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Cross-layer communication over fading channels with adaptive decision feedback

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***Abstract*—In this paper, cross-layer design of transmitting data packets over AWGN fading channel with adaptive decision feedback is considered. The transmitter decides the number of packets to transmit and the threshold of the decision feedback based on the queue length and the channel state. The transmit power is chosen such that the probability of error is below a pre-specified threshold. We model the system as a Markov decision process and use ideas from lattice theory to establish qualitative properties of optimal transmission strategies. In particular, we show that: (i) if the channel state remains the same and the number of packets in the queue increase, then the optimal policy either transmits more packets or uses a smaller decision feedback threshold or both; and (ii) if the number of packets in the queue remain the same and the channel quality deteriorates, then the optimal policy either transmits fewer packets or uses a larger threshold for the decision feedback or both. We also show under rate constraints that if the channel gains for all channel states are above a threshold, then the "or" in the above characterization can be replaced by "and". Finally, we present a numerical example showing that adaptive decision feedback significantly improves the power-delay trade-off as compared with the case of no feedback.**

***Index Terms*—Power-delay trade-off, cross-layer design, fading channels, decision feedback, Markov decision processes,**

I. Introduction

Wireless communication is a ubiquitous component of almost all modern technologies. Emerging applications such as vehicle-to-vehicle communication, automated driving, industrial control, virtual reality, and tactile Internet have strict requirements in terms of latency and reliability. Meeting such ultra-reliable low-latency requirements over time-varying fading channels in an energy efficient manner is one of the key challenges in the design of next generation wireless networks [1].

Typically, a higher layer application generates traffic which is queued in a buffer; packets from the buffer are transmitted over the wireless fading channel, and then passed on to a higher layer application at the receiver. Such models have two competing objectives: minimizing the power consumed to transmit the packets and minimizing the end-to-end delay experienced by the packets. There is a fundamental trade-off between the two: delay may be reduced by increasing the transmission rate which, in turn, results in an increase in the power required to maintain the same probability of error. In the presence of fading, it is also possible to adapt the transmission rate (or, equivalently, the transmit power), based on the state of the channel.

The power-delay trade-off has been investigated for various models [2]–[7]. These include minimizing transmit power under delay constraints [2]–[4], optimizing power and rate under delay constraints [5], minimizing the delay under constraint on transmit power [6], or general throughput maximization under a cost constraint [7]. All these models are analyzed using Markov decision theory and for some of the models, qualitative properties of optimal strategies are also established.

In all of the papers above, rate adaptation was the only mechanism of trading off transmit power and delay. Another mechanism for adaptation was proposed in [8], which considered a communication model where the receiver uses decision feedback [9], i.e., if the received symbol is too far away from the transmitted codewords, the receiver may declare an erasure and request a retransmission. It was shown in [8] that this additional degree of freedom improves the power-delay trade-off for a stylized model with binary transmit decisions (transmit or not transmit) and binary decision feedback (to use or not use decision feedback). They modeled the problem as a Markov decision process with two dimensional state (queue length and channel state) and two dimensional actions (number of packets to transmit and the decision feedback to use) and characterized the structural properties of the optimal transmission strategies.

Our main contribution is to generalize the result of [8] to a more realistic system model where we do not restrict the number of transmitted packets or decision feedback to binary values. The proof technique of [8] relied on exhaustive enumeration of all possible combinations of actions and does not work for the more general case that we consider. We use ideas from lattice theory to develop monotonicity properties of arg min of functions defined over partially ordered sets and then use these properties to establish the structural properties of optimal transmission strategies. We then present a numerical example to show that the presence of decision feedback can significantly improve the power-delay trade-off.

II. Model and Problem Formulation

A. The communication system

Consider a communication system show in Fig.1. A source generates stochastic data packets that have to be transmitted over an i.i.d. AWGN fading channel. The transmitter has a buffer where the data packets are queued. The system operates in discrete time slots. The data packets that arrive in a slot are available only at the beginning of the next slot.

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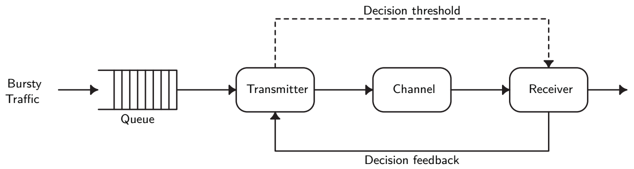

Fig. 1: Model of a transmitter with decision feedback

At the beginning of a slot, the transmitter chooses some data packets from the queue, encodes them, and transmits the encoded symbol at some energy level over the AWGN channel.

The receiver receives an attenuated and noise corrupted copy of transmitted symbol. The receiver uses decision feedback to decode. If the log likelihood ratio of the hypothesis that the received signal comes from the closest codeword is less than a threshold, then the receiver declares an erasure and sends a negative acknowledgment to the transmitter. If not, the receiver declares the closest codeword as the decoded symbol and sends an acknowledgment to transmitter.

If the transmitter receives an acknowledgment, it removes the transmitter packets from the queue; if it receives a negative acknowledgment, it keeps the transmitted packets in the queue. At the end of the slot, the system incurs a delay penalty that depends on the number of packets remaining in the queue.

Time slots are indexed by $k \in \mathbb{N}$. The system variables are denoted as follows:

- $Q_k \in \mathbb{Z}_{\geq 0}$ denotes the number of packets in the queue at the beginning of slot k.
- $A_k \in \mathbb{Z}_{\geq 0}$ denotes the number of packets that arrive at the beginning of slot k.
- $U_k \in F(Q_k) := \{0, \dots, Q_k\}$ denotes the number of packets transmitted during slot k.
- $S_k \in \mathcal{S}$ denotes the state of the channel during slot k. The fading coefficient in state $s \in \mathcal{S}$ is denoted by $H(s)$. For ease of analysis, we define $h(s) = 1/H(s)$. We assume that $\mathcal{S}$ is a finite and totally ordered set and the function $h : \mathcal{S} \to \mathbb{R}_{\geq 0}$ is strictly increasing and convex. Thus, a smaller value of the state of the channel indicates a better channel quality.
- $\Pi_k \in \mathbb{R}_{\geq 0}$ denotes the energy used to transmit during slot k.
- $T_k \in \mathcal{T}$ denotes the threshold used for the decision feedback, where $\mathcal{T}$ is a closed and convex subset of $\mathbb{R}_{\geq 0}$.
- $\mathfrak{E} \in \{0, 1\}$ denotes the decision feedback generated by the receiver, where $\mathfrak{E} = 1$ denotes an acknowledgment and $\mathfrak{E} = 0$ denotes a negative acknowledgment.

