Estimates near the boundary for critical SQG

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ABSTRACT. We obtain estimates near the boundary for the critical dissipative SQG equation in bounded domains, with the square root of the Dirichlet Laplacian dissipation. We prove that global regularity up to the boundary holds if and only if a certain quantitative vanishing of the scalar at the boundary is maintained.

1. Introduction

The Surface Quasigeostrophic equation (SQG) of geophysical origin ([**16**]) was proposed as a two dimensional model for the study of inviscid incompressible formation of singularities ([**3**], [**7**]). The equation has been studied extensively. Blow up from smooth initial data is still an open problem, although the original blow-up scenario of [**7**] has been ruled out analytically ([**13**]) and numerically ([**6**]). The addition of fractional dissipation produces globally regular solutions if the power of the Laplacian is larger or equal than one half. When the linear dissipative operator is precisely the square root of the Laplacian, the equation is commonly referred to as the "critical dissipative SQG", or "critical SQG". The global regularity of solutions for critical SQG in the whole space or on the torus was obtained independently in [**1**] and [**18**] by very different methods. Several subsequent proofs were obtained (see [**11**] and references therein).

The critical SQG equation in bounded domains is given by

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda_D \theta = 0 \tag{1}$$

with

$$u = \nabla^{\perp} \Lambda_D^{-1} \theta. \tag{2}$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded open set with smooth boundary, Λ_D is the square root of the Laplacian with vanishing Dirichlet boundary conditions, and $\nabla^{\perp} = J\nabla$ with J an invertible antisymmetric matrix. The local existence and uniqueness of solutions of (1) given in [**5**] is

PROPOSITION 1. *Let $d = 2$, and let $\theta_0 \in H_0^1(\Omega) \cap H^2(\Omega) = \mathcal{D}(\Lambda_D^2)$. There exists $T > 0$ and a unique solution of (1) with initial datum θ_0 satisfying*

$$\theta \in L^{\infty}(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2\left(0, T; \mathcal{D}\left(\Lambda_D^{2.5}\right)\right). \tag{3}$$

Local existence of solutions of the same type holds also for supercritical SQG in bounded domains ([**9**]). Weak solutions exist globally ([**4**]), even without dissipation ([**8**]), but are not known to be unique. However, if the initial data are interior Lipschitz continuous, then weak solutions are globally interior Lipschitz continuous. A priori bounds for smooth solutions were given in [**5**] and a construction was given in [**17**]. Let

$$d(x) = dist(x, \partial\Omega) \tag{4}$$

denote the distance from x to the boundary of Ω.

The main result of [**17**] is

THEOREM 1. *Let $\theta_0 \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ and let $0 < T \le \infty$. There exists $\theta(x,t)$, a solution of (1) on the time interval $[0, T)$, with initial data $\theta(x, 0) = \theta_0(x)$ and a constant Γ_1 depending only on Ω such that*

$$\|\theta(\cdot, t)\|_{L^{\infty}(\Omega)} \le \|\theta_0\|_{L^{\infty}(\Omega)}, \tag{5}$$

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and

$$\sup_{0\le t<T}\sup_{x\in\Omega} d(x)|\nabla_x\theta(x,t)| \le \Gamma_1\left[\sup_{x\in\Omega} d(x)|\nabla_x\theta_0(x)| + \left(1+\|\theta_0\|_{L^\infty(\Omega)}\right)^4\right] := M \tag{6}$$

hold.

This result holds in any dimension d. Interior Lipschitz regularity is obtained using nonlinear lower bounds for the square root of the Dirichlet Laplacian ([**5**]) and commutator estimates. The main obstacle to obtain regularity up to the boundary is the absence of translation invariance, which is most sharply felt near the boundary. The nonlinear lower bounds for the square root of the Dirichlet Laplacian ([**4**], [**5**]) are similar to those available in the whole space ([**10**]), but have a cut-off due to the boundary. The lack of translation invariance is manifested in the commutator estimates, where the commutator between the square root of the Laplacian and differentiation is of the order $d(x)^{-2}$ pointwise.

In this paper we investigate the behavior of solutions near the boundary. The local solutions obtained in Proposition 1 belong to $C^\alpha(\Omega)$ up to the boundary, for any $0<\alpha<1$, but this fact follows from embedding of $H^2(\Omega)\subset C^\alpha(\Omega)$ in $d=2$ and the control of the $H^2(\Omega)$ norm is only for short time. An interesting recent work [**19**] in the spirit of [**1**] shows that a solution-dependent $C^\alpha(\Omega)$ regularity holds as long as the solution is sufficiently smooth. Unfortunately, as we mentioned earlier, smooth solutions can be guaranteed to exist only for a short time.

The currently available quantitative global in time information for solutions with smooth initial data is comprised of following three components:

I) Energy bounds, which imply that $\theta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;\mathcal{D}(\Lambda_D^{\frac12}))$,
II) A maximum principle, which implies $\theta\in L^\infty(0,T;L^\infty(\Omega))$, and,
III) For solutions constructed by a judicious method mentioned above, the interior Lipschitz bound (6).

No uniqueness is guaranteed. The velocity is given by rotated Riesz transforms. It is known ([**2**]) that if θ vanishes at the boundary and belongs to C^α then its Dirichlet Riesz transforms are in $C^\alpha(\Omega)$. If θ belongs to C^α and vanishes at the boundary, then the stream function $\psi=\Lambda_D^{-1}\theta$ belongs to $C^{1,\alpha}$ and vanishes at the boundary, and therefore so do its tangential derivatives. Thus, the normal component of the velocity vanishes at the boundary, but no rate is available if θ belongs to C^α.

In this work we show that the problem of controlling the Hölder continuity of the solution up to the boundary depends solely on quantitative bounds on the the vanishing of θ at the boundary. We prove two results detailing this fact. We consider

$$b_1(x,t)=\frac{\theta(x,t)}{w_1(x)} \tag{7}$$

where w_1 is the normalized positive first eigenfunction of the Dirichlet Laplacian, which is known to be smooth and to vanish as $d(x)$ at the boundary. In Theorem 3 we show that for solutions constructed from smooth initial data obeying the a priori information detailed above (I, II, III), and for any $p>d$, there exists a time T_0 and a constant B, depending only on $\|\theta_0\|_{L^\infty}$, M (of 6) and the initial norm $\|b_1(0)\|_{L^p(\Omega)}$, such that

$$\sup_{0\le t\le T_0}\|b_1(t)\|_{L^p(\Omega)}\le B \tag{8}$$

holds. This is a local existence theorem, local because the control of $\|b_1(t)\|_{L^p(\Omega)}$ is maintained for finite time, although the interior Lipschitz bound and the L^∞ bound are global.

Our second main result, Theorem 4, shows that if the bound (8) holds for some interval of time, then the solutions constructed in ([**17**]) are in $C^\alpha(\Omega)$ on that interval of time. The Hölder exponent α is explicit, it is given by $\alpha<1-\frac{d}{p}$ where p is the exponent in (8). Thus, the condition (8), which can be maintained for short time, is sufficient for global Hölder regularity up to the boundary. This condition also implies a quantitative vanishing of the normal component of velocity at the boundary, $u\cdot N=O(d(x)^\alpha)$ with rate depending on M and B.

The boundedness of $\|b_1\|_{L^p(\Omega)}$ is a weaker condition than $\theta \in W_0^{1,p_1}(\Omega)$, $p_1 > p > d$. Thus our condition is necessary for regularity. Local well-posedness in $W_0^{1,p_1}(\Omega)$ is not known. The previously known local existence theory was established in the domain of the Laplacian, which is a strictly smaller space. If the solution is in $W_0^{1,p}(\Omega)$, $p > d$, then by embedding results, it is in C^α up to the boundary for any $0 < \alpha < 1 - \frac{d}{p}$.

In order to prove our results, we obtain key quantitative bounds using B. We show first that if B is finite, then the velocity is bounded (Proposition 6). By contrast, if only the available a priori information (I, II, III) is used, then the velocity logarithmically diverges with the distance to the boundary (Proposition 5).

Secondly, we obtain bounds for the finite difference quotients of velocity which diverge at the boundary with a sublinear power of the distance, $d(x)^{-\frac{d}{p}}$, (Proposition 8), as opposed to $d(x)^{-1}$ in the case of the a priori global information, as was shown in [**5**]. A quantitative rate of vanishing of the normal component of velocity is proved in Proposition 9.

Thirdly, we obtain bounds for the commutator between finite differences and Λ_D which diverge subquadratically near the boundary, $d(x)^{-1-\frac{d}{p}}$, (Proposition 10) as opposed to quadratically $d(x)^{-2}$, which is the case in which only the global a priori information (I, II, III) is used.

These three elements, together with the strong boundary repulsive damping effect of the square root of the Laplacian, form the basis of the proof of persistence of C^α regularity, with $\alpha < 1 - \frac{d}{p}$.

In the whole space, any C^α, $\alpha > 0$ regularity can be upgraded to Lipschitz regularity (and further to C^∞ ([**12**])). In bounded domains, while any interior C^α regularity can be upgraded to interior Lipschitz regularity ([**5**]), in general, the problem of global Lipschitz regularity up to the boundary is open. The passage to Lipschitz bounds up to the boundary is not achievable with our tools, even conditioned on knowledge of linear vanishing of θ (i.e. even assuming a time-independent bound for b_1 in L^∞). This is due to the fact that the commutator between derivatives and Λ_D still costs $d(x)^{-1}$ near the boundary.

