Design of Nonsubsampled Graph Filter Banks via Lifting Schemes

Junzheng Jiang, David B. Tay, Qiyu Sun, and Shan Ouyang

***Abstract*—Graph filter banks play a crucial role in the vertex and spectral representation of graph signals. The notion of two-channel nonsubsampled graph filter banks (NSGFBs) on an undirected graph was introduced recently. The absence of downsampling/upsampling operators allows greater flexibility in the design of NSGFBs that achieve perfect reconstruction. However the design of NSGFBs that take the spectral response into account has not been adequately addressed yet. Based on the polynomial/rational lifting scheme, this letter presents a simple method to design NSGFBs with good spectral response and perfect reconstruction. Experimental results will demonstrate the effectiveness of the proposed method in tailoring the spectral responses of the lifted NSGFBs. Application of the NSGFB to denoising will also be considered.**

***Index Terms*—Graph signal processing, nonsubsampled graph filter bank, lifting scheme, Laplacian matrix.**

I. Introduction

GRAPH signal processing (GSP) is a research field that is gaining prominence and finds a variety of applications where the data is defined over an irregular domain, e.g. sensor networks and social networks [1], [2]. One of the main ideas behind GSP is the exploitation of pairwise relationships between data values defined over nodes via a graph model. Some of the fundamental principles from traditional signal processing have been extended to the graph domain, giving rise to the graph Fourier transform [3], graph filter [4], graph wavelet filter bank [5]–[9], etc. Graph filter banks (GFBs), in particular, is a topic that has drawn significant attention as they play a crucial role in the vertex and spectral representation of graph signals. GFBs with downsampling/upsampling (DU) operators are, in general, difficult to analyse and design, except in cases with special graphs, such as bipartite graphs [6], [7].

In [9], the notion of two-channel nonsubsampled graph filter banks (NSGFB) was introduced by Jiang, Cheng and Sun. A two-channel NSGFB consists of an analysis filter bank $\{\mathbf{H}_0, \mathbf{H}_1\}$ and a synthesis filter bank $\{\mathbf{G}_0, \mathbf{G}_1\}$. The absence of the DU operators leads to the following perfect reconstruction (PR) condition [9]

$$\mathbf{G}_0\mathbf{H}_0 + \mathbf{G}_1\mathbf{H}_1 = \mathbf{I}. \tag{I.1}$$

The analysis filter bank $\{\mathbf{H}_0, \mathbf{H}_1\}$ of an NSGFB on a graph $\mathcal{G}$ is said to be *normal* if (i) the lowpass graph filter passes the weighted constant signal, i.e. $\mathbf{H}_0\mathbf{D}_{\mathcal{G}}^{1/2}\mathbf{1} = \mathbf{D}_{\mathcal{G}}^{1/2}\mathbf{1}$; and (ii) the highpass graph filter blocks that signal, i.e. $\mathbf{H}_1\mathbf{D}_{\mathcal{G}}^{1/2}\mathbf{1} = \mathbf{0}$, where $\mathbf{D}_{\mathcal{G}}$ is the degree matrix of the graph $\mathcal{G}$ and $\mathbf{1} = [1\ \ldots\ 1]^T$. In [9], the authors proposed two methods to design normal NSGFBs satisfying the PR condition (I.1) but do not explicitly take the spectral response into account. The spectral approach to graph signal transforms provides a representation that is similar to the Fourier transform for regular-domain signals. A spectral/frequency domain interpretation of the transforms allows a distinction between low-frequency and high-frequency components of a graph signal. Another advantage with the spectral approach is that the spectral filter $h(\lambda)$ (λ spectral variable) can be designed independently of the graph structure [5]–[8]. When the filter is applied to a specific graph, the filter adjusts itself to the structure of the graph when λ is substituted with a graph matrix, e.g. Laplacian $\mathbf{L}$, which encodes the structure of the graph.

In this letter, we propose a simple yet flexible method to design PR NSGFBs that is based on the lifting scheme applied to the analysis filters. It allows the tailoring of the spectral response via a lifting filter. Analysis filters with good frequency characteristics can be designed by optimizing the lifting filters. Transform based on lifting has also been previously proposed in [11] for graph signals. Subsampling is however used in [11] and some edge information are lost because of this. The filters in [11] are specified in the vertex domain and shaping the spectral response is not readily achievable. In this work there is no loss of edge information (as there is no subsampling) and the control the spectral characteristics of the filters, e.g. low-pass, is easily achieved.

