Take $f \in V(\phi)$ and let M_f be as in (2.2). Then $g \in M_f$ if and only if there exist $\xi_i \in \{-1,1\}, i \in I$, such that $g = \sum_{i\in I} \xi_i f_i$, where $f_i, i \in I$, are nonseparable signals in (2.6) and (2.7).

The proof of Theorem 2.3 depends on Theorem 3.2 and it will be given in Subsection 6.3.

3. Phaseless sampling and reconstruction

A set $\Lambda \subset \mathbb{R}$ is said to have sampling density $D(\Lambda)$ if

$$D(\Lambda) = \lim_{b-a\to+\infty} \frac{\#(\Lambda\cap[a,b])}{b-a}, \tag{3.1}$$

where $\#E$ is the cardinality of a set E. In this section, we consider the problem whether a signal f in the shift-invariant space $V(\phi)$ can be recovered, up to a sign, from its phaseless samples taken on a discrete set with finite sampling density. By Theorem 2.1, a necessary condition is that the signal f is nonseparable.

For the case that the generator ϕ has support length $L = 1$, it follows from (2.4) that any nonseparable signal in $V(\phi)$ is determined, up to a sign, from its phaseless samples on $t_0 + \mathbb{Z} \subset \mathbb{R}$ with sampling density one, where $t_0 \in (0,1)$ is so chosen that $\phi(t_0) \neq 0$. In the next theorem, we show that any nonseparable signal in a shift-invariant space generated by a compactly supported function with support length $L \ge 2$ can be reconstructed from its phaseless samples taken on a discrete set with finite sampling density.

Theorem 3.1. *Let ϕ and X be as in Theorem 2.3. Then any nonseparable signal $f \in V(\phi)$ is determined, up to a sign, from its phaseless samples $|f(t)|, t \in X+\mathbb{Z}$, taken on the shift-invariant set $X+\mathbb{Z}$.*

The proof of the above theorem on phaseless sampling and reconstruction, with detailed arguments given in Subsection 6.2, depends on the following characterization of nonseparable signals.

Theorem 3.2. *Let ϕ be a real-valued continuous function satisfying (2.1), (2.5) and (2.8), and $f(t) = \sum_{k\in\mathbb{Z}} c(k)\phi(t-k)$ be a nonzero signal in $V(\phi)$. Then f is nonseparable if and only if*

$$\sum_{l=0}^{L-2} |c(k+l)|^2 \neq 0 \tag{3.2}$$

for all $K_-(f) - L + 1 < k < K_+(f) + 1$, where $K_-(f) = \inf\{k : c(k) \neq 0\}$ and $K_+(f) = \sup\{k : c(k) \neq 0\}$.

The nonsingularity of all $L \times L$ submatrices of the matrix Φ in (2.8) is also known as its full sparkness ([8,22]). The full sparkness requirement (2.8) on the matrix Φ implies that ϕ has linearly independent shifts, i.e., the linear map from sequences to signals in $V(\phi)$,

$$\big(c(k)\big)_{k=-\infty}^{\infty} \longmapsto \sum_{k=-\infty}^{\infty} c(k)\phi(t-k),$$

is one-to-one ([35,49]). Conversely, if ϕ has linearly independent shifts and it is a continuous solution of the refinement equation ([20,37])

$$\phi(t) = \sum_{n=0}^{N} a(n)\phi(2t-n) \quad \text{and} \quad \int_{\mathbb{R}} \phi(t)dt = 1, \tag{3.3}$$

where $\sum_{n=0}^{N} a(n) = 2$, then Φ in (2.8) is of full spark for almost all $(x_1, \cdots, x_{2N-1}) \in (0,1)^{2N-1}$, see [49, Theorem A.2]. This together with Theorem 3.1 leads to the following result for wavelet signals, cf. [53, Theorem 1] and Corollary 6.3 for bandlimited signals.

Corollary 3.3. *Let ϕ be a continuous solution of the refinement equation (3.3) with linearly independent shifts. Then there exists a set $X \subset (0,1)$ containing $2N-1$ distinct points such that any nonseparable signal in $V(\phi)$ is determined, up to a sign, from its phaseless samples taken on $X + \mathbb{Z}$.*

For the refinement equation (3.3), under the assumption that

$$\sum_{n=0}^{N} a(n) z^n = (1+z)Q(z) \tag{3.4}$$

for some polynomial Q having positive coefficients and its zeros with strictly negative real part, the corresponding matrix Φ in (2.8) is of full spark whenever $x_m \in (0,1), 1 \le m \le 2N-1$, are distinct ([29,30]). It is well known that the B-spline B_N of order $N \ge 2$ satisfies the refinement equation (3.3) with $Q(z)$ in (3.4) given by $2^{-N+1}(1+z)^{N-1}$. This together with Theorem 3.1 yields the following result for spline signals, cf. [52].

Corollary 3.4. *Let $N \ge 2$ and X contain $2N-1$ distinct points in $(0,1)$. Then any nonseparable spline signal in $V(B_N)$ is determined, up to a sign, from its phaseless samples taken on the shift-invariant set $X + \mathbb{Z}$.*

Remark 3.5. Let ϕ be as in Theorem 3.2. For a signal $f = \sum_{k\in\mathbb{Z}} c(k)\phi(\cdot - k) \in V(\phi)$, define

$$S_f = \inf_{K_-(f)-L+1<k<K_+(f)+1} \sum_{l=0}^{L-2} |c(k+l)|^2. \tag{3.5}$$

By Theorem 3.2, we obtain that $S_f = 0$ if f is separable, and that $S_f > 0$ if f is a nonseparable signal with compact support. The quantity S_f can be used to measure the distance from a signal f to the set of all separable signals in $V(\phi)$, cf. Theorem 4.1.

