

Solvable base change

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Abstract. We determine the image and the fibers for solvable base change.

1. Introduction

The reciprocity conjectures formulated by Langlands give a parametrization of cusp forms associated to $GL(n)$ over a global field K by n -dimensional complex representations of the Langlands group attached to K . The Langlands group, whose existence is yet to be shown, is a vast generalization of the absolute Galois group or the Weil group of K , and can be considered in analogy with these latter groups. In this analogy, the theory of base change amounts to restriction of parameters on the Galois theoretic side.

For cyclic extensions of number fields of prime degree, the existence and the characterization of the image and fibers of base change for $GL(2)$ was done by Langlands [9] following earlier work of Saito and Shintani. This was used by Langlands to establish Artin's conjecture for a class of octahedral two-dimensional representations of the absolute Galois group of a number field. The work of Saito, Shintani and Langlands was generalized by Arthur and Clozel to $GL(n)$, for all n . In [1], they proved the existence and characterized the image of the base change transfer for cyclic extensions of number fields of prime degree. However, the proof in the general, cyclic case contained a mistake.

The theorem of Langlands, Arthur and Clozel, gives inductively the existence of the base change transfer corresponding to a solvable extension of number fields for $GL(n)$. The problem of characterizing the image and fibers of base change for cyclic extensions of non-prime degree was considered by Lapid and Rogawski in [12]. This led them to conjecture the non-existence of certain types of cusp forms on $GL(n)$, and they proved this conjecture when $n = 2$. It was shown in [15] that the conjecture of Lapid and Rogawski allows a characterization of the image and fibers of the base change map for solvable extensions of number fields. In this article, our main aim is to prove the conjecture of Lapid and Rogawski for all n .

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In order to make this paper more self-contained, we have included here complete proofs, more detailed than in the originals, of the theorem of Lapid and Rogawski characterizing cusp forms on $\mathrm{GL}(n)$ whose Galois conjugate by a generator of a cyclic Galois group differs from the original form by an Abelian twist (Theorem 2), which was in their paper conditional on Theorem 1; and of the theorem of one of us (Rajan) characterizing the image and fiber of base change in a solvable extension (Theorem 3).

1.1. Main theorem. For a number field F , let $\mathbf{A}_F$ denote the adèle ring of F and C_F the group of idele classes of F . Given a representation π of $\mathrm{GL}(n, \mathbf{A}_F)$ and σ an automorphism of F , define ${}^\sigma\pi$ (or $\sigma(\pi)$) to be the representation $g \mapsto \pi(\sigma^{-1}(g))$ for $g \in \mathrm{GL}(n, \mathbf{A}_F)$. Given an extension E/F of number fields, if ω is an idele class character of E , denote by ω_F its restriction to the idele class group C_F of F . If ω is a character of C_F , define $\omega^E = \omega \circ N_{E/F}$, where $N_{E/F} : C_E \rightarrow C_F$ is the norm map on the idele classes of E .

Let E/F be a cyclic extension of number fields of degree d and σ denote a generator for $\mathrm{Gal}(E/F)$. By $\varepsilon_{E/F}$, we mean an idele class character of C_F corresponding to the extension E/F , i.e., a character of C_F of order d , vanishing on the subgroup of norms $N_{E/F}(C_E)$ coming from E .

The primary aim of this paper is to establish the following conjecture of Lapid and Rogawski [12] for $\mathrm{GL}(n)$ for all n , proved by them for $\mathrm{GL}(2)$:

Theorem 1 ([12, Statement A, p. 178]). *Let E/F be a cyclic extension of number fields of degree $d > 1$ and σ denote a generator for $\mathrm{Gal}(E/F)$. Let ω be an idele class character of E such that its restriction to $C_F \subset C_E$ is $\varepsilon_{E/F}$. Then there does not exist any cuspidal automorphic representation Π of $\mathrm{GL}(n, \mathbf{A}_E)$ such that*

$$(1.1) \quad {}^\sigma\Pi \simeq \Pi \otimes \omega.$$

We note that the theorem is obvious if d does not divide n , as one sees by considering the restriction to C_F of the central character of Π .

1.2. Galois conjugate cusp forms up to twisting by a character. From Theorem 1, Lapid and Rogawski derive a structure theorem for cusp forms π on $\mathrm{GL}(n, \mathbf{A}_E)$, whose Galois conjugate ${}^\sigma\pi$ differs from π up to twisting by a character ω .

Given a cyclic extension E/F of number fields and an automorphic representation π of $\mathrm{GL}(n, \mathbf{A}_F)$, denote by $\mathrm{BC}_F^E(\pi)$ the base change lift of π to an automorphic representation of $\mathrm{GL}(n, \mathbf{A}_E)$. For a cuspidal automorphic representation η on $\mathrm{GL}(n, \mathbf{A}_E)$, let $\mathrm{AI}_E^F(\eta)$ denote the automorphic representation of $\mathrm{GL}(nd, \mathbf{A}_F)$, where $d = \deg(E/F)$, the existence of which was proved for $nd = 2$ in [8], and for general n, d in [1]. For a field F let G_F denote the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$ where $\bar{F}$ denotes an algebraic closure of F . At the level of Galois representations, base change corresponds to the restriction of representations from G_F to G_E , and automorphic induction corresponds to the induction of representations of G_E to G_F .

Theorem 2 ([12, Statement B, p. 179]). *Let E be a number field and σ an automorphism of E of order d . Let F be the field left fixed by the subgroup of automorphisms of E generated by σ . Let ω be an idele class character of E and let π be a cuspidal representation of $\mathrm{GL}(n, \mathbf{A}_E)$ such that $\sigma(\pi) \simeq \pi \otimes \omega$. Let K/F be the extension corresponding to the*

character ω_F of C_F , and let $L = KE$. Then:

- (1) $K \cap E = F$ and $[K : F]$ divides n . Let $r = n/[K : F]$ and let τ be the unique extension of σ to L trivial on K .
- (2) There exists a cuspidal representation π_0 of $\mathrm{GL}(r, \mathbf{A}_K)$ such that

$$\pi = \mathrm{AI}_L^E(\mathrm{BC}_K^L(\pi_0) \otimes \psi),$$

where ψ is a Hecke character of L such that $\tau(\psi)\psi^{-1} = \omega^L$.

- (3) Conversely, given π_0 and ψ as in (2), the representation

$$\pi = \mathrm{AI}_L^E(\mathrm{BC}_K^L(\pi_0) \otimes \psi)$$

satisfies $\sigma(\pi) \simeq \pi \otimes \omega$. However, π need not be cuspidal.

1.3. Solvable base change. The following theorem characterizing the image and fibers of the base change transfer for solvable extensions of number fields was established in [15]:

Theorem 3. *Let E/F be a solvable extension of number fields, and let Π be a unitary, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_E)$.*

- (1) *Suppose Π is $\mathrm{Gal}(E/F)$ -invariant. Then there exist a $\mathrm{Gal}(E/F)$ -invariant Hecke character ψ of E and a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_E)$ such that*

$$\mathrm{BC}_F^E(\pi) \simeq \Pi \otimes \psi.$$

Further, ψ is unique up to base change to E of a Hecke character of F .

- (2) *Suppose there exist cuspidal automorphic representations π, π' of $\mathrm{GL}_n(\mathbf{A}_F)$ such that*

$$\mathrm{BC}_F^E(\pi) = \mathrm{BC}_F^E(\pi') = \Pi.$$

Then there exists a character χ of C_F corresponding via class field theory to a character of $\mathrm{Gal}(E/F)$ such that

$$\pi' \simeq \pi \otimes \chi.$$

Moreover, if χ is non-trivial, the representations π and $\pi \otimes \chi$ are distinct.

Theorem 3 follows by an inductive argument from Theorem 2.

Remark. Suppose E/F is a solvable extension with the property that invariant idele class characters of E descend to F . Then any invariant, unitary, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_E)$ lies in the image of the base change map $\mathrm{BC}_{E/F}$. In particular, we recover the classical formulation that invariant, unitary, cuspidal automorphic representations descend if E/F is cyclic.

Remark. The motivation for this theorem stems from the following analogous Galois theoretic situation: let E/F be a Galois extension of number fields, and $\rho : G_E \rightarrow \mathrm{GL}_n(\mathbb{C})$ an irreducible representation of G_E . Suppose that ρ is invariant under the action of G_F on the collection of representations of G_E . By an application of Schur's lemma, it can be seen that ρ extends as a projective representation, say $\tilde{\eta}$ to G_F . By a theorem of Tate on the vanishing of $H^2(G_F, \mathbb{C}^\times)$ ([16]), this representation can be lifted to a linear representation η of G_F . This implies that $\rho \otimes \chi$ descends to a representation of G_F for some character χ of G_E .

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2. Proof of Theorem 2

In this section, assuming the validity of Theorem 1, we give a proof of Theorem 2 modifying the arguments given in [12].

Let E/F be a cyclic extension of degree d , and let σ denote a generator for $\text{Gal}(E/F)$. We want to classify the idele class characters ω of E and cusp forms π on $\text{GL}(n, \mathbf{A}_E)$ satisfying the condition $\sigma(\pi) \simeq \pi \otimes \omega$. At the level of central characters, this implies $\sigma(\chi_\pi) = \chi_\pi \omega^n$, where χ_π denotes the central character of π . In particular, this implies that the restriction ω_F , of ω to C_F , has finite order dividing n . Let K/F be the cyclic extension of F corresponding to ω_F .

We first show that $K \cap E = F$. We prove this by induction on the pair (d, n) , assuming the validity of the claim for all extensions E/F of degree less than d , or for cusp forms on GL_m for $m < n$. When $n = 1$, the assertion is clearly true. So, we now assume that $d > 1$ and $n > 1$.

We initially rule out the following case: $K \subset E$ and K is not equal to F . The case $E = K$ is ruled out by Theorem 1. Let $F \subset K \subset E' \subset E$ be an extension of fields such that the degree of E/E' is p , for some rational prime p . Let $m = [E' : F]$. The group $\text{Gal}(E/E')$ is generated by σ^m . We observe

$$\sigma^m(\pi) \simeq \sigma^{m-1}(\pi \otimes \omega) \simeq \cdots \simeq \pi \otimes \psi,$$

where $\psi = \omega \sigma(\omega) \cdots \sigma^{m-1}(\omega)$. The restriction $\psi_{E'}$ of ψ to E' satisfies

$$\psi_{E'} = \omega_F \circ N_{E'/F}.$$

Since $K \subset E'$, $\psi_{E'}$ is trivial. Consequently, there exists an idele class character η of C_E such that $\frac{\eta}{\sigma^m(\eta)} = \psi$. Hence,

$$\sigma^m(\pi \otimes \eta) \simeq \sigma^m(\pi) \otimes \sigma^m(\eta) \simeq \pi \otimes \psi \otimes \sigma^m(\eta) \simeq \pi \otimes \eta.$$

By the descent theorem for cusp forms invariant with respect to a cyclic extension of prime degree [1], there exists a cuspidal automorphic representation ρ' of $\text{GL}_n(\mathbf{A}_{E'})$ such that

$$\text{BC}_{E'}^E(\rho') = \pi \otimes \eta.$$

Let σ' denote the restriction of σ to E' . Then

$$\text{BC}_{E'}^E(\sigma'(\rho')) \simeq \sigma(\pi \otimes \eta) \simeq \pi \otimes \omega \otimes \sigma(\eta) \simeq \text{BC}_{E'}^E(\rho') \otimes \sigma(\eta) \eta^{-1} \omega.$$

We observe now that $\sigma(\eta) \eta^{-1} \omega$ is invariant under $\text{Gal}(E/E') = \langle \sigma^m \rangle$. From the definition of ψ , it follows that $\sigma(\psi) \psi^{-1} = \sigma^m(\omega) \omega^{-1}$. Hence,

$$\begin{aligned} \sigma^m(\sigma(\eta) \eta^{-1} \omega) &= \sigma(\sigma^m \eta) \sigma^m(\eta)^{-1} \sigma^m(\omega) \\ &= \sigma(\eta) \sigma(\psi)^{-1} \psi \eta^{-1} \sigma(\psi) \psi^{-1} \omega \\ &= \sigma(\eta) \eta^{-1} \omega. \end{aligned}$$

Hence there exists an idele class character θ of E' such that $\theta \circ N_{E/E'} = \sigma(\eta)\eta^{-1}\omega$. We have

$$\mathrm{BC}_{E'}^E(\sigma'(\rho')) \simeq \mathrm{BC}_{E'}^E(\rho' \otimes \theta).$$

From the characterization of the fibers of base change for cyclic extensions of prime degree

$$\sigma'(\rho') \simeq \rho' \otimes \theta \otimes \varepsilon_{E/E'}^i$$

for some integer i . Let $\chi = \theta \otimes \varepsilon_{E/E'}^i$. Thus there exists an idele class character χ of E' such that

$$\sigma'(\rho') \simeq \rho' \otimes \chi$$

such that the base change of the character χ to E is $\sigma(\eta)\eta^{-1}\omega$. Hence $\chi_F^p = \omega_F$, and χ_F defines an extension K' of F containing K . Since the degree $[E' : F]$ is less than that of $[E : F]$, by the inductive hypothesis $K' \cap E' = F$. Since $K \subset E'$, we have

$$K = K \cap E' \subset K' \cap E' = F.$$

Hence we have ruled out the case that $K \subset E$ and $K \neq F$.