The paper is organized as follows. After recalling basic facts in Section 2 we prove in Section 3 a remarkable generalization of the Córdoba-Córdoba inequality ([**14**]) which was obtained in bounded domains in [**4**]. This new pointwise inequality involves weights w,

$$\Phi'(b)\Lambda_D(wb) - \Lambda_D(w\Phi(b)) \geq (\Lambda_D(w))\left(b\Phi'(b) - \Phi(b)\right) \tag{9}$$

(see (32, 33)) and is valid for any convex function Φ of one variable which satisfies $\Phi(0) = 0$, any smooth function b and any smooth positive function w which vanishes at $\partial\Omega$. The inequality implies a comparison principle for solutions of drift diffusion equations with Dirichlet square root Laplacian and may have independent interest. We use it with $b = \frac{\theta}{w_1}$ and prove that B of (8) persists to be finite if the drift is the sum of a regular function in L^∞ whose normal component vanishes at the boundary and a small L^∞ function. In Section 4 we derive bounds for the Dirichlet Riesz transforms and in Section 5 we obtain bounds for finite differences of the Dirichlet Riesz transforms. Section 6 is devoted to the improved bounds on the commutator between local finite differences and Λ_D, and Section 7 contains the bound for the Hölder seminorms near the boundary.

2. Preliminaries

We consider $\Omega \subset \mathbb{R}^d$ a bounded open set with smooth boundary. The $L^2(\Omega)$ - normalized eigenfunctions of $-\Delta$ are denoted w_j, and its eigenvalues counted with their multiplicities are denoted λ_j:

$$-\Delta w_j = \lambda_j w_j. \tag{10}$$

It is well known that $0 < \lambda_1 \leq ... \leq \lambda_j \to \infty$ and that $-\Delta$ is a positive selfadjoint operator in $L^2(\Omega)$ with domain $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. The ground state w_1 is positive and

$$c_0 d(x) \leq w_1(x) \leq C_0 d(x) \tag{11}$$

holds for all $x \in \Omega$, where $c_0,\ C_0$ are positive constants depending on Ω. Functional calculus can be defined using the eigenfunction expansion. In particular

$$(-\Delta)^{\beta} f = \sum_{j=1}^{\infty} \lambda_j^{\beta} f_j w_j \tag{12}$$

with

$$f_j = \int_{\Omega} f(y) w_j(y) dy$$

for $f \in \mathcal{D}\left((-\Delta)^{\beta}\right) = \{f \mid \ (\lambda_j^{\beta} f_j) \in \ell^2(\mathbb{N})\}$. We denote by

$$\Lambda_D^s = (-\Delta)^{\frac{s}{2}}\,, \tag{13}$$

the fractional powers of the Dirichlet Laplacian, with $0 \le s \le 2$ and with $\|f\|_{s,D}$ the norm in $\mathcal{D}\left(\Lambda_D^s\right)$:

$$\|f\|_{s,D}^2 = \sum_{j=1}^{\infty} \lambda_j^s f_j^2. \tag{14}$$

It is well-known that

$$\mathcal{D}\left(\Lambda_D\right) = H_0^1(\Omega).$$

Note that in view of the identity

$$\lambda^{\frac{s}{2}} = c_s \int_0^{\infty} (1 - e^{-t\lambda}) t^{-1-\frac{s}{2}} dt, \tag{15}$$

with

$$1 = c_s \int_0^{\infty} (1 - e^{-\tau}) \tau^{-1-\frac{s}{2}} d\tau,$$

valid for $0 \le s < 2$, we have the representation

$$\left(\left(\Lambda_D\right)^s f\right)(x) = c_s \int_0^{\infty} \left[f(x) - e^{t\Delta} f(x)\right] t^{-1-\frac{s}{2}} dt \tag{16}$$

for $f \in \mathcal{D}\left((-\Lambda_D)^s\right)$. We use precise upper and lower bounds for the kernel $H_D(t,x,y)$ of the heat operator,

$$(e^{t\Delta} f)(x) = \int_{\Omega} H_D(t,x,y) f(y) dy. \tag{17}$$

These are as follows ([**15**],[**20**],[**21**]). There exists a time $T > 0$ depending on the domain Ω and constants c, C, k, K, depending on T and Ω such that

$$\begin{array}{l} c \min\left(\frac{w_1(x)}{|x-y|}, 1\right) \min\left(\frac{w_1(y)}{|x-y|}, 1\right) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{kt}} \le \\ H_D(t,x,y) \le C \min\left(\frac{w_1(x)}{|x-y|}, 1\right) \min\left(\frac{w_1(y)}{|x-y|}, 1\right) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}} \end{array} \tag{18}$$

holds for all $0 \le t \le T$. Moreover

$$\frac{|\nabla_x H_D(t,x,y)|}{H_D(t,x,y)} \le C \left\{ \begin{array}{ll} \frac{1}{d(x)}, & \text{if } \sqrt{t} \ge d(x), \\ \frac{1}{\sqrt{t}}\left(1 + \frac{|x-y|}{\sqrt{t}}\right), & \text{if } \sqrt{t} \le d(x) \end{array} \right. \tag{19}$$

holds for all $0 \le t \le T$. Note that

$$H_D(t,x,y) = \sum_{j=1}^{\infty} e^{-t\lambda_j} w_j(x) w_j(y), \tag{20}$$

and therefore long time $t \geq T$ estimates are rather straightforward. The gradient bounds (19) result by symmetry in

$$\frac{|\nabla_y H_D(t,x,y)|}{H_D(t,x,y)} \leq C \begin{cases} \frac{1}{d(y)}, & \text{if } \sqrt{t} \geq d(y), \\ \frac{1}{\sqrt{t}}\left(1 + \frac{|x-y|}{\sqrt{t}}\right), & \text{if } \sqrt{t} \leq d(y). \end{cases} \tag{21}$$

We use as well the bounds ([**5**])

$$\nabla_x \nabla_x H_D(x,y,t) \leq C t^{-1-\frac{d}{2}} e^{-\frac{|x-y|^2}{\bar{K}t}} \tag{22}$$

valid for $t \leq cd(x)^2$ and $0 < t \leq T$, and

$$\nabla_x \nabla_x H_D(x,y,t) \leq C d(x)^{-2} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{\bar{K}t}} \tag{23}$$

for $t \geq cd(x)^2$, which follow from the upper bounds (18), (19). Important additional bounds we use are

$$\int_\Omega |(\nabla_x + \nabla_y) H_D(x,y,t)|\, dy \leq C t^{-\frac{1}{2}} e^{-\frac{d(x)^2}{\bar{K}t}}, \tag{24}$$

with pointwise version

$$|(\nabla_x + \nabla_y) H_D(x,y,t)| \leq c t^{-\frac{d+1}{2}} e^{-\frac{d(x)^2}{\bar{K}t}}, \tag{25}$$

and

$$\int_\Omega |\nabla_x(\nabla_x + \nabla_y) H_D(x,y,t)|\, dy \leq C t^{-1} e^{-\frac{d(x)^2}{\bar{K}t}}, \tag{26}$$

with pointwise version

$$|\nabla_x(\nabla_x + \nabla_y) H_D(x,y,t)| \leq C t^{-\frac{d+2}{2}} e^{-\frac{d(x)^2}{\bar{K}t}}. \tag{27}$$

valid for $t \leq cd(x)^2$ and $0 < t \leq T$. These bounds reflect the fact that translation invariance is remembered in the solution of the heat equation with Dirichlet boundary data for short time, away from the boundary. They were proved in [**5**], [**8**].

The following elementary lemma is used in several instances:

LEMMA 1. *Let $\rho > 0$, $p > 0$. Then*

$$\int_0^\infty t^{-1-\frac{m}{2}} \left(\frac{p}{\sqrt{t}}\right)^j e^{-\frac{p^2}{Kt}} dt \leq C_{K,m,j} p^{-m} \tag{28}$$

if $m \geq 0$, $j \geq 0$, $m + j > 0$, and

$$\int_0^{\rho^2} t^{-1} e^{-\frac{p^2}{Kt}} dt = \int_{\frac{p^2}{K\rho^2}}^\infty x^{-1} e^{-x} dx \tag{29}$$

if $m = 0$ and $j = 0$, with constants $C_{K,m,j}$ independent of ρ and p. Note that when $m + j > 0$, $\rho = \infty$ is allowed. Note also that the right-hand side of (29) is exponentially small if $\rho \leq \epsilon p$.

We recall from [**4**] that the Córdoba-Córdoba inequality ([**14**]) holds in bounded domains. In fact, more is true: there is a lower bound that provides a strong boundary repulsive term:

PROPOSITION 2. *Let Ω be a bounded domain with smooth boundary. Let $0 \leq s < 2$. There exists a constant $c > 0$ depending only on the domain Ω and on s, such that, for any Φ, a C^2 convex function satisfying $\Phi(0) = 0$, and any $f \in C_0^\infty(\Omega)$, the inequality*

$$\Phi'(f)\Lambda_D^s f - \Lambda_D^s(\Phi(f)) \geq \frac{c}{d(x)^s}\left(f\Phi'(f) - \Phi(f)\right) \tag{30}$$

holds pointwise in Ω.

We specialize from now on to $s = 1$. We use in particular the result above in the form ([**4**])

$$D(f)(x) = \left(f\Lambda_D f - \frac{1}{2}\Lambda_D \left(f^2\right) \right)(x) \ge \gamma_1 \frac{f^2(x)}{d(x)} \tag{31}$$

with $\gamma_1 > 0$ depending only on Ω.

3. Weighted estimates

Let $w(x)$ be a function which is positive in Ω and belongs to $\mathcal{D}(\Lambda_D)$, for instance $w(x) = w_1(x)$.