II. Preliminaries and Lifting Scheme

Let $\mathcal{G} = (V, E, \mathbf{W})$ be an undirected weighted graph with no self-loops and multiple edges, where $V = \{1, 2, \ldots, N\}$ is the set of nodes, E is the set of edges, and $\mathbf{W} = [w_{ij}]_{1\le i,j\le N}$ is the weighted adjacency matrix. The degree matrix $\mathbf{D}_{\mathcal{G}}$ is a diagonal

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matrix whose i-th diagonal entry is given by $d_{ii} = \sum_{j \in V} w_{ij}$. The symmetrically normalized Laplacian matrix is defined as $\mathbf{L}_{\mathcal{G}}^{\text{sym}} \equiv \mathbf{I} - \mathbf{D}_{\mathcal{G}}^{-1/2}\mathbf{W}\mathbf{D}_{\mathcal{G}}^{-1/2}$. The eigendecomposition of $\mathbf{L}_{\mathcal{G}}^{\text{sym}}$ is given by $\mathbf{L}_{\mathcal{G}}^{\text{sym}} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$, where $\mathbf{U}$ is the orthogonal eigenvector matrix and $\mathbf{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_N)$ is a diagonal eigenvalue matrix with eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_N \leq 2$. The eigenvalues of $\mathbf{L}_{\mathcal{G}}^{\text{sym}}$ can be interpreted as spectral frequencies [5]–[7].

A signal on the graph $\mathcal{G}$ is represented by a vector $\mathbf{x} = [x_1 \ \cdots \ x_N]^T$, where each element x_i represents a numerical quantity associated with node i. The graph Fourier transform $\widehat{\mathbf{x}}$ of a graph signal $\mathbf{x}$ is given by $\widehat{\mathbf{x}} = \mathbf{U}^T\mathbf{x}$. A graph filter is represented by a matrix $\mathbf{H}$ that acts on an input signal $\mathbf{x}$ to give an output signal $\mathbf{y} = \mathbf{H}\mathbf{x}$. A common class of filters are functions of the normalized Laplacian matrix $\mathbf{L}_{\mathcal{G}}^{\text{sym}}$,

$$\mathbf{H} = h(\mathbf{L}_{\mathcal{G}}^{\text{sym}}) := \mathbf{U}h(\mathbf{\Lambda})\mathbf{U}^T, \tag{II.1}$$

where h is a function defined on $[0, 2]$ and $h(\mathbf{\Lambda})$ is a diagonal matrix with diagonal entries $h(\lambda_1), \ldots, h(\lambda_N)$. For a filter $\mathbf{H}$ of the form (II.1) and an input $\mathbf{x}$, the graph Fourier transform of the corresponding output $\mathbf{H}\mathbf{x}$ is given by $\widehat{\mathbf{H}\mathbf{x}} = h(\mathbf{\Lambda})\widehat{\mathbf{x}}$. Hence the filter $\mathbf{H}$ has frequency response $h(\lambda_i)$ at the spectral frequencies $\lambda_i, 1 \leq i \leq N$ [1], [3], [5]–[7]. Due to the above observation, the function h is known as the *spectral response function* of the filter $\mathbf{H}$ in (II.1).

The lifting scheme, pioneered by Sweldens [10], can improve the frequency response of the subband filters by using lifting filters. The lifting in [10] was developed for regular domain filter banks with critical subsampling. We, however adapt the lifting for filter banks without subsampling, and operating on signals defined over irregular domains, i.e. graphs. One of the main advantages of lifting is that the PR constraint is structurally imposed, and does need to be explicitly considered in the design process. In other techniques, such as the complementary filter method [14], the constraint needs to be explicitly considered, which will complicate the design process. Motivated by this fact, we apply the lifting scheme to improve the spectral response of the filters in NSGFBs. There are two approaches to improve the frequency response characteristic via the lifting scheme. The first is to apply lifting to the synthesis filter bank,

$$\mathbf{G}_0 = \mathbf{G}_0^P + \mathbf{R}\mathbf{H}_1^P, \ \ \mathbf{G}_1 = \mathbf{G}_1^P - \mathbf{R}\mathbf{H}_0^P, \tag{II.2}$$