4. Stable reconstruction from phaseless samples

In this section, we consider the scenario that the available data

$$z_\epsilon(\gamma) = |f(\gamma)|^2 + \epsilon(\gamma),\ \gamma \in X + \mathbb{Z}, \tag{4.1}$$

are phaseless samples of a signal

$$f = \sum_{k\in\mathbb{Z}} c(k)\phi(\cdot - k) \in V(\phi) \tag{4.2}$$

taken on a shift-invariant set $X + \mathbb{Z}$ corrupted by additive noises $\epsilon = (\epsilon(\gamma))_{\gamma\in X+\mathbb{Z}}$ with bounded level

$$\|\epsilon\|_\infty = \sup\big\{|\epsilon(\gamma)| : \ \gamma \in X + \mathbb{Z}\big\}.$$

Based on the constructive proof of Theorem 3.1 in Subsection 6.2, we propose an MAPS algorithm to find an approximation

Algorithm 1 MAPS Algorithm.

Inputs: The finite set X; support length of the generator L; noisy phaseless sampling data $\big(z_\epsilon(x_m+l)\big)_{x_m\in X, l\in\mathbb{Z}}$, $1\le m\le 2L-1$.
Instructions:
1) Minimizing locally: For any $k'\in\mathbb{Z}$, let

$$c_{\epsilon,k'} = (c_{\epsilon,k'}(k))_{k\in\mathbb{Z}} \tag{4.4}$$

have zero components except that $c_{\epsilon,k'}(k)$, $k'-L+1\le k\le k'$, are solutions of the minimization problem

$$\min \sum_{m=1}^{2L-1} \Big|\Big|\sum_{k=k'-L+1}^{k'} c(k)\phi(x_{m,k'}-k)\Big| - \sqrt{z_\epsilon(x_{m,k'})}\Big|^2, \tag{4.5}$$

where $x_m\in X$ and $x_{m,k'}=x_m+k'$, $1\le m\le 2L-1$.
2) Adjusting Phase: For $k'\in\mathbb{Z}$, multiplying $c_{\epsilon,k'}$ by $\delta_{\epsilon,k'}\in\{-1,1\}$ so that

$$\langle \delta_{\epsilon,k'}c_{\epsilon,k'}, \delta_{\epsilon,k'+1}c_{\epsilon,k'+1}\rangle \ge 0 \text{ for all } k'\in\mathbb{Z}. \tag{4.6}$$

3) Sewing:

$$c_\epsilon(k) = \frac{1}{L}\sum_{k'=k}^{k+L-1} \delta_{\epsilon,k'}c_{\epsilon,k'}(k),\ \ k\in\mathbb{Z}, \tag{4.7}$$

to obtain an approximation of amplitude vector $\big(c(k)\big)_{k\in\mathbb{Z}}$.
Outputs: Amplitude vector $\big(c_\epsilon(k)\big)_{k\in\mathbb{Z}}$, and the reconstructed signal $f_\epsilon=\sum_{k\in\mathbb{Z}} c_\epsilon(k)\phi(\cdot-k)$.

$$f_\epsilon = \sum_{k\in\mathbb{Z}} c_\epsilon(k)\phi(\cdot-k)\in V(\phi) \tag{4.3}$$

to the signal f in (4.2) when the noisy phaseless samples in (4.1) are available.

The proposed MAPS algorithm consists of the following three parts: (i) solving the **m**inimization problem (4.5) to obtain local approximations $c_{\epsilon,k'}$, $k'\in\mathbb{Z}$, of $\delta_{k'}c$ on $k'+[-L+1,0]$, up to a phase $\delta_{k'}\in\{-1,1\}$; (ii) **a**djusting **p**hases to obtain local approximations $\delta_{\epsilon,k'}c_{\epsilon,k'}$ to either c or $-c$ on $k'+[-L+1,0]$; and (iii) **s**ewing $\delta_{\epsilon,k'}c_{\epsilon,k'}$, $k'\in\mathbb{Z}$, together to get an approximation c_ϵ to either c or $-c$.

From implementation of the MAPS algorithm, we can reconstruct signals in $V(\phi)$ almost in real time from their phaseless samples, cf. [18,49] and references therein on local and distributed reconstruction. Moreover, the MAPS algorithm has linear computational complexity $O(K_2-K_1)$ to reconstruct nonseparable signals $f=\sum_{k=K_1}^{K_2} c(k)\phi(\cdot-k)\in V(\phi)$ approximately, up to a sign, from their noisy phaseless samples on $(X+\mathbb{Z})\cap[K_1,K_2+L]$. In the realistic model for sampling in a shift-invariant space, the generator ϕ does not have large support length L. Hence the minimization problem (4.5) of size L can be solved by many algorithms available in a stable way [15–17,25,28,34,40–42,47].

In the noiseless sampling environment (i.e., $\epsilon=\mathbf{0}$), the proposed MAPS algorithm provides a perfect reconstruction to a nonseparable signal, up to a sign. In a noisy sampling environment, we show in the following theorem that the MAPS algorithm (4.4)–(4.7) provides, up to a sign, a stable approximation to the original nonseparable signal f.

Theorem 4.1. *Let ϕ and X be as in Theorem 2.3, $f(t)=\sum_{k\in\mathbb{Z}} c(k)\phi(t-k)$ in (4.2) be a nonseparable real-valued signal with S_f in (3.5) being positive, and let $\big(z_\epsilon(\gamma)\big)_{\gamma\in X+\mathbb{Z}}$ be the noisy phaseless samples of the form (4.1) with bounded noise level $\|\epsilon\|_\infty$. Assume that $f_\epsilon(t)=\sum_{k\in\mathbb{Z}} c_\epsilon(k)\phi(t-k)$ is the signal in (4.3) reconstructed by the MAPS algorithm (4.4)–(4.7). If*

$$\|\epsilon\|_\infty \le \frac{S_f}{48L\|(\Phi_L)^{-1}\|^2}, \tag{4.8}$$

then there exists $\delta \in \{-1, 1\}$ *such that*

$$|c_\epsilon(k) - \delta c(k)| \le \|(\Phi_L)^{-1}\| \sqrt{8L\|\epsilon\|_\infty},\ k \in \mathbb{Z}, \tag{4.9}$$

where $\|A\| := \sup_{\|x\|_2=1} \|Ax\|_2$ *for a matrix* A *and*

$$\|(\Phi_L)^{-1}\| := \sup_{m_0<\ldots<m_{L-1}} \left\| \Big(\big(\phi(x_{m_l} + n)\big)_{0\le l,n\le L-1} \Big)^{-1} \right\|. \tag{4.10}$$

The proof of Theorem 4.1 includes an approximation property of vectors $c_{\epsilon,k'}, k' \in \mathbb{Z}$, in the first step of the MAPS algorithm (4.4)–(4.7), and existence of phase adjustment in the second step. The detailed arguments will be given in Subsection 6.4.