Let $N = K \cap E$. We want to show $N = F$. Assume now that ω_F is not trivial. The field K associated to ω_F is a non-trivial cyclic extension of F . By what we have shown above, $K \cap E$ is a proper subfield of K . Let $K' \subset K$ be a subfield of K containing N such that its degree over N is a rational prime p . We have

$$\pi \simeq \sigma^d(\pi) \simeq \pi \otimes \omega \sigma(\omega) \cdots \sigma^{d-1}(\omega) \simeq \pi \otimes \omega_F \circ N_{E/F}.$$

By class field theory, the character $\omega_F \circ N_{E/F}$ corresponds to the cyclic extension $L = KE$ over E . The field $E' := EK'$ is an extension of E of degree p contained inside L . The isomorphism $\pi \simeq \pi \otimes \omega_F \circ N_{E/F}$ implies an isomorphism $\pi \simeq \pi \otimes (\omega_F \circ N_{E/F})^k$ for any natural number k . Taking $k = [L : E']$, we have $\pi \simeq \pi \otimes \varepsilon_{E'/E}$, where $\varepsilon_{E'/E}$ is an idele class character of E corresponding via class field theory to the extension E'/E . By the characterization of automorphic induction [1], there exists a cusp form π' on $\mathrm{GL}(n/p, \mathbf{A}_{E'})$ such that

$$\pi \simeq \mathrm{AI}_{E'}^E(\pi').$$

Let $\sigma' \in \mathrm{Gal}(E'/F)$ be an extension of σ to E' . Then

$$\mathrm{AI}_{E'}^E(\sigma'(\pi')) \simeq \sigma(\pi) \simeq \pi \otimes \omega \simeq \mathrm{AI}_{E'}^E(\pi' \otimes \omega'),$$

where $\omega' = \omega \circ N_{E'/E}$ is the base change of ω to E' . From the characterization of the fibers of automorphic induction with respect to a cyclic extension of prime degree, it follows that there exists an automorphism $\tau \in \mathrm{Gal}(E'/E)$ such that $\tau\sigma'(\pi') \simeq \pi' \otimes \omega'$. The automorphism $\tau\sigma'$ of E' extends σ . Renaming $\tau\sigma'$ as σ' , we see that there exists an automorphism σ' of E' extending σ such that

$$\sigma'(\pi') \simeq \pi' \otimes \omega'.$$

Let F' be the fixed field of σ' . Since the fixed field of σ is F , $E \cap F' = F$. Further, d divides the degree of E' over F' as $\sigma'|_E = \sigma$, and $[E' : F] = dp$. There are two cases: either $F' = F$ or $F' \neq F$.

Suppose $F' = F$. Then E' is a cyclic extension of F of degree dp , and $\omega'_F = \omega_F^p$. Let K'' be the cyclic extension of F cut out by ω_F^p . By induction, applied to (π', σ', ω') , $K'' \cap E' = F$. Since p divides $[K : N]$ and K'' is the unique subfield of index p in $K \supset F$, it follows that $N \subset K''$. Hence $N = K \cap E = K'' \cap E' = F$.

Before taking up the case $F' \neq F$, we now rule out the case $F' = F$ by showing that π is not cuspidal in this case. By construction, $K' \subset K$ and $E' = EK'$ is contained inside the field $L = KE$. The field K'' is contained inside K . Hence $K''E' \subset KE$. Counting degrees

$$\begin{aligned} [K''E' : E] &= [K''E' : E'][E' : E] = p[K'' : K'' \cap E'] \\ &= p[K'' : F] = [K : F] = [KE : E], \end{aligned}$$

we get $L = KE = K''E'$. Since $\omega_F \circ N_{E/F}$ corresponds to the extension L of E , the character $\omega'_F \circ N_{E'/F} = \omega_F^p \circ N_{E'/F}$ corresponds to the extension $L = KE = K''E'$. The above process can be continued, and the representation π' (and hence π) is automorphically induced from a cuspidal representation π_L of $\mathrm{GL}(n/pr, \mathbf{A}_L)$, where $r = [L : E']$. The equation

$$\sigma'(\pi') \simeq \pi' \otimes \omega'$$

implies that π_L satisfies the condition

$$\sigma''(\pi_L) \simeq \pi_L \otimes \omega^L$$

for some automorphism σ'' of L extending σ' on E' , and of order strictly greater than d , the order of σ . Now ω_K^L is the trivial character. The automorphism σ''^d is a non-trivial automorphism of L trivial on E , as σ'' extends σ on E . But

$$\sigma''^d(\pi_L) \simeq \pi_L \otimes \omega^L \circ N_{L/K} \simeq \pi_L.$$

This implies that π is not cuspidal, contrary to our hypothesis. Hence this rules out the case that $F' = F$.

Hence we are in the situation that $F' \neq F$. Since d divides $[E' : F']$ and $[E' : F] = dp$, $[E' : F'] = d$, we see that $[F' : F] = p$; moreover, we saw that $F' \cap E = F$. The extension E' over F is a compositum of the linearly disjoint extensions E and F' over F . The character $\omega'_{F'} = \omega_F \circ N_{F'/F}$, corresponds to the extension KF' which contains the compositum $NF' \subset E'$. By the induction hypothesis, $KF' \cap E' = F'$. Hence $NF' = F'$, and this implies that $N = K \cap E = F$.

We now show that $F' = K'$. If $K' \neq F'$, the degree of the compositum $[K'F' : F] = p^2$. Since $KF' \cap E' = F'$, $E' = EK' \supset E$ and $K' \subset K$,

$$K'F' \cap E = F' \cap E = F.$$

Since K' and F' are both contained inside E' and $[E : F] = d$, this implies $[E' : F] \geq dp^2$. This contradicts the fact that the degree of E' over F is dp . Hence $F' = K'$.

We also obtain that the representation π is automorphically induced from a cuspidal representation π' of $\mathrm{GL}(n/p, \mathbf{A}_{E'})$ satisfying

$$\sigma'(\pi') \simeq \pi' \otimes \omega'.$$

The field E' is also the compositum of the fields E and K' . The extension of E defined by the character $\omega'_{F'} \circ N_{E'/F'}$ is L . This can be continued, and we obtain a cuspidal representation π_L of $\mathrm{GL}(n/r, \mathbf{A}_L)$ such that

$$\pi \simeq \mathrm{AI}_L^E(\pi_L),$$

where $r = [L : E]$. The representation π_L satisfies,

$$\tau(\pi_L) \simeq \pi_L \otimes \omega^L,$$

where τ is the unique automorphism of L extending σ such that the fixed field of L by τ is equal to K .

Since ω_K^L is trivial, there exists a Hecke character ψ of L such that $\tau(\psi)\psi^{-1} = \omega^L$. Then

$$\tau(\pi_L \otimes \psi^{-1}) \simeq \pi_L \otimes \omega^L \otimes \tau(\psi)^{-1} \simeq \pi_L \otimes \psi^{-1}.$$

Hence the cuspidal representation $\pi_L \otimes \psi^{-1}$ is invariant with respect to the cyclic automorphism group $\text{Gal}(L/K)$ generated by τ . To complete the proof of Theorem 2, we now have to establish descent for cyclic extensions.

Let E/F be a cyclic extension of degree d and let π be a cuspidal representation of $\text{GL}_n(\mathbf{A}_E)$ invariant under the action of $\text{Gal}(E/F) = \langle \sigma \rangle$. Choose an extension $F \subset E' \subset E$ such that $[E : E'] = p$ for some rational prime p . By the descent for cyclic extensions of prime degree, there exists a cuspidal representation π' of $\text{GL}_n(\mathbf{A}_{E'})$ which base changes to π . We need to show that π' is left invariant by $\text{Gal}(E'/F)$. Let σ' denote the restriction of σ to E' . Suppose $\sigma'(\pi') \simeq \pi' \otimes \varepsilon$, where ε is an idele class character of E' which corresponds to the extension E/E' via class field theory. Since E/F is cyclic, $\varepsilon = \eta \circ N_{E'/F}$ for some idele class character η of F defining the cyclic extension E/F by class field theory. If p is prime to $[E' : F]$, $\sigma'^p(\pi') \simeq \pi' \otimes \varepsilon^p \simeq \pi'$, but σ'^p generates $\text{Gal}(E'/F)$, so π' is invariant by $\text{Gal}(E'/F)$.

Assume p^2 divides d . Since η is of order d , $\varepsilon_F = \eta^{d/p}$ defines a non-trivial cyclic extension F'' of degree p of F contained inside E . Then $F'' = E^{\sigma^p} \subset E' = E^{\sigma^{d/p}}$, so $E' \cap F'' \neq F$. This contradicts the first part of Theorem 2 proved above.

Hence π' is left invariant by $\text{Gal}(E'/F)$, and by induction can be descended to a cuspidal representation of $\text{GL}_n(\mathbf{A}_F)$.

Hence we obtain

$$\pi \simeq \text{AI}_L^E(\text{BC}_K^L(\pi_0)),$$

where π_0 is a cuspidal representation of $\text{GL}(n/r, \mathbf{A}_K)$ which is a descent of π_L .

Conversely, it is easy to see from the functorial properties of base change and automorphic induction that any automorphic representation as in part (3) of Theorem 2 satisfies $\sigma(\pi) \simeq \pi \otimes \omega$.

3. Proof of Theorem 3

In this section, we deduce Theorem 3 from Theorem 2, following the arguments given in [15].

Lemma 1. *Let E/F be a solvable extension of number fields. Suppose π is a cuspidal automorphic representation of $\text{GL}_n(\mathbf{A}_F)$ such that its base change to E remains cuspidal. Let χ be a non-trivial idele class character on F such that the composition $\chi \circ N_{E/F}$ is trivial, where $N_{E/F} : C_E \rightarrow C_F$ is the norm map on the idele classes. Then π and $\pi \otimes \chi$ are not isomorphic.*

Proof. Suppose $\pi \otimes \chi$. This implies $\pi \simeq \pi \otimes \chi^k$ for any natural number k . Hence we can assume that χ is of prime order. In this case, χ cuts out a cyclic extension E' of prime degree contained in F . By the characterization of automorphically induced representations [1], it follows that the base change of π to E' is not cuspidal. This contradicts the assumption that the base change of π to E is cuspidal. $\square$

We first prove part (2) of Theorem 3, characterizing the fibers of the base change lift for solvable extensions of number fields. Suppose Π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_E)$. Let π and π' be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ which base change to Π . We need to show that $\pi' \simeq \pi \otimes \chi$ for some Hecke character χ on F such that $\chi \circ N_{E/F} = 1$.

By the results of [1], the theorem is true for cyclic extensions of prime degree. Let $E \supset E_1 \supset F$ be a tower of Galois extensions of F , where E_1/F is of prime degree. Let $\mathrm{BC}_F^{E_1}(\pi) = \pi_1$ and $\mathrm{BC}_F^{E_1}(\pi') = \pi'_1$. By induction, we assume that the theorem is true for the extension E/E_1 . We have $\pi'_1 \simeq \pi_1 \otimes \chi_1$ for some Hecke character χ_1 on E_1 such that $\chi_1 \circ N_{E/E_1} = 1$. Let σ be a generator of $\mathrm{Gal}(E_1/F)$. We have

$$\pi_1 \otimes \chi_1 \simeq \pi'_1 \simeq {}^\sigma \pi'_1 \simeq {}^\sigma \pi'_1 \otimes {}^\sigma \chi_1 \simeq \pi_1 \otimes {}^\sigma \chi_1.$$

Hence

$$\pi_1 \simeq \pi_1 \otimes {}^{1-\sigma} \chi_1.$$

If $\chi_1 \neq {}^\sigma \chi_1$, let f denote the order of ${}^{1-\sigma} \chi_1$, p a prime dividing f , and let

$$\nu = ({}^{1-\sigma})^{f/p} \chi_1.$$

Observe that ν is a non-trivial character of $\mathrm{Gal}(E/E_1)$ of order p satisfying

$$\pi_1 \simeq \pi_1 \otimes \nu.$$

It follows from the characterization of automorphic induction that π_1 is automorphically induced from a cuspidal representation π_ν belonging to the class field E_ν defined by ν . But $E_\nu \subset E$, and it follows that Π is not cuspidal, contrary to our assumption on Π . Hence we have that χ_1 is invariant by $\mathrm{Gal}(E_1/F)$ and descends to an idele class character χ' of C_F such that $\chi_1 = \chi' \circ N_{E_1/F}$. Then

$$\mathrm{BC}_F^{E_1}(\pi \otimes \chi') \simeq \pi_1 \otimes \chi_1 \simeq \pi'_1 \simeq \mathrm{BC}_F^{E_1}(\pi').$$

Hence we have a Hecke character θ corresponding to a character of $\mathrm{Gal}(E_1/F)$ such that

$$\pi' \simeq \pi \otimes \chi' \theta = \pi \otimes \chi,$$

and $\chi = \chi' \theta$ defines a character of $\mathrm{Gal}(E/F)$. This proves part (2) of Theorem 3, as the distinction between π and $\pi \otimes \chi$ follows from the properties of automorphic induction.