where $\{\mathbf{H}_0^P, \mathbf{H}_1^P\}$ and $\{\mathbf{G}_0^P, \mathbf{G}_1^P\}$ are the prototype analysis and synthesis filter banks respectively, and $\mathbf{R}$ is a lifting filter. The second is to apply lifting to the analysis filter bank,

$$\mathbf{H}_0 = \mathbf{H}_0^P + \mathbf{G}_1^P\mathbf{R}, \ \ \mathbf{H}_1 = \mathbf{H}_1^P - \mathbf{G}_0^P\mathbf{R}. \tag{II.3}$$

If the prototype NSGFB is assumed to satisfy the PR condition (I.1), then with the analysis filter bank in (II.3) and prototype synthesis filter bank $\{\mathbf{G}_0^P, \mathbf{G}_1^P\}$, we have $\mathbf{G}_0^P\mathbf{H}_0 + \mathbf{G}_1^P\mathbf{H}_1 = \mathbf{I} + (\mathbf{G}_0^P\mathbf{G}_1^P - \mathbf{G}_1^P\mathbf{G}_0^P)\mathbf{R}$. Hence the resultant NSGFB satisfies (I.1) provided that the prototype synthesis filters $\mathbf{G}_0^P$ and $\mathbf{G}_1^P$ are commutative, i.e., $\mathbf{G}_0^P\mathbf{G}_1^P = \mathbf{G}_1^P\mathbf{G}_0^P$. Following a similar argument, we can show that the resultant NSGFB with the prototype analysis filters $\{\mathbf{H}_0^P, \mathbf{H}_1^P\}$ which commute, i.e. $\mathbf{H}_0^P\mathbf{H}_1^P = \mathbf{H}_1^P\mathbf{H}_0^P$, and the synthesis filters $\{\mathbf{G}_0, \mathbf{G}_1\}$ in (II.2), also satisfies (I.1).

Proposition II.1: Let the prototype NSGFB satisfies (I.1), i.e., $\mathbf{G}_0^P\mathbf{H}_0^P + \mathbf{G}_1^P\mathbf{H}_1^P = \mathbf{I}$. If the prototype synthesis filters $\{\mathbf{G}_0^P, \mathbf{G}_1^P\}$ are commutative, then the lifted NSGFB with the analysis filters in (II.3) satisfies (I.1). Conversely, if the prototype analysis filters $\{\mathbf{H}_0^P, \mathbf{H}_1^P\}$ are commutative, then the lifted NSGFB with the synthesis filters in (II.2) satisfies (I.1).

The commutativity between prototype synthesis filters $\{\mathbf{G}_0^P, \mathbf{G}_1^P\}$ follows if they both can be diagonalized by a common nonsingular matrix. In [9], a representative class of analysis filter bank is the spline filter bank $\{\mathbf{H}_{0,n}^{\text{spln}}, \mathbf{H}_{1,n}^{\text{spln}}\}$ of order $n \geq 1$ given by $\mathbf{H}_{0,n}^{\text{spln}} \equiv (\mathbf{I} - \mathbf{L}_{\mathcal{G}}^{\text{sym}}/2)^n$, $\mathbf{H}_{1,n}^{\text{spln}} \equiv (\mathbf{L}_{\mathcal{G}}^{\text{sym}}/2)^n$. The corresponding minimum degree synthesis filter bank $\{\mathbf{G}_{0,n}^{\text{spln}}, \mathbf{G}_{1,n}^{\text{spln}}\}$ is designed via the Bezout identity, where $\mathbf{G}_{0,n}^{\text{spln}} \equiv P_n(\mathbf{L}_{\mathcal{G}}^{\text{sym}}/2)$, $\mathbf{G}_{1,n}^{\text{spln}} \equiv P_n(\mathbf{I} - \mathbf{L}_{\mathcal{G}}^{\text{sym}}/2)$, and P_n is the unique polynomial solution of degree $n-1$ to the equation $(1-t)^n P_n(t) + t^n P_n(1-t) = 1$. Now the synthesis filters are polynomials of the symmetric Laplacian matrix. Therefore, they share the same orthogonal eigenvector space, and this makes them commutative. A similar argument can be made for the commutativity of the prototype analysis filters.