LEMMA 2. *Let Φ be a convex function of one variable with $\Phi(0) = 0$. Let b be a continuous function in the open set Ω, with $wb \in \mathcal{D}(\Lambda_D)$. Then*

$$\Phi'(b(x))\Lambda_D(wb)(x) - \Lambda_D(w\Phi(b))(x) = (\Lambda_D w)(x)\left(b(x)\Phi'(b(x)) - \Phi(b(x))\right) + D_\Phi(x) \tag{32}$$

with

$$D_\Phi(x) = c\int_0^\infty t^{-\frac{3}{2}} \int_\Omega w(y)H_D(x,y,t)\left[\Phi(b(y)) - \Phi(b(x)) - \Phi'(b(x))(b(y)-b(x))\right] dydt \tag{33}$$

Proof. Let $\phi(x) = \Phi(b(x))$ and $\widetilde{\phi}(x) = \Phi'(b(x))$. We have

$$\begin{aligned}
&(\widetilde{\phi}\Lambda_D(wb) - \Lambda_D(w\phi))(x) = c\textstyle\int_0^\infty t^{-\frac{3}{2}}\left[\widetilde{\phi}(x)w(x)b(x) - \int_\Omega \widetilde{\phi}(x)w(y)H_D(x,y,t)b(y)dy\right]dt\\
&-c\textstyle\int_0^\infty t^{-\frac{3}{2}}\left[w(x)\phi(x) - \int_\Omega w(y)H_D(x,y,t)\phi(y)dy\right]dt\\
&= (b(x)\widetilde{\phi}(x) - \phi(x))c\textstyle\int_0^\infty t^{-\frac{3}{2}}\left[w(x) - \int_\Omega w(y)H_D(x,y,t)dy\right]dt + D_\Phi(x)\\
&= (b(x)\widetilde{\phi}(x) - \phi(x))\Lambda_D(w)(x) + D_\Phi(x)
\end{aligned}$$

REMARK 1. *We note that $D_\Phi \ge 0$ for convex functions Φ because the integrand is nonnegative, in view of $H_D \ge 0$. Convexity of Φ is not needed for the statement of the lemma, but the lemma is used only when D_Φ is nonnegative.*

Let us consider now an evolution equation

$$\partial_t\theta + v\cdot\nabla\theta + \Lambda_D\theta = 0 \tag{34}$$

with $v = v(x,t)$ a divergence-free vector field tangent to the boundary of Ω. Let us consider a smooth enough weight $w(x,t) > 0$ which vanishes at the boundary of Ω and compute the evolution of $\Phi(b(x,t))$ where Φ is a nonnegative convex function of one variable, with $\Phi(0) = 0$, and where

$$b(x,t) = \frac{\theta(x,t)}{w(x,t)}. \tag{35}$$

In view of (32), we obtain the remarkable equation

$$(\partial_t + v\cdot\nabla + \Lambda_D)(w\Phi(b)) + ((\partial_t + v\cdot\nabla + \Lambda_D)w)\left(b\Phi'(b) - \Phi(b)\right) + D_\Phi = 0, \tag{36}$$

where D_Φ is defined above in (33). Denoting

$$L_v = \partial_t + v\cdot\nabla + \Lambda_D \tag{37}$$

we have thus

$$L_v(w\Phi(b)) + (L_v(w))(b\Phi'(b) - \Phi(b)) + D_\Phi = 0 \tag{38}$$

for $w > 0$ and Φ convex with $\Phi(0) = 0$. There are several important consequences of this identity. In view of the fact that

$$\int_\Omega \Lambda_D(w\Phi(b))dx = \int_\Omega w\Phi(b)\Lambda_D(1)dx, \tag{39}$$

where $\Lambda_D(1)$ is defined by duality and $w\Phi(b) \in \mathcal{D}(\Lambda_D)$, and the lower bound ([**5**])

$$\Lambda_D(1)(x) \ge c_0\frac{1}{w_1(x)}, \tag{40}$$

we have that

$$\int_\Omega \Lambda_D(w\Phi(b))dx \geq c_0 \int_\Omega \left(\frac{w(x)}{w_1(x)}\right) \Phi(b(x))dx. \tag{41}$$

Therefore, from (38) we obtain

$$\frac{d}{dt}\int_\Omega w\Phi(b)dx + c_0\int_\Omega \left(\frac{w(x)}{w_1(x)}\right)\Phi(b(x))dx + \int_\Omega (L_v(w))(b\Phi'(b) - \Phi(b))dx + \int_\Omega D_\Phi(x,t)dx \leq 0. \tag{42}$$

Let us take now Φ to be (a smooth convex approximation of) the function

$$\Phi_B(b) = (b - B)_+ \tag{43}$$

where B is a large fixed number. Notice that in this case

$$b\Phi_B'(b) - \Phi_B(b) = BH(b - B), \tag{44}$$

where $H(x)$ is the Heaviside function. Because $b\Phi_B'(b) - \Phi_B(b) \geq 0$, if $L_v(w) \geq 0$, then

$$\frac{d}{dt}\int_\Omega w(x,t)\Phi_B(b(x,t))dx \leq 0. \tag{45}$$

It follows that, If $\Phi_B(b(x,0)) = 0$, then $\Phi_B(b(x,t)) = 0$ for $t \geq 0$. Applying this reasoning to the functions b defined above in (35) as well as to $b_- = \frac{-\theta}{w_1}$, we obtain

$$|\theta(x,t)| \leq Bw(x,t). \tag{46}$$

REMARK 2. *This shows that if $L_v(w) \geq 0$ and $|\theta_0(x)| \leq Bw(x,0)$, then (46) holds.*

THEOREM 2. *Let θ solve (34) where v is a continuous, divergence-free field, tangent to the boundary. Assume that there exists a constant $\gamma(t)$ such that*

$$v \cdot \nabla w_1 + \gamma(t)w_1 \geq 0 \tag{47}$$

holds for $x \in \Omega$ and $t \geq 0$. Assume that the initial data θ_0 obeys

$$|\theta_0(x)| \leq Bw_1(x) \tag{48}$$

for all $x \in \Omega$. Then

$$|\theta(x,t)| \leq Bw_1(x)e^{-t\sqrt{\lambda_1} + \int_0^t \gamma(s)ds} \tag{49}$$

holds for all $x \in \Omega$ and all $t \geq 0$.

Proof. Consider

$$w(x,t) = e^{-t\sqrt{\lambda_1} + \int_0^t \gamma(s)ds}w_1. \tag{50}$$

Note that the assumption (47) implies that

$$L_v(w(x,t)) \geq 0 \tag{51}$$

Then we use (46) and conclude the proof.

If v is bounded and if its normal component vanishes of first order at the boundary then the condition (47) is satisfied.

PROPOSITION 3. *Condition (47) is satisfied if v is bounded,*

$$\|v(t)\|_{L^\infty} \leq V(t), \tag{52}$$

and has a normal component which vanishes to first order near the boundary of Ω,

$$|v(x,t) \cdot N(x)| \leq V(t)d(x), \tag{53}$$

where $N(x)$ is a continuous unit vector defined near the boundary $\partial\Omega$ and extending the normal at $\partial\Omega$.

Indeed, for any smooth vector field $T(x)$ defined near the boundary and tangent to the boundary, we have by the smoothness of w_1 and its equivalence to the distance to the boundary,

$$T(x) \cdot \nabla w_1(x) \le C w_1(x) \tag{54}$$

near the boundary Ω. This inequality is true because $T \cdot \nabla w_1$ is continuously differentiable with bounded derivatives in $\overline{\Omega}$ (hence Lipschitz continuous) and vanishes at the boundary, so it is bounded by a multiple of $d(x)$, and hence by a multiple of $w_1(x)$. Then we decompose $v = (v \cdot T)T + (v \cdot N)N = v_T + v_N$ with T smooth near the boundary and tangent to the boundary, and use the fact that

$$|v_N(x)| \le C w_1(x), \tag{55}$$

near the boundary, which follows by the assumption (53). The fact that $|v \cdot \nabla w_1| \le C w_1$ away from the boundary follows from the boundedness of v. This concludes the proof of Proposition 3.

REMARK 3. *Theorem 2 can be proved also using*

$$\Phi(b) = b^{2m}. \tag{56}$$

We note that in this case

$$b\Phi'(b) - \Phi(b) = (2m-1)\Phi(b). \tag{57}$$

We take $w = w_1$ and use the fact that

$$L_v(w_1) = v \cdot \nabla w_1 + \sqrt{\lambda_1} w_1, \tag{58}$$

and returning to (42) we obtain

$$\frac{d}{dt}\int_\Omega w_1(x)\Phi(b(x,t))dx \le (2m-1)(\gamma(t) - \sqrt{\lambda_1})\int_\Omega w_1(x)\Phi(b(x,t))dx \tag{59}$$

where

$$\gamma(t) = \sup_{x\in\Omega}\left(-\frac{v(x,t)\cdot\nabla w_1(x)}{w_1(x)}\right) \tag{60}$$

Integrating in time, taking $2m$ roots and then the limit $m \to \infty$, we arrive at

$$\|b(t)\|_{L^\infty} \le \|b_0\|_{L^\infty} e^{-t\sqrt{\lambda_1} + \int_0^t \gamma(s)ds}. \tag{61}$$

Note that if (47) holds then (61) is precisely (49).

We consider now the case of fixed m.

PROPOSITION 4. *Let $m \ge 1$ be an integer, let v be a bounded divergence-free function which can be decomposed*

$$v = v_r + v_s \tag{62}$$

with $v_r(x,t)$ obeying $\gamma_r \in L^1[0,T]$, where $\gamma_r(t)$ is defined as in (60) by

$$\sup_{x\in\Omega}\left(-\frac{v_r(x,t)\cdot\nabla w_1(x)}{w_1(x)}\right) = \gamma_r(t) \tag{63}$$

and with

$$\|v_s(t)\|_{L^\infty} \le \frac{c_0}{(2m-1)\|\nabla w_1\|_{L^\infty}} \tag{64}$$

where c_0 is the constant from (42). Then

$$\int_\Omega w_1(x)\left(\frac{\theta(x,t)}{w_1(x)}\right)^{2m} dx \le e^{(2m-1)\left(-t\sqrt{\lambda_1} + \int_0^t \gamma_r(s)ds\right)} \int_\Omega w_1(x)\left(\frac{\theta_0(x)}{w_1(x)}\right)^{2m} dx \tag{65}$$

holds for $t \in [0,T]$.