Define the reconstruction error of the MAPS algorithm by

$$E(\epsilon) := \min_{\delta\in\{-1,1\}} \|f_\epsilon(t) - \delta f(t)\|_\infty. \tag{4.11}$$

Then there exists a positive constant C by Theorem 4.1 such that

$$E(\epsilon) \le L\|\phi\|_\infty \min_{\delta\in\{-1,1\}} \max_{k\in\mathbb{Z}} |c_\epsilon(k) - \delta c(k)| \le C\sqrt{\|\epsilon\|_\infty}. \tag{4.12}$$

This together with (4.8) implies that there is no resonance phenomenon for the phaseless sampling and reconstruction model (4.1) if the noise level $\|\epsilon\|_\infty$ is sufficiently small. Moreover, numerical simulations in the next section show that the upper bound estimate in (4.12) for the reconstruction error $E(\epsilon)$ is suboptimal as it is about of the order $\sqrt{\|\epsilon\|_\infty}$.

5. Numerical simulations

In this section, we demonstrate the performance of the MAPS algorithm on reconstructing a cubic spline signal

$$f(t) = \sum_{k=K_1}^{K_2} c(k) B_4(t - k) \tag{5.1}$$

with finite duration, where B_4 is the cubic B-spline in (1.2) and integers K_1, K_2 satisfy $K_1 \le K_2$. Our noisy phaseless samples are taken on $X_K + \mathbb{Z}$,

$$z_\epsilon(\gamma) = |f(\gamma)|^2 + \|f\|_\infty^2 \epsilon(\gamma) \ge 0,\ \gamma \in X_K + \mathbb{Z}, \tag{5.2}$$

where $\epsilon(\gamma) \in [-\varepsilon, \varepsilon]$ are randomly selected with noise level $\varepsilon > 0$, and

$$X_K = \left\{ \frac{m}{K+1} : 1 \le m \le K \right\},\ K \ge 7. \tag{5.3}$$

The set X_K with $K = 7$ can be used as the set X in (2.8) and also in Theorem 3.1. In our simulations,

$$c(k) \in [-1, 1] \setminus [-0.1, 0.1],\ K_1 \le k \le K_2, \tag{5.4}$$

are randomly selected. Denote the signal reconstructed by the MAPS algorithm from the noisy phaseless samples (5.2) by

$$f_\epsilon(t) = \sum_{k=K_1}^{K_2} c_\epsilon(k) B_4(t-k), \tag{5.5}$$

cf. Theorem 4.1. Define an amplitude reconstruction error by

$$e(\epsilon) := \min_{\delta\in\{-1,1\}} \max_{k\in\mathbb{Z}} |c_\epsilon(k) - \delta c(k)|. \tag{5.6}$$

As $B_4(t) \geq 0$ and $\sum_{k\in\mathbb{Z}} B_4(t-k) = 1$ for all $t \in \mathbb{R}$, we have

$$E(\epsilon) = \min_{\delta\in\{-1,1\}} \max_{t\in\mathbb{R}} |f_\epsilon(t) - \delta f(t)| \leq e(\epsilon), \tag{5.7}$$

where $E(\epsilon)$ is the signal reconstruction error defined in (4.11), cf. (4.12). For the phaseless sampling and reconstruction model (5.2) with small noise level ε, it follows from Theorem 4.1 that the amplitude reconstruction error $e(\epsilon)$ in (5.6) and signal reconstruction error $E(\epsilon)$ in (4.11) are $O(\sqrt{\varepsilon})$. It is confirmed in the numerical simulations for nonseparable cubic spline signals, see Fig. 1.

The MAPS algorithm may not recover a nonseparable signal in a shift-invariant space if the noise level ε is not sufficiently small. Presented in Fig. 1 are the success rate in percentage and the average amplitude error after 1000 trials for different noise levels ε, where the MAPS algorithm to recover cubic spline signals f in (5.1) with $c(k), k \in \mathbb{Z}$, in (5.4) from noisy samples in (5.2) is considered to save the phase successfully if $e(\epsilon)$ in (5.6) satisfies $e(\epsilon) < 0.1$. In the simulations, a successful recovery implies that $c_\epsilon(k)$ and $c(k), K_1 \leq k \leq K_2$, have the same signs,

$$c_\epsilon(k)c(k) > 0 \text{ for all } K_1 \leq k \leq K_2.$$

The success rate of the MAPS algorithm can be improved if we have phaseless samples on a discrete set with high sampling density. Presented in Fig. 1 is the success rate in percentage to recover splines f in (5.1), up to a sign, from noisy phaseless samples in (5.2) taken on $X_K + \mathbb{Z}, 7 \leq K \leq 15$, where the noise level ε, the original signal f and the success threshold are the same as before. In addition to the improvement on success rate, our simulations also indicate that the amplitude reconstruction error in (5.6) decreases when the sampling density K increases, cf. [3, Theorem 3] for oversampling in a shift-invariant space.

The MAPS algorithm is applicable even if the original signal f is separable. Denote by g_ϵ the signal constructed from the MAPS algorithm. Our simulations show that the reconstruction error $\inf_{|g|=|f|} \|g_\epsilon - g\|_\infty$ is about $O(\sqrt{\varepsilon})$, cf. (5.7), and hence the signal g_ϵ provides a good approximation to a signal g in Theorem 2.3, not necessarily the original signal f. Presented in Fig. 2 is the performance of the MAPS algorithm when the amplitude coefficients of the original cubic spline f in (5.1) satisfy $c(k) \in [-1, 1]$ for all $K_1 \leq k \leq K_2$, cf. (5.4).