III. Design Methodology

The role of the analysis filter bank is to decompose the input signal into different spectral frequency bands, while the role of the synthesis filter bank is to reconstruct the signal. The frequency characteristic of the former is therefore more important than the latter in many applications. With the aim to achieve good frequency characteristics for the analysis filters, we concentrate on the lifting scheme (II.3) so that the spectral responses of the lifted analysis filters approximate the desired spectral responses $d_0(\lambda)$ and $d_1(\lambda)$. Typical examples of the desired responses are the ideal brick-wall low-pass/high-pass functions $I_0(\lambda)$ and $I_1(\lambda)$ given by

$$I_0(\lambda) \equiv \begin{cases} 1 \text{ if } \lambda \in [0,1] \\ 0 \text{ if } \lambda \in (1,2] \end{cases} \text{ and } I_1(\lambda) \equiv \begin{cases} 0 \text{ if } \lambda \in [0,1) \\ 1 \text{ if } \lambda \in [1,2]. \end{cases}$$

Consider an NSGFB with its analysis filter bank $\{\mathbf{H}_0^P, \mathbf{H}_1^P\}$ and synthesis filter bank $\{\mathbf{G}_0^P, \mathbf{G}_1^P\}$ being of the form (II.1), i.e. $\mathbf{H}_i^P = h_i^P(\mathbf{L}_{\mathcal{G}}^{\text{sym}})$ and $\mathbf{G}_i^P = g_i^P(\mathbf{L}_{\mathcal{G}}^{\text{sym}})$, $i = 0, 1$ for some functions $h_i^P(\lambda)$, $g_i^P(\lambda)$ on the interval $\lambda \in [0, 2]$. In this section, we design lifting filters of the form $\mathbf{R} = r(\mathbf{L}_{\mathcal{G}}^{\text{sym}})$ so that the spectral response of the lifted analysis filter bank in (II.3), i.e. $\{h_0(\lambda), h_1(\lambda)\}$, approximates the desired responses $\{d_0(\lambda), d_1(\lambda)\}$. One can readily verify that the lifted analysis filters responses are given by $h_0(\lambda) = h_0^P(\lambda) + g_1^P(\lambda)r(\lambda)$ and $h_1(\lambda) = h_1^P(\lambda) - g_0^P(\lambda)r(\lambda)$. Hence the design of the lifting filter $\mathbf{R}$ reduces to finding a function $r(\lambda)$ so that $h_i(\lambda)$ is close to $d_i(\lambda), i = 0, 1$, with respect to a chosen error criterion.

A least squares formulation of the design problem is to determine the lifting filter $r(\lambda)$ that minimizes

$$\phi(r) = \int_0^{\mu_p} \left|h_0(\lambda) - d_0(\lambda)\right|^2 d\lambda + \int_{\mu_s}^{2} \left|h_1(\lambda) - d_1(\lambda)\right|^2 d\lambda, \tag{III.1}$$

where $0 < \mu_p \leq 2$ and $0 \leq \mu_s < 2$. The first term in the objective functional $\phi(r)$ measures the difference between the lifted spectral response $h_0(\lambda)$ to the desired spectral response $d_0(\lambda)$ on $[0, \mu_p]$, while the second measures the difference between the lifted spectral response $h_1(\lambda)$ to the desired spectral response $d_1(\lambda)$ on $[\mu_s, 2]$. The edge frequency parameters μ_s and μ_p can be prescribed by the designer.

We will address the design of the filter $r(\lambda)$ with two functional forms. The first is the polynomial form and the second is the rational form.

1) Polynomial Filters: The lifting filter is given by

$$r(\lambda) = r_0 + \sum_{l=1}^{L} r_l \lambda^l = \mathbf{c}_\lambda^T \mathbf{r}, \tag{III.2}$$

where $\mathbf{c}_\lambda \equiv [1\ \lambda\ \cdots\ \lambda^L]^T$ and $\mathbf{r} = [r_0\ r_1\ \cdots\ r_L]^T$. Define the spectral response difference as $e_0(\lambda) \equiv d_0(\lambda) - h_0^P(\lambda)$ and $e_1(\lambda) \equiv d_1(\lambda) - h_1^P(\lambda)$. Applying (III.2), we can recast the minimization problem (III.1) as a least squares problem with respect to the coefficients vector $\mathbf{r}$,