REMARK 4. *Note that the right hand side of (64) depends only on Ω and m.*

Proof. The proof follows along the same lines as above. We take Φ as in (56), $w = w_1$, and using the decomposition we have that

$$L_v(w_1) = v_s \cdot \nabla w_1 + \left(\sqrt{\lambda_1} + \left(\frac{v_r \cdot \nabla w_1}{w_1}\right)\right) w_1. \tag{66}$$

Consequently, from (63) and (64) we have

$$(2m-1)L_v(w_1) \geq -c_0 + (2m-1)(\sqrt{\lambda_1} - \gamma_r(t))w_1(x). \tag{67}$$

We use this inequality and (57) in (38), integrate in time, and deduce (65).

We record here a lemma relating weighted and unweighted norms of b:

LEMMA 3. *Let $m > p \geq 1$. Then, there exists a constant $C_{m,p}$ depending only on Ω, m and p such that*

$$\|b\|_{L^p(\Omega)} \leq C_{m,p} \left(\int_\Omega w_1(x) b^{2m}(x) dx\right)^{\frac{1}{2m}} \tag{68}$$

holds for any b. Conversely, let $p \geq 2m-1 \geq 1$ and let $b_1 = \frac{\theta}{w_1}$. Then

$$\left(\int_\Omega w_1(x) b_1^{2m}(x) dx\right)^{\frac{1}{2m}} \leq \|\theta\|_{L^\infty(\Omega)}^{\frac{1}{2m}} \|b_1\|_{L^p(\Omega)}^{\frac{2m-1}{m}} |\Omega|^{\frac{p+1-2m}{2mp}} \tag{69}$$

Proof. The first inequality uses just the Hölder inequality for the functions $w_1(x)^{\frac{p}{2m}}|b(x)|^p$ and $w_1(x)^{-\frac{p}{2m}}$, with exponents $\frac{2m}{p}$, $\frac{2m}{2m-p}$, and

$$A_{m,p} = \int_\Omega w_1(x)^{-\frac{p}{2m-p}} dx < \infty \tag{70}$$

which holds because $\frac{p}{2m-p} < 1$. Then $C_{m,p} = A_{m,p}^{\frac{2m-p}{2mp}}$. The second inequality is straighforward.

4. Bounds for Riesz transforms

We consider u given in (2),

$$u = \nabla^\perp \Lambda_D^{-1} \theta.$$

where we recall that $\nabla^\perp = J\nabla$ with J an invertible antisymmetric matrix. We are interested in estimates of u in terms of θ.

PROPOSITION 5. *Let u be given by (2) and let θ be bounded and interior Lipschitz, i.e., obeying*

$$d(y)|\nabla\theta(y)| \leq M. \tag{71}$$

Then, there exist constants C depending only on the domain Ω such that

$$|u(x)| \leq CM + C\|\theta\|_{L^\infty}\left(1 + \log\left(\frac{C}{d(x)}\right)\right). \tag{72}$$

As a consequence, there exist constants $\gamma > 0$ and C, depending only on the domain Ω, M and $\|\theta\|_{L^\infty}$ such that

$$\int_\Omega e^{\gamma|u(x)|} dx \leq C. \tag{73}$$

REMARK 5. *The bound (72) does not use any information about vanishing of θ at the boundary, but it uses (71) which follows in our case from a priori bounds (6). The bound is in particular true for $\theta = 1$, where we know that the Riesz transform is in general only BMO and is not bounded all the way to the boundary.*

Proof. In view of the representation

$$\Lambda_D^{-1} = c\int_0^\infty t^{-\frac{1}{2}} e^{t\Delta}dt \tag{74}$$

we have that

$$u(x) = c\int_0^\infty t^{-\frac{1}{2}}dt\int_\Omega \nabla_x^\perp H_D(x,y,t)\theta(y)dy. \tag{75}$$

We split

$$u = u^{in} + u^{out} \tag{76}$$

with

$$u^{in}(x) = c\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega \nabla_x^\perp H_D(x,y,t)\theta(y)dy \tag{77}$$

with $\rho = \rho(x)$ a length scale smaller than the distance from x to the boundary of Ω:

$$\rho \le \epsilon d(x). \tag{78}$$

We split further

$$\begin{aligned} u^{in}(x) &= \int_0^{\rho^2} t^{-\frac{1}{2}}\int_\Omega (\nabla_x^\perp + \nabla_y^\perp)H_D(x,y,t)\theta(y)dydt + v(x)\\ &= u_1^{in}(x) + v(x)\end{aligned} \tag{79}$$

and then

$$\begin{aligned} u_1^{in}(x) &= \int_0^{\rho^2} t^{-\frac{1}{2}}\int_\Omega (\nabla_x^\perp + \nabla_y^\perp)H_D(x,y,t)(\phi(y) + (1-\phi(y)))\theta(y)dydt\\ &= v_1 + v_2\end{aligned} \tag{80}$$

where ϕ is a standard cutoff compactly supported in a ball of radius $\ell = \frac{\eta d(x)}{4}$ around x and identically one in the ball of radius $\frac{\ell}{4}$. Above $\epsilon > 0$ and $\eta > 0$ are small numbers at our disposal.

We use (25) to bound

$$t^{-\frac{1}{2}}\left|(\nabla_x^\perp + \nabla_y^\perp)H_D(x,y,t)\right| \le C(d(x))^{-(d+2)}$$

and thus

$$|v_1(x)| = \left|\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega (\nabla_x^\perp + \nabla_y^\perp)H_D(x,y,t)\phi(y)\theta(y)dy\right| \le C\epsilon^2\eta^d\|\theta\|_{L^\infty}. \tag{81}$$

Here we estimate the volume of the support of ϕ by $C\eta^d d(x)^d$. For v_2 we use the bounds (19) and (21) and the fact that $|x-y| \ge \eta d(x)/8$ on the support of $1-\phi$ to obtain

$$|v_2(x)| \le C\frac{\epsilon^2}{\eta^2}\|\theta\|_{L^\infty}. \tag{82}$$

Here we used the fact that $t^{-\frac{d+2}{2}}e^{-\frac{|x-y|^2}{Kt}} \le C|x-y|^{-(d+2)}$, and

$$\rho^2\int_{|x-y|\ge\eta d(x)/8} |x-y|^{-(d+2)}dy \le C\frac{\epsilon^2}{\eta^2}.$$

Now we write

$$v(x) = -c\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega \nabla_y^\perp H_D(x,y,t)(\phi(y) + 1 - \phi(y))\theta(y)dy = v_3 + v_4 \tag{83}$$

We observe that v_4 is estimated exactly like v_2,

$$|v_4(x)| \le C\frac{\epsilon^2}{\eta^2}\|\theta\|_{L^\infty} \tag{84}$$

In v_3 we integrate by parts

$$v_3(x) = c\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega H_D(x,y,t)\nabla^\perp(\phi(y)\theta(y))dy. \tag{85}$$

We use here $|\nabla\phi| \le C(\eta d(x))^{-1}$, and (71). Because on the support of ϕ we have $d(x) \le 2d(y)$, we deduce that

$$|v_3(x)| \le C\frac{M+\eta^{-1}\|\theta\|_{L^\infty}}{d(x)}\int_0^{\rho^2(x)} t^{-\frac{1}{2}}dt \le C\epsilon(M+\eta^{-1}\|\theta\|_{L^\infty}), \tag{86}$$

and consequently, from (81), (82), (84) and (86) that

$$|u^{in}(x)| \le C\left(\epsilon M + (\frac{\epsilon^2}{\eta^2}+\epsilon^2\eta^d+\frac{\epsilon}{\eta})\|\theta\|_{L^\infty}\right) \tag{87}$$

We write u^{out} as

$$u^{out}(x) = u_T(x) + u^T(x) \tag{88}$$

where

$$u_T(x) = \int_T^\infty t^{-\frac{1}{2}}\int_\Omega \nabla_x^\perp H_D(x,y,t)\theta(y)dy \tag{89}$$

and

$$u^T(x) = \int_{\rho^2(x)}^T t^{-\frac{1}{2}}\int_\Omega \nabla_x^\perp H_D(x,y,t)\theta(y)dy. \tag{90}$$

Because (20) u_T is smooth, and in particular

$$\|u_T\|_{L^\infty} \le C\|\theta\|_{L^\infty} \tag{91}$$

We take $\epsilon = \eta = 1$. From (87) and (91) we have

$$|u^{in}(x)| + |u_T(x)| \le C(\|\theta\|_{L^\infty} + M). \tag{92}$$

On the other hand, from (11) and (18) we bound

$$\frac{1}{d(x)}H_D(x,y,t) \le C\frac{1}{|x-y|}t^{-\frac{d}{2}}e^{\frac{|x-y|^2}{Kt}} \tag{93}$$

and using (19) we obtain

$$\begin{array}{l} |u^T(x)| \le c\int_{\rho^2}^T t^{-\frac{1}{2}}dt\int_\Omega t^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}}\frac{1}{|x-y|}|\theta(y)|dy \\ \le C\|\theta\|_{L^\infty}\left(\int_{\mathbb{R}^2}|y|^{-1}e^{-|y|^2}dy\right)\int_{\rho^2}^T t^{-1}dt \\ = C\log\left(\frac{T}{\rho^2(x)}\right)\|\theta\|_{L^\infty}. \end{array} \tag{94}$$

For this result we used $\epsilon = \eta = 1$, but ϵ may be used to show that the dependence on M is logarithmic for large M. This concludes the proof of Proposition 5.