We now move on to proving part (1) of Theorem 3. We prove a preliminary lemma, which also proves the uniqueness assertion about ψ in Theorem 3.

Lemma 2. *Let E/F be a solvable extension, and let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Suppose χ is a $\mathrm{Gal}(E/F)$ invariant idele class character of E such that both Π and $\Pi \otimes \chi$ are in the image of base change from F . Then χ lies in the image of base change.*

Proof. The proof is by induction. The lemma is true for cyclic extensions of prime degree, as any invariant character descends. Assume we have $E \supset E_1 \supset F$, with E_1/F cyclic of prime degree p . By induction, $\chi = \chi_1 \circ N_{E/E_1}$ for some idele class character χ_1 of C_{E_1} .

Suppose π_1 and π'_1 are cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_{E_1})$ which base change respectively to Π and $\Pi \otimes \chi$. Since both π'_1 and $\pi_1 \otimes \chi_1$ base change to $\Pi \otimes \chi$, by the description of the fibers of base change proved above, we obtain

$$\pi'_1 \simeq \pi_1 \otimes \chi_1 \eta_1$$

for some Hecke character η_1 of C_{E_1} vanishing on $N_{E/E_1} C_E$.

Assume further, as we may from the hypothesis, that both π_1 and π'_1 lie in the image of base change from F to E_1 . For any $\sigma \in \mathrm{Gal}(E_1/F)$,

$$\pi_1 \otimes \chi_1 \eta_1 \simeq \pi'_1 \simeq {}^\sigma \pi'_1 \simeq \pi_1 \otimes {}^\sigma (\chi_1 \eta_1).$$

Hence, $\pi_1 \simeq \pi_1 \otimes \nu$, where $\nu = {}^\sigma (\chi_1 \eta_1) (\chi_1 \eta_1)^{-1}$. Since χ is $\mathrm{Gal}(E/F)$ -invariant, we have ${}^\sigma \chi_1 = \chi_1 \varepsilon_1^i$ and ${}^\sigma \eta_1 = \varepsilon_1^j$ for some integers i, j , where ε_1 is associated to a cyclic extension of E_1 contained inside E . Hence $\nu = \varepsilon_1^l$ for some integer l . Since Π is cuspidal, the cuspidality criterion for automorphic induction implies that $\nu = 1$. Hence we get that $\chi_1 \eta_1$ is invariant by $\mathrm{Gal}(E_1/F)$. By induction, $\chi_1 \eta_1$ lies in the image of base change from F to E_1 , and it follows that χ also lies in the image of base change from F to E . $\square$

With this lemma, we now proceed to the proof of part (1) of Theorem 3. The proof is by induction on the degree of the extension E over F . By the results of [1], it is true for extensions of prime degree. We now assume there is a sequence of fields

$$E \supset E' \supset F,$$

where E'/F is a cyclic extension of prime degree p . By the inductive hypothesis, there exists a $\mathrm{Gal}(E/E')$ -invariant idele class character ψ_0 of E , and a cuspidal automorphic representation π' of $\mathrm{GL}(n, \mathbf{A}_{E'})$ such that

$$\Pi \otimes \psi_0 = \mathrm{BC}_{E'}^E(\pi').$$

Let τ' be a generator of $\mathrm{Gal}(E'/F)$, and let τ be an element of $\mathrm{Gal}(E/F)$ lifting τ' . Then

$$\mathrm{BC}_{E'}^E({}^{\tau'} \pi') \simeq {}^\tau \Pi \otimes {}^\tau \psi_0 \simeq \Pi \otimes {}^\tau \psi_0 \simeq (\Pi \otimes \psi_0) \otimes {}^\tau \psi_0 \psi_0^{-1}.$$

Since $\mathrm{Gal}(E/E')$ is a normal subgroup of $\mathrm{Gal}(E/F)$, for any $\sigma \in \mathrm{Gal}(E/E')$,

$$\sigma {}^\tau \psi_0 = {}^{\tau(\tau^{-1}\sigma\tau)} \psi_0 = {}^\tau \psi_0.$$

Hence ${}^\tau \psi_0 \psi_0^{-1}$ is $\mathrm{Gal}(E/E')$ -invariant. Since both $\Pi \otimes \psi_0$ and $(\Pi \otimes \psi_0) \otimes {}^\tau \psi_0 \psi_0^{-1}$ lie in the image of base change from E' to E , by Lemma 2 there exists an idele class character χ' of E' such that ${}^\tau \psi_0 \psi_0^{-1} = \chi' \circ N_{E/E'}$. Hence,

$$\mathrm{BC}_{E'}^E({}^{\tau'} \pi' \otimes \chi'^{-1}) \simeq \Pi \otimes \psi_0.$$

By part (2) of Theorem 3, characterizing the fibers of the base change lift, we conclude that there is an idele class character χ'' corresponding via class field theory to a character of $\mathrm{Gal}(E/E')$ such that

$$(3.1) \quad {}^{\tau'} \pi' \simeq \pi' \otimes \chi' \chi'' = \pi' \otimes \eta',$$

where $\eta' = \chi' \chi''$. Further,

$$(3.2) \quad \eta' \circ N_{E/E'} = \chi' \circ N_{E/E'} = {}^\tau \psi_0 \psi_0^{-1}.$$

Write the elements of

$$\mathrm{Gal}(E/F) = \{\tau^{-i}\sigma \mid 0 \leq i < p, \sigma \in \mathrm{Gal}(E/E')\}.$$

We have for $x \in C_E$,

$$\begin{aligned} \eta'_F \circ N_{E/F}(x) &= \eta' \left(\prod_{i=0}^{p-1} \prod_{\sigma \in \mathrm{Gal}(E/E')} \tau^{-i}\sigma(x) \right) = \prod_{i=0}^{p-1} \tau^i \eta'(N_{E/E'}(x)) \\ &= \prod_{i=0}^{p-1} \tau^i (\tau(\psi_0)\psi_0^{-1})(x) \\ &= \tau^p(\psi_0)\psi_0^{-1}(x). \end{aligned}$$

Since ψ_0 is $\mathrm{Gal}(E/E')$ -invariant and $\tau^p \in \mathrm{Gal}(E/E')$, it follows that $\eta'_F \circ N_{E/F}$ is trivial. Hence by part (1) of Theorem 2, and equation (3.1), η'_F is trivial.

Let α be an idele class character of E' , satisfying $\alpha\tau'(\alpha)^{-1} = \eta'$. By equation (3.1),

$$\tau'(\pi' \otimes \alpha) = \tau'(\pi') \otimes \alpha\eta'^{-1} = \pi' \otimes \alpha.$$

Hence $\pi' \otimes \alpha$ is $\mathrm{Gal}(E'/F)$ -invariant, and descends to F . Thus, $\Pi \otimes \psi_0 \otimes (\alpha \circ N_{E/E'})$ descends.

To finish the proof, we have to check that $\psi_0 \otimes (\alpha \circ N_{E/E'})$ is $\mathrm{Gal}(E/F)$ -invariant. For this it is enough to check that $\psi_0 \otimes (\alpha \circ N_{E/E'})$ is τ -invariant:

$$\begin{aligned} \tau(\psi_0 \otimes (\alpha \circ N_{E/E'})) &= \tau(\psi_0) \otimes \tau(\alpha) \circ N_{E/E'} \\ &= \tau(\psi_0) \otimes (\alpha \circ N_{E/E'})(\eta' \circ N_{E/E'})^{-1} \\ &= \psi_0 \otimes (\alpha \circ N_{E/E'}), \end{aligned}$$

where the last equality follows from equation (3.2).

4. Trace formula

We want to prove Theorem 1 ruling out the existence of a cuspidal representation Π of $\mathrm{GL}(n, \mathbf{A}_E)$ satisfying equation (1.1),

$${}^\sigma\Pi \simeq \Pi \otimes \omega,$$

where ω is an idele class character of E such that its restriction to C_F corresponds by Artin reciprocity to a primitive character of the cyclic group $\mathrm{Gal}(E/F)$.

If Π satisfies (1.1), it will, for a suitable choice of a function $\phi \in C_c^\infty(\mathrm{GL}(n, \mathbf{A}_E))$, contribute a non-zero term to the trace

$$(4.1) \quad \mathrm{Trace}(I_\theta(R_{\mathrm{disc}} \otimes \omega^{-1})(\phi)).$$

This follows from strong multiplicity one for cuspidal representations. Here R_{disc} is the discrete part of the representation of $\mathrm{GL}(n, \mathbf{A}_E)$ on

$$\mathcal{A}_2 := L^2(\mathrm{GL}(n, E)A_G \backslash \mathrm{GL}(n, \mathbf{A}_E)).$$

Here A_G is $\mathbb{R}_+^\times$ embedded diagonally into $\mathrm{GL}(n, E_w)$ at all Archimedean places w of E . The operator I_θ is given by $f(g) \mapsto f(\sigma^{-1}(g))$, where $f \in \mathcal{A}_2$ and σ is our chosen generator of $\Sigma = \mathrm{Gal}(E/F)$.

There is a general formula for the trace in equation (4.1) due to Kottwitz–Shelstad [7] and Mœglin–Waldspurger [14]. (In fact, this trace must be completed by Arthur’s “discrete terms”, which we will describe presently in our case.) The formula is

$$(4.2) \quad T_{\text{disc}}(\phi \times \theta; \omega^{-1}) = \sum_{\mathbf{G}' \subset \mathcal{E}} \iota(\mathbf{G}') ST_{\text{disc}}^{\mathbf{G}'}(\phi^{\mathbf{G}'}).$$

Here $\mathbf{G}'$ runs over the elliptic endoscopic data consisting of triples of the form $(G', \mathcal{G}', \tilde{s})$: these will be reviewed in the next paragraph; G' is a reductive F -group and $\phi^{\mathbf{G}'}$ is a function on $G'(\mathbf{A}_F)$ associated to ϕ .

The so-called “stable discrete trace formula” $ST_{\text{disc}}^{\mathbf{G}'}$ will be very simple in our case, as G' will be a group $\text{GL}(m)$: see Section 7. The terms in $T_{\text{disc}}(\phi \times \theta; \omega^{-1})$ are as follows:

- (1) The traces $\text{Trace}(I_\theta(\Pi \otimes \omega^{-1})(\phi))$ for a cuspidal representation Π of $\text{GL}(n, \mathbf{A}_E)$ such that ${}^\sigma \Pi \simeq \Pi \otimes \omega$. The operator I_θ sends an automorphic form $f(g) \mapsto f(\sigma^{-1}(g))$, $g \in \text{GL}(n, \mathbf{A}_E)$. The cusp forms occur with multiplicity one and I_θ is an intertwining operator sending Π to ${}^\sigma \Pi$.
- (2) Similar traces where Π belongs to the discrete spectrum (and is not cuspidal) (see [13]). This means that $n = ab$, and that there exists π_a , a cuspidal representation of $\text{GL}(a, \mathbf{A}_E)$, such that Π is a quotient of the representation

$$\rho = \pi_a | \cdot |^{\frac{b-1}{2}} \boxplus \pi_a | \cdot |^{\frac{b-3}{2}} \boxplus \cdots \boxplus \pi_a | \cdot |^{\frac{1-b}{2}},$$

where $| \cdot |$ denotes the idele norm, seen as a character of $\text{GL}(a, \mathbf{A}_E)$ via the determinant; and the notation $\boxplus$ denotes, as usual, parabolic induction, here from the parabolic subgroup of $\text{GL}(n)$ of type $(a, \dots, a)$. Now if ${}^\sigma \Pi \simeq \Pi \otimes \omega$, the same is true of ρ . Since the representation π_a is almost tempered, this implies that ${}^\sigma \pi_a \simeq \pi_a \otimes \omega$. By induction (since $a < n$), this is impossible.