$$\min_{\mathbf{r}} \mathbf{r}^T \mathbf{A} \mathbf{r} - 2\mathbf{b}^T \mathbf{r} + c_0, \tag{III.3}$$

where

$$\mathbf{A} = \int_0^{\mu_p} \mathbf{c}_\lambda |g_1^P(\lambda)|^2 \mathbf{c}_\lambda^T d\lambda + \int_{\mu_s}^{2} \mathbf{c}_\lambda |g_0^P(\lambda)|^2 \mathbf{c}_\lambda^T d\lambda, \tag{III.4}$$

$$\mathbf{b} = \int_0^{\mu_p} g_1^P(\lambda) e_0(\lambda) \mathbf{c}_\lambda d\lambda - \int_{\mu_s}^{2} g_0^P(\lambda) e_1(\lambda) \mathbf{c}_\lambda d\lambda, \tag{III.5}$$

and $c_0 = \int_0^{\mu_p} |e_0(\lambda)|^2 d\lambda + \int_{\mu_s}^2 |e_1(\lambda)|^2 d\lambda$. If either (i) the spectral response $g_1^P(\lambda) \not\equiv 0$ on $[0, \mu_p]$; or (ii) $g_0^P(\lambda) \not\equiv 0$ on $[\mu_s, 2]$, then it can be verified that the matrix $\mathbf{A}$ in (III.4) is positive definite and therefore non-singular. Hence the optimal solution of the problem (III.3) is given by $\mathbf{r}^* = \mathbf{A}^{-1}\mathbf{b}$.

The lifted analysis filter bank $\{\mathbf{H}_0, \mathbf{H}_1\}$ in (II.3) may not be normal even if the prototype analysis filter bank $\{\mathbf{H}_0^P, \mathbf{H}_1^P\}$ is. However, it will be normal if the lifting filter $\mathbf{R}$ satisfies $\mathbf{R}\mathbf{D}_{\mathcal{G}}^{1/2}\mathbf{1} = \mathbf{0}$. The above requirement is met for a lifting filter $\mathbf{R}$ of the form (II.1) if its spectral response function r satisfies $r(0) = 0$, which becomes $r_0 = 0$ for a polynomial lifting filter $r(\lambda)$ of the form (III.2). Similar to the optimal solution $\mathbf{r}^* = \mathbf{A}^{-1}\mathbf{b}$, for unconstrainted problem (III.3), we can show that $\tilde{\mathbf{r}}^* = \tilde{\mathbf{A}}^{-1}\tilde{\mathbf{b}}$ is the optimal solution of the constrained problem (III.3) with $r_0 = 0$, where $\tilde{\mathbf{r}} = [r_1\ \cdots\ r_L]^T$, $\tilde{\mathbf{c}}_\lambda = [\lambda\ \cdots\ \lambda^L]^T$, and $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{b}}$ are defined by (III.4) and (III.5) respectively, but with $\mathbf{c}_\lambda$ replaced by $\tilde{\mathbf{c}}_\lambda$.

2) Rational Filters: We now consider the design of rational lifting filters, i.e. $r(\lambda) = \frac{a(\lambda)}{b(\lambda)}$, where $a(\lambda)$ and $b(\lambda)$ are polynomials with $b(0) = 1$. As the filter function is rational, a formulation using (III.1) would involve integrals which are difficult to evaluate analytically. An alternative formulation of the minimization problem in (III.1), which replaces the integral with a weighted discrete sum, is proposed as follows:

$$\min_{a(\lambda),b(\lambda)} \sum_{\lambda_i \in [0,\mu_p]} w_i \left| g_1^P(\lambda_i) \frac{a(\lambda_i)}{b(\lambda_i)} - e_0(\lambda_i) \right|^2 + \sum_{\lambda_i \in [\mu_s,2]} w_i \left| g_0^P(\lambda_i) \frac{a(\lambda_i)}{b(\lambda_i)} + e_1(\lambda_i) \right|^2 \tag{III.6}$$

where $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 2$ and $w_1, \ldots, w_n$ are weights specified by users according to the importance of different λ_i over the intervals. One typical example is with the uniform partition: $\lambda_i = \frac{2(i-1)}{n}$ and $w_i = \frac{2}{n}, 1 \le i \le n$ Another example is with the spectral partition (with $n = N$): $\lambda_i \in \Lambda$ and $w_i = \lambda_{i+1} - \lambda_i$, $1 \le i \le N$, where Λ is the set of eigenvalues of $\mathbf{L}_{\mathcal{G}}^{\text{sym}}$, and $\lambda_{N+1} = 2$.