The next proposition uses information about vanishing of θ at the boundary.

PROPOSITION 6. *Let θ be bounded and interior Lipschitz, i.e., obeying (71). Let*

$$b_1(x) = \frac{\theta(x)}{w_1(x)}. \tag{95}$$

Let u be given by (2). For any $p > d$, there exist constants C depending only on the domain Ω and p such that

$$\|u\|_{L^\infty} \le CM + C\|\theta\|_{L^\infty}\left(1+\log\|b_1\|_{L^p(\Omega)}\right). \tag{96}$$

Proof. We proceed like in the proof of Proposition 5, and in particular we use the bound (92). We bound u^T differently. We take a small number δ and we split

$$u^T = u_1 + u_2 \tag{97}$$

where

$$u_1(x) = \int_{\rho^2}^T t^{-\frac{1}{2}}\int_{\Omega\cap\ |x-y|\le\delta} \nabla_x^\perp H_D(x,y,t)\theta(y)dy \tag{98}$$

and

$$u_2(x) = \int_{\rho^2}^T t^{-\frac{1}{2}} \int_{\Omega\cap\ |x-y|\ge\delta} \nabla_x^\perp H_D(x,y,t)\theta(y)dy \tag{99}$$

For u_2 we have, using (19), the bound (93) and Lemma 1 with $j=0$ and $m=d-1$,

$$|u_2(x)| \le C\|\theta\|_{L^\infty} \int_{\Omega\cap\ |x-y|\ge\delta} |x-y|^{-d} e^{-\frac{|x-y|^2}{2KT}} dy \le C\|\theta\|_{L^\infty} e^{-\frac{\delta^2}{2KT}} \log\left(\frac{C}{\delta}\right). \tag{100}$$

In order to estimate u_1 we write

$$\theta(y) = b_1(y)w_1(y) \tag{101}$$

and use

$$w_1(y) \le Cd(y) \le C(d(x)+|x-y|) \tag{102}$$

and (93) to estimate

$$\begin{array}{l} |u_1(x)| \le C\int_{\rho^2}^T t^{-\frac{1}{2}} \int_{\Omega\cap\ |x-y|\le\delta} |b_1(y)| \left(1+\frac{d(x)}{|x-y|}\right) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}} dy \\ \le C\int_{\Omega\cap\ |x-y|\le\delta} |b_1(y)||x-y|^{-(d-1)}dy + C\frac{1}{\epsilon}\int_{\rho^2}^T t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}} dt \int_{\Omega\cap\ |x-y|\le\delta} |b_1(y)||x-y|^{-1}dy \\ \le C(1+\frac{1}{\epsilon})\int_{\Omega\cap\ |x-y|\le\delta} |b_1(y)||x-y|^{-(d-1)}dy \end{array} \tag{103}$$

where we used Lemma 1 with $j=0$ and $m=d-1$ in the first term of the second inequality and $d(x)\le \epsilon^{-1}\sqrt{t}$ and Lemma 1 with $j=0$ and $m=d-2$ in the second term. If $d=2$ we treat the second term of the second inequality directly, ignoring the exponential and integrating

$$d(x)\int_{\rho^2}^T t^{-\frac{3}{2}} dt \le \frac{2}{\epsilon}.$$

From the bounds above we obtain

$$|u^T(x)| \le C\|\theta\|_{L^\infty} e^{-\frac{\delta^2}{2KT}} \log\left(\frac{C}{\delta}\right) + C\left(1+\frac{1}{\epsilon}\right)\int_{\Omega\cap\ |x-y|\le\delta} |b_1(y)||x-y|^{-(d-1)}dy. \tag{104}$$

We take now $\eta=\epsilon=1$. Putting together the estimates (92) and (104) we obtain

$$|u(x)| \le C(M+\|\theta\|_{L^\infty}) + C\|\theta\|_{L^\infty}\log\left(\frac{C}{\delta}\right) + C\int_{\Omega\cap\ |x-y|\le\delta} |b_1(y)||x-y|^{-(d-1)}dy. \tag{105}$$

The estimate (96) follows by appropriately choosing δ small enough. This ends the proof of Proposition 6. Clearly, the condition $b_1\in L^p(\Omega)$, $p>d$ can be relaxed to

$$\lim_{\delta\to 0}\int_{\Omega\cap|x-y|\le\delta} |b_1(y)||x-y|^{-(d-1)}dy = 0. \tag{106}$$

We show now a decomposition of the type (62).

PROPOSITION 7. *If $p>d$ and if $b_1=\frac{\theta}{w_1}\in L^p(\Omega)$ then, for any $c_r>0$, there exists $\tau>0$ depending only on Ω, the norm $\|b_1\|_{L^p(\Omega)}$, the constant M of (71) and on $\|\theta\|_{L^\infty}$ such that*

$$u_s(x) := \int_0^\tau t^{-\frac{1}{2}} dt \int_\Omega \nabla_x^\perp H_D(x,y,t)\theta(y)dy \tag{107}$$

obeys

$$\|u_s\|_{L^\infty} \le c_r. \tag{108}$$

Proof. We note that

$$u_s(x) = u^{in}(x) + u^{\tau}(x) \tag{109}$$

where u^{in} is defined in (77) and hence obeys (87) and where u^{τ} is u^T of (90) for $T = \tau$. We choose $\eta > 0$ of order one, then ϵ to be small enough such that $\epsilon\eta^{-1}$ also is small enough, so that from (87) we obtain

$$|u^{in}(x)| \leq \frac{c_r}{4}. \tag{110}$$

With these choices of ϵ and η we use the fact that

$$\int_{\Omega\cap|x-y|\leq\delta} |b_1(y)||x-y|^{-(d-1)}dy \leq \|b_1\|_{L^p(\Omega)}\delta^{1-\frac{d}{p}} \tag{111}$$

to choose δ small enough so that

$$C\left(1+\frac{1}{\epsilon}\right)\int_{\Omega\cap\ |x-y|\leq\delta} |b_1(y)||x-y|^{-1}dy \leq \frac{c_r}{4}. \tag{112}$$

Now, once these choices have been made , we choose τ small enough so that

$$C\|\theta\|_{L^\infty}e^{-\frac{\delta^2}{2K\tau}}\log\left(\frac{C}{\delta}\right) \leq \frac{c_r}{2}. \tag{113}$$

The result then follows from the bounds (110), (112), (113) and (104). This concludes the proof of Proposition 7.

We state now the result of local control of $\|b_1\|_{L^p}$.

THEOREM 3. *Let $\theta_0 \in H_0^1(\Omega)\cap W^{1,\infty}(\Omega)$. Let $m > d$. There exists a time T_0 depending only on $\|\theta_0\|_{L^\infty}$, $\sup_{x\in\Omega} d(x)|\nabla\theta_0(x)|$ and $\|b_1(0)\|_{L^{2m}(w_1dx)}$ and a solution $\theta(x,t)$ of (1) obeying (6) and*

$$\|b_1(t)\|_{L^{2m}(w_1dx)} = \left(\int_\Omega w_1(x)b_1(x,t)^{2m}dx\right)^{\frac{1}{2m}} \leq C \tag{114}$$

for $t \leq T_0$. Consequently, for $d < p < m$ there exists B_p such that

$$\sup_{0\leq t\leq T_0}\|b_1(t)\|_{L^p(\Omega)} \leq B_p \tag{115}$$

holds.

Proof. The Proposition 4 and Proposition 7 are used in conjunction with Theorem 1 and Lemma 3.

REMARK 6. *We observe that if θ_0 vanishes at the boundary of the order $d(x)^\beta$ with $\beta > 1-\frac{1}{m}$ then $b_1(0)\in L^{2m}(w_1dx)$.*

5. Bounds for finite differences of Riesz transforms

We consider now finite differences

$$\delta_h u(x) = c\int_0^\infty t^{-\frac{1}{2}}dt\int_\Omega \delta_h^x\nabla_x^\perp H_D(x,y,t)\theta(y)dy \tag{116}$$

with $|h| \leq \frac{d(x)}{16}$.

DEFINITION 1. *Let us consider a small length ℓ_0, and take $0 \leq \ell \leq \ell_0$. We consider a ball B centered at a point x_0 with $d(x_0) \geq 2\ell$ and of radius ℓ. We take a smooth nonnegative function $\phi = \Psi\left(\frac{|x-x_0|}{\ell}\right)$ with Ψ a smooth, nonincreasing function of $z \in \mathbb{R}_+$, $\Psi(z) = 1$ for $z \leq \frac{5}{16}$ and $\Psi(z) = 0$ for $z \geq \frac{7}{16}$. We also take a function $\chi = \Psi\left(\frac{|x-x_0|}{2\ell}\right)$, noting that $0 \leq \phi \leq \chi \leq 1$, $\chi\phi = \phi$ and that the support of $1-\chi$ is included in $|x-x_0| \geq \frac{5\ell}{8}$, and that of $\nabla\chi$ in $\frac{5\ell}{8} \leq |x-x_0| \leq \frac{7\ell}{8}$, so they both are disjoint from the support of ϕ which is included in $|x-x_0| \leq \frac{7\ell}{16}$. We refer to ϕ as a "standard cutoff with scale ℓ" and center x_0, and to χ as its "companion".*

PROPOSITION 8. *Let ϕ be a standard cutoff with scale ℓ and companion χ, let $p > d$, and let u be given by (2). Then for any $\epsilon > 0$, there exists $\delta(\epsilon)$ with $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$, and a constant C_ϵ depending only on Ω, ϵ and p such that*

$$\begin{array}{l} |\phi(x)\delta_h u(x)| \\ \le \sqrt{\epsilon d(x) D(\chi\delta_h\theta)} + C_\epsilon |h| d(x)^{-\frac{d}{p}}\|b_1\|_{L^p(\Omega)} + \delta(\epsilon)\phi(x)|\delta_h\theta(x)| \end{array} \tag{117}$$

holds pointwise.