6. Proofs

In Section 6.1, we introduce nonseparability of a real-valued signal in a linear space, give a proof of Theorem 2.1, and establish the equivalence between complement property for ideal sampling functionals and nonseparability of all signals in a linear space. In Section 6.2, we characterize all nonseparable signals in a shift-invariant space and use them to prove Theorems 3.1 and 3.2. The proofs of Theorems 2.3 and 4.1 are given in Sections 6.3 and 6.4 respectively.

6.1. Nonseparability and complement property

In this subsection, we consider phase retrievability of signals in a linear space.

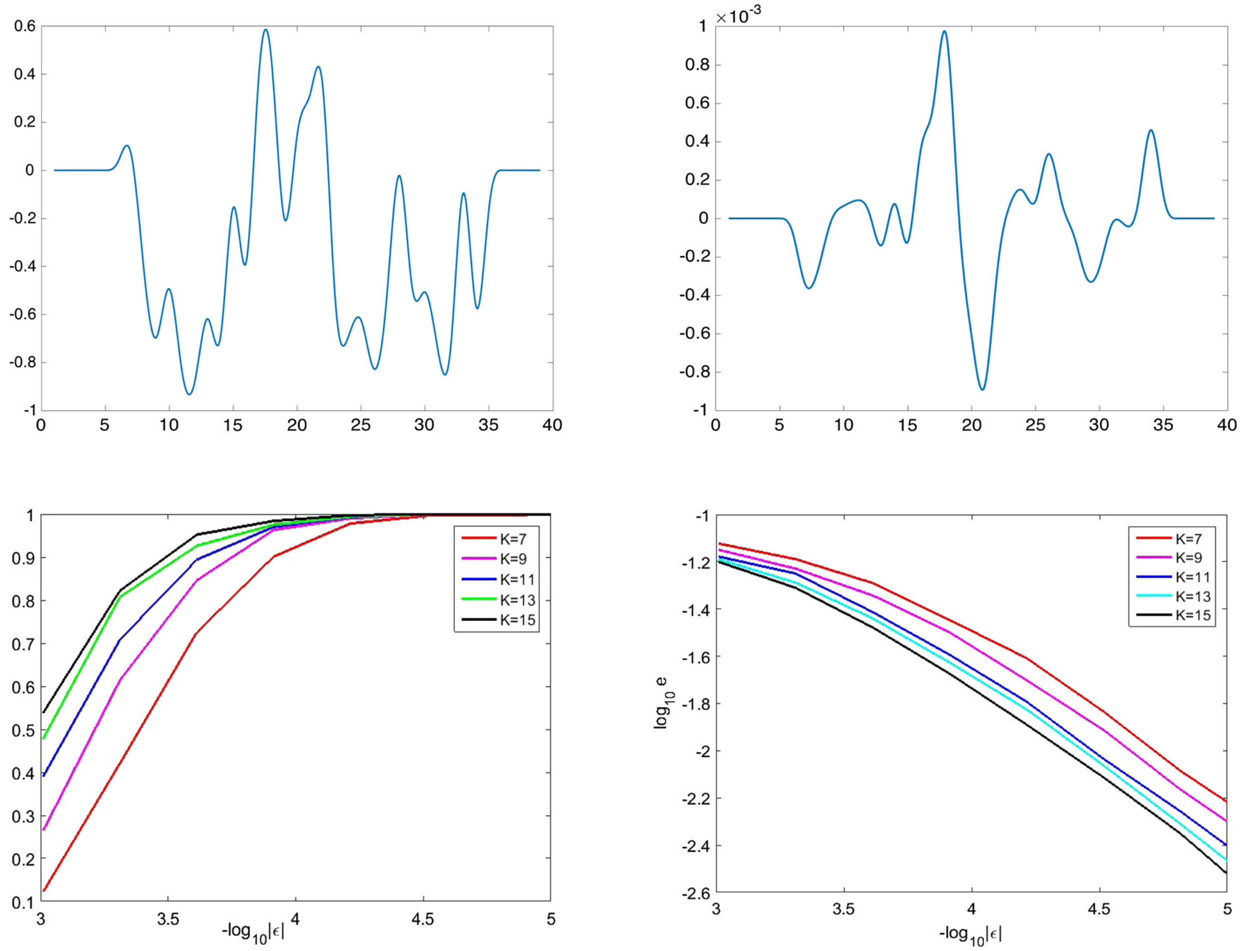

Fig. 1. Plotted on the top left is a nonseparable cubic spline f with $K_1 = 5, K_2 = 32$ and $c(k), k \in \mathbb{Z}$, in (5.4). On the top right is the difference between the signal f on the top left and the signal f_ϵ reconstructed by the MAPS algorithm from the noisy samples (5.2) with $\varepsilon = 10^{-5}$ and $K = 7$, where the amplitude reconstruction error $e(\epsilon)$ is 0.0014. Plotted on the bottom left is the success rate against noise level $-\log_{10}\varepsilon$ to recover a nonseparable cubic spline f by the MAPS algorithm for 1000 trails, with $c(k), k \in \mathbb{Z}$, randomly selected as in (5.4) and odd integers $7 \le K \le 15$. On the bottom right is the average error $\log_{10} e(\epsilon)$ against noise level $-\log_{10}\varepsilon$ in the logarithmic scale for a nonseparable cubic spline f running our MAPS algorithm for 1000 trails, where the error $e(\epsilon)$ is counted in the average only when phases are saved successfully.