For ease of notation, we define $\mathcal{Q} = \mathbb{Z}_{\geq 0}$ and $\mathcal{U} = \mathbb{Z}_{\geq 0}$ to denote the space of realizations for Q_k and U_k. The various components of the communication system are described below.

1) Assumptions on the primitive random variables: We assume that the data arrival process $\{A_k\}_{k\geq 1}$ and the channel state process $\{S_k\}_{k\geq 1}$ are independent and identically distributed exogenous processes that are independent of each other. Let P_A and P_S denote the probability mass functions of A_k and S_k, $k \in \mathbb{N}$. We assume that P_A is weakly decreasing and convex, i.e., the difference between consecutive probability mass points is also decreasing.

2) Network layer modeling: We assume that each packet has N bits. The queue dynamics are given by:

$$Q_{k+1} = \begin{cases} Q_k - U_k + A_k, & \text{if} \quad \mathfrak{E}_k = 1; \\ Q_k + A_k, & \text{if} \quad \mathfrak{E}_k = 0. \end{cases} \tag{1}$$

Packets that remain in the queue at the end of slot k incur a delay penalty of $d(Q_{k+1} - A_k)$, where $d : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is weakly increasing and convex function with $d(0) = 0$.

3) Physical layer modeling: We assume that the channel is a narrow-band block-fading AWGN channel in which the coherence time of the channel equals to the time slot. Thus, the fading gain remains constant for the duration of the time slot and changes independently across time. Each transmission uses a block code of length M. Let $\mathbf{X}_k = [X_{k1}, \dots, X_{kM}]$ and $\mathbf{Y}_k = [Y_{k1}, \dots, Y_{kM}]$ denote the transmitted and received signals at slot k. These two signals are related according to

$$\mathbf{Y}_k = H(S_k)\mathbf{X}_k + \mathbf{Z}_k$$

where $\mathbf{Z}_k = [Z_{k1}, \dots, Z_{kM}]$ is the channel noise. We assume that $\{\mathbf{Z}_k\}_{k\geq 0}$ is an i.i.d. process and the variables $Z_{k1}, \dots, Z_{kM}$ are independent zero-mean Gaussian random variables with unit variance. The average transmitted power during slot k is given by $\Pi_k = \big(\sum_{m=1}^{M} \mathbb{E}[X_{km}^2]\big)/M$ where expectation is with respect to the measure induced on the message.

Let P_e and P_r denote the probability of (block) error and the probability of erasure (i.e., $\mathfrak{E} = 0$), respectively. By substituting the error exponent for AWGN channels [10, Eq. (124)] in the expressions for probability of error for adaptive feedback [9, Theorem 2], we can bound P_e and P_r as following:

$$P_e \leq p_e(U_k, T_k, \Pi_k, S_k) \quad \text{and} \quad P_r \leq p_r(U_k, T_k, \Pi_k, S_k),$$

where $p_e(u, t, \pi, s)$ is given by

$$\exp\Big(-\tfrac{\rho}{2} M \log\big(1 + \tfrac{\pi H(s)}{(1+\rho)}\big) + \rho N u - \tfrac{\rho}{1+\rho} M t\Big), \tag{2}$$

and $p_r(u, t, \pi, s)$ is given by

$$\exp\Big(-\tfrac{\rho}{2} M \log\big(1 + \tfrac{\pi H(s)}{(1+\rho)}\big) + \rho N u + \tfrac{1}{1+\rho} M t\Big), \tag{3}$$

where ρ is a parameter between 0 and 1. Note that

$$p_r(u, t, \pi, s) = \exp(Mt) p_e(u, t, \pi, s). \tag{4}$$

We assume that M is large enough so that the upper bounds $p_e(U_k, T_k, \Pi_k, S_k)$ and $p_r(U_k, T_k, \Pi_k, S_k)$ are tight approximations for P_e and P_r.

B. The performance metrics and the optimization problem

There are three performance metrics of interest at each time:

1) The probability P_e of block error, which we assume can be approximated by the upper bound $p_e(U_k, T_k, \Pi_k, S_k)$.
2) The power Π_k used for transmissions.
3) The delay $d(Q_{k+1} - A_k)$ incurred for holding the packets in the queue.

We are interested in characterizing the power-delay trade-off under the assumption that the probability of block error P_e is equal to some pre-specified value ε, i.e., $P_e = \varepsilon$.

There are three decision variables: the number U_k of packets, the threshold T_k of decision feedback, and the power Π_k used for transmission. We impose the following simplifying assumption which implies that Π_k is a function of U_k and T_k.

We assume that (recall that $h(s) = 1/H(s)$):

$$\Pi_k = \phi(U_k, T_k)h(S_k), \tag{5}$$

where

$$\phi(u,t) = (1+\rho)\Big[\exp\Big(-\frac{2\log\varepsilon}{\rho M} + 2\frac{N}{M}u - \frac{2}{1+\rho}t\Big) - 1\Big].$$

Such a choice for Π_k ensures that for any (S_k, U_k, T_k), we always satisfy

$$p_e(U_k, T_k, \Pi_k, S_k) = \varepsilon. \tag{6}$$

Note that, by construction, $\phi(u,t)$ is strictly increasing in u and strictly decreasing in t. Furthermore, by (4) and (6), the retransmission probability is given by

$$p_r(U_n, T_n, \Pi_n, S_n) = \exp(MT_n)\varepsilon =: p(T_n). \tag{7}$$

By construction, $p(t)$ is strictly increasing in t. As a consequence of (5), we are left with only two decision variables: U_k and T_k. We assume that both of these are chosen at the transmitter and communicated noiselessly to the receiver over a control channel. Both the variables are chosen as a function of Q_k and S_k, i.e.,

$$(U_k, T_k) = g_k(Q_k, S_k),$$

where for any $q \in \mathcal{Q}$ and $s \in \mathcal{S}$, $g_k(q,s) \in F(q) \times \mathcal{T}$. The function g_k is called the *communication rule* at slot k and the collection $g = (g_1, g_2, \dots)$ is called the *communication policy*.

