Coupled Control Systems: Periodic Orbit Generation with Application to Quadrupedal Locomotion

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***Abstract*—A robotic system can be viewed as a collection of lower-dimensional systems that are coupled via reaction forces (Lagrange multipliers) enforcing holonomic constraints. Inspired by this viewpoint, this paper presents a novel formulation for nonlinear control systems that are subject to coupling constraints via virtual "coupling" inputs that abstractly play the role of Lagrange multipliers. The main contribution of this paper is a process—mirroring solving for Lagrange multipliers in robotic systems—wherein we isolate subsystems free of coupling constraints that provably encode the full-order dynamics of the coupled control system from which it was derived. This dimension reduction is leveraged in the formulation of a nonlinear optimization problem for the isolated subsystem that yields periodic orbits for the full-order coupled system. We consider the application of these ideas to robotic systems, which can be decomposed into subsystems. Specifically, we view a quadruped as a coupled control system consisting of two bipedal robots, wherein applying the framework developed allows for gaits (periodic orbits) to be generated for the individual biped yielding a gait for the full-order quadrupedal dynamics. This is demonstrated on a quadrupedal robot through simulation and walking experiments on rough terrains.**

***Index Terms*—Robotics, cooperative control, optimization.**

I. Introduction

To achieve dynamic walking on high-dimensional robotic systems, methods that assume simplified models have been applied, such as embedding the central pattern generators to multi-legged locomotion [4]. Through another methodology — dimension reduction, hybrid zero dynamics (HZD) has proven to be a successful approach as a result of its ability to make theoretic guarantees [19] and yield walking for complex humanoids [13], [17] without assuming model simplifications. The main idea behind this approach is that the full-order dynamics of the robot can be reduced to a lower-dimensional surface on which the system evolves. The system can then be studied via the low-dimensional dynamic representation and, importantly, guarantees made can be translated back to the full-order dynamics, i.e., periodic orbits (or walking gaits) in the low-dimensional system imply corresponding periodic orbits in the full-order system. The goal of this paper is to capture this dimension reduction in a more general context—that of *coupled control systems*, which shows the ability to decompose a complex system into low-dimensional subsystems.

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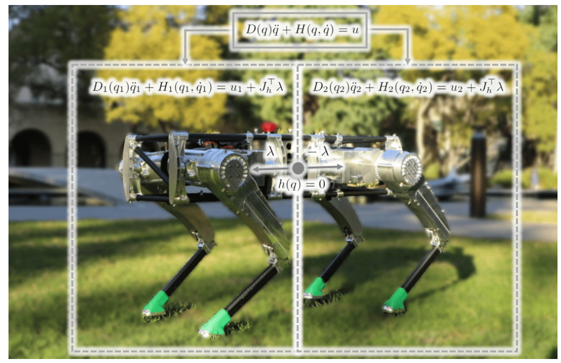

Fig. 1. Conceptual illustration of the full body dynamics decomposition, where the 3D quadruped — the Vision 60 — is decomposed into two constrained 3D bipedal robots.

Another means of dimension reduction for robotic systems comes from isolating subsystems and coupling these subsystems at the level of reaction forces, i.e., Lagrange multipliers that enforce holonomic constraints. This is the idea underlying the highly efficient method for calculating the dynamics of robotic systems: Spatial vector algebra [5]. For example, a double pendulum can be decomposed into two single pendula connected via a constraint at the pivot joint [6]. More generally, one can consider two equivalent ways of expressing the dynamics of a robotic system [12]:

$$\underbrace{D(q)\ddot{q} + H(q,\dot{q}) = u}_{\text{Full-Order Dynamics}} \Leftrightarrow \underbrace{\begin{cases} D_i(q_i)\ddot{q}_i + H_i(q_i,\dot{q}_i) = u_i + J_{h_i}^\top \lambda \\ \text{s.t.} \quad h_i(q) = 0 \end{cases}}_{\text{Reduced-Order Coupled Dynamics}}$$

for $i = 1, 2$, where h is a coupling (holonomic) constraint that is enforced via the Lagrange multiplier λ allowing for the higher-dimensional q to be decomposed into lower-dimensional q_i, i.e., $q = (q_1, q_2)$. For example, a quadrupedal robot can be decomposed into two bipeds as in Fig. 1. Thus, if one can make guarantees on the reduced-order coupled systems, they can be translated to the full-order dynamics.