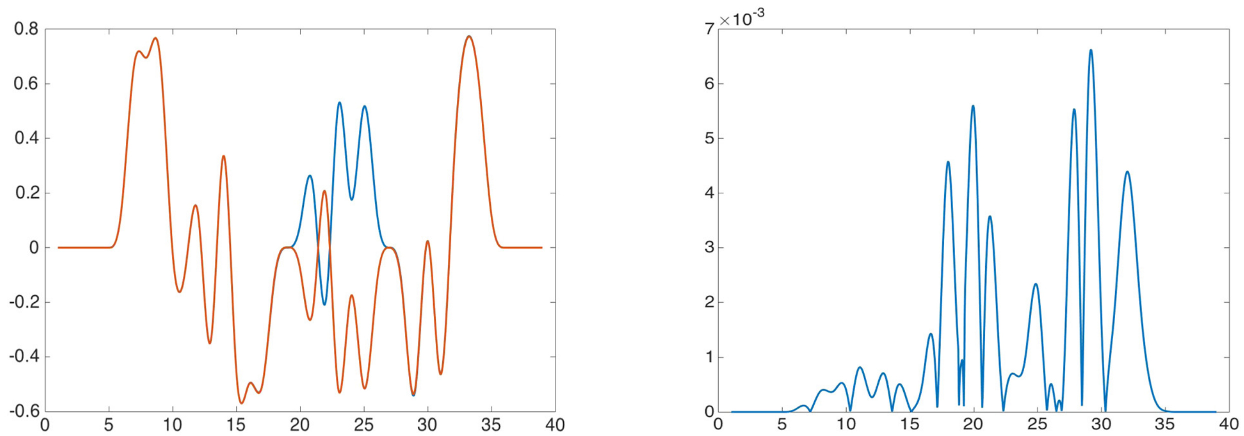

Fig. 2. Plotted on the left is the original cubic spline f (in blue) and the constructed signal g_ϵ (in red) via the MAPS algorithm, where $K_1 = 5$, $K_2 = 32$, $\varepsilon = 10^{-5}$ and $c(k) \in [-1, 1], 5 \le k \le 32$. On the right is the difference $|g_\epsilon - g|$ between the signal g_ϵ and a signal g in Theorem 2.3. The corresponding reconstruction error $\inf_{|g|=|f|} \|g_\epsilon - g\|_\infty$ is 0.0066. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Definition 6.1. Let V be a linear space of real-valued continuous signals on a set Y. A signal $f \in V$ is said to be nonseparable if there do not exist nonzero signals f_1 and f_2 in V such that

$$f = f_1 + f_2 \quad \text{and} \quad f_1 f_2 = 0. \tag{6.1}$$

The following is a generalization of Theorem 2.1.

Theorem 6.2. *Let V be a linear space of real-valued continuous signals on a set Y. Then a signal $f \in V$ is determined, up to a sign, by its magnitude measurements $|f(t)|, t \in Y$, if and only if it is nonseparable.*

Proof. ($\Longrightarrow$) Suppose, on the contrary, that there exist nonzero signals $f_1, f_2 \in V$ such that $f = f_1 + f_2$ and $f_1 f_2 = 0$. Set $g = f_1 - f_2 \in V$. Then $g \neq \pm f$ and $|g| = |f_1| + |f_2| = |f|$. This is a contradiction.

($\Longleftarrow$) Assume that f is nonseparable and $g \in V$ satisfies $|g| = |f|$. Set $g_1 := (f + g)/2$ and $g_2 := (f - g)/2 \in V$. Then $f = g_1 + g_2$ and $g_1 g_2 = 0$. This together with the nonseparable assumption on f implies that either $g_1 = 0$ or $g_2 = 0$. Hence $g = \pm f$ and the sufficiency is proved. $\square$

Observe that any bandlimited signal does not have a decomposition of the form (6.1), as it is analytic on $\mathbb{R}$. Therefore by Theorem 6.2 we have the following corollary, cf. [53, Theorem 1].

Corollary 6.3. *Any real-valued bandlimited signal is determined, up to a sign, by its magnitude measurements on the real line.*

In this subsection, we next consider linear spaces V such that *all* signals in V are determined, up to a sign, from their magnitude measurements on the real line.

Definition 6.4. Let V be a linear space of real-valued continuous signals on a set Y. We say that V has complement property if for any subset $A \subset Y$, there do not exist two nonzero signals f and g in V such that

$$f(t) = 0 \text{ on } A \quad \text{and} \quad g(t) = 0 \text{ on } Y \backslash A. \tag{6.2}$$

The above concept for ideal sampling functionals on a linear space is similar to the complement property for frames in a Hilbert space [10,13,14] and continuous frames in a Banach space [7]. As shown below, it characterizes the phase retrievability of all signals in that linear space.

Theorem 6.5. *Let V be a linear space of real-valued continuous signals on a set Y. Then all signals in V are determined, up to a sign, from their magnitude measurements on Y if and only if V has the complement property* (6.2).

Proof. We follow the arguments used in [7,10,13,14], and include a detailed proof for convenience.

($\Longrightarrow$) Suppose, on the contrary, that there exist a set A and two nonzero signals $f, g \in V$ satisfying (6.2). Then signals $f + g$ and $f - g$ have the same magnitude measurements $|f(t)| + |g(t)|, t \in Y$, but $f + g \neq \pm(f - g)$. This is a contradiction.

($\Longleftarrow$) Suppose, on the contrary, that there exist signals $f, g \in V$ such that $|f| = |g|$ and $f \neq \pm g$. Set $h_1 = f + g$ and $h_2 = f - g$. Then h_1 and h_2 are nonzero signals in V satisfying $h_1 h_2 = 0$. Hence (6.2) holds with f and g replaced by h_1 and h_2 respectively, and the set A by the support of h_2. This is a contradiction. $\square$

Combining Theorems 6.2 and 6.5, we have the following result about nonseparability of signals in a linear space and complement property for ideal sampling functionals.

Corollary 6.6. *Let V be a linear space of real-valued continuous signals on a set Y. Then V has the complement property (6.2) if and only if all signals in V are nonseparable.*

6.2. Proofs of Theorems 3.1 and 3.2

Theorems 3.1 and 3.2 follow from the following equivalences for nonseparable signals.