We assume that there is a cost λ_d for each unit of delay and a cost λ_π for each unit of power. Then, the per-step cost incurred by the system is given by

$$C_k = \lambda_\pi \Pi_k + \lambda_d d(Q_{k+1} - A_k).$$

For ease of notation, we define

$$\begin{aligned} c(q,s,u,t) &= \mathbb{E}[C_k \mid Q_k = q, S_k = s, U_k = u, T_k = t] \\ &= \lambda_\pi \phi(u,t)h(s) + \lambda_d\big[p(t)d(q) + (1-p(t))d(q-u)\big]. \end{aligned} \tag{8}$$

For ease of reference, we restate the assumptions on the different components of the cost and the probability distribution:

(A1) The probability mass function P_A is weakly decreasing and convex.
(A2) $d(q)$ is weakly increasing and convex in q and $d(0) = 0$.
(A3) $h(s)$ is strictly increasing and convex in s.
(A4) $\phi(u,t)$ is strictly increasing in u and strictly decreasing in t.
(A5) $p(t)$ is strictly increasing in t.

The system runs for a finite horizon K. The performance of any communication policy g is given by

$$J(g) = \mathbb{E}\Big[\sum_{k=1}^{K} c(Q_k, S_k, U_k, T_k)\Big], \tag{9}$$

where the expectation is with respect to the joint measure on the system variables induced by the choice of g.

We are interested in the following optimization problem.

Problem 1 *For the model described above, given an horizon K, choose a communication policy $g = (g_1, \dots, g_K)$ to minimize $J(g)$ given by* (9).

Problem 1 is a finite horizon Markov decision process (MDP). In Sec. III-A, we present use standard results from Markov decision theory [11] to obtain a dynamic program. Our main contribution is to use the dynamic program to identify qualitative properties of the value function and the optimal policy, which we present in Sec. III-B. One of the challenges in identifying such qualitative properties is that the action space $\mathcal{U} \times \mathcal{T}$ is not a totally ordered set. The proofs of the qualitative properties utilize results on partially ordered sets from lattice theory, which we summarize in Sec. V. The proofs themselves are presented in Sec. VI.

III. The Main Results

A. Dynamic Programming Decomposition

Proposition 1 *Consider the functions $\{V_k \colon \mathcal{Q} \times \mathcal{S} \to \mathbb{R}\}_{k \ge 1}$, $\{\overline{W}_k \colon \mathcal{Q} \times \mathcal{U} \times \mathcal{T} \to \mathbb{R}\}_{k \ge 1}$, and $\{W_k \colon \mathcal{Q} \times \mathcal{S} \times \mathcal{U} \times \mathcal{T} \to \mathbb{R}\}_{k \ge 1}$ defined as follows: For any $q \in \mathcal{Q}$ and $s \in \mathcal{S}$,*

$$V_{K+1}(q,s) = 0$$

and for $k \in \{K, K-1, \dots, 1\}$, recursively define

$$\begin{aligned} \overline{W}_k(q,u,t) = \sum_{(s,a) \in \mathcal{S} \times \mathbb{Z}_{\ge 0}} P_S(s) P_A(a) \big[&(1-p(t))V_{k+1}(q-u+a, s) \\ &+ p(t) V_{k+1}(q+a, s)\big], \end{aligned} \tag{10}$$

$$W_k(q,s,u,t) = c_k(q,s,u,t) + \overline{W}_k(q,u,t), \tag{11}$$

$$V_k(q,s) = \min_{u \in F(q), t \in \mathcal{T}} W_k(q,s,u,t). \tag{12}$$

Let $g_k(q,s)$ denote the arg min at stage k. Then, the policy $g = (g_1, \dots, g_K)$ is optimal for Problem 1. □

B. Qualitative properties of the solution

Theorem 1 *For any time slot k, the value function $V_k(q,s)$ satisfies the following properties:*

1) *For any $s \in \mathcal{S}$, $V_k(q,s)$ is weakly increasing in q.*
2) *For any $q \in \mathcal{Q}$, $V_k(q,s)$ is weakly increasing in s.* □

The proof is presented in Section VI-A.

Theorem 1 establishes the following qualitative properties of the value function. If the channel state remains the same, increasing the number of packets in the queue increases the optimal cost to go. Similarly, if the number of packets in the queue remain the same, going to a worse channel state increases the optimal cost to go.

Theorem 2 *For any $k \in \{1, \dots, K\}$, we have the following.*

1) *For any $s \in \mathcal{S}$ and $q_1, q_2 \in \mathcal{Q}$ such that $q_1 \le q_2$, let $(u_1, t_1) = g_k(q_1, s)$ and $(u_2, t_2) = g_k(q_2, s)$. Then, $u_1 \le u_2$ or $t_1 \ge t_2$.*
2) *For any $q \in \mathcal{Q}$ and $s_1, s_2 \in \mathcal{S}$ such that $s_1 \le s_2$, let $(u_1, t_1) = g_k(q, s_1)$ and $(u_2, t_2) = g_k(q, s_2)$. Then, $u_1 \ge u_2$ or $t_1 \le t_2$.* □

Note that the "or" in the above theorem is an inclusive or. The proof is presented in Sec. VI-B.

Theorem 2 establishes the following qualitative properties of the optimal policy. If the channel state remains the same and the number of packets in the queue increase, then the optimal policy either transmits more packets or uses a smaller threshold for decision feedback or both. If the number of packets in the queue remain the same and the channels state increases (i.e., the channel quality deteriorates), then the optimal policy either transmits fewer packets or uses a larger threshold for decision feedback or both.

Theorem 3 *Suppose the cost function satisfies the following property:*

(P) *for any $(q, s) \in \mathcal{Q} \times \mathcal{S}$, and any $u_1, u_2 \in \mathcal{U}$ and $t_1, t_2 \in \mathcal{T}$ such that $u_1 \le u_2$ and $t_1 \le t_2$, we have*

$$c(q, s, u_1, t_2) + c(q, s, u_2, t_1) \le c(q, s, u_2, t_2) + c(q, s, u_1, t_1).$$

Then, the "or" in Theorem 2 can be replaced by "and". □

The proof is presented in Sec. III-B.

Theorem 4 *The following two conditions are sufficient for property* (P) *to hold.*

1) *There exists a differentiable function $D : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that at $q \in \mathbb{Z}_{\ge 0}$, $D(q) = d(q)$ and $D'(q) \ge 1$ (where $D'(q)$ denotes the derivative of q).*
2) *The rate of transmission (which equals Nu/M) is upper bounded by a constant, which we denote by $R_{\max}$.*
3) *For all $s \in \mathcal{S}$,*

$$h(s) \le \frac{\lambda_d}{\lambda_\pi} \frac{M^2}{4N} \varepsilon^{1+2/\rho M} \exp(-2R_{max}).$$ □

The proof is presented in Appendix C.