Above $D(\chi\delta_h\theta)$ is given in (31).

Proof. We start with bouunds for the gradient. We use the representation

$$\nabla u(x) = \nabla u^{in}(x) + \nabla u^{out}(x) \tag{118}$$

where

$$\nabla u^{in}(x) = c\int_0^{\rho^2} t^{-\frac12}dt\int_\Omega \nabla_x\nabla_x^\perp H_D(x,y,t)\theta(y)dy \tag{119}$$

and $\rho = \rho(x) \le \epsilon d(x)$. For ∇u^{out} we split in three parts. The inner portion of the integral

$$\int_{\rho^2}^{\epsilon d(x)^2} t^{-\frac12}dt\int_{|x-y|\le d(x)} \nabla_x\nabla_x^\perp H_D(x,y,t)\theta(y)dy$$

is bounded using (22) and ignoring the exponential, yielding

$$\begin{array}{l} \int_{\rho^2}^{\epsilon d(x)^2} t^{-\frac{d+3}{2}}dt\int_{|x-y|\le d(x)}|b_1(y)|(d(x)+|x-y|)dy \\ \le C\rho^{-(d+1)}d(x)\int_{|x-y|\le d(x)}|b_1(y)|dy \le Cd(x)^{-\frac{d}{p}}\left(\frac{d(x)}{\rho}\right)^{d+1}\|b_1\|_{L^p}. \end{array} \tag{120}$$

We use $|\nabla_x\nabla_x H_D(x,y,t)| \le Ct^{-\frac{d}{2}}d(x)^{-2}$ for $t \ge cd^2(x)$ and $|x-y| \le d(x)$ to bound the integral

$$\int_{\epsilon d(x)^2}^\infty t^{-\frac{d+1}{2}}dt\int_{|x-y|\le d(x)} d(x)^{-2}|b_1(y)|(d(x)+|x-y|)dy \le C_\epsilon d(x)^{-\frac{d}{p}}\|b_1\|_{L^p}. \tag{121}$$

From $|\nabla_x\nabla_x H_D(x,y,t)| \le Cd(x)^{-2}t^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}}$ for $t \ge Cd(x)^2$, $|x-y| \ge d(x)$ and Lemma 1 with $m = d-1$ we obtain

$$\begin{array}{l} \int_0^\infty t^{-\frac{d+1}{2}}\int_{|x-y|\ge d(x)}\frac{1}{d(x)^2}|b_1(y)|(d(x)+|x-y|)e^{-\frac{|x-y|^2}{Kt}}dydt \\ \le Cd(x)^{-1}\int_{|x-y|\ge d(x)}|b_1(y)||x-y|^{-d+1}(1+\frac{|x-y|}{d(x)})dy \\ \le Cd(x)^{-\frac{d}{p}}\|b_1\|_{L^p}. \end{array} \tag{122}$$

We used for the last integral $d > 2$. In the case $d = 2$ we take advantage of the fact that H_D is smooth for large time to bound

$$\begin{array}{l} d(x)^{-2}\int_0^T t^{-\frac32}dt\int_{|x-y|\ge d(x)}|x-y|e^{-\frac{|x-y|^2}{Kt}}|b_1(y)|dy \\ \le CTd(x)^{-2}\int_{|x-y|\ge d(x)}|x-y|^{-2}|b_1(y)|dy \\ \le Cd(x)^{-\frac{2}{p}}\|b_1\|_{L^p}. \end{array} \tag{123}$$

We have thus

$$|\nabla u^{out}(x)| \le C_\epsilon d(x)^{-\frac{d}{p}}\left(1+\left(\frac{d(x)}{\rho}\right)^{d+1}\right)\|b_1\|_{L^p(\Omega)} \tag{124}$$

with C depending on p, ϵ and Ω only.

We split

$$\delta_h u = \delta_h u^{in} + \delta_h u^{out} \tag{125}$$

with

$$\delta_h u(x)^{in} = c\int_0^{\rho^2} t^{-\frac12}dt\int_\Omega \delta_h^x\nabla_x^\perp H_D(x,y,t)\theta(y)dy \tag{126}$$

with ρ satisfying $\rho \le \epsilon d(x)$. From the bound (124) we have

$$|\delta_h u^{out}(x)| \le C_\epsilon |h| d(x)^{-\frac{d}{p}}\left(1+\left(\frac{d(x)}{\rho}\right)^{d+1}\right)\|b_1\|_{L^p(\Omega)} \tag{127}$$

with a constant depending only on Ω, ϵ and p. Note that if

$$\rho(x) = \epsilon d(x) \tag{128}$$

then the estimate becomes

$$|\delta_h u^{out}(x)| \le C_\epsilon |h| d(x)^{-\frac{d}{p}}\|b_1\|_{L^p(\Omega)} \tag{129}$$

with a constant depending only on Ω, ϵ and p.

We take a standard cutoff ϕ with scale ℓ and its companion χ and write

$$\phi(x)\delta_h u^{in}(x) = u_h(x) + v_h(x) \tag{130}$$

with

$$u_h(x) = c\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega \nabla_x^\perp H(x,y,t)(\chi(y)\delta_h\theta(y) - \chi(x)\delta_h\theta(x))dy \tag{131}$$

and where

$$v_h(x) = e_1(x) + e_2(x) + e_3(x) + e_5(x) + \phi(x)\delta_h\theta(x)e_4(x) \tag{132}$$

with

$$e_1(x) = c\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega \nabla_x^\perp (H_D(x+h,y,t) - H_D(x,y,t))\phi(x)(1-\chi(y))\theta(y)dy, \tag{133}$$

$$e_2(x) = c\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega \nabla_x^\perp (H_D(x+h,y,t) - H_D(x,y-h,t))\phi(x)\chi(y)\theta(y)dy, \tag{134}$$

$$e_3(x) = c\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega \nabla_x^\perp H_D(x,y,t)(\chi(y+h) - \chi(y))\phi(x)\theta(y+h)dy, \tag{135}$$

and

$$e_4(x) = c\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega \nabla_x^\perp H_D(x,y,t)dy. \tag{136}$$

We used here the facts that $\phi = \chi\phi$ and that $(\chi\theta)(\cdot)$ and $(\chi\theta)(\cdot + h)$ are compactly supported in Ω and hence

$$\int_\Omega \nabla_x^\perp H_D(x,y-h,t)\phi(x)\chi(y)\theta(y)dy = \int_\Omega \nabla_x^\perp H_D(x,y,t)\phi(x)\chi(y+h)\theta(y+h)dy.$$

From (22) we have

$$|e_1(x)| \le C\rho^2(x)|h|\int_0^1 d\lambda\int_A \frac{1}{|x+\lambda h - y|^{d+3}}(d(x) + |x-y|)|b_1(y)|dy$$

where $A = \{y\in\Omega\mid\ |x_0 - y| \ge \frac{5\ell}{8}\}$ is the support of $1-\chi(y)$. Because x belongs to the support of ϕ, it follows that

$$|e_1(x)| \le C|h|d(x)^{-\frac{d}{p}}\|b_1\|_{L^p(\Omega)}. \tag{137}$$

We bound e_3 using $|\nabla\chi| \le C\ell^{-1}$, Lemma 1 with $m = d$ in conjunction with (19):

$$|e_3(x)| \le C|h|\int_\Omega |\nabla\chi(y)|\frac{1}{|x-y|^d}|b_1(y)|dy,$$

and consequently, because x belongs to the support of ϕ,

$$|e_3(x)| \le C|h|d(x)^{-\frac{d}{p}}\|b_1\|_{L^p(\Omega)}. \tag{138}$$

Regarding e_4, in view of

$$\int_\Omega \nabla_y^\perp H_D(x,y,t)dy = 0 \tag{139}$$

we have

$$e_4(x) = c\int_0^{\rho^2} t^{-\frac{1}{2}}\int_\Omega \left(\nabla_x^\perp + \nabla_y^\perp\right) H_D(x,y,t)dy.$$

From (24) and Lemma 1 with $m = j = 0$, choosing $\epsilon = \epsilon(\delta)$ small enough in $\rho = \epsilon d(x)$, we obtain

$$|e_4(x)| \le \delta. \tag{140}$$

In order to estimate e_2 we write

$$H_D(x+h,y,t) - H_D(x,y-h,t) = h\cdot\int_0^1 (\nabla_x+\nabla_y)H_D(x+\lambda h, y+(\lambda-1)h,t)d\lambda \tag{141}$$

and use (27) to obtain

$$\begin{array}{l}|e_2(x)|\\ \le |h|\int_0^1 d\lambda\int_0^{\rho^2} t^{-\frac{1}{2}}dt\int_\Omega |\nabla_x^\perp(\nabla_x+\nabla_y)H_D(x+\lambda h, y+(\lambda-1)h)||\chi(y)|(d(x)+|x-y|)|b_1(y)|dy\\ \le C|h|\|b_1\|_{L^p}d(x)^{1+d(1-\frac{1}{p})}\int_0^{\rho^2} t^{-\frac{d+3}{2}}e^{-\frac{d(x)^2}{4Kt}}dt,\end{array}$$

and therefore

$$|e_2(x)| \le C|h|d(x)^{-\frac{d}{p}}\|b_1\|_{L^p(\Omega)}. \tag{142}$$

Finally, by a Schwartz inequality,

$$|u_h(x)| \le C\sqrt{\rho D(\chi\delta_h\theta)}. \tag{143}$$

This concludes the proof of the proposition.

We give also a bound for the normal component of the velocity at the boundary.