Theorem 6.7. *Let ϕ, X be as in Theorem 2.3, and $f(t) = \sum_{k\in\mathbb{Z}} c(k)\phi(t-k)$ be a nonzero real-valued signal in $V(\phi)$. Then the following are equivalent.*

(i) The signal f is nonseparable.
(ii) $\sum_{l=0}^{L-2} |c(k+l)|^2 \neq 0$ for all $K_-(f) - L + 1 < k < K_+(f) + 1$, where $K_-(f) = \inf\{k : c(k) \neq 0\}$ and $K_+(f) = \sup\{k : c(k) \neq 0\}$.
(iii) The signal f is determined, up to a sign, from its phaseless samples $|f(t)|, t \in X + \mathbb{Z}$, taken on the shift-invariant set $X + \mathbb{Z}$.

Proof. The implication iii)$\Longrightarrow$i) follows immediately from Theorem 2.1. Then it remains to prove i)$\Longrightarrow$ii) and ii)$\Longrightarrow$iii).

i)$\Longrightarrow$ii): Set $K_\pm = K_\pm(f)$. For $K_- - L + 1 < k < K_- + 1$ or $K_+ - L + 1 < k < K_+ + 1$, the conclusion $\sum_{l=0}^{L-2} |c(k+l)|^2 \neq 0$ follows from the definitions of K_- and K_+. Then it remains to establish the statement ii) for $K_- < k < K_+ - L + 2$. Suppose, on the contrary, that

$$\sum_{l=0}^{L-2} |c(k_1+l)|^2 = 0 \tag{6.3}$$

for some $K_- < k_1 < K_+ - L + 2$. Set $f_1(t) := \sum_{l=K_-}^{k_1-1} c(l)\phi(t-l)$ and $f_2(t) := \sum_{l=k_1+L-1}^{K_+} c(l)\phi(t-l)$. Then

$$f = f_1 + f_2 \quad \text{and} \quad f_1 f_2 = 0 \tag{6.4}$$

by (6.3) and the observation that f_1 and f_2 are supported in $(-\infty, k_1+L-1]$ and $[k_1+L-1, \infty)$ respectively. Clearly, f_1 and f_2 are nonzero signals in $V(\phi)$. This together with (6.4) implies that f is separable, which contradicts to the assumption i).

ii)$\Longrightarrow$iii): To prove this implication, we need a lemma.

Lemma 6.8. *Let ϕ and X be as in Theorem 2.3. Then for any $l \in \mathbb{Z}$ and signal $g(t) = \sum_{k\in\mathbb{Z}} d(k)\phi(t-k) \in V(\phi)$, coefficients $d(k), l - L + 1 \leq k \leq l$, are completely determined, up to a sign, by its phaseless samples $|g(x_m + l)|,\ x_m \in X$.*

The above lemma follows immediately from [10, Theorem 2.8] and the observation that

$$g(x_m + l) = \sum_{k=l-L+1}^{l} d(k)\phi(x_m + l - k), \quad x_m \in X.$$

Take a particular integer $K_- - 1 < k_0 < K_+ + 1$ with $c(k_0) \neq 0$. Without loss of generality, we assume that

$$c(k_0) > 0, \tag{6.5}$$

otherwise replacing f by $-f$.

Using (6.5) and applying Lemma 6.8 with g and l replaced by f and k_0 respectively, we conclude that $c(k_0 - L + 1), \cdots, c(k_0)$ are completely determined by phaseless samples $|f(X + k_0)|$ of the signal f on $X + k_0$. Now we prove the following claim:

$$c(k),\ k \le k_0,\ \text{are determined by } |f(X+k)|,\ k \le k_0 \tag{6.6}$$

by induction. Inductively we assume that $c(k), k_0-p-L+1 \le k \le k_0$, are determined from $|f(X+k)|, k_0-p \le k \le k_0$. The inductive proof is complete if $k_0 - p - L + 1 \le K_-$. Otherwise $k_0 - p - L + 1 > K_-$ and

$$\sum_{l=0}^{L-2} |c(k_0 - p - L + l + 1)|^2 \ne 0 \tag{6.7}$$

by the assumption ii). Applying Lemma 6.8 with g and l replaced by f and $k_0 - p - 1$ respectively, we conclude that $c(k_0 - p - L), \cdots, c(k_0 - p - 1)$ are determined, up to a sign, by $|f(X + k_0 - p - 1)|$. This together with (6.7) and the inductive hypothesis implies that $c(k_0 - p - L), \cdots, c(k_0 - p - 1)$ are completely determined by $|f(X + k)|, k_0 - p - 1 \le k \le k_0$. Thus the inductive argument can proceed.

Using the similar argument, we can show that $c(k), k \ge k_0$, are determined by $|f(X + k)|, k \ge k_0$. This together with (6.6) completes the proof. □

6.3. Proof of Theorem 2.3

The sufficiency follows as $f_i, i \in I$, have mutually disjoint supports. To prove the necessity, we need a lemma.

Lemma 6.9. *Let ϕ be as in Theorem 2.3. Then for any nonzero signal $f \in V(\phi)$ there exist nonseparable signals $f_i \in V(\phi), i \in I$, satisfying (2.6) and (2.7). Moreover the decomposition (2.6) and (2.7) is unique.*

Proof. Write $f = \sum_{k\in\mathbb{Z}} c(k)\phi(\cdot - k)$ and set

$$\mathcal{L} := \big\{l \in \mathbb{Z} : (c(l), \dots, c(l + L - 2)) \ne \mathbf{0}\big\}. \tag{6.8}$$

The set $\mathcal{L}$ can be decomposed into maximal mutually disjoint sets of consecutive integers. The above decomposition is unique and it can be described by existence of $b_i, b'_i \in \mathbb{Z} \cup \{-\infty, +\infty\}, i \in I$, such that

$$\mathcal{L} = \bigcup_{i\in I} \big((b_i, b'_i) \cap \mathbb{Z}\big) = \bigcup_{i\in I} \big\{b_i + 1, \dots, b'_i - 1\big\} \tag{6.9}$$

and

$$\text{intervals } [b_i, b'_i),\ i \in I,\ \text{are mutually disjoint.} \tag{6.10}$$