Remark 1 The first condition in Theorem 4 covers a large family of delay functions of interest, e.g., $d(q) = q$ or $d(q) = q^2$. Recall that $h(s)$ denotes the reciprocal of the fading gain when the channel is in state s. Thus, the third condition in Theorem 4 states that fading gain for all channel states must be greater than a threshold, where the threshold depends on the design parameters $(\lambda_d, \lambda_\pi, N, M)$. □

C. Significance of the results

The results of Theorems 2 and 3 may appear to be "obvious", but it is important to formally establish them because there are models where such qualitative properties do not hold [12]. Knowing that the optimal policy is monotone has two advantages. First, the monotonicity of optimal policy can be exploited for more efficient planning and learning algorithms (e.g., monotone dynamic programming [11], renewal Monte Carlo [13], and structure aware reinforcement leanring [14]). Second, monotone policies are easier to implement in either hardware or software than general policies.

IV. An example to illustrate the benefit of decision feedback

In this section, we present a numerical example to illustrate the value of decision feedback for improving the power-delay trade-off. For simplicity of exposition, we consider an infinite horizon discounted setup. For any time-homogeneous transmission policy g, define the expected discounted transmitted power as

$$\Pi^{(g)}(q, s) = (1-\beta)\mathbb{E}\Big[\sum_{k=1}^{\infty} \beta^{k-1} \phi(U_k, T_k) h(S_k) \;\Big|\; Q_1 = q, S_1 = s\Big]$$

and the expected discounted delay as

$$D^{(g)}(q, s) = (1-\beta)\mathbb{E}\Big[\sum_{k=1}^{\infty} \beta^{k-1} D(Q_{k+1} - A_k) \;\Big|\; Q_1 = q, S_1 = s\Big].$$

For fixed (q, s), the optimal power-delay trade-off is given by

$$\mathsf{P}(\alpha) = \min_g \{\Pi^{(g)}(q, s) \mid D^{(g)}(q, s) \le \alpha\}. \tag{13}$$

We compare the power-delay trade-off of a system with adaptive feedback with the power-delay trade-off of the system without adaptive feedback. The latter is obtained by restricting attention to policies which choose $T_k = 0$ and setting $p(0) = 0$. We compute the power-delay trade-off for both systems for the following choice of parameters: $N = 10$, $M = 100$, $\varepsilon = 10^{-6}$, $\rho = 1$, $\beta = 0.99$. Data packets arrive according to a deterministic process of 5 packets per unit time. The channel has two states and both states are equally likely with $H(1) = 0.1$ and $H(2) = 0.9$ (thus, $h(1) = 10$ and $h(2) = 10/9$). The delay function is $d(q) = q$. The state space of the dynamic program is $\mathbb{Z}_{\ge 0} \times \mathbb{R}_{\ge 0}$. To obtain a numerical solution, we truncate the model to $\{0, 1, \dots, B\} \times \{0, \tau_1, \tau_2, \dots, \tau_{10}\}$,

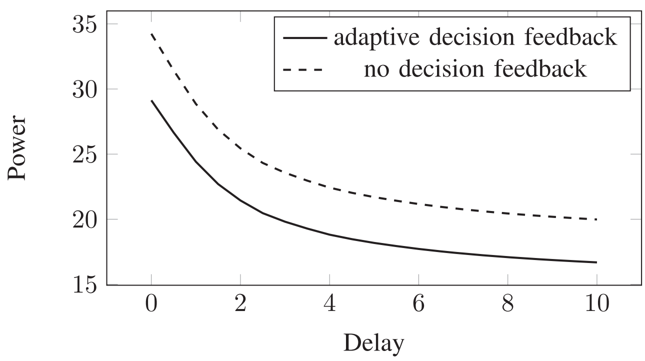

Fig. 2: Power-delay trade-off with and without adaptive decision feedback

where $B = 100$ and $\{\tau_1, \ldots, \tau_{10}\}$ are chosen such that $p(t)$ takes values $\{0, 0.1, \ldots, 1\}$. The power-delay function $\mathsf{P}(\alpha)$ is computed by solving the constrained MDP using linear programming [15]. The code of this numerical example is available at [16].

The power-delay trade-off for $(q, s) = (0, 1)$ are shown in Figure 2, which shows the advantage of using adaptive decision feedback.

V. Preliminaries on lattice theory

In this section, we present some basic definitions from lattice theory [17]. These definitions are required in the proof of structural results of the optimal transmission and feedback policies.

A. Partially ordered sets, monotone functions, and functions with monotone differences

Given a binary relation $\preceq$ on a set $\mathcal{X}$, we say that $(\mathcal{X}, \preceq)$ is a *partially ordered set* if the relation $\preceq$ is

1) reflexive, i.e., for any $x \in \mathcal{X}$, $x \preceq x$;
2) anti-symmetric, i.e., for any $x, y \in \mathcal{X}$, if $x \preceq y$ and $y \preceq x$, then $x = y$; and
3) transitive, i.e., for any $x, y, z \in \mathcal{X}$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

For two elements $x, y \in \mathcal{X}$, if $x \preceq y$ and $x \neq y$, then we say $x \prec y$. The negation of $x \prec y$ is denoted by $x \not\prec y$.

A partially ordered set $(\mathcal{X}, \preceq)$ is said to be *totally ordered* if for any two elements x and y, either $x \preceq y$ or $y \preceq x$.

Remark 2 In general, given two elements x and y in a partially ordered set $(\mathcal{X}, \preceq)$, it is possible that $x \not\preceq y$ and $y \not\preceq x$. Such elements are said to be incomparable. Due to the existence of incomparable elements, $x \not\preceq y$ does not imply $y \preceq x$. □

Given two partially ordered sets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$, a function $f \colon \mathcal{X} \to \mathcal{Y}$ is called *increasing* if for any $x_1, x_2 \in \mathcal{X}$, $x_1 \preceq_{\mathcal{X}} x_2$ implies that $f(x_1) \preceq_{\mathcal{Y}} f(x_2)$. The function f is called *non-decreasing* if for any $x_1, x_2 \in \mathcal{X}$, $x_1 \preceq_{\mathcal{X}} x_2$ implies that $f(x_1) \not\succ_{\mathcal{Y}} f(x_2)$. It follows from Remark 2 that non-decreasing function is not necessarily increasing. Similar definitions hold for *decreasing* and *non-increasing* functions.

Given two partially ordered sets $(\mathcal{X}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}, \preceq_{\mathcal{Y}})$, a function $f \colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is said to have *increasing differences* if for any $x_1, x_2 \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{Y}$ such that $x_1 \preceq_{\mathcal{X}} x_2$ and $y_1 \preceq_{\mathcal{Y}} y_2$, we have

$$f(x_2, y_2) - f(x_1, y_2) \geq f(x_2, y_1) - f(x_1, y_1). \tag{14}$$

If the inequality in (14) is reversed, the function f is said to have *decreasing differences*.