The presence of the denominator term is the main hurdle to solving problem (III.6). If the denominator is removed, the original problem reduces to a bi-quadratic problem. The polynomials $a(\lambda)$ and $b(\lambda)$ can then be optimized alternatively with an iterative algorithm. This could lead to a solution that is biased as the approximation error in the modified problem is different to the original problem. We present an approach inspired by [12] to solve the minimization problem (III.6).

Firstly, we approximate the original problem (III.6) via

$$\min_{a(\lambda),b(\lambda)} \sum_{\lambda_i \in [0,\mu_p]} \frac{w_i}{|\hat{b}(\lambda_i)|^2} \left| g_1^P(\lambda_i) a(\lambda_i) - e_0(\lambda_i) b(\lambda_i) \right|^2 + \sum_{\lambda_i \in [\mu_s,2]} \frac{w_i}{|\hat{b}(\lambda_i)|^2} \left| g_0^P(\lambda_i) a(\lambda_i) + e_1(\lambda_i) b(\lambda_i) \right|^2 \tag{III.7}$$

which is a relaxation where pseudo denominators $\hat{b}(\lambda_i)$ are fixed during each iteration of the optimization. The values $|\hat{b}(\lambda_i)|$ will be updated at the beginning of the next iteration, which is not the same as $|b(\lambda_i)|$ in general. Our aim is to reach $|\hat{b}(\lambda_i)| \approx |b(\lambda_i)|$ via some iterative process.

Write $a(\lambda) = \sum_{k=0}^{K_a} p_k \lambda^k$ and $b(\lambda) = 1 + \sum_{k=1}^{K_b} q_k \lambda^k$, and set vectors $\mathbf{c}_p(\lambda) = [1\,\lambda\,\cdots\,\lambda^{K_a}]^T$, $\mathbf{c}_q(\lambda) = [\lambda\,\cdots\,\lambda^{K_b}]^T$, $\mathbf{p} = [p_0\,p_1\,\cdots\,p_{K_a}]^T$ and $\mathbf{q} = [q_1\,\cdots\,q_{K_b}]^T$. Then the minimization problem (III.7) can be reformulated as

$$\min_{\mathbf{p},\mathbf{q}} \sum_{\lambda_i \in [0,\mu_p]} \frac{w_i}{|\hat{b}(\lambda_i)|^2} \Big| g_1^P(\lambda_i) \mathbf{c}_p^T(\lambda_i) \mathbf{p} - e_0(\lambda_i) \mathbf{c}_q^T(\lambda_i) \mathbf{q} - e_0(\lambda_i) \Big|^2 + \sum_{\lambda_i \in [\mu_s,2]} \frac{w_i}{|\hat{b}(\lambda_i)|^2} \Big| g_0^P(\lambda_i) \mathbf{c}_p^T(\lambda_i) \mathbf{p} + e_1(\lambda_i) \mathbf{c}_q^T(\lambda_i) \mathbf{q} + e_1(\lambda_i) \Big|^2. \tag{III.8}$$

When $|\hat{b}(\lambda_i)| = |b(\lambda_i)|, 1 \le i \le n$, the minimization problems (III.8) and (III.6) are equivalent to each other. The closer $|\hat{b}(\lambda_i)|$ is to $|b(\lambda_i)|$, the closer the solution of (III.8) will be to the solution of (III.6). In view of this, we propose an iterative method where at each iteration, $|\hat{b}(\lambda_i)|$ is updated so that it is closer to $|b(\lambda_i)|$. Specifically, we update the pseudo denominators at the m-th iteration as follows: $|\hat{b}_m(\lambda_i)|^2 = |b_m(\lambda_i)|^2 + \rho(m)$, where $\rho(m)$, $m \ge 0$ is a sequence of decreasing positive numbers and $\rho(m) \to 0$ as $m \to \infty$. Since $\rho(m) \to 0$, the approximation gap between the problem (III.8) and the original one in (III.6) decreases with each subsequent iteration. At each iteration, the coefficients vectors $\mathbf{p}$ and $\mathbf{q}$ are obtained by solving the least squares problem (III.8). This iterative algorithm is summarized in Algorithm 1.