For instance, one may verify that the unique decomposition corresponding to the set $\mathcal{L} = \{-1, 0, 1, 2, 3, 5, 6, 7, 10\}$ is

$$\mathcal{L} = \big((-2, 4) \cap \mathbb{Z}\big) \cup \big((4, 8) \cap \mathbb{Z}\big) \cup \big((9, 11) \cap \mathbb{Z}\big) = \{-1, 0, 1, 2, 3\} \cup \{5, 6, 7\} \cup \{10\}.$$

By (6.8), (6.9) and (6.10), we have

$$c(k) = 0 \quad \text{for all} \quad k \notin \cup_{i\in I}(b_i + L - 2, b_i') \cap \mathbb{Z}. \tag{6.11}$$

Define

$$f_i = \sum_{b_i+L-2<k<b_i'} c(k)\phi(\cdot - k), \; i \in I. \tag{6.12}$$

Then the decomposition (2.6) holds by (6.11) and (6.12), and the mutually disjoint property (2.7) follows from (6.10) and the observation that $f_i, i \in I$, have support intervals $[b_i + L - 1, b_{i'} + L - 1]$. Observe from (6.9) that $K_+(f_i) = b_i' - 1$ and $K_-(f_i) = b_i + L - 1, i \in I$. This together with Theorem 3.2 implies that $f_i, i \in I$, are nonseparable. Therefore $f_i, i \in I$, in (6.12) are nonseparable signals satisfying (2.6) and (2.7).

Now it remains to prove uniqueness of the decomposition (2.6) and (2.7). Suppose that $g_j \in V(\phi), j \in J$, are nonseparable signals with their support intervals $[a_j, a_j']$ satisfying

$$f = \sum_{j\in J} g_j \tag{6.13}$$

and

$$[a_j, a_j') \cap [a_{j'}, a_{j'}') = \emptyset \text{ for all distinct } j, j' \in J. \tag{6.14}$$

Then it suffices to prove that $J = I$ and for any $j \in J$ there exists a unique $i \in I$ such that $g_j = f_i$, where $f_i, i \in I$, are given in (6.12). By (2.1), (2.5), (6.13) and (6.14), we have

$$g_j = \sum_{a_j-1<k<a_j'-L+1} c(k)\phi(\cdot - k) \tag{6.15}$$

and

$$c(k) = 0 \text{ for all } k \notin \cup_{j\in J}(a_j - 1, a_j' - L + 1). \tag{6.16}$$

Applying (6.15), (6.16) and Theorem 3.2, we obtain

$$\mathcal{L} = \cup_{j\in J}\big((a_j - L + 1, a_j' - L + 1) \cap \mathbb{Z}\big), \tag{6.17}$$

where the set $\mathcal{L}$ is given in (6.8). This together with (6.14) leads to another decomposition of the set $\mathcal{L}$ that satisfies (6.9) and (6.10). Due to the uniqueness of such a decomposition, we have that $J = I$ and for any $j \in J$ there exists a unique $i \in I$ such that $(a_j, a_j') = (b_i + L - 1, b_i' + L - 1)$, where $b_i, b_i', i \in I$, are given in (6.9). This together with (6.15) completes the proof. □

Now we start the proof of Theorem 2.3.

Proof of Theorem 2.3. Without loss of generality, we assume that $f \neq 0$. Write $g = \sum_{k\in\mathbb{Z}} d(k)\phi(\cdot - k)$ and $f = \sum_{k\in\mathbb{Z}} c(k)\phi(\cdot - k)$. By Lemma 6.8, for any $l \in \mathbb{Z}$ there exists $\delta_l \in \{-1, 1\}$ such that

$$d(l + n) = \delta_l c(l + n), \; 0 \le n \le L - 1. \tag{6.18}$$

Set $\mathcal{L} := \{l \in \mathbb{Z} : (c(l), \dots, c(l + L - 2)) \neq \mathbf{0}\}$ as in (6.8). Then it follows from (6.18) that

$$\delta_{l-1} = \delta_l \quad \text{for all} \quad l \in \mathcal{L}. \tag{6.19}$$

As in (6.9), we write $\mathcal{L}$ as the union of open intervals $(a_i, a'_i) \cap \mathbb{Z}, i \in I$, with $[a_i, a'_i), i \in I$, being mutually disjoint. Thus $\delta_l = \delta_{l'}$ for all $l, l' \in (a_i - 1, a'_i) \cap \mathbb{Z}$, which implies that the existence of $\xi_i \in \{-1, 1\}$ with

$$d(k) = \xi_i c(k) \text{ for all } a_i + L - 2 < k < a'_i \text{ with } i \in I. \tag{6.20}$$

By (6.11) and (6.18), we have

$$d(k) = 0 \text{ for all } k \notin \cup_{i\in I}\big((a_i + L - 2, a'_i) \cap \mathbb{Z}\big). \tag{6.21}$$

Therefore the conclusion $g = \sum_{i\in I} \xi_i f_i$ follows from (6.12), (6.20), (6.21) and Lemma 6.9. □

6.4. Proof of Theorem 4.1

To prove Theorem 4.1, we first show that the vector $c_{\epsilon,k'}$ obtained in the first step approximates the original vector c on $[k' - L + 1, k']$, up to a sign depending on k'.