The study of coupled control systems has a long and rich history from which the framework presented in this paper has taken inspiration. The most prevalent example is that of multi-robot systems [11], the consensus problem [14], [20], and interconnected systems [2]. In the context of mechanical and robotic systems on graphs, network synchronization has been studied [3], [16]. Port-Hamiltonian systems also capture the notion of coupling present in general mechanical systems [18]. Finally, the coordination of quadruped and human reaction forces has recently been studied [7]. While not explicitly discussed due to space constraints, many of these formulations fit within the general setting of coupled control systems presented here.

This paper generalizes the aforementioned methods — zero dynamics and system decomposition through coupling constraints — and unifies them through a novel formulation: *coupled control systems*. We then utilize zero dynamics to reduce to a subsystem dependent on coupling constraints which is then eliminated via coupling relations to yield the final isolated subsystem. The main result of this paper is that solutions to the isolated subsystem are solutions to the full-order system, and thus periodic orbits on the subsystem yield periodic orbits on the full-order system. This result is leveraged to construct a nonlinear optimization problem utilizing collocation methods to generate these periodic solutions.

Our motivating application is gait (periodic orbit) generation for quadrupedal robots. Previously, HZD methods were applied to quadrupedal walking [9]; yet the high complexity of this system made it computationally expensive to generate gaits when compared to their bipedal analogs. To address this shortcoming, recent work has aimed at decomposing quadruped into bipedal robots [10]—it is this methodology that this paper formalizes and extends. In this paper, we consider quadrupedal robots utilizing the coupled control system paradigm, wherein this system can be reduced to lower-dimensional isolated subsystems on which periodic orbits (gaits) can be generated. We demonstrate the results through the realization of these generated gaits experimentally to achieve stable walking on rough terrains.

II. Coupled Control Systems

This section introduces the notion of coupled control systems, for which a collection of differential equations are coupled via algebraic coupling condition. The goal is to present the basic paradigm used throughout the paper.

We first introduce a bidirectional graph $\mathbf{\Gamma} = (\mathbf{N}, \mathbf{E})$ where the vertices $\mathbf{N} = \{1, 2\}$ represent the indices of the subsystems and edges $\mathbf{E} = \{(1, 2), (2, 1)\}$ represent their connections. We then denote $\mathcal{X} = \{\mathcal{X}_i\}_{i\in\mathbf{N}}$ as a set of internal states, $\mathcal{Z} = \{\mathcal{Z}_i\}_{i\in\mathbf{N}}$ as a set of coupled states, and $\mathcal{U} = \{\mathcal{U}_i\}_{i\in\mathbf{N}}$ as a set of admissible control inputs. In addition, we assume $i \neq j \in \mathbf{N}$ and $e = (i, j), \bar{e} = (j, i) \in \mathbf{E}$ throughout the paper.

We can now define the main object of interest.

Definition 1. A ***coupled control system (CCS)*** $\mathcal{C}_{\mathcal{C}}$ is defined on a graph $\mathbf{\Gamma}$ and a conditional expression:

$$\mathcal{C}_{\mathcal{C}} \triangleq \begin{cases} \dot{x}_i = f_i(x_i, z_i) + g_i(x_i, z_i)u_i + \breve{g}_e(x_i, z_i, z_j)\lambda_e \\ \dot{z}_i = p_i(x_i, z_i) + q_i(x_i, z_i)u_i + \breve{q}_e(x_i, z_i, z_j)\lambda_e \\ \text{s.t.} \quad c_e(z_i, z_j) = -c_{\bar{e}}(z_j, z_i) \equiv 0 \\ \qquad \lambda_e = -\lambda_{\bar{e}}, \end{cases} \tag{1}$$

where, $x_i \in \mathcal{X}_i, z_i \in \mathcal{Z}_i, u_i \in \mathcal{U}_i$, $c_e(z_i, z_j) \equiv 0$ is a *coupling constraint* enforced by the *coupling inputs* λ_e, and $\equiv$ represents the identical equality of functions.