B. Lattice and submodular functions

Given a partially ordered set $(\mathcal{X}, \preceq)$ and a subset $\mathcal{S}$ of $\mathcal{X}$, an element $x \in \mathcal{X}$ is called a *lower bound* of $\mathcal{S}$ if, for any element $s \in \mathcal{S}$, we have $x \preceq s$. If b is a lower bound of $\mathcal{S}$ such that for any other lower bound x, $x \preceq b$, then b is called *greatest lower bound*. Upper bound and least upper bound are defined analogously.

A partially ordered set $(\mathcal{X}, \preceq)$ is called a *lattice* if every two-element subsets $\{x, y\}$ has a join (i.e. least upper bound denoted by $x \vee y$) and a meet (i.e. greatest lower bound denoted by $x \wedge y$) in $\mathcal{X}$. A lattice $(\mathcal{X}, \preceq)$ is said to be *complete* if the set $\mathcal{X}$ has a least upper bound and a greatest lower bound.[1]

Given a lattice $(\mathcal{X}, \preceq_{\mathcal{X}})$, a function $f \colon \mathcal{X} \to \mathbb{R}$ is called *submodular* if for any $x_1, x_2 \in \mathcal{X}$, we have

$$f(x_1 \wedge x_2) + f(x_1 \vee x_2) \leq f(x_1) + f(x_2).$$

C. Nested increasing subsets, nested submodularity, and nested monotone differences

Given a partially ordered set $(\mathcal{X}, \preceq_{\mathcal{X}})$ and a lattice $(\mathcal{Y}, \preceq_{\mathcal{Y}})$, a family $\{\mathcal{S}_x\}_{x \in \mathcal{X}}$ of subsets of $\mathcal{Y}$ is called *nested increasing* if each subset $\mathcal{S}_x$ is a totally ordered set; for all $x_1, x_2 \in \mathcal{X}$ such that $x_1 \preceq_{\mathcal{X}} x_2$, we have $\mathcal{S}_{x_1} \subseteq \mathcal{S}_{x_2}$; and for any $y_1 \in \mathcal{S}_{x_1}$ and $y_2 \in \mathcal{S}_{x_2} \setminus S_{x_1}$, we have $y_1 \preceq_{\mathcal{Y}} y_2$.

Given a partially ordered set $(\mathcal{X}, \preceq_{\mathcal{X}})$, a lattice $(\mathcal{Y}, \preceq_{\mathcal{Y}})$, and a family $\{\mathcal{S}_x\}_{x \in \mathcal{X}}$ of nested increasing subsets of $\mathcal{Y}$, a function $f \colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ with domain $\{(x, y) : x \in \mathcal{X}, y \in \mathcal{S}_x\}$ is called *nested submodular* if the section $f(x, \cdot)$ is submodular on $\mathcal{S}_x$, for all x.

Given a partially ordered set $(\mathcal{X}, \preceq_{\mathcal{X}})$, a lattice $(\mathcal{Y}, \preceq_{\mathcal{Y}})$, and a family $\{\mathcal{S}_x\}_{x \in \mathcal{X}}$ of nested increasing subsets of $\mathcal{Y}$, a function $f \colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ with domain $\{(x, y) : x \in \mathcal{X}, y \in \mathcal{S}_x\}$ is said to have *increasing differences on nested increasing subsets* if for any $x_1, x_2 \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{S}_{x_1}$ such that $x_1 \preceq_{\mathcal{X}} x_2$ and $y_1 \preceq_{\mathcal{Y}} y_2$, Eq. (14) holds.

If the inequality in (14) is reversed, the function f is said to have *decreasing differences on nested increasing subsets*.

D. Minimizing a function over a lattice

Consider a function $f \colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ with domain $\{(x, y) : x \in \mathcal{X}, y \in \mathcal{S}_x\}$, where $(\mathcal{X}, \preceq_{\mathcal{X}})$ is a partially ordered set, $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ is a lattice and $\{\mathcal{S}_x\}_{x \in \mathcal{X}}$ is a family of nested increasing subsets. Suppose that for all x, the $\min_{y \in \mathcal{S}_x} f(x, y)$ exists and the (partially-ordered) set $\arg\min_{y \in \mathcal{S}_x} f(x, y)$ has a minimum element. Define

$$g(x) := \min\Big\{\arg\min_{y \in \mathcal{S}_x} f(x, y)\Big\}. \tag{15}$$

The next result identifies sufficient conditions for g to be monotone.

Theorem 5 *The following properties hold:*

1) *If f has increasing differences, then g is non-increasing.*
2) *If f has increasing differences and, for all $x \in \mathcal{X}$, the function $f(x, \cdot) \colon \mathcal{S}_x \to \mathbb{R}$ is submodular, then g is decreasing.*
3) *If f has decreasing differences on the nested increasing subsets $\{S_x\}_{x \in \mathcal{X}}$, then g is non-decreasing.*
4) *If f has decreasing differences on the nested increasing subsets $\{S_x\}_{x \in \mathcal{X}}$ and, for all $x \in \mathcal{X}$, the function $f(x, \cdot) \colon \mathcal{S}_x \to \mathbb{R}$ is submodular, then g is increasing.* □

[1]A lattice $(\mathcal{X}, \preceq)$ with finite number of elements is always complete, but lattice with countably or uncountably infinite elements need not be complete.

The proof is omitted due to space constraints.

Theorem 5 is similar in spirit to the monotonicity results presented in [17]. However, the analysis in [17] did not consider the setup with nested increasing sets.

VI. Proof of Main Results

A. Proof of Theorem 1

We first observe that as an immediate consequence of properties (A2) and (A3), we have the following.

Lemma 1 *The cost function $c(q,s,u,t)$ satisfies the following properties:*

1) *For any fixed $(s,u,t) \in \mathcal{S} \times \mathcal{U} \times \mathcal{T}$, $c(q,s,u,t)$ is increasing and convex in $q \in \mathcal{Q}$, $q \geq u$.*
2) *For any fixed $(q,u,t) \in \mathcal{Q} \times \mathcal{U} \times \mathcal{T}$, $c(q,s,u,t)$ is increasing and convex in $s \in \mathcal{S}$.* □

We now prove the two parts of Theorem 1 separately.

a) Monotonicity in q: We prove this part by backward induction. The results hold trivially for V_{K+1}. This forms the basis of induction. Now assume that for any fixed s, $V_{k+1}(q,s)$ is weakly increasing in q. Now consider two $q_1, q_2 \in \mathcal{Q}$ such that $q_1 < q_2$. Let $(u_2,t_2) = g_k(q_2,s)$. We now consider two cases: $u_2 \in F(q_1)$ and $u_2 \notin F(q_1)$.