- (3) There are now the discrete terms defined by Arthur, which do not come from the discrete spectrum. We first consider the simplest case. Here $\Pi = \pi_1 \boxplus \cdots \boxplus \pi_t$, where π_i is a cuspidal representation of $\text{GL}(n_i, \mathbf{A}_E)$, and $\sum_{i=1}^t n_i = n$. We assume then ${}^\sigma \Pi \simeq \Pi \otimes \omega$; of course

$${}^\sigma \Pi = {}^\sigma \pi_1 \boxplus \cdots \boxplus {}^\sigma \pi_t,$$

and this equivalence implies that there is an element $s \in W_M$, the Weyl group corresponding to the Levi component $\text{GL}(n_1) \times \cdots \times \text{GL}(n_t)$ such that

$${}^\sigma \pi_i \simeq \pi_{s(i)} \omega.$$

We must further assume that s is “regular”, i.e., $\mathfrak{a}_M^s = \mathfrak{a}_G$, where $\mathfrak{a}_M$ (resp. $\mathfrak{a}_G$) denote the Lie algebra of the split component of the center of M and G respectively. This implies that M is homogeneous ($n = ab$) and that ${}^\sigma \pi_i \simeq \pi_{s(i)} \omega$, where s is a cyclic permutation of order b . The corresponding term is the trace of the product of $\Pi \omega^{-1}(\phi)$, and of an intertwining operator associated to $s \times \sigma$, defined by Arthur, acting on the space of Π . Its precise form will be irrelevant.

- (4) Finally, we can build similar terms with Π_i cuspidal replaced by a residual, discrete spectrum representations as in (2) above.

We note that all the representations of type (1, 2) occur with multiplicity 1. Furthermore, their Hecke eigenvalues are independent from those of the representations of type (3, 4): fix

a finite set of primes S , containing the Archimedean primes, and consider a finite set of such representations, unramified outside S . Let T be a sufficiently large set of (finite) primes disjoint from S . Our representations, via their Hecke matrices, define characters of the tensor product, over the primes in T , of the local, unramified Hecke algebras. Then these characters, indeed for representations of different types $(1, 2, 3, 4)$, are all distinct.

In the next paragraph, we compute the right-hand side, i.e., the endoscopic terms.

For more information on the endoscopic stabilization of the trace formula, and in particular the use of formula (4.2), we refer to [14]; in particular Sections I.6.4 and X.5.9. Suffice it to say here that if Π is a representation of $\mathrm{GL}(n, \mathbf{A}_E)$ occurring in the left-hand side of (4.2), i.e., in the discrete part of Arthur's trace formula as reviewed above, there will be an endoscopic group G' , and a representation π' of $G'(\mathbf{A}_F)$ such that π' and Π are associated, i.e., the Hecke matrices of Π are deduced at almost all primes from those of π' in a prescribed manner, determined by the endoscopic datum, given in [13, Section 6.4]. In our case, there will be a unique datum $\mathbf{G}'$ (or none at all) and the relation between π' and Π will be quite explicit.

5. Endoscopic data

We now consider the right-hand side of equation (4.2). We must first describe the endoscopic data. We use Waldspurger's formalism for base change [2, 18].

We consider $\mathrm{GL}(n)/E$ as an F -group by restriction of scalars and denote it by G . We will sometimes denote by G_0 the group $\mathrm{GL}(n)$ over E . The generator σ of $\Sigma = \mathrm{Gal}(E/F)$ acts on G by F -automorphisms; as such we denote it by θ . We fix an isomorphism $\sigma \mapsto \iota(\sigma)$ between Σ and $\mathbb{Z}/d\mathbb{Z}$. For $w \in W_F$, $\iota(w)$ is then defined by composition.

The connected component of identity of the dual group of G is

$$\hat{G} = \mathrm{GL}(n, \mathbb{C})^d = \prod_{i \in \mathbb{Z}/d\mathbb{Z}} \mathrm{GL}(n, \mathbb{C});$$

the F -structure on G gives an action of $\mathrm{Gal}(\bar{F}/F)$ on $\hat{G}$ quotienting through Σ :

$$(5.1) \quad \sigma(g_1, \dots, g_d) = (g_{1+\iota(\sigma)}, \dots, g_{d+\iota(\sigma)}).$$

Then ${}^L G = \hat{G} \rtimes W_F$, the action of W_F being so obtained. On the other hand, θ defines an automorphism $\hat{\theta}$ of $\hat{G}$,

$$\hat{\theta}(g_1, \dots, g_d) = (g_2, \dots, g_d, g_1).$$

Suppose we are given a character ω of $\mathbf{A}_E^\times$, which defines via the determinant an abelian character of $G(\mathbf{A}_F) = \mathrm{GL}(n, \mathbf{A}_E)$. By a result of Langlands [11] we can see ω as an element $\mathbf{a} \in H^1(W_F, Z(\hat{G}))$. Note that $Z(\hat{G}) = (\mathbb{C}^\times)^d$, the action of W_F being given by equation (5.1). In general, the element $\mathbf{a}$ is only defined modulo the group

$$\ker^1(F, Z(\hat{G})) = \ker \left(H^1(W_F, Z(\hat{G})) \rightarrow \bigoplus_v H^1(W_{F_v}, Z(\hat{G})) \right),$$

where v ranges over the places of F (see [7, 18]).

In our case, however, Shapiro's lemma implies that

$$H^1(W_F, Z(\hat{G})) = H^1(W_E, \mathbb{C}^\times),$$

with trivial action of W_E . Hence,

$$H^1(W_F, Z(\hat{G})) = \text{Hom}_{ct}(W_E, \mathbb{C}^\times) = \text{Hom}_{ct}(C_E, \mathbb{C}^\times),$$

where C_E is the group of idele classes. Similarly, for a place v of F ,

$$H^1(W_{F_v}, Z(\hat{G})) = \bigoplus_{w|v} H^1(W_{E_w}, \mathbb{C}^\times).$$

Thus, $\ker^1(F, Z(\hat{G}))$ is the group of idele class characters that are locally trivial, so

$$\ker^1(F, Z(\hat{G})) = \{1\}.$$

Now an endoscopic datum for $(G, \theta, \mathbf{a})$ is a triple, $\mathbf{G}' = (G', \mathcal{G}', \tilde{s} = s\hat{\theta})$ subject to the following conditions:

- (E1) G' is a quasisplit connected reductive group over F .
- (E2) $\tilde{s} = s\hat{\theta}$ is a semisimple element in $\hat{G} \rtimes \Theta$, where $\Theta = \langle \hat{\theta} \rangle \simeq \mathbb{Z}/d\mathbb{Z}$.
- (E3) $\mathcal{G}' \subset {}^L G$ is a closed subgroup.
- (E4) There exists a split exact sequence

$$1 \rightarrow \hat{G}_{\tilde{s}} \rightarrow \mathcal{G}' \rightarrow W_F \rightarrow 1,$$

where $\hat{G}_{\tilde{s}}$ is the neutral connected component of the centralizer of $\tilde{s}$ and $\mathcal{G}' \rightarrow W_F$ is induced by the map ${}^L G \rightarrow W_F$. In particular, $\mathcal{G}' \cap \hat{G} = \hat{G}_{\tilde{s}}$. For this action of W_F , $\hat{G}_{\tilde{s}}$ is a dual group of G' .

- (E5) For $(g, w) \in \mathcal{G}'$,

$$s\hat{\theta}(g)w(s)^{-1} = a(w)g,$$

where $a(w)$ is a 1-cocycle of W_F with values in $Z(\hat{G})$ and defining $\mathbf{a}$.

We will denote by H the group G' . We note that any semisimple $\tilde{s} = s\hat{\theta}$ is conjugate to an element $\tilde{s}$ such that $s = (s_0, 1, \dots, 1)$. In this case, $\hat{H} = \hat{G}_{\tilde{s}} = \hat{G}_{0, s_0}$ is diagonally embedded in $\hat{G}$. Here $\hat{G}_{0, s_0}$ is the centralizer of s_0 in $\hat{G}_0$, which is connected. Thus,

$$\hat{G}_{\tilde{s}} = \{(h, \dots, h) \mid h \in \hat{H}\} = \{\text{diag}(h) \mid h \in \hat{H}\},$$

where $\text{diag} : \text{GL}(n, \mathbb{C}) \rightarrow \text{GL}(n, \mathbb{C})^d$ is the diagonal map. We look for

$$\xi : \mathcal{G}' \rightarrow {}^L G,$$

where $\mathcal{G}'$ admits an exact sequence (E4). Thus for $h \in \hat{H}$,

$$\xi : (h, 1) \mapsto (\text{diag}(h), 1),$$

while for $w \in W_F$,

$$\xi : (1, w) \mapsto (n(w), w) = (n(w), 1)(1, w).$$

Here we have chosen a splitting $n : W_F \rightarrow \mathcal{G}'$ for $\mathcal{G}'$. Let us denote by $h \mapsto {}^w h$ the action of W_F on $\hat{H}$ coming from (E4). Then

$$({}^w h, 1) = (n, w)(h, 1)(n, w)^{-1},$$

where we are writing for short $n = n(w)$ and $(h, 1) = (\text{diag}(h), 1)$. Hence

$$({}^w h, 1) = (n, 1)(1, w)(h, 1)(1, w)^{-1}(n, 1)^{-1} = (n, 1)(h, 1)(n, 1)^{-1},$$

since h , being diagonal in $\hat{G}$, is invariant by the action (5.1) of W_F . Write

$$n = n(w) = (n_1, \dots, n_d),$$

so

$$(5.2) \quad n \text{diag}(h)n^{-1} = (n_1 h n_1^{-1}, \dots, n_d h n_d^{-1}) = \text{diag}(h')$$

for some $h' \in \hat{H}$.

We now assume that $s_0 = (s_1, \dots, s_a)$ is given by diagonal scalar matrices s_i of degree b_i with distinct eigenvalues t_i . Then

$$\hat{H} = \prod_{i=1}^a \text{GL}(b_i) \subset \text{GL}(n).$$

Write $a = a_1 + \dots + a_r$ ($a_k \geq 1$) with

$$b_1 = b_2 = \dots = b_{a_1} < b_{a_1+1} = \dots = b_{a_1+a_2} < \dots$$

Since n_i normalizes $\hat{H}$,

$$n_i \in \prod_{k=1}^r \text{GL}(b_k)^{a_k} \rtimes \mathfrak{S}_{a_k}$$

with obvious notation. We choose explicitly as representatives of the Weyl group $\mathfrak{S}_{a_k}$ the obvious block matrices with blocks of size b_k equal to either 0 or 1.

Write $W = \prod_{k=1}^r \mathfrak{S}_{a_k}$, so that the normalizer of $\hat{H}$ is $\hat{H}W$. By equation (5.2),

$$\text{Ad}(n_i)h = \text{Ad}(n_j)h \quad \text{for all } i, j \text{ and all } h \in \hat{H}.$$

Hence $n_i = h_i \tau$, where $\tau \in W$ is independent of i and $h_i \in \hat{H}$; moreover,

$$\text{Ad}(h_i)h = \text{Ad}(h_j)h \quad \text{for all } i, j \text{ and all } h \in \hat{H}.$$

Thus $h_i = (z_{ij})h$ with $(z_{ij}) \in Z(\hat{H}) = (\mathbb{C}^\times)^a$. Hence we can write

$$n(w) = (z_i(w)h(w)\tau(w))_i, \quad \text{where } z_i(w) = (z_{i1}(w), \dots, z_{ia}(w)) \in (\mathbb{C}^\times)^a.$$

In the stabilization of the trace formula, we are only interested in the elliptic endoscopic data, i.e., those such that the neutral component of $Z(\hat{H})^{W_F}$ and of $Z(\hat{G})^{W_F, \hat{\theta}}$ coincide. The second group is equal to $\mathbb{C}^\times$ embedded diagonally in $\text{GL}(n, \mathbb{C})^d$. We have $Z(\hat{H}) = \prod_{k=1}^r (\mathbb{C}^\times)^{a_k}$ and $n(w)$ acts by $\tau(w) \in \prod_k \mathfrak{S}_{a_k}$. Thus $Z(\hat{H})^{W_F}$ is the set of fixed points of the $\tau(w)$, $w \in W_F$. In particular, it contains the product $\prod_k \mathbb{C}^\times$, embedded diagonally in $\prod_k \text{GL}(b_k)$.