PROPOSITION 9. *Let T be a C^1 divergence-free vector field tangent to the boundary of Ω. Let $N = -T^\perp$. Let $0<\alpha<1$ and $p>d$. There exist constants $\ell_0>0$ and C depending on the domain Ω, on $p>d$ and on α such that*

$$|u(x)\cdot N(x)| \le C\left(d(x)^{1-\frac{d}{p}}\|b_1\|_{L^p} + d(x)^\alpha\|\theta\|_{C^\alpha(\Omega)}\right)\|T\|_{L^\infty} + Cd(x)^{2-\frac{d}{p}}\|b_1\|_{L^p}\|\nabla T\|_{L^\infty} \tag{144}$$

holds for $d(x)\le \ell_0$.

Proof. In view of (124), the fact that $T = N^\perp$ is tangent to the boundary, and $T\cdot\nabla_x H_D(x,y,t) = 0$ for $x\in\partial\Omega$, we have

$$|u^{out}(x)\cdot N(x)| \le Cd(x)^{1-\frac{d}{p}}\|b_1\|_{L^p} \tag{145}$$

where

$$u^{out}(x) = \int_{cd^2(x)}^\infty t^{-\frac{1}{2}}dt\int_\Omega \nabla_x^\perp H_D(x,y,t)\theta(y)dy. \tag{146}$$

We consider now u^{in} and we write

$$\begin{array}{l}u^{in}(x)\cdot N(x) = -\int_0^{cd^2(x)} t^{-\frac{1}{2}}dt\int_\Omega T(x)\cdot\nabla_x H_D(x,y,t)\theta(y)dy\\ = \int_0^{cd^2(x)} t^{-\frac{1}{2}}dt\int_\Omega (T(y)-T(x))\cdot\nabla_x H_D(x,y,t)\theta(y)dy\\ -\int_0^{cd^2(x)} t^{-\frac{1}{2}}dt\int_{|x-y|\ge cd(x)} T(y)\cdot(\nabla_x+\nabla_y)H_D(x,y,t)\theta(y)dy\\ -\int_0^{cd^2(x)} t^{-\frac{1}{2}}dt\int_{|x-y|\le cd(x)} T(y)\cdot(\nabla_x+\nabla_y)H_D(x,y,t)\theta(y)dy\\ +\int_0^{cd^2(x)} t^{-\frac{1}{2}}dt\int_\Omega T(y)\cdot\nabla_y H_D(x,y,t)(\theta(y)-\theta(x))dy\\ = U_1+U_2+U_3+U_4.\end{array} \tag{147}$$

Because $|T(x)-T(y)|\le C|x-y|$ we have that

$$|U_1| \le \|\nabla T\|_{L^\infty}\int_0^{cd^2(x)} t^{-\frac{1}{2}}dt\int_\Omega t^{-\frac{d}{2}}\frac{|x-y|}{t^{\frac{1}{2}}}e^{-\frac{|x-y|^2}{Kt}}(d(x)+|x-y|)|b_1(y)|dy$$

and therefore

$$|U_1| \le Cd(x)^{2-\frac{d}{p}}\|b_1\|_{L^p}\|\nabla T\|_{L^\infty} \tag{148}$$

holds. For U_2 we use the bounds (19), (21), $\theta = w_1 b$ (102) and Lemma 1 to obtain

$$|U_2| \le C \int_{|x-y|\ge d(x)} |x-y|^{-d} \left(d(x) + |x-y|\right) |b_1(y)| dy \le C d(x)^{1-\frac{d}{p}} \|b_1\|_{L^p} \tag{149}$$

with the understanding that if $p = \infty$ then $d(x)^{1-\frac{d}{p}}$ is replaced by $d(x)\log\left(\frac{\operatorname{diam}\Omega}{d(x)}\right)$. For U_3 we use the bound (25) and Lemma 1 to write

$$d(x)\int_0^{cd^2(x)} t^{-\frac{1}{2}} \int_{|x-y|\le d(x)} |(\nabla_x + \nabla_y) H_D(x,y,t)||b_1(y)| dy \le C d(x)^{1-\frac{d}{p}} \|b_1\|_{L^p}. \tag{150}$$

Finally, for U_4 we use (21) and the fact that

$$\int_0^{cd^2(x)} t^{-1-\frac{d}{2}} \int_{\mathbb{R}^d} |x-y|^{\alpha} e^{-\frac{|x-y|^2}{t}} dy \le C d(x)^{\alpha} \tag{151}$$

to obtain

$$|U_4| \le C d(x)^{\alpha} \|\theta\|_{C^{\alpha}}. \tag{152}$$

This concludes the proof of the lemma.

6. Commutators

We consider the finite difference

$$(\delta_h \Lambda_D \theta)(x) = (\Lambda_D\theta)(x+h) - (\Lambda_D \theta)(x) \tag{153}$$

with $|h| \le \frac{d(x)}{32}$. We use a standard cutoff with scale ℓ, ϕ and its companion χ.

PROPOSITION 10. *We consider the commutator*

$$C_h(\theta) = \phi(x)(\delta_h\Lambda_D\theta)(x) - \phi(x)\Lambda_D(\chi\delta_h\theta)(x). \tag{154}$$

There exists a constant Γ_0 such that the commutator $C_h(\theta)$ obeys

$$|C_h(\theta)(x)| \le \Gamma_0 \frac{|h|}{d(x)} \|b_1\|_{L^p(\Omega)} d(x)^{-\frac{d}{p}} \tag{155}$$

for $|h| \le \frac{\ell}{16}$, $\theta \in H_0^1(\Omega)\cap L^{\infty}(\Omega)$ and $b_1 = \frac{\theta}{w_1} \in L^p(\Omega)$ with $p > d$. The constant is bounded as $p \to \infty$ and if $p = \infty$ the estimate is

$$|C_h(\theta)(x)| \le \Gamma_0 \frac{|h|}{d(x)} \|b_1\|_{L^{\infty}(\Omega)}. \tag{156}$$

Proof. We compute the commutator as follows

$$\begin{array}{l}
(\phi\delta_h\Lambda_D\theta)(x) - \phi(\Lambda_D\chi\delta_h\theta)(x)\\
= c\int_0^{\infty} t^{-\frac{3}{2}} dt \int_{\Omega} (H_D(x,y,t) - H_D(x+h,y,t))\phi(x)(1-\chi(y))\theta(y) dy\\
-c\int_0^{\infty} t^{-\frac{3}{2}} dt \int_{\Omega} (H_D(x+h,y,t) - H_D(x,y-h,t))\phi(x)\chi(y)\theta(y) dy\\
-c\int_0^{\infty} t^{-\frac{3}{2}} dt \int_{\Omega} H_D(x,y,t)\phi(x)(\delta_h\chi)(y)\theta(y+h) dy\\
= E_1(x) + E_2(x) + E_3(x).
\end{array} \tag{157}$$

We use (157). We observe by triangle inequlaity $d(y) \le d(x) + |x-y|$ and thus

$$|\theta(y)| \le C|b_1(y)|(d(x) + |x-y|) \tag{158}$$

holds for any $x, y \in \Omega$. For $E_1(x)$ we use the inequalities (18), (19), and Lemma 1 with $m = d+2$ when $t \le d(x)^2$, and $m = d+1$ when $t \ge d(x)^2$, together with $d(x)^{-1}H_D \le C|x-y|^{-1}t^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}}$. Substituting (158) for θ, we deduce

$$|E_1(x)| \le C|h| \int_{\Omega} |x-y|^{-(d+2)}(d(x) + |x-y|)|b_1(y)|\phi(x)|1-\chi(y)| dy$$

and then, using a Hölder inequality we obtain

$$E_1(x) \le C\frac{|h|}{d(x)}\|b_1\|_{L^p} d(x)^{-\frac{d}{p}}. \tag{159}$$

For E_2 we use (141) like in the proof of the estimate (142) and (25), together with Lemma 1 with $m = d+2$, $j = 0$, and bounds

$$\begin{array}{l} \int_0^\infty t^{-\frac{3}{2}} t^{-\frac{d+1}{2}} e^{-\frac{d(x)^2}{Kt}} dt \int_{|x-y|\le d(x)} |b_1(y)|(d(x)+|x-y|)dy \\ \le C(d(x))^{-2-d} d(x) \int_{|x-y|\le d(x)} |b_1(y)|dy \\ \le C d(x)^{-1-\frac{d}{p}} \|b_1\|_{L^p} \end{array} \tag{160}$$

and

$$\begin{array}{l} \int_0^\infty t^{-\frac{3}{2}} \int_{|x-y|\ge d(x)} (t^{-\frac{1}{2}} + \frac{1}{d(x)} + \frac{1}{d(y)}) H_D(x,y,t)|b_1(y)|(d(x)+|x-y|)dydt \\ \le C\int_0^\infty t^{-\frac{3}{2}} \int_{|x-y|\ge d(x)} (t^{-\frac{1}{2}} + \frac{1}{|x-y|}) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}} |b_1(y)|(d(x)+|x-y|)dy \\ \le C\int_{|x-y|\ge d(x)} \frac{1}{|x-y|^{d+2}} |b_1(y)|(d(x)+|x-y|)dy \le C d(x)^{-1-\frac{d}{p}} \|b_1\|_{L^p} \end{array} \tag{161}$$

to obtain

$$|E_2(x)| \le C\frac{|h|}{d(x)}\|b_1\|_{L^p} d(x)^{-\frac{d}{p}}. \tag{162}$$

For E_3 we have

$$|E_3(x)| \le |h|\int_0^\infty t^{-\frac{3}{2}} dt \int_\Omega H_D(x,y,t)\phi(x)|\nabla\chi(y)|(d(x)+|x-y|)b_1(y)dy$$

and from Lemma 1 with $m = d, d+1$, $j = 0$ we obtain

$$|E_3(x)| \le C\frac{|h|}{d(x)}\|b_1\|_{L^p} d(x)^{-\frac{d}{p}}.$$

This concludes the proof.