1) First consider $u_2 \in F(q_1)$. From the induction hypothesis and (10), we have
$$\overline{W}_k(q_2,u_2,t_2) \geq \overline{W}_k(q_1,u_2,t_2). \quad (16)$$
From Lemma 1 and (16), we have
$$W_k(q_2,s,u_2,t_2) \geq W_k(q_1,s,u_2,t_2). \quad (17)$$
Now, consider
$$V_k(q_2,s) = W_k(q_2,s,u_2,t_2) \overset{(a)}{\geq} W_k(q_1,s,u_2,t_2) \overset{(b)}{\geq} V_k(q_1,s), \quad (18)$$
where (a) follows from (17) and (b) follows from (12).
2) Now consider the case $u_2 \notin F(q_1)$. Let $u_1 = q_1 - \min\{q_2 - u_2, q_1\}$. By construction, $u_1 \in F(q_1)$ and $q_1 - u_1 \leq q_2 - u_2$. Now, by the induction hypothesis, we have $V_{k+1}(q_2 - u_2 + a, s) \geq V_{k+1}(q_1 - u_1 + a, s)$ and $V_{k+1}(q_2 + a, s) \geq V_{k+1}(q_1 + a, s)$. Substituting in (10), we have
$$\overline{W}_k(q_2,u_2,t) \geq \overline{W}_k(q_1,u_1,t). \quad (19)$$
By a similar argument, we have that
$$p(t)d(q_2) + (1-p(t))d(q_2 - u_2) \geq p(t)d(q_1) + (1-p(t))d(q_1 - u_1). \quad (20)$$
Since $u_1 \in F(q_1)$ and $u_2 \notin F(q_1)$, we have $u_1 < u_2$ and hence
$$\phi(u_2,t)h(s) \geq \phi(u_1,t)h(s) \quad (21)$$
Combining (20) and (21), we get
$$c(q_2,s,u_2,t) \geq c(q_1,s,u_1,t). \quad (22)$$
Combining (19) and (22), we get
$$W_k(q_2,s,u_2,t) \geq W_k(q_1,s,u_1,t). \quad (23)$$
Now, consider
$$V_k(q_2,s) = W_k(q_2,s,u_2,t_2) \overset{(c)}{\geq} W_k(q_1,s,u_1,t_2) \overset{(d)}{\geq} V_k(q_1,s), \quad (24)$$
where (c) follows from (23) and (d) follows from (12).

Eqs. (18) and (24) show that $V_k(q,s)$ is weakly increasing in q. Thus, by the principle of induction, the property holds for all k.

b) Monotonicity in s: For any fixed (q,u,t), Lemma 1 and (11) imply that $W_k(q,s,u,t)$ is strictly increasing in s. The pointwise minimum of strictly increasing functions is strictly increasing. Thus, $V_k(q,s)$ is strictly increasing in s.

B. Proof of Theorem 2

The proof of Theorem 2 relies on establishing that the action-value function $W_k(q,s,u,t)$ has monotone differences in appropriate components and use Theorem 5 to establish that the optimal policy is monotone.

We start by defining a partial order on the space of actions.

Definition 1 (Partial order on $\mathcal{U} \times \mathcal{T}$) Let $\preceq_{\mathcal{A}}$ denote a partial order on $\mathcal{U} \times \mathcal{T}$ such that for any $(u_1,t_1),(u_2,t_2) \in \mathcal{U} \times \mathcal{T}$, we say $(u_1,t_1) \preceq_{\mathcal{A}} (u_2,t_2)$ if $u_1 \leq u_2$ and $t_1 \geq t_2$. □

Since both $\mathcal{U}$ and $\mathcal{T}$ are totally ordered sets, the partially ordered set $(\mathcal{U} \times \mathcal{T}, \preceq_{\mathcal{A}})$ is a lattice. Furthermore, $\{F(q) \times \mathcal{T}\}_{q \in \mathcal{Q}}$ is a family of nested increasing subsets of $\mathcal{U} \times \mathcal{T}$.

Remark 3 Property (P) in Theorem 2 states that for any fixed $(q,s) \in \mathcal{Q} \times \mathcal{S}$, the cost function $c(q,s,u,t)$ is submodular on $F(q) \times \mathcal{T}$ with respect to the partial order defined above. □

Let $L(q'|q,u,t) = \mathbb{P}(Q_{k+1} \geq q' \mid Q_k = q, U_k = u, T_k = t)$ denote the reverse cumulative density function for the queue length, which can also be written as
$$L(q'|q,u,t) = p(t)\mathbb{P}(A \geq q' - q) + (1-p(t))\mathbb{P}(A \geq q' - q + u). \quad (25)$$

Lemma 2 *For any $q' \in \mathcal{Q}$, the reverse cumulative density function $L(q'|q,u,t)$ has decreasing differences on $\mathcal{Q} \times (\mathcal{U} \times \mathcal{T})$ on nested increasing sets $\{F(q) \times \mathcal{T}\}_{q \in \mathcal{Q}}$.* □

The proof is omitted due to space constraints.

Lemma 3 *The cost function satisfies the following properties:*

1) *For any fixed $s \in \mathcal{S}$, the cost function $c(q,s,u,t)$ has decreasing differences on $\mathcal{Q} \times (\mathcal{U} \times \mathcal{T})$ on nested increasing sets $\{F(q) \times \mathcal{T}\}_{q \in \mathcal{Q}}$.*
2) *For any fixed $q \in \mathcal{Q}$, the cost function $c(q,s,u,t)$ has increasing differences on $\mathcal{S} \times (F(q) \times \mathcal{T})$.* □

The proof is presented in Appendix A.

Lemma 4 *For any $k \in \{1,\dots,K\}$, the action-value function $W_k(q,s,u,t)$ has the following properties:*

1) For fixed $s \in \mathcal{S}$, $W_k(q,s,u,t)$ has decreasing differences on $\mathcal{Q} \times (\mathcal{U} \times \mathcal{T})$ on nested increasing subsets $\{F(q) \times \mathcal{T}\}_{q \in \mathcal{Q}}$.

2) For fixed $q \in \mathcal{Q}$, $W_k(q,s,u,t)$ has increasing differences on $\mathcal{S} \times (\mathcal{U} \times \mathcal{T})$. □

The proof is presented in Appendix B.