If H is elliptic, we see that $r = 1$, so $\hat{H} = \text{GL}(b)^a$ is homogeneous. Furthermore, W_F acts on $(\mathbb{C}^\times)^a$ via $\tau(w) \in \mathfrak{S}_a$. The image of W_F by $w \mapsto \tau(w)$ must therefore be a transitive subgroup of $\mathfrak{S}_a$.

So far we have shown that $\hat{H} = \text{GL}(b)^k$, and

$$n(w) = (n_i(w)), \quad \text{where } n_i(w) = z_i(w)h(w)\tau(w),$$

with $z_i(w) \in Z(\hat{H}) \simeq (\mathbb{C}^\times)^a$.

The group $\mathcal{G}' = \hat{H} \rtimes W_F$ is defined as a semi-direct product, by the conjugation action of $n(w)$ on $\hat{H}$. Dually, $H \times_F \bar{F} \simeq \mathrm{GL}(b)^a / \bar{F}$, where the rational structure will be described presently. In particular, the derived subgroup of H is simply connected. This implies (see [7, Section 2.2]), that $\mathcal{G}'$ is an L -group, i.e., that for a suitable choice of section the action of W_F on $\hat{H}$ preserves a Borel subgroup and a splitting.

We have seen that $n(w) = (z_i(w)h(w)\tau(w))_i$ acts by conjugation on $\hat{H}$. If $h(w) = 1$, this is easily seen to preserve a splitting. Conversely, if $n(w)$ preserves a splitting, it can be conjugated within $\hat{H}$ to a section preserving the trivial splitting: one then checks that the $h(w) \in \hat{H}$ must act trivially by conjugation, so we may assume $h(w) \equiv 1$. With this section (if it is one), $\mathcal{G}' \cong {}^L H$ is naturally embedded in ${}^L G$, whence a homomorphism of L -groups

$$\xi : {}^L H \rightarrow {}^L G.$$

The contribution of this endoscopic datum will be deduced from ξ .

Since now $n(w) = (z_i(w)\tau(w))_i$, we must still check the cocycle relation

$$n(ww') = n(w).wn(w'),$$

where the action of w is given by the structure of ${}^L G$. If $w \in W_F$ is sent to $\sigma^k \in \Sigma$, with $k = \iota(w)$, this says that

$$n_i(ww') = n_i(w)n_{i+k}(w').$$

Write $z_i(w) = (z_{i,\alpha}(w))$ according to the decomposition $Z(\hat{H}) = (\mathbb{C}^\times)^a$ for $\alpha = 1, \dots, a$. Thus

$$z_i(ww')\tau(ww') = z_i(w)\tau(w)z_{i+k}(w')\tau(w'),$$

i.e.,

$$z_i(ww') = z_i(w).\tau z_{i+k}(w')\tau^{-1},$$

with $\tau = \tau(w) \in \mathfrak{S}_a$. Now $\tau((z_\alpha)) = z_{\tau^{-1}\alpha}$, so the cocycle relation reads

$$(5.3) \quad z_{i,\alpha}(ww') = z_{i,\alpha}(w)z_{i+k,\tau^{-1}\alpha}(w'),$$

where $k = \iota(w)$.

6. Endoscopy, with character

We now have to introduce the character ω in the endoscopic computations. This intervenes through formula (E5) in the definition of endoscopic datum. We want to make the element $\mathbf{a} \in H^1(W_F, Z(\hat{G}))$, or rather a representative $a \in Z^1(W_F, Z(\hat{G}))$, explicit. We write for $w \in W_F$,

$$a(w) = (a_i(w)), \quad a_i(w) \in \mathbb{C}^\times.$$

Since ω is a character of C_E , it can be identified with an element of $H^1(W_E, \mathbb{C}^\times)$. We now need Shapiro's lemma. For its explicit description, we follow Langlands [10] (see also Serre [17]). Recall that

$$W_E \backslash W_F \simeq \mathrm{Gal}(E/F) \simeq \mathbb{Z}/d\mathbb{Z},$$

the isomorphism sending the generator σ of Σ to 1. We choose a representative $\sigma \in W_F$ of this generator, which we also denote by σ . Now $\{\sigma, \sigma^2, \dots, \sigma^d\}$ are representatives of $W_E \backslash W_F$. Note that $\sigma^d \neq 1$ as follows from class field theory, cf. (7.4), (7.5) below.

For any $w \in W_F$,

$$\sigma^i w = \delta_i(w) \sigma^j,$$

where $j \equiv i + k \pmod{d}$ if $k = \iota(w)$ (see equation (5.1)) and $\delta_i(w) \in W_E$. We set

$$a_i(w) = \omega(\delta_i(w)).$$

For $w \in W_E$, $\sigma^i w = \delta_i(w) \sigma^i$, so

$$(6.1) \quad a_i(w) = (\omega(\sigma^i w \sigma^{-i})) = \omega(\sigma^i w) = \omega \circ \sigma^i(w),$$

since the lifting σ acts by conjugation, on the abelianized Weil group C_E of E , through its image in $\text{Gal}(E/F)$.

Consider our chosen lift $\sigma \in W_F$. Then

$$\sigma^i \sigma = \sigma^{i+1} = \delta_i(\sigma) \sigma^{[i+1]},$$

where $[i + 1]$ is the representative of $i + 1$ in $\{1, \dots, d\}$. The foregoing equation implies

$$(6.2) \quad \delta_i(\sigma) = 1 \quad \text{for } i = 1, \dots, d-1, \quad \delta_d(\sigma) = \sigma^d \in W_E.$$

This defines completely $a(w)$. Now,

$$\phi : W_F \rightarrow Z(\hat{G}) \rtimes W_F, \quad w \mapsto (a_i(w), w),$$

defines an L -parameter for the L -group of $\text{Res}_{E/F} \text{GL}(1)$, corresponding to the character ω .

We now have to introduce the condition

$$\omega|_{A_F^\times} = \varepsilon_{E/F}.$$

In cohomological terms, this is given by the corestriction

$$\text{Cor} : H^1(W_E, \mathbb{C}^\times) \rightarrow H^1(W_F, \mathbb{C}^\times)$$

dual to the transfer map $W_F/W_F^{\text{der}} \rightarrow W_E/W_E^{\text{der}}$. Explicitly, the transfer is given as follows ([17, Chapter VII, Section 8]): it associates to $w \in W_F$ the image in W_E/W_E^{der} of $\prod_i \delta_i(w)$. Our condition is therefore, for $w \in W_F$,

$$(6.3) \quad \prod_i a_i(w) = \prod_i \omega(\delta_i(w)) = \varepsilon_{E/F}(w),$$

where both sides are seen as characters of the Weil groups (recall that $\varepsilon_{E/F}$ is here seen as a character of the Galois group). For $w \in W_E$, $\varepsilon_{E/F}(w) = 1$ and the left-hand side is

$$\prod_i \sigma^i \omega(w) = \omega(N_{E/F} z),$$

where $z \in C_E$ is the image of w . Since ω restricts trivially to $N_{E/F}(C_E) \subset C_F$, relation (6.3) is satisfied.

On the other hand, $(\delta_i(\sigma)) = (1, \dots, 1, \sigma^d)$, thus equation (6.3) is equivalent to

$$a(\sigma) = (1, \dots, \zeta),$$

where $\zeta = \varepsilon_{E/F}(\sigma)$ is a primitive root of unity of order d . (In particular, $\omega(\sigma^d) = \zeta$ when σ^d is seen as an element of $W_{E/F}$, hence of C_E , etc.)

Now consider condition (E5) on the endoscopic group

$$(6.4) \quad s\hat{\theta}(g)w(s)^{-1} = a(w)g, \quad (g, w) \in \mathcal{G}'.$$

For $w = 1$, this is the condition defining $\hat{H} = \hat{G}'$. Consider the image $(n(w), w)$ of the section n for $w \in W_F$. We have

$$n(w) = (z_i(w)\tau(w))_i.$$

Write for simplicity τ, z_i, a_i for $\tau(w), z_i(w), a_i(w)$, respectively. If $k = \iota(w)$, then

$$w(s) = w(s_0, 1, \dots, 1) = (1, \dots, s_0, \dots, 1),$$

where s_0 occurs at the place $l = d + 1 - k$ with the convention that $l = 1$ if $k = 0$. Equation (6.4) reads

$$(6.5) \quad (s_0, 1, \dots, 1)(z_2\tau, \dots, z_1\tau)(1, \dots, 1, s_0^{-1}, \dots, 1) = ((a_i z_i \tau))_i.$$

Thus,

$$(6.6) \quad \begin{aligned} s_0 z_2 \tau &= a_1 z_1 \tau, \\ z_3 \tau &= a_2 z_2 \tau, \\ &\vdots \\ z_{l+1} \tau s_0^{-1} &= a_l z_l \tau, \\ &\vdots \\ z_1 \tau &= a_d z_d \tau. \end{aligned}$$

(If $k = 0$, $s_0 z_2 \tau s_0^{-1} = a_1 z_1 \tau$, etc.; if $k = 1$, $l = d$ and the last equation is $z_1 \tau s_0^{-1} = a_d z_d \tau$.) Write $\tau s_0^{-1} = \tau s_0^{-1} \tau^{-1}$. Then equation (6.6) is equivalent to

$$(6.7) \quad \begin{aligned} s_0 z_2 &= a_1 z_1, \\ z_3 &= a_2 z_2, \\ &\vdots \\ z_{l+1} \tau s_0^{-1} &= a_l z_l, \\ &\vdots \\ z_1 &= a_d z_d. \end{aligned}$$

Note that all these elements are contained in $Z(\hat{H})$, hence commute. (For $k = 0$, the first line is $s_0 z_0 \tau s_0^{-1} = a_1 z_1$; for $k = 1$, the last line is $z_1 \tau s_0^{-1} = a_d z_d$.) Taking the product, we see that

$$s_0 \tau s_0^{-1} = \prod a_i = \varepsilon_{E/F}(w),$$

by equation (6.3), so $s_0 = \varepsilon_{E/F}(w) \tau s_0$.

Now $s_0 \in Z(\hat{H}) = (\mathbb{C}^\times)^a$, and $\tau \in \mathfrak{S}_a$. Write $s_0 = (s_{0,\alpha})$, $\alpha = 1, \dots, a$. Thus we have $\tau s_0 = (s_{0,\tau^{-1}\alpha})$, whence

$$(6.8) \quad s_{0,\tau\alpha} = \varepsilon_{E/F}(w) s_{0,\alpha}.$$

Assume that $w \in W_F$ is sent to the chosen generator $\sigma \in \Sigma$, so $\varepsilon_{E/F}(w) = \zeta$. Recall that $s_0 \in Z(\hat{H})$ is given by block-diagonal matrices $s_{0,\alpha}$ of the size b with distinct eigenvalues. Equation (6.8) now implies that the $s_{0,\alpha}$ can be partitioned into $a' = \frac{a}{d}$ subsets of the form

$$(6.9) \quad (s_1, \zeta s_1, \dots, \zeta^{d-1} s_1; s_2, \zeta s_2, \dots, \zeta^{d-1} s_2; \dots);$$

the entries being block-diagonal, we assimilate them to scalars. In particular, $d \mid a$, so $d \mid n$. The scalars s_j ($j = 1, \dots, a'$) verify $s_j \neq \mu s_{j'}$ for any $\mu \in \mu_d(\mathbb{C})$. Equation (6.8) now uniquely determines $\tau_1 = \tau(\sigma)$: it is a product of a' d -cycles.

Consider now an arbitrary element $w \in W_F$. If $w \in W_E$, $\varepsilon_{E/F}(w) = 1$ and equation (6.8) implies that $\tau = 1$, the eigenvalues being distinct. Thus W_F acts via $\Sigma = W_F/W_E$ and $\tau(w) = \tau_1^{(w)}$. The image of W_F is therefore a cyclic subgroup of $\mathfrak{S}_a$, of order d , preserving the strings of length d in equation (6.9). The ellipticity of H now implies that this action is transitive, so $a = d$.