Proposition 6.10. *Let $c, z_\epsilon, \epsilon, X, \|(\Phi_L)^{-1}\|$ be as in Theorem 4.1. Then for any $k' \in \mathbb{Z}$, there exists $\delta_{k'} \in \{-1, 1\}$ such that*

$$\sum_{k=k'-L+1}^{k'} |c_{\epsilon,k'}(k) - \delta_{k'} c(k)|^2 \le 8L\|(\Phi_L)^{-1}\|^2 \|\epsilon\|_\infty. \tag{6.22}$$

Proof. Set $x_{m,k'} = x_m + k' \in X + k', 1 \le m \le 2L - 1$. Then

$$\begin{aligned}
&\sum_{m=1}^{2L-1}\Bigg(\Big|\sum_{k=k'-L+1}^{k'} c_{\epsilon,k'}(k)\phi(x_{m,k'} - k)\Big| - \Big|\sum_{k=k'-L+1}^{k'} c(k)\phi(x_{m,k'} - k)\Big|\Bigg)^2 \\
&\le 2\sum_{m=1}^{2L-1}\Bigg(\Big|\sum_{k=k'-L+1}^{k'} c_{\epsilon,k'}(k)\phi(x_{m,k'} - k)\Big| - \sqrt{z_\epsilon(x_{m,k'})}\Bigg)^2 \\
&\quad + 2\sum_{m=0}^{2L-1}\Bigg(\sqrt{z_\epsilon(x_{m,k'})} - \Big|\sum_{k=k'-L+1}^{k'} c(k)\phi(x_{m,k'} - k)\Big|\Bigg)^2 \\
&\le 4\sum_{m=1}^{2L-1}\Big||f(x_{m,k'})| - \sqrt{z_\epsilon(x_{m,k'})}\Big|^2 \le 8L\|\epsilon\|_\infty,
\end{aligned}$$

where the second inequality holds by (4.5), and the third estimate follows from the triangle inequality $|\sqrt{x^2 + y} - |x|| \le \sqrt{|y|}$ for all $x \in \mathbb{R}$ and $y \ge -x^2$. Therefore there exists a subset $\mathcal{M} \subset \{1, \ldots, 2L - 1\}$ such that

$$\sum_{m\in\mathcal{M}}\Big(\sum_{k=k'-L+1}^{k'}\big(c_{\epsilon,k'}(k)-c(k)\big)\phi(x_{m,k'}-k)\Big)^2$$
$$+\sum_{m\in\{1,\ldots,2L-1\}\setminus\mathcal{M}}\Big(\sum_{k=k'-L+1}^{k'}\big(c_{\epsilon,k'}(k)+c(k)\big)\phi(x_{m,k'}-k)\Big)^2\le 8L\|\epsilon\|_\infty.$$

This together with the definition of $\|(\Phi_L)^{-1}\|$ in (4.10) completes the proof. $\square$

To prove Theorem 4.1, we adjust phases of $c_{\epsilon,k'}$, $k'\in\mathbb{Z}$, obtained in the first step so that the phase adjusted vectors $\delta_{\epsilon,k'}c_{\epsilon,k'}$, $k'\in\mathbb{Z}$, approximate the original vector c on $[k'-L+1,k']$, up to a sign independent on k'.

Proposition 6.11. *Let $\delta_{k'}\in\{-1,1\}, k'\in\mathbb{Z}$, be as in Proposition 6.10. If (4.8) holds for some $\delta_{\epsilon,k'}\in\{-1,1\}, k'\in\mathbb{Z}$ in (4.6), then*

$$\delta_{\epsilon,k'}\delta_{\epsilon,k'+1}=\delta_{k'}\delta_{k'+1} \tag{6.23}$$

for all $k'\in\mathbb{Z}$ with $\sum_{k=-L+2}^{0}|c(k+k')|^2\neq 0$.

Proof. For any $k'\in\mathbb{Z}$,

$$\begin{aligned}
&\Big|\langle\delta_{k'}c_{\epsilon,k'},\delta_{k'+1}c_{\epsilon,k'+1}\rangle-\sum_{k=k'-L+2}^{k'}|c(k)|^2\Big|\\
&\le\sum_{k=k'-L+2}^{k'}|\delta_{k'}c_{\epsilon,k'}(k)-c(k)||c(k)|+\sum_{k=k'-L+2}^{k'}|\delta_{k'+1}c_{\epsilon,k'+1}(k)-c(k)||c(k)|\\
&\quad+\sum_{k=k'-L+2}^{k'}|\delta_{k'}c_{\epsilon,k'}(k)-c(k)||\delta_{k'+1}c_{\epsilon,k'+1}(k)-c(k))|\\
&\le 4\sqrt{2L\|\epsilon\|_\infty}\big\|(\Phi_L)^{-1}\big\|\Big(\sum_{k=k'-L+2}^{k'}|c(k)|^2\Big)^{1/2}+8L\big\|(\Phi_L)^{-1}\big\|^2\|\epsilon\|_\infty\\
&<\sum_{k=k'-L+2}^{k'}|c(k)|^2,
\end{aligned}$$

where the second estimate follows from Proposition 6.10, and the last inequality holds by the assumption (4.8) on the noise level $\|\epsilon\|_\infty$. Therefore the vectors $\delta_{k'}c_{\epsilon,k'}$ and $\delta_{k'}c_{\epsilon,k'+1}$ have positive inner product. This together with (4.6) proves (6.23). $\square$

We finish this subsection with the proof of Theorem 4.1.

Proof of Theorem 4.1. Set $K_\pm=K_\pm(f)$. By Theorem 3.2 and Proposition 6.11, there exists $\delta\in\{-1,1\}$ such that

$$\delta_{\epsilon,k'}=\delta\delta_{k'} \tag{6.24}$$

for all $k'\in(K_--1,K_++L)$. For $k\in\mathbb{Z}$, we obtain from (4.6), (4.7), (6.24) and Proposition 6.10 that

$$
\begin{aligned}
|c_\epsilon(k) - \delta c(k)| &\le \frac{1}{L}\sum_{k'=k}^{k+L-1} |c_{\epsilon,k'}(k) - \delta_{k'} c(k)| + \frac{1}{L}\sum_{k'=k}^{k+L-1} |\delta_{k'}\delta_{\epsilon,k'} - \delta||c(k)| \\
&\le \|(\Phi_L)^{-1}\|\sqrt{8L\|\epsilon\|_\infty}.
\end{aligned}
$$

This completes the proof. □

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