7. SQG: Hölder bounds

We consider the equation (1) with u given by (2) and with initial data $\theta_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We note we have

$$\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}. \tag{163}$$

We prove the following result.

THEOREM 4. *Let $\theta(x,t)$ be a solution of (1) in the bounded domain with smooth boundary Ω, obeying (6) on a time interval $[0,T]$. Assume that*

$$\sup_{0\le t\le T} \|b_1(t)\|_{L^p(\Omega)} \le B \tag{164}$$

holds with $p > d$. Then for $0 < \alpha < 1 - \frac{d}{p}$ there exists a constant K, depending only on the domain Ω and p, such that

$$\sup_{0\le t\le T} \sup_{x\in\Omega} \sup_{|h|\le \frac{d(x)}{32}} \frac{|\delta_h\theta(x,t)|}{|h|^\alpha} \le 2\|\theta_0\|_{C^\alpha} + KB(M+1) \tag{165}$$

holds, where M is the a priori bound in (1). Moreover, the velocity u is bounded $u \in L^\infty(0,T; L^\infty(\Omega))$, obeying (96) and the normal component of the velocity vanishes near the boundary of order $d(x)^\alpha$, obeying (144).

REMARK 7. *In view of Theorem 3 there exists $T_0 > 0$ such that condition (164) is satisfied on $[0,T_0]$. We recall that (6) holds unconditionally, in view of Theorem 1.*

Proof. We take

$$|h| \le \frac{\ell}{16}. \tag{166}$$

We take x with $d(x) \le 2\ell_0$. From the SQG equation we obtain the equation

$$\frac{1}{2}\left(\partial_t + u\cdot\nabla\right)|\delta_h\theta|^2 + (\delta_h\theta)\delta_h\Lambda_D\theta = -(\delta_h\theta)\delta_h u\cdot\nabla\theta(x+h). \tag{167}$$

We use a standard cutoff ϕ with scale ℓ, and companion χ. We multiply by ϕ^2 and obtain

$$\frac{\phi^2}{2}(\partial_t + u\cdot\nabla)(|\delta_h\theta|^2) + \phi^2\delta_h\theta\Lambda_D(\chi\delta_h\theta) = -(\phi\delta_h\theta)C_h(\theta) - (\phi\delta_h\theta)\phi\delta_h u\cdot\nabla\theta(x+h) \tag{168}$$

where $C_h(\theta)$ is the commutator given above in (154).

Multiplying by $|h|^{-2\alpha}$ where $\alpha > 0$ is smaller than $1 - \frac{d}{p}$, we obtain

$$\frac{\phi^2}{2}\left(\partial_t + u\cdot\nabla\right)(f^2) + \phi^2 f\Lambda_D(\chi f) = -|h|^{-\alpha}\phi f C_h(\theta) - |h|^{-\alpha}\phi f\phi\delta_h u\cdot\nabla\theta(x+h) \tag{169}$$

where

$$f(x,t;h) = f = |h|^{-\alpha}\delta_h\theta(x,t). \tag{170}$$

The first term in the right hand side of (169) is bounded using the commutator estimate (155).

$$\begin{array}{l}\phi|f||h|^{-\alpha}|C_h(\theta)| \le \phi|f||h|^{-\alpha}\Gamma_0\frac{|h|}{d(x)}Bd(x)^{-\frac{d}{p}}\\ \le (C\Gamma_0 B|h|^{1-\alpha-\frac{d}{p}})\frac{1}{d(x)}\phi|f|\\ \le \frac{1}{d(x)}\left[C\Gamma_0 B\ell^{1-\alpha-\frac{d}{p}}\right]\phi|f|.\end{array} \tag{171}$$

In view of (6) we have that

$$|\nabla\theta(x+h)| \le M\frac{1}{d(x)}. \tag{172}$$

The second term in the right hand side of (169) is estimated using (117) with $\delta = \delta(\epsilon) \le \frac{\gamma_1}{8M}$ with ϵ sufficiently small (depending on γ_1 and M but not on B). We obtain

$$\begin{array}{l}\phi|f||h|^{-\alpha}|\phi\delta_h u||\nabla\theta(x+h)|\\ \le Md(x)^{-1}\phi|f||h|^{-\alpha}\left[\sqrt{\epsilon d(x)D(\chi\delta_h\theta)} + C_\epsilon|h|d(x)^{-\frac{d}{p}}B + \frac{\gamma_1}{8M}\phi|\delta_h\theta(x)|\right]\\ \le \frac{1}{2}D(\chi f) + \frac{\gamma_1}{4d(x)}\phi^2|f|^2 + C_\epsilon BM|h|^{1-\alpha}d(x)^{-1-\frac{d}{p}}\phi|f|.\end{array} \tag{173}$$

where $D(g)$ is given in (31) and where we also used $M^2\epsilon \le \frac{\gamma_1}{8}$. Therefore, if we have

$$0 < \alpha < 1 - \frac{d}{p} \tag{174}$$

we obtain from (169), (171), (173) that

$$\frac{\phi^2}{2}\left(\partial_t + u\cdot\nabla\right)(f^2) + \phi^2 f\Lambda_D(\chi f) \le \frac{1}{2}D(\chi f) + \frac{\gamma_1}{4d(x)}\phi^2|f|^2 + \frac{1}{d(x)}\left[K_1 B(M+1)\ell^{1-\alpha-\frac{d}{p}}\right]\phi|f| \tag{175}$$

holds for $|h| \le \frac{\ell}{16}$. Note that K_1 does not depend on ℓ nor on h and that, in view of (174) we may take $|h|$ and $\ell > 0$ as small as we wish.

The rest of the argument is by contradiction. We fix $T > 0$ and take $0 < \ell < \ell_0$. We consider the compact region

$$A_\ell = \{x \in \Omega \mid \ \ell \le d(x) \le 2\ell\}. \tag{176}$$

We assume by contradiction that there exists $x_1 \in A_\ell$, $t_0 \in [0,T)$ and h_0 with $|h_0| \le \frac{d(x_1)}{32}$ such that

$$|h_0|^{-\alpha}|\delta_{h_0}\theta(x_1,t_0)| \ge 2\|\theta_0\|_{C^\alpha} + KB(M+1)\ell_0^{1-\alpha-\frac{d}{p}}. \tag{177}$$

with $K = \frac{5K_1}{\gamma_1}$ where K_1 appears in (175). We may assume without loss of generality that t_0 is the infimum of such t that (177) holds for some $x_1 \in A_\ell$. The prefactor 2 in front of $\|\theta_0\|_{C^\alpha}$ was put there for convenience, in order to make sure that $t_0 > 0$ (it could have been any number larger than one). We fix h_0 and t_0 and take $x_0 \in A_\ell$ to be a point where the maximum of the function $f^2(x, t_0; h_0)$ is achieved in the region A_ℓ. We know that θ is interior Lipschitz, so $|h_0| > 0$ and f^2 is Lipschitz continuous there. Therefore (177) holds with x_0 replacing x_1. We take a standard cutoff with scale ℓ, center x_0 and companion χ. We use the inequality (30):

$$\chi f \Lambda_D(\chi f) - \frac{1}{2}\Lambda_D(\chi^2 f^2) = D(\chi f) \ge \gamma_1 (d(x))^{-1}\chi^2 |f|^2, \tag{178}$$

valid pointwise. We also use the fact that $\Lambda_D(\chi^2 f^2)(x_0) > 0$ because $\chi^2 f^2$ is maximized at x_0. In fact, more is true,

$$\Lambda_D(\chi^2 f^2)(x_0) \ge \chi^2(x_0) f^2(x_0) \Lambda_D 1.$$

Indeed, for any function g in the domain of Λ_D which achieves its maximum at $x_0 \in \Omega$ we have

$$(\Lambda_D g)(x_0) = c\int_0^\infty t^{-\frac{3}{2}} \left(g(x_0) - \int_\Omega H_D(x,y,t) g(y) dy \right) \ge g(x_0)\Lambda_D 1.$$

Using $\phi(x_0) = \chi(x_0) = 1$ we have from (175), (177) and (178) and the fact that $u \cdot \nabla f^2 = 0$ at an interior local maximum,

$$\begin{aligned}
&\tfrac{1}{2}\partial_t f^2 \le -\chi f \Lambda_D \chi f + \tfrac{1}{2} D(\chi f) + \tfrac{\gamma_1}{4d(x_0)}\phi^2 f^2 + \tfrac{1}{d(x_0)} K_1 B(M+1)\ell^{1-\alpha-\frac{d}{p}}\phi|f| \\
&\le -\tfrac{1}{2}\Lambda_D(\chi^2 f^2) - \tfrac{1}{2} D(\chi f) + \tfrac{\gamma_1}{4d(x_0)}\phi^2 f^2 + \tfrac{1}{d(x_0)} K_1 B(M+1)\ell^{1-\alpha-\frac{d}{p}}\phi|f| \\
&\le -\tfrac{\gamma_1}{4d(x_0)}\phi^2 f^2 + \tfrac{1}{d(x_0)} K_1 B(M+1)\ell^{1-\alpha-\frac{d}{p}}\phi|f| \\
&\le -\tfrac{\gamma_1}{4d(x_0)} B(M+1)\ell_0^{1-\alpha-\frac{d}{p}}\phi|f|(K - \tfrac{4K_1}{\gamma_1}) \\
&\le -\tfrac{1}{4d(x_0)} B(M+1)\ell_0^{1-\alpha-\frac{d}{p}} K_1 < 0,
\end{aligned} \tag{179}$$

which is a contradiction. Thus (165) holds, and the proof of the uniform bound on the Hölder norm is concluded. The fact that u obeys (96) follows from Lemma 6 and the vansihing of the normal component of velocity follows from Proposition 9, in view of the bound (165).

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