Since we have determined $n(w)$ up to central elements in $\hat{H}$, we have now computed the F -group H . Indeed, $\hat{H} = \mathrm{GL}(b) \times \dots \times \mathrm{GL}(b)$ ($a = d$ factors) and W_F acts via Σ , cyclically permuting the factors. This implies that H is isomorphic to $\mathrm{Res}_{E/F}(\mathrm{GL}(b)/E)$. This embedding into $\hat{G}$ is given on $\hat{H}$ by

$$(h_1, \dots, h_d) \mapsto \mathrm{diag}(h_1 \oplus \dots \oplus h_d) \in \mathrm{GL}(n, \mathbb{C})^d.$$

We can now summarize the main result of this section:

Proposition 4. *The following statements hold:*

- (1) *If $d \nmid n$, there exists no elliptic endoscopic group for (θ, ω) .*
- (2) *If $n = db$, there exists (at least) one endoscopic datum for (θ, ω) given by the foregoing construction.*

To complete the proof of the proposition, we still have to show that we can choose the $z_i(w)$ so as to satisfy equations (E5) and (6.5). We will obtain in fact a more precise result.

The permutation τ associated to $w \in W_F$ is a cyclic permutation on the indices α . We have $\tau(z_\alpha)\tau^{-1} = (z_{\tau^{-1}\alpha})$. We now assume that for $\iota(w) = k$, $\tau^{-1}(\alpha) = \alpha - k$. Relation (5.3) now reads

$$(6.10) \quad z_{i,\alpha}(ww') = z_{i,\alpha}(w)z_{i+k,\alpha-k}(w').$$

We consider all indices as elements of $\mathbb{Z}/d\mathbb{Z}$. Now fix $\beta \pmod{d}$. Now equation (6.10) yields for $i + \alpha = \beta$,

$$z_{i,\beta-i}(ww') = z_{i,\beta-i}(w)z_{i+k,\beta-i-k}(w').$$

Set $\zeta_i^\beta(w) = z_{i,\beta-i}(w)$: we now have

$$\zeta_i^\beta(ww') = \zeta_i^\beta(w)\zeta_{i+k}^\beta(w').$$

This can be understood as follows. Let $\hat{H}_0 = \mathrm{GL}(b)^d$ be the “absolute” dual group of H . We have $\hat{H}_0^d \subset \hat{G} = \mathrm{GL}(n)^d$. The elements $z(w) = (z_{i,\alpha}(w))$ belong to the center $\hat{U} = \prod_{i,\alpha} \mathbb{C}^\times$ of $(\hat{H}_0)^d$. The datum (now fixed) τ determines an action of W_F on $\hat{U}$, by $(z_{i,\alpha}) \mapsto (z_{i+k,\alpha-k})$. By (6.13), $z_{i,\alpha}$ is a 1-cocycle of W_F , with values in $\hat{U}$, for this action.

For β fixed, the factor of $\hat{U}$ given by $i + \alpha = \beta$, isomorphic to $(\mathbb{C}^\times)^d$, is preserved by this action, and the resulting action of W_F on $(\mathbb{C}^\times)^d$ is the action defined by seeing $(\mathbb{C}^\times)^d$ as the dual group of $\mathrm{Res}_{E/F}(\mathrm{GL}(1))$. However, $H^1(W_F, (\mathbb{C}^\times)^d)$, for W_F acting by its action on the dual group of $\mathrm{Res}_{E/F}(\mathrm{GL}(1))$ is equal to $H^1(W_E, \mathbb{C}^\times) = \mathrm{Hom}(C_E, \mathbb{C}^\times)$ by Shapiro’s lemma. Thus each character η_β of C_E defines such a cocycle, by $\zeta_i^\beta(w) = \eta_\beta(\delta_i(w))$. We can then set

$$(6.11) \quad z_{i,\alpha}(w) = \eta_{i+\alpha}(\delta_i(w)),$$

and we see that a section (and therefore a subgroup) is defined by the choice of the characters η_β . Call

$$\xi_1 : {}^L H \rightarrow {}^L G$$

the corresponding morphism.

For further reference, we note that $\hat{U} \subset \hat{G}$ is the centralizer, in $\hat{G}$, of $\hat{H}$ (for its diagonal embedding.)

We still have to fulfill the condition given by equation (6.7). Assume first that $w \in W_E$. Then $\tau = 1$, and the condition is simply

$$z_{i+1}(w) = a_i(w)z_i(w).$$

Now $z_i = (z_{i,\alpha})$ with $z_{i,\alpha} = \eta_{i+\alpha}(\delta_i(w)) = \eta_{i+\alpha} \circ \sigma^i(w)$ by equation (6.11) and the description, before (6.1), of the cocycle associated to a character. Applying the same to ω and $a(w)$ ($w \in W_E$), we see that the condition is

$$\eta_{i+1+\alpha} \circ \sigma^{i+1}(w) = \omega \circ \sigma^i(w) \eta_{i+\alpha} \circ \sigma^i(w)$$

(w being seen as an element of $C_E \cdots$), i.e.,

$$\eta_{\beta+1} \circ \sigma = \omega \eta_\beta, \quad \beta = 1, \dots, d.$$

We can now write the condition of equation (6.7) for $w = \sigma$. Recall from equation (6.2) that $(\delta_i(\sigma)) = (1, \dots, 1, \sigma^d)$ with $\sigma^d \in W_E$. We still have $z_{i,\alpha}(w) = \eta_{i+\alpha}(\delta_i(w))$, whence

$$\begin{aligned} z_{i,\alpha}(\sigma) &= 1, \quad 1 \leq i \leq d-1, \\ z_{d,\alpha}(\sigma) &= \eta_\alpha(\sigma^d). \end{aligned}$$

Moreover, as before,

$$\eta_{\alpha+1} \circ \sigma(\sigma^d) = \omega(\sigma^d) \eta_\alpha(\sigma^d),$$

and $\sigma \sigma^d \sigma^{-1} = \sigma^d$, $\omega(\sigma^d) = \zeta$, whence

$$(z_{d,\alpha}(\sigma))_\alpha = (\eta, \zeta \eta, \dots, \zeta^{d-1} \eta),$$

where $\eta = \eta_1(\sigma^d)$. Write $z_i = (z_{i,\alpha}) = (z_{i,\alpha}(\sigma))$. Then equation (6.7) for σ reads

$$\begin{aligned} (6.12) \quad s_0 z_2 &= z_1, \\ z_3 &= z_2, \\ &\vdots \\ z_d &= z_{d-1}, \\ \tau_{s_0}^{-1} z_1 &= \zeta z_d, \end{aligned}$$

and is obviously not satisfied. Recall that for $k = k(\sigma) = 1$ we have set

$$(\tau_{s_0})_\alpha = s_{0,\tau^{-1}\alpha} = s_{0,\alpha-1}$$

and, cf. equation (6.8), $s_{0,\tau\alpha} = s_{0,\alpha+1} = \zeta s_{0,\alpha}$. We can choose

$$s_0 = (1, \zeta, \dots, \zeta^{d-1}).$$

Substituting $s_0 {}^\tau s_0^{-1} = \zeta$ in equation (6.12), we get

$$(6.13) \quad \begin{aligned} s_0 z_2 &= z_1, \\ z_3 &= z_2, \\ &\vdots \\ z_d &= z_{d-1}, \\ s_0^{-1} z_1 &= z_d. \end{aligned}$$

Now we can replace the z_i by cohomologous elements for the action of W_F on $\hat{U}$ described above, and giving the cocycle relation (5.3), i.e.,

$$w(z_{i\alpha}) = z_{i+k, \alpha-k}.$$

A coboundary is given for $w \in W_F$, $k = \iota(w)$, by

$$\zeta_{i\alpha}(w) = v_{i\alpha} v_{i+k, \alpha-k}^{-1}.$$

In particular, for $w = \sigma$,

$$\zeta_{i\alpha}(\sigma) = v_{i\alpha} v_{i+1, \alpha-1}^{-1}.$$

Fix $\beta \in \mathbb{Z}/d\mathbb{Z}$. Proceeding as we did for W_E , let us write $u_i^\beta := \zeta_{i\alpha}(\sigma)$ with $\beta = i + \alpha$. Thus we have $u_i^\beta = v_i^\beta (v_{i+1}^\beta)^{-1}$, with $v_i^\beta = v_{i, \beta-i}$. The u_i^β must then satisfy for each β the condition

$$(6.14) \quad \prod_i u_i^\beta = 1.$$

Replacing the cocycle z_i by the cohomologous cocycle $z_i \zeta_i(\sigma) = (z_{i\alpha} u_i^{i+\alpha})_\alpha$, our equations then become, with $\alpha = 1, \dots, d$,

$$\begin{aligned} s_{0,\alpha} u_2^{2+\alpha} &= u_1^{1+\alpha}, \\ u_3^{3+\alpha} &= u_2^{2+\alpha}, \\ &\vdots \\ u_{d-1}^{d-1+\alpha} &= u_{d-2}^{d-2+\alpha}, \\ z_{d,\alpha} u_d^\alpha &= u_{d-1}^{d-1+\alpha}, \\ s_{0,\alpha}^{-1} u_1^{1+\alpha} &= z_{d,\alpha} u_d^\alpha. \end{aligned}$$

Write $u = u_2 = (u^\alpha)$. Then the $d - 3$ lines yield

$$u_{i+1}^{1+\alpha} = u_i^\alpha, \quad i = 2, \dots, d-2,$$

so

$$(6.15) \quad u_i^\alpha = u^{\alpha-i+2}, \quad i = 2, \dots, d-1.$$

Thus

$$\begin{aligned} s_{0,\alpha} u^{2+\alpha} &= u_1^{1+\alpha}, \\ z_{d,\alpha} u_d^\alpha &= u^{\alpha+2}, \\ s_{0,\alpha}^{-1} u_1^{1+\alpha} &= z_{d,\alpha} u_d^\alpha; \end{aligned}$$

these equations are obviously compatible. We determine u_1 and u_d by the first equations. Thus,

$$u_1^\alpha = s_{0,\alpha-1} u^{\alpha+1}, \quad u_d^\alpha = z_{d,\alpha}^{-1} u^{\alpha+2}.$$

However, we must choose the u_i^β verifying, for each β , equation (6.14). (Note that we have retained for the upper indices the notation α , naturally stemming from formulas (6.13). We now use, more naturally, β .) The product is then

$$\prod_{i=1}^d u_i^\beta = s_{0,\beta-1} z_{d,\beta}^{-1} u^{\beta+1} \left(\prod_{i=2}^{d-1} u^{\beta+2-i} \right) u^{\beta+2}.$$

The range of upper indices affecting u is the translate by β of

$$\{1, 0, -1, -2, \dots, 3-d, 2\} = \{1, 2, \dots, d\} \bmod d.$$

The condition is therefore

$$\prod_{i=1}^d u^i = z_{d,\beta} s_{0,\beta-1}^{-1},$$

which must be true for any β . But

$$z_{d,\beta} s_{0,\beta-1}^{-1} = \eta \zeta^{\beta-1} (\zeta^{\beta-2})^{-1} = \eta \zeta,$$

independently of β .

We can now define the new cocycle in $Z^1(W_F, \hat{U})$ by multiplying the previous map by the coboundary just obtained. It defines a new section $\mathbf{s}_2$, which verifies the defining condition (6.7), obviously for $w \in W_E$, for all powers of σ , and therefore for $w \in W_F$. This proves the second part of Proposition 4, and moreover it exhibits an explicit section.

Recall that ξ_1 was defined by a cocycle $z(w)$, which we now denote by $z^1(w)$:

$$(6.16) \quad \begin{aligned} \xi_1 : h &\mapsto \text{diag}(h) \quad (h \in \hat{H}), \\ w &\mapsto (\mathbf{s}_1(w), w) = (z^1(w)\tau, w) \quad (w \in W), \end{aligned}$$

so $(h, w) \mapsto (hz^1(w)\tau, w)$. Similarly, we define ξ_2 , replacing z^1 by z^2 , where z^2 is our cohomologous cocycle

$$z^2(w)_i = (z^1(w)_i) v_i (\tau v \tau^{-1})_{i+k}^{-1}.$$

We have moreover:¹⁾

Lemma 3. *The new section $\mathbf{s}_2$ is conjugate in $\hat{G}$ to the section $\mathbf{s}_1$ given by the $z_{i,\alpha}$. In particular, ξ_1 and ξ_2 define (up to conjugation by an element of $\hat{G}$) the same embedding ${}^LH \rightarrow {}^LG$.*

This is clear: if we define by $\mathbf{s}_2$ the new section, so

$$\mathbf{s}_1(w) = z^1(w)\tau, \quad \mathbf{s}_2(w) = z^2(w)\tau,$$

with $\tau = \tau(w)$, and, with $k = \iota(w)$,

$$z^2(w)_i = (z^1(w)_i) v_i (\tau v \tau^{-1})_{i+k}^{-1},$$

¹⁾ We could avoid this verification by using [3, Lemma 4.5]. It will be clearer explicitly to exhibit the conjugation.

with $v = (v_i) \in \hat{U}$, τ acting diagonally. Thus,

$$\mathbf{s}_2(w) = z^1(w)v\tau(v_{i+k})^{-1}$$

so

$$\mathbf{s}_2(w) = v(z^1(w)\tau)w(v)^{-1},$$

and $(\mathbf{s}_2(w), w)$ is conjugate by $\hat{G}$ to $(\mathbf{s}_1(w), w)$.