Remark: Once the filter response function are obtained, the transformation matrix corresponding to the actual filtering can be obtained by substituting λ with $\mathbf{L}_{\mathcal{G}}^{\text{sym}}$. Polynomial or rational filter functions can be implemented efficiently without the need for eigendecomposition - see [1], [12] for details. Note also that the designed filter functions are universal filters, and not limited to a specified graph (which has a specified eigenvalue spectrum), as the filters are defined for all possible eigenvalues, i.e. any graph.

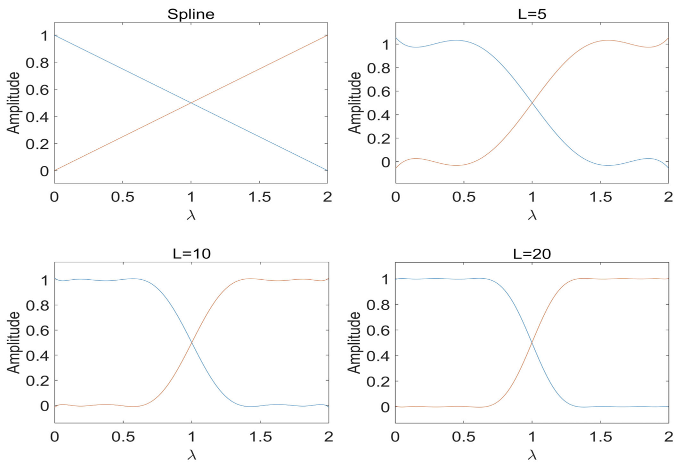

Fig. 1. Spectral response of the spline prototype and the lifted analysis filters with various lifting filters degree L.

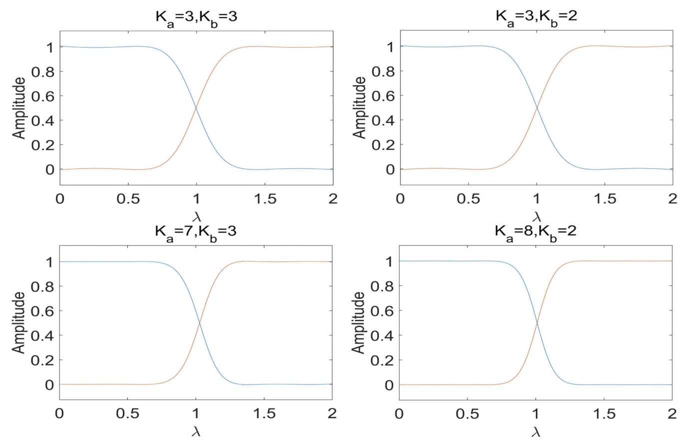

Fig. 2. Spectral response of analysis filters using rational lifting filters with various degrees (K_a, K_b).

Algorithm 1: Design of the Rational Lifting Filter.

Inputs: stop criterion ε, iteration limit M, orders K_a, K_b, weights w_i, and gap sequence $\boldsymbol{\rho} = (\rho(m))_{m \geq 0}$.
Initialization: Initialize $\mathbf{q}_0$ as a random vector with length K_b, set $m = 0$.
Iteration:
1) Given $\mathbf{q}_m$, compute
$|\hat{b}_m(\lambda_i)|^2 = |1 + \mathbf{c}_q^T(\lambda_i)\mathbf{q}_m|^2 + \rho(m), 1 \leq i \leq n.$
2) Solve the least squares problem (III.8) to get $\mathbf{p}$ and $\mathbf{q}$. Denote the solution by $\mathbf{p}_m, \mathbf{q}_{m+1}$.
3) If $\|\mathbf{q}_{m+1} - \mathbf{q}_m\|_\infty \leq \varepsilon$ or the number of iteration exceeds M, terminate the iteration and output $\mathbf{p}_m, \mathbf{q}_{m+1}$ and m. Otherwise, set $m = m + 1$ and return to Step **1)**.
Outputs: $\mathbf{p}_m, \mathbf{q}_{m+1}$ and m.