PROOF (THEOREM 2 PART 1) The results of Theorem 2-part (1), follows from Theorem 5-part (3). By Lemma 4-part (1), we know $W_k(q,s,u,t)$ has decreasing difference on nested increasing subsets $\{F(q) \times \mathcal{T}\}_{q \in \mathcal{Q}}$. By applying Theorem 5-part (3) on $W_k(q,s,u,t)$, we prove the result. ■

PROOF (THEOREM 2 PART 2) The results of Theorem 2-part (2), follows from Theorem 5-part (1). By Lemma 4-part (2), we know $W_k(q,s,u,t)$ has increasing difference on $\mathcal{S} \times (\mathcal{U} \times \mathcal{T})$. By applying Theorem 5-part (1) on $W_k(q,s,u,t)$, we prove the result. ■

C. Proof of Theorem 3

In the proof of Theorem 2, we showed action-value function $W_k(q,s,u,t)$ has monotone difference. The proof of Theorem 3 relies on using these results and additionally prove that $W_k(q,s,u,t)$ is submodular in (u,t) with respect to $\preceq_{\mathcal{A}}$.

Lemma 5 *For any $q', q \in \mathcal{Q}$ and $s \in \mathcal{S}$, reverse cumulative density function $L(q'|q,u,t)$ is submodular in $\mathcal{U} \times \mathcal{T}$ with respect to $\preceq_{\mathcal{A}}$.* □

The proof is omitted due to space constraints.

An implication of Lemma 5 is the following lemma.

Lemma 6 *Given the condition* (P)*, for any $k \in \{1,\dots,K\}$, $q \in \mathcal{Q}$, and $s \in \mathcal{T}$ the action-value function $W_k(q,s,u,t)$ is submodular in $\mathcal{Q} \times \mathcal{T}$ with respect to $\preceq_{\mathcal{A}}$.* □

The proof is omitted due to space constraints.

PROOF (THEOREM 3 PART 1) The results of Theorem 3-part (1), follows from Theorem 5-part (4). By Lemma 4-part (1), we know $W_k(q,s,u,t)$ has decreasing difference on nested increasing subsets $\{F(q) \times \mathcal{T}\}_{q \in \mathcal{Q}}$. Furthermore, by Lemma 6, we know $W_k(q,s,u,t)$ is submodular in $\mathcal{U} \times \mathcal{T}$ for fixed q, s. Now, by applying Theorem 5-part (4) on $W_k(q,s,u,t)$, we prove the result. ■

PROOF (THEOREM 3 PART 2) The results of Theorem 3-part (2), follows from Theorem 5-part (2). By Lemma 4-part (2), we know $W_k(q,s,u,t)$ has increasing difference on $\mathcal{S} \times (\mathcal{U} \times \mathcal{T})$. Furthermore, by Lemma 6, we know $W_k(q,s,u,t)$ is submodular in $\mathcal{U} \times \mathcal{T}$ for fixed q, s. Now, by applying Theorem 5-part (2) on $W_k(q,s,u,t)$, we prove the result. ■

VII. CONCLUSION

In this paper, we consider a cross-layer design of transmitting stochastic traffic over AWGN fading channels with adaptive decision feedback. We formulate the problem as a Markov decision process and use ideas from lattice theory to characterize the qualitative properties of the optimal policies. We illustrate via a numerical example that adaptive decision feedback significantly improves the power-delay trade-off as compared to no feedback.

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APPENDIX A
PROOF OF LEMMA 3

We prove the two parts separately.

a) Decreasing differences on $\mathcal{Q} \times (\mathcal{U} \times \mathcal{T})$: We want to show that for any $q_1, q_2 \in \mathcal{Q}$ and $(u_1,t_1),(u_2,t_2) \in F(q_1) \times \mathcal{T}$ such that $q_1 \le q_2$ and $(u_1,t_1) \preceq_{\mathcal{A}} (u_2,t_2)$ (i.e., $u_1 \le u_2$ and $t_1 \ge t_2$), we have

$$\begin{aligned} &c(q_2,s,u_2,t_2) - c(q_1,s,u_2,t_2) \\ &\qquad \le c(q_2,s,u_1,t_1) - c(q_1,s,u_1,t_1). \end{aligned} \tag{26}$$

Define $\Delta_0 = d(q_2) - d(q_1)$, $\Delta_1 = d(q_2 - u_1) - d(q_1 - u_1)$, and $\Delta_2 = d(q_2 - u_2) - d(q_1 - u_2)$. Since $d(\cdot)$ is weakly increasing and convex, we have $\Delta_0 \ge \Delta_1 \ge \Delta_2$.

Now, consider left hand side of (26).

$$\begin{aligned} &c(q_2,s,u_2,t_2) - c(q_1,s,u_2,t_2) \\ &\quad = p(t_2)\Delta_0 + (1-p(t_2))\Delta_2 \ = p(t_2)[\Delta_0 - \Delta_2] + \Delta_2 \end{aligned}$$

$$\overset{(a)}{\leq} p(t_1)[\Delta_0 - \Delta_2] + \Delta_2 \quad = p(t_1)\Delta_0 + (1 - p(t_1))\Delta_2$$
$$\overset{(b)}{\leq} p(t_1)\Delta_0 + (1 - p(t_1))\Delta_1$$
$$= c(q_2, s, u_1, t_1) - c(q_1, s, u_1, t_1)$$

where (a) follows from (A5) and (b) follows from $\Delta_1 \geq \Delta_2$. Thus, Eq. (26) holds and thus $c(q, s, u, t)$ has the stated property.

b) Increasing differences on $\mathcal{S} \times (\mathcal{U} \times \mathcal{T})$: Consider $s_1, s_2 \in \mathcal{S}$ and $(u_1, t_1), (u_2, t_2) \in \mathcal{U} \times \mathcal{T}$ such that $s_1 \leq s_2$ and $(u_1, t_1) \preceq_{\mathcal{A}} (u_2, t_2)$, i.e., $u_1 \leq u_2$ and $t_1 \geq t_2$. We want to show that

$$c(q, s_2, u_2, t_2) - c(q, s_1, u_2, t_2) \geq c(q, s_2, u_1, t_1) - c(q, s_1, u_1, t_1). \quad (27)$$

From (9), we get that (27) is equivalent to

$$\phi(u_2, t_2)[h(s_2) - h(s_1)] \geq \phi(u_1, t_1)[h(s_2) - h(s_1)]. \quad (28)$$

From (A3), $h(s_2) - h(s_1) \geq 0$. From (A4), we get $\phi(u_2, t_2) \geq \phi(u_1, t_2) \geq \phi(u_1, t_1)$. Thus, Eq. (28) holds and consequently, so does (27).