In particular, ξ_1 and ξ_2 send ‘‘Hecke matrices’’ ($\hat{H}$ -conjugacy classes in ${}^L H$) to the same conjugacy classes in ${}^L G$, so they yield the same endoscopic transfer from automorphic representations of $H(\mathbf{A}_F)$ to automorphic representations of $G(\mathbf{A}_F)$.

Recall that two endoscopic data

$$\mathbf{G}'_1 = (G'_1, \mathcal{G}'_1, \tilde{s}_1) \quad \text{and} \quad \mathbf{G}'_2 = (G'_2, \mathcal{G}'_2, \tilde{s}_2)$$

are equivalent if there exists $g \in \hat{G}$ such that

$$g\mathcal{G}'_1 g^{-1} = \mathcal{G}'_2, \quad g\tilde{s}'_1 g^{-1} = x\tilde{s}_2$$

for an element $x \in Z(\hat{G})$. (Recall that $\tilde{s}_i = (s_i, \hat{\theta})$ with $s_i \in \hat{G}$.)

Proposition 5 (Waldspurger). *Assume that $d \mid n$. Then there exists only one equivalence class of elliptic endoscopic data for (G, θ, a) .*

Consider first the case of our particular cocycle a . It follows from our analysis that we must take $s_0 = (1, \zeta, \dots, \zeta^{d-1})$ (in fact, block matrices of size b) up to a scalar. For two choices we therefore have $g\tilde{s}_1 g^{-1} = x\tilde{s}_2$, $x \in Z(\hat{G})$ (take $g \in \text{GL}(n)^d$ diagonal). We can therefore assume that s_0 is fixed. Then $\hat{H} \subset \hat{G}$ is well defined.

If $\mathcal{G}' \subset \hat{G} \rtimes W_F$ is an endoscopic subgroup, its fiber $\mathcal{G}'_w$ over $w \in W_F$ is equal to $\hat{G}_{\tilde{s}} \mathbf{s}(w)$ for any section $\mathbf{s}$. In particular, it is equal to $\hat{H} \mathbf{s}(w)$ in our case, with of course $\hat{H} \subset \hat{G}$ given by the diagonal embedding.

Our previous analysis shows that a section $\mathbf{s}$ is of the form $\mathbf{s}(w) = z(w)\tau$, $z(w)$ being an element of $Z^1(W_F, \hat{U})$. If $\mathbf{s}, \mathbf{s}'$ are two sections, $\mathbf{t} = \mathbf{s}(\mathbf{s}')^{-1}$ is an element of $Z^1(W_F, \hat{U})$. Now $\mathbf{s}, \mathbf{s}'$ have to verify (E5). This implies that $s\hat{\theta}(\mathbf{t}(w))s^{-1} = \mathbf{t}(w)$, i.e., $\mathbf{t}(w) \in \hat{H}$ (diagonally embedded.) Thus $\mathbf{s}(w) \in \hat{H}\mathbf{s}'(w)$.

However, the description of the endoscopic subgroups also depended on the choice of a representative a of the cocycle $\mathbf{a} \in Z^1(W_F, Z(\hat{G}))$. Assume then that $\mathcal{G} = \{(\mathbf{s}(w), w)\}$ verifies (E5) for a cocycle b , and $\mathcal{G}' = \{(\mathbf{s}'(w), w)\}$ for a cocycle a . Then $a(w) = b(w)x/w(x)$ for an element $x \in Z(\hat{G})$. We first note that the description of s_0 and $\hat{H}$ was independent of the explicit cocycle, cf. (6.7). Thus we may, as before, assume s_0 fixed. Conjugation by $y \in \hat{G}$ sends $(g, w) \in \mathcal{G}$ to $(ygw(y)^{-1}, w) \in \mathcal{G}'$. Thus, assuming (E5),

$$(6.17) \quad s\hat{\theta}(g)w(s)^{-1} = b(w)g,$$

we must deduce

$$s\hat{\theta}(gyw(y)^{-1})w(s)^{-1} = b(w)(x/wx)(ygw(y)^{-1}).$$

Assume $y \in Z(\hat{G})$. The last equation is easily seen, given (6.17), to be equivalent to

$$(6.18) \quad \hat{\theta}(y/wy) = (x/wx)(y/wy).$$

Note that W_F now acts via Σ . It is easily seen that (6.18) is verified if it is verified for the generator σ . Now $\hat{\theta}(y/\sigma y)(\sigma y/y)$ is of the form $\hat{\theta}(u)/u$ for $u = y/\sigma y$; these elements u are exactly the elements of $Z(\hat{G})$ verifying $\prod u_i = 1$. We have to solve (replacing u by u^{-1}):

$$(6.19) \quad \hat{\theta}(u)/u = \sigma x/x.$$

Recall that $\hat{\theta}$ and σ act by the same permutation $(x_1, x_2, \dots, x_d) \mapsto (x_2, x_3, \dots, x_1)$ on $Z(\hat{G})$. A solution is $u = x$, but x does not in general satisfy $\prod x_i = 1$. However, if we set $u_i = vx_i$ for $v \in \mathbb{C}^\times$, u now verifies (6.19) and is an adequate element if $v^d \prod x_i = 1$.

We have shown that conjugation by some element $y \in Z(\hat{G})$ sends the fiber $\mathcal{G}_w$ to the fiber $\mathcal{G}'_w$. Since these fibers are principal homogeneous spaces under $\hat{H}$, which commutes with y , they are conjugate. Since $y\tilde{s}y^{-1} = y\hat{\theta}(y)^{-1}\tilde{s}$, the second condition for equivalence is also satisfied. This concludes the proof.

7. Proof of Theorem 1

7.1. We now have to understand the effect of our homomorphism ξ_1 of L -groups on the data pertinent to the stabilization, i.e., on the data composed of Hecke matrices for almost all primes. Note that

$${}^L H = \mathrm{GL}(b, \mathbb{C})^d \rtimes W_F$$

is not a direct product, so a ‘‘Hecke matrix’’ at a prime v is in fact a conjugacy class in $\hat{H} \times \mathrm{Frob}_v$ under the conjugation action of $\hat{H}$.

For this we must first consider a simple case. Assume that $\omega = 1$, so we are in the case of non-twisted base change, i.e., characterizing the representations Π_n of $\mathrm{GL}(n, \mathbf{A}_E)$ such that ${}^\sigma \Pi_n \simeq \Pi_n$.

Recall [1, 4] the two natural operations associated to (cyclic) base change. The first is automorphic restriction, denoted earlier by BC_F^E , sending representations of $\mathrm{GL}(n, \mathbf{A}_F)$ to representations of $\mathrm{GL}(n, \mathbf{A}_E)$. It is associated to the diagonal embedding ${}^L G_0 \rightarrow {}^L G$, where ${}^L G_0$ is the L -group of $\mathrm{GL}(n)/F$, ${}^L G$ the L -group of $\mathrm{Res}_{E/F}(\mathrm{GL}(n)/E)$, so $\hat{G} = \mathrm{GL}(n, \mathbb{C})^d$:

$$(g, w) \mapsto (\mathrm{diag}(g), w) \quad (w \in W_F).$$

Suppose $n = db$. The second operation is automorphic induction, denoted by AI_E^F , sending representations of $\mathrm{GL}(b, \mathbf{A}_E)$ to those of $\mathrm{GL}(n, \mathbf{A}_F)$. The associated embedding ${}^L H \rightarrow {}^L G_0$ is given by

$$(g_1, \dots, g_d) \mapsto g_1 \oplus \dots \oplus g_d \quad (g_i \in \mathrm{GL}(b, \mathbb{C}))$$

and

$$(1, w) \mapsto (\tau(w), w),$$

where $\tau(w) \in \mathfrak{S}_d$ (realized as before by block-scalar matrices in $\mathrm{GL}(n, \mathbb{C})$), and

$$\tau(\oplus g_i) \tau^{-1} = (g_{i+k}),$$

where $k = i(w)$, so $\tau^{-1}i = i + k$.

We simply write Res and Ind for these two operations, the fields being here F and E . This corresponds to our constructions in [1], taking $\omega = 1$. The corresponding operations are described in [1, Chapter 3], see also [4]. They are well-defined for representations that are ‘‘induced from cuspidal’’ ([1, Sections 3.1, 3.6]), i.e., induced from unitary cuspidal representations.

Composing these two operations, we get a homomorphism of L -groups

$$\mu_1 : \mathrm{GL}(b)^d \rtimes W_F \rightarrow \mathrm{GL}(n)^d \rtimes W_F,$$

given by

$$(7.1) \quad \begin{aligned} (g_1, \dots, g_d) &\mapsto \mathrm{diag}(g_1 \oplus \dots \oplus g_d), \\ (1, w) &\mapsto (\mathrm{diag} \tau(w), w). \end{aligned}$$

Recall also that for π_i ($i = 1, \dots, r$) representations of $\mathrm{GL}(n_i, \mathbf{A}_E)$, there is an associated representation $\boxplus \pi_i$ of $\mathrm{GL}(n, \mathbf{A}_E)$ obtained by parabolic induction ($n = \sum n_i$). We recall the following well-known result:

Proposition 6. *For π_b a representation of $\mathrm{GL}(b, \mathbf{A}_E)$ induced from cuspidal,*

$$\mathrm{Res} \circ \mathrm{Ind}(\pi_b) = \pi_b \boxplus \sigma \pi_b \boxplus \dots \boxplus \sigma^{d-1} \pi_b.$$

Consider finite primes $w \mid v$ of E over F , where all data are unramified. If $t_{w'}$ is the Hecke matrix of π_b at such a prime w' , the matrix T_v of $\mathrm{Ind}(\pi_b)$ at v is

$$T_v = \bigoplus_{w' \mid v} (t_{w'}^{1/f} \oplus \zeta t_{w'}^{1/f} \oplus \dots \oplus \zeta^{f-1} t_{w'}^{1/f}),$$

where ζ is a primitive root of unity of order $f = [E_w : F_v]$. The Hecke matrix T_w of $\mathrm{Res}(\pi)$ for a representation π of $\mathrm{GL}(n, \mathbf{A}_F)$ is T_v^f . Thus the Hecke matrix of $\mathrm{Res} \circ \mathrm{Ind}(\pi_b)$ at a prime w is $\bigoplus_{w' \mid v} (t_{w'} \oplus \dots \oplus t_{w'})$, equal to $\bigoplus_{\sigma \in \mathrm{Gal}(E/F)} t_{\sigma w}$, the Hecke matrix of the right-hand side. Since the representations on the two sides of the equality are induced from cuspidal, they are equal.

Now there exists an obvious homomorphism ${}^L H \rightarrow {}^L G$ realizing the operation²⁾

$$\pi_b \mapsto \pi_b \boxplus \sigma \pi_b \boxplus \dots \boxplus \sigma^{d-1} \pi_b = \sigma \pi_b \boxplus \dots \boxplus \sigma^{d-1} \pi_b \boxplus \pi_b.$$

Let $\hat{H}^d = \prod_{k=1}^d \hat{H}$. First $(\pi_1, \dots, \pi_d) \mapsto \pi_1 \boxplus \dots \boxplus \pi_d$ is given by

$$(7.2) \quad \hat{H}^d \rightarrow \hat{G}, \quad (g_{ki}) \mapsto \left(\bigoplus_k g_{ki} \right)_i,$$

where k corresponds to the factor in $\hat{H}^d$ (or to π_k) and i is the index associated to restriction of scalars in $\hat{H} = \mathrm{GL}(b)^d$, $\hat{G} = \mathrm{GL}(n)^d$. It is obviously compatible with the operation of the Weil group. On the other hand $\pi \mapsto {}^\sigma \pi$ is given by the map $\hat{H} \mapsto \hat{H}$,

$$(g_i) \mapsto (g_{i+1}).$$

So the composite operation, $\pi \mapsto \sigma \pi \boxplus \dots \boxplus \sigma^{d-1} \pi \boxplus \pi$, is given by

$$\mathrm{GL}(b)^d \rightarrow \mathrm{GL}(n)^d, \quad (g_i) \mapsto \left(\bigoplus_k g_{i+k} \right)_i.$$

It is equivariant for the action of W_F acting (via the restriction of scalars) on both sides. Thus we get

$$\mu_0 : \mathrm{GL}(b)^d \rtimes W_F \rightarrow \mathrm{GL}(n)^d \rtimes W_F, \quad (g_i, w) \mapsto \left(\left(\bigoplus_k g_{i+k} \right)_i, w \right).$$

²⁾ We have change the ordering to simplify the indexes in the following formulas.