IV. Design Demonstrations

The prototype filters used are given by $\mathbf{H}_i^P = \mathbf{H}_{i,1}^{\text{spln}}$ and $\mathbf{G}_i^P = \mathbf{G}_{i,1}^{\text{spln}} = \mathbf{I}$, $i = 0, 1$. In this section, we demonstrate the design of both rational and polynomial lifting filters so that the lifted analysis filter bank $\{\mathbf{H}_0, \mathbf{H}_1\}$ achieves a good approximation to the ideal low-pass/high-pass filter bank.

We first consider polynomial lifting filters of degree L with coefficients determined by $\mathbf{r}^* = \mathbf{A}^{-1}\mathbf{b}$. Fig. 1 shows the spectral responses of the prototype analysis filter bank and the lifted analysis filter banks with different degrees $L = 5, 10, 20$. The edge parameters used in (III.1) are $\mu_p = 0.7$ and $\mu_s = 1.3$. The error measure (III.1) value is $\phi(r) = 5.7 \times 10^{-2}$ when no lifting is used. With lifting, the error measures are $\phi(r) = 1.2 \times 10^{-3}$, 6.3×10^{-5} and 2.1×10^{-6} when the lifting filters degrees are $L = 5, 10, 20$ respectively. This shows that the lifted analysis filter bank $\{\mathbf{H}_0, \mathbf{H}_1\}$ better approximates the ideal filter bank with a higher degree L. However, with a higher degree, the complexity is higher and there is reduced vertex localization.

We next consider rational lifting filters designed using Algorithm 1. The edge parameters used are $\mu_p = 0.7, \mu_s = 1.3$. The discrete frequencies and weights selected using the uniform partition: $\lambda_i = \frac{2(i-1)}{n}$ $(1 \leq i \leq n)$ and $w_i = \frac{2}{n}$ with $n = 200$. The gap sequence $\rho(m)$, $m \geq 0$ is given by $\rho(m + 1) = \rho(m)/m^{1.5}$ with $\rho(0) = 0.01$. Fig. 2 shows the spectral responses of the lifted analysis filters with $(K_a, K_b) = (3, 3), (3, 2), (7, 3)$ and $(8, 2)$, where K_a (resp. K_b) is the degree of the numerator polynomial (resp. denominator polynomial) of the rational function $r(\lambda)$. The error measures are $\phi(r) = 3.4 \times 10^{-5}$, 3.3×10^{-5}, 1.7×10^{-6} and 2.4×10^{-7} when the lifting filter degrees are $(K_a, K_b) = (3, 3), (3, 2), (7, 3)$ and $(8, 2)$, respectively. This shows that, compared to polynomial lifting filters, rational lifting filters achieves a better approximation to the ideal filter bank, even with small degrees.

TABLE I
Denoising Performance of the Lifting NSGFB

σ	1/32	1/16	1/8	1/2
Noisy	27.88	21.88	15.85	3.79
Spline	26.36	19.82	14.58	7.94
Polynomial $L = 5$	27.30	23.78	19.77	8.64
Rational $K_a = 3, K_b = 2$	27.95	24.19	19.83	8.51
NSGFB-B1 [9]	27.42	21.43	16.43	6.44
NSGFB-L1 [9]	26.44	20.19	15.78	8.66

Application in denoising: The graph is a random geometric graph with $N = 4096$ vertices, generated using the GSPBOX [13]. The test signal is obtained by combining the first 20% eigenvectors of the normalized Laplacian matrix of the graph. The random noise is generated from a uniform distribution over the interval $[-\sigma, \sigma]$. Table I lists the signal-to-noise ratios (SNRs) of the denoised and noisy signals under different noise levels σ. It is observed that the lifting NSGFBs achieve better denoising performance than that of the spline and filter banks from [9]. This can be attributed to the better spectral selectivity of the lifting NSGFB.

V. Conclusion

This letter proposed a simple method to design two-channel nonsubsampled graph filter banks with perfect reconstruction and good frequency characteristics. The method is based on the lifting scheme and it was shown the design process amounts to the design of a lifting filter. Algorithms to design polynomial and rational lifting filters were presented. Experimental results demonstrate the effectiveness of the method in tailoring the spectral response of the graph filters. Application of the filter banks to graph signal denoising showed the advantage of having better frequency selectivity.

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