Appendix B
Proof of Lemma 4

We first state a basic inequality [11, Lemma 4.7.2]

Lemma 7 *Let $\{x_i\}_{i\geq 0}$ and $\{y_i\}_{i\geq 0}$ be real-valued non-negative sequences satisfying $\sum_{i\geq j} x_i \geq \sum_{i\geq j} y_i, \quad \forall j \in \mathbb{Z}_{\geq 0}$. Then, for any increasing real-valued sequence $\{v_i\}_{i\geq 0}$, we have $\sum_{i=0}^{\infty} x_i v_i \geq \sum_{i=0}^{\infty} y_i v_i$.* □

We prove the two parts separately.

a) Decreasing differences on $\mathcal{Q} \times (\mathcal{U} \times \mathcal{T})$: Consider $q_1, q_2 \in \mathcal{Q}$ and $(u_1, t_1), (u_2, t_2) \in F(q_1) \times \mathcal{T}$ such that $q_1 \leq q_2$ and $(u_1, t_1) \preceq_{\mathcal{A}} (u_2, u_2)$ (i.e., $u_1 \leq u_2$ and $t_1 \geq t_2$). We use $P_Q(\ell | q, u, t)$ as a short hand for $\mathbb{P}(Q_{k+1} = \ell \mid Q_k = q, U_k = u, T_k = t)$.

Observe that for any $q' \in \mathcal{Q}$, Lemma 2 implies that

$$L(q' | q_2, u_2, t_2) + L(q' | q_1, u_1, t_1). \leq L(q' | q_2, u_1, t_1) + L(q' | q_1, u_2, t_2).$$

Or, equivalently,

$$\sum_{\ell \geq q'} [P_Q(\ell | q_2, u_2, t_2) + P_Q(\ell | q_1, u_1, t_1)] \leq \sum_{\ell \geq q'} [P_Q(\ell | q_2, u_1, t_1) + P_Q(\ell | q_1, u_2, t_2)]. \quad (29)$$

Each term in the square bracket of the above equation is positive. Pick any $s \in \mathcal{S}$. From (29), Lemma 7, and Theorem 1, we get that

$$\sum_{q'=0}^{\infty} [P_Q(q' | q_2, u_2, t_2) + P_Q(q' | q_1, u_1, t_1)] V_{k+1}(q', s) \leq \sum_{q'=0}^{\infty} [P_Q(q' | q_2, u_1, t_1) + P_Q(q' | q_1, u_2, t_2)] V_{k+1}(q', s). \quad (30)$$

Multiplying both sides by $P_S(s)$ and summing over s, we get

$$\overline{W}_k(q_2, u_2, t_2) + \overline{W}_k(q_1, u_1, t_1) \leq \overline{W}_k(q_2, u_1, t_1) + \overline{W}_k(q_1, u_2, t_2). \quad (31)$$

Thus, $\overline{W}_k(q, u, t)$ has decreasing differences on $\mathcal{Q} \times (\mathcal{U} \times \mathcal{T})$ on nested increasing subsets $\{F(q) \times \mathcal{T}\}_{q \in \mathcal{Q}}$. The result then follows from Lemma 3 and (11).

b) Increasing differences on $\mathcal{S} \times (\mathcal{U} \times \mathcal{T})$: This is an immediate consequence of Lemma 3 and (11).

Appendix C
Proof of Theorem 4

We want to show that for fixed $q \in \mathcal{Q}$, $s \in \mathcal{S}$, $c(q, s, u, t)$ is submodular on $F(q) \times \mathcal{T}$ with respect to the partial order $\preceq_{\mathcal{A}}$.

Since $\mathcal{T}$ is a continuous set, sub-modularity of $c(q, s, u, t)$ is equivalent to $\frac{\partial}{\partial t} c(q, s, u, t)$ being increasing in u for all t. We first derive the expression for some partial derivatives.

1) Let $A = 2\log\varepsilon/\rho M$, $B = 2N/M$, $K = (1 + \rho)$, $C = 2/K$. Then $\phi(u, t)$ can be written as

$$\phi(u, t) = K[\exp(-A + Bu - Ct) - 1].$$

Thus,

$$\frac{\partial \phi(u, t)}{\partial t} = -2\exp(-A + Bu - Ct).$$

2) Recall $p(t) = \exp(Mt)\varepsilon$. Thus

$$\frac{\partial p(t)}{\partial t} = \varepsilon M \exp(Mt).$$

Now,

$$\begin{aligned} \frac{\partial}{\partial t} c(q, s, u, t) &= \lambda_\pi h(s) \frac{\partial}{\partial t} \phi(u, t) \\ &\quad + \lambda_d \frac{\partial}{\partial t} p(t)[d(q) - d(q - u)] \\ &= -2\lambda_\pi h(s) \exp(-A + Bu - Ct) \\ &\quad + \lambda_d \varepsilon M \exp(Mt)[d(q) - d(q - u)]. \end{aligned} \quad (32)$$

Let $c_t(q, s, u, t)$ denote the RHS of (32). We want to show that for fixed q, s, t, $c_t(q, s, u, t)$ is increasing in $u \in F(q) = \{0, 1, \ldots, q\}$. To show this, we expand the domain of $c_t(q, s, u, t)$ to $[0, q]$ and define it as

$$\frac{\partial}{\partial t} c(q, s, u, t) = -2\lambda_\pi h(s) \exp(-A + Bu - Ct) + \lambda_d \varepsilon M \exp(Mt)[D(q) - D(q - u)]. \quad (33)$$

Then,

$$\frac{\partial}{\partial u} c_t(q, s, u, t) = -2\lambda_\pi B h(s) \exp(-A + Bu - Ct) + \lambda_d \varepsilon M \exp(Mt) D'(q - u).$$

Note that $\exp(-A + Bu - Ct) \leq \exp(-A + 2R_{\max})$ and $\exp(Mt) \geq 1$. Under the condition $D'(q) \geq 1$, we have:

$$\frac{\partial}{\partial u} c_t(q, s, u, t) \geq -2\lambda_\pi B h(s) \exp(-A + 2R_{\max}) + \lambda_d \varepsilon M.$$

Under the condition on $h(s)$, we get $\partial c_t(q, s, u, t)/\partial u \geq 0$. Thus, $c(q, s, u, t)$ is sub-modular on $F(q) \times \mathcal{T}$ with respect to $\preceq_{\mathcal{A}}$.