Since the two homomorphisms of L -groups μ_0 and μ_1 have the same effect on representations, they should be conjugate by an element in $\hat{G} = \mathrm{GL}(n)^d$. We proceed to exhibit this conjugation. We first consider the connected dual groups. We seek $P = (P_i) \in \mathrm{GL}(n)^d$ such that

$$P\mu_0(g)P^{-1} = \mu_1(g), \quad g = (g_i) \in \mathrm{GL}(b)^d.$$

Thus,

$$P_i \left(\bigoplus_k g_{i+k} \right) P_i^{-1} = g_1 \oplus \cdots \oplus g_d$$

for each i . If Q is (a block-scalar matrix) associated to a permutation $\tau \in \mathfrak{S}_d$,

$$Q \left(\bigoplus_k g_k \right) Q^{-1} = (g_{\tau^{-1}(k)}).$$

Thus P_i must be the permutation matrix associated to τ , where $\tau(k) = i + k$.

To avoid confusion, we now replace our indices k by α (in conformity with the previous section) and write $k = \iota(w)$, $w \in W_F$. The conjugation of the homomorphisms on W_F gives

$$(P, 1)(1, w)(P, 1)^{-1} = (\mathrm{diag} \tau(w), w) \quad (w \in W_F),$$

so

$$(7.3) \quad P_i P_{i+k}^{-1} = \tau(w).$$

The left-hand side is associated to $\tau(\alpha) = \alpha - k$. Thus (with the previous choices) we must take

$$\tau(w) = \tau_1^{\iota(w)}, \quad \tau_1(\alpha) = \alpha - 1$$

(compare with the formula for τ preceding (7.1)).

We have therefore proved:

Lemma 4. *With the above notation,*

$$P\mu_0 P^{-1} = \mu_1,$$

where $P = (P_i) \in \mathrm{GL}(n, \mathbb{C})^d$ and P_i is the permutation matrix associated to $\alpha \mapsto i + \alpha$.

7.2. Now return to the homomorphism μ_0 given in (7.1) realizing the operation

$$\pi \mapsto \bigoplus_{\alpha=1}^d \sigma^\alpha \pi$$

(π being a representation of $\mathrm{GL}(d, \mathbf{A}_E)$). Let $\eta_1, \dots, \eta_d$ be characters of C_E , associated to the parameters $\eta_\alpha(\delta_i(w)) \in (\mathbb{C}^\times)^d$ as discussed before equation (6.11). Now the homomorphisms (7.2) can be multiplied by the homomorphisms associated with the η_α , so we see that

$$(g_{\alpha i}) \mapsto \left(\bigoplus_{\alpha} g_{\alpha i} \eta_\alpha(\delta_i(w)) \right)_i$$

corresponds to $(\pi_\alpha) \mapsto \boxplus_\alpha \pi_\alpha \otimes \eta_\alpha$. In particular, $\pi \mapsto \boxplus_\alpha \sigma^\alpha \pi \otimes \eta_\alpha$, is then given by

$$\xi_0 : \mathrm{GL}(b)^d \rtimes W_F \rightarrow \mathrm{GL}(n)^d \rtimes W_F, \quad (g_i, w) \mapsto [(g_{\alpha+i} \eta_\alpha(\delta_i(w)))_i, w].$$

Conjugating by P , we obtain a homomorphism ξ_1 . On $\mathrm{GL}(b)^d$, it coincides with η_1 . In order to make it explicit, we must compute, for $w \in W_F$,

$$(P, 1) \left[\left(\bigoplus_{\alpha} \eta_{\alpha}(\delta_i(w)) \right)_i, w \right] (P^{-1}, 1).$$

The i -th component is

$$P_i \left(\bigoplus_{\alpha} \eta_{\alpha}(\delta_i(w)) \right) P_{i+k}^{-1} = P_i \left(\bigoplus_{\alpha} \eta_{\alpha}(\delta_i(w)) \right) P_i^{-1} \tau(w)$$

by (7.3). The conjugation by P_i is the permutation $\alpha \mapsto i + \alpha$, so this is

$$\left(\bigoplus_{\alpha} \eta_{i+\alpha}(\delta_i(w)) \right) \tau(w).$$

In conclusion:

Lemma 5. *The map $\pi \mapsto \boxplus_{\alpha} \sigma^{\alpha} \pi \otimes \eta_{\alpha}$ is associated to the homomorphism of L -groups*

$$\begin{aligned} \xi_1 : \mathrm{GL}(b)^d \rtimes W_F &\rightarrow \mathrm{GL}(n)^d \rtimes W_F, \\ \xi_1(g_1, \dots, g_d) &= \mathrm{diag}(g_1 \oplus \dots \oplus g_d), \\ \xi_1(w) &= \left[\left(\left(\bigoplus_{\alpha} \eta_{i+\alpha}(\delta_i(w)) \right) \tau(w) \right)_i, w \right]. \end{aligned}$$

We note that this is the homomorphism ξ_1 obtained from endoscopy as in equations (6.16) and (6.11).

7.3. We can now complete the proof of Theorem 1. Assume first d does not divide n . We then have

$$T_{\mathrm{disc}}(\phi \times \theta; \omega^{-1}) = 0.$$

The cuspidal representations occurring in the discrete trace have multiplicity one, and their families of Hecke eigenvalues (away from a finite set S of primes) are linearly independent, and independent from those of other representations. Of course only the cuspidal representations such that ${}^{\sigma}\Pi \simeq \Pi \otimes \omega$ contribute. We conclude that there are no such representations, as was of course clear from the consideration of the central characters (see the remark after Theorem 1).

Consider now the case when d divides n . There is only, up to equivalence, one endoscopic datum, defining the endoscopic group H ; if π_b is a cuspidal representation of $H(\mathbf{A}_F)$, the associated map on Hecke matrices sends (up to conjugation) $t_v(\pi_b)$ to

$$t_v(\sigma \pi_b \otimes \eta_1 \boxplus \sigma^2 \pi_b \otimes \eta_2 \boxplus \dots \boxplus \pi_b \otimes \eta_d)$$

at primes where all data are unramified, as follows from the conjugation of ξ_1 and μ_0 . This then remains true if π_b is any automorphic representation of $\mathrm{GL}(b, \mathbf{A}_E)$, in particular for those appearing in the discrete trace formula for H . Indeed, let π be such a representation. Then π is a subquotient – in fact a quotient if π is one of the “discrete” representations of $H(\mathbf{A}_F)$ described, for G , in Section 4 – of a representation $\pi_1 \boxplus \pi_2 \boxplus \dots \boxplus \pi_r$ induced from (non-necessarily unitary) cuspidal representations of $\mathrm{GL}(b_k, \mathbf{A}_E)$, $k = 1, 2, \dots, r$. If π_v is unram-

ified, the $\pi_{i,v}$ are unramified and the Hecke matrix of π_v is the same as that of the induced representation. Consider the Levi subgroup $M_H = \prod_{k=1}^r \text{Res}_{E/F} \text{GL}(b_k)$ of H , and similarly $M_G = \prod_{k=1}^r \text{Res}_{E/F} \text{GL}(n_k)$, where $n_k = db_k$. From the formulas in Lemma 5, one immediately deduces that ξ_1 “commutes with induction”: the homomorphism ξ_1 is defined, by the same formulas, for $\text{Res}_{E/F} \text{GL}(b_k)$; call it ξ_1^k . Then induction from a parabolic subgroup being given, both for G and H , by direct sums of Hecke matrices, the Hecke matrix of the representation of $G(\mathbf{A}_F)$ deduced from π is the sum of the Hecke matrices of the representations deduced, via ξ_1^k , from the π_k ; each of these is given by the previous formula. (Note that Proposition 6 remains true as stated when π_b is not unitary.)

Now consider the identity given by the stable trace formula

$$(7.4) \quad T_{\text{disc}}(\phi \times \theta; \omega^{-1}) = \iota(\mathbf{G}') S T_{\text{disc}}^{G'}(\phi^{G'}).$$

By Proposition 5, there is only one datum, with $G' = H$, on the right. Since $G'(\mathbf{A}_F) = H(\mathbf{A}_F)$ is $\text{GL}(b, \mathbf{A}_E)$, its trace formula is stable. Thus the right-hand side of (7.4) reads

$$(7.5) \quad T_{\text{disc}}(\phi \times \theta; \omega^{-1}) = \iota(\mathbf{G}') T_{\text{disc}}^H(\phi^H).$$

Here $\iota(\mathbf{G}')$ is a positive, non-zero rational number and T_{disc}^H has been described in Section 4.

Formula (7.5) can be restricted to smaller spaces. Let $F_\infty = \prod_v F_v$, where v runs over the set V_∞ of Archimedean places of F , and let $\mathcal{Z}_G$ be the center of the enveloping algebra of $G(F_\infty)$, and similarly $\mathcal{Z}_H$. There is a natural map $\mathcal{Z}_G \rightarrow \mathcal{Z}_H$ ([14, pp. 442–443]) whence a map $v' \rightarrow v$ between infinitesimal characters for H and G . (This map depends on ξ and ω .) Furthermore, as can be deduced from the description given in [14, p. 442], $\mathcal{Z}_H$ is, in our case, finite over the image of $\mathcal{Z}_G$; thus there are a finite number of infinitesimal characters v' associated to v .³⁾

Fix v , and let $V \supset V_\infty$ be a finite set of places of F . We assume all data, i.e., E/F , ω , ξ_1 (and therefore G, H) unramified outside V . We consider a decomposed function $\phi = \bigotimes \phi_v$ on $G(\mathbf{A}_F)$, with ϕ_v in the unramified Hecke algebra for $v \notin V$, ϕ_v smooth compactly supported if $v \in V$, and K_∞ -finite if $v \in V_\infty$, K_∞ being a maximal compact subgroup of $G(F_\infty)$. The datum of ξ_1 then determines a function ϕ^H , with ϕ_v^H unramified if $v \notin V$. The left-hand side of (7.6) is now, if we consider the representations of infinitesimal character v , a finite sum

$$(7.6) \quad T_{\text{disc}}(\phi \times \theta; \omega^{-1})_v = \sum_{\Pi} a(\Pi) \text{trace}(I_\theta(\Pi_V \otimes \omega^{-1})(\phi_V)) c_\Pi(\phi^V),$$

which we write more simply

$$(7.7) \quad T_{\text{disc}}(\phi \times \theta; \omega^{-1})_v = \sum_{\Pi} b(\Pi) c_\Pi(\phi^V).$$

Here the $a(\Pi)$ are the constants affecting Arthur’s discrete terms; c_Π is the character of $\mathcal{H}^V$ (the unramified Hecke algebra outside V) determined by Π . Now $T_{\text{disc}}(\phi \times \theta; \omega^{-1})_v$ is the product of $\iota(\mathbf{G}')$ by the sum, over v' , of the $T_{\text{disc}}(\phi^H)_{v'}$ ([14, X.8.1]), and similarly,

$$(7.8) \quad T_{\text{disc}}(\phi^H)_{v'} = \sum_{\pi} b_H(\pi) c_\pi(\phi^{H,V}).$$

³⁾ The datum of ξ_1 and ω also determines, from the central character of A_G defining our space of automorphic forms on $G(\mathbf{A}_F)$ (Section 4), a central character of A_H . See [13, p. 441]. We assume the central character for G trivial, but this will not be the case for H in general. However the spaces of automorphic forms associated to two characters of A_H are “essentially” isomorphic (twist by a character of the determinant.)

Now assume that a character of $\mathcal{H}^V$ occurs non-trivially in (7.7). It must be obtained by composition from a character of $\mathcal{H}_H^V$ occurring in (7.8). This implies that, at each prime outside V , Π_v is the representation $\sigma\pi_v \otimes \eta_{1,v} \boxplus \sigma^2\pi_v \otimes \eta_{2,v} \boxplus \cdots \boxplus \pi \otimes \eta_{d,v}$ or, possibly, its unique unramified subquotient. This is true for all representations π in (7.8), by the previous remark. By the fundamental results of Jacquet and Shalika [5, 6], Π cannot be cuspidal. However, if Π is our putative cuspidal representation of $G(\mathbf{A}_F) = \mathrm{GL}(n, \mathbf{A}_E)$ such that ${}^\sigma\Pi \cong \Pi \otimes \omega$, it is easy, using strong multiplicity one, to find a function ϕ_V on $G(F_V)$ such that the corresponding term in (7.6) does not vanish. This concludes the proof of Theorem 1.

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