We additionally denote $x = (x_1^\top, x_2^\top)^\top \in \mathcal{X}, z = (z_1^\top, z_2^\top)^\top \in \mathcal{Z}, u = (u_1^\top, u_2^\top)^\top \in \mathcal{U}$ and $\lambda = (\lambda_e^\top, \lambda_{\bar{e}}^\top)^\top$ throughout the paper.

Solutions. We define solutions to coupled control systems by assuming the existence of feedback control laws: $u(x, z) \triangleq \{u_1(x_1, z), u_2(x_2, z)\}$. Applying these controllers to (1) yields a ***coupled dynamical system (CDS)***:

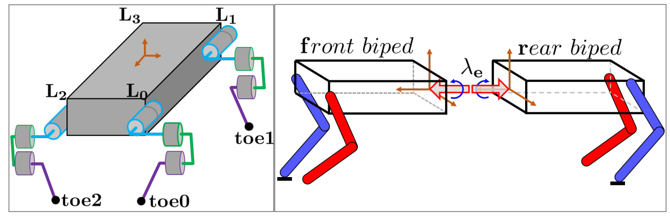

Fig. 2. Left: the configuration of the quadruped, each leg of which has a point contact toe. Right: the decomposition of a quadrupedal robot into two bipedal systems.

$$\mathcal{D}_{\mathcal{C}} \triangleq \begin{cases} \dot{x}_i = f_i^{\text{cl}}(x_i, z) + \breve{g}_e(x_i, z)\lambda_e \\ \dot{z}_i = p_i^{\text{cl}}(x_i, z) + \breve{q}_e(x_i, z)\lambda_e \\ \text{s.t.} \quad c_e(z) \equiv 0, \quad \lambda_e = -\lambda_{\bar{e}} \end{cases} \tag{2}$$

where, $f_i^{\text{cl}} \triangleq f_i(x_i, z_i) + g_i(x_i, z_i)u_i(x_i, z)$, and $p_i^{\text{cl}} \triangleq p_i(x_i, z_i) + q_i(x_i, z_i)u_i(x_i, z)$. Then the ***solution*** of the coupled dynamic system, $\mathcal{D}_{\mathcal{C}}$, is a set of solutions:

$$\Big\{(x_1(t), z_1(t), \lambda_e(t)), (x_2(t), z_2(t), \lambda_{\bar{e}}(t))\Big\} \text{ s.t. (2) } \forall t \in \mathbf{I} \subset \mathbb{R}$$

with initial condition: $\{(x_1(0), z_1(0), \lambda_e(0)), (x_2(0), z_2(0), \lambda_{\bar{e}}(0))\}$, and $\mathbf{I} \subset \mathbb{R}$ is the time interval of their existence. Per the above notation, we will sometimes denote the solutions by $(x(t), z(t), \lambda(t))$ with initial condition $(x(0), z(0), \lambda(0))$.

Coupling constraints. Importantly, the solutions must satisfy the coupling constraints at all time. Therefore,

$$c_e(z) \equiv 0 \;\Rightarrow\; \dot{c}_e(z, \dot{z}) \equiv 0 \tag{3}$$

$$\Rightarrow \underbrace{\frac{\partial c_e(z_i, z_j)}{\partial z_i}}_{\triangleq J_c^{(i,j)}(z)} \dot{z}_i + \underbrace{\frac{\partial c_e(z_i, z_j)}{\partial z_j}}_{\triangleq J_c^{(j,i)}(z)} \dot{z}_j \equiv 0$$

$$\Rightarrow \dot{c}_e(x, z) = J_c^{(i,j)}(z)\left(p_i^{\text{cl}}(x_i, z) + \breve{q}_e(x_i, z)\lambda_e\right) + J_c^{(j,i)}(z)\left(p_j^{\text{cl}}(x_j, z) + \breve{q}_{\bar{e}}(x_j, z)\lambda_{\bar{e}}\right) \equiv 0. \tag{4}$$

Hence, to solve for the coupling inputs λ_e that satisfy the coupling constraints, it is necessary to solve an equation that depends on the states of both subsystems. To address this, we present a method for isolating a subsystem via conditions on the controllers of the other systems in the next section. Before doing this, we utilize the following example to illustrate the concepts of coupled control systems.

Application to quadrupedal robots. The motivating application considered here, is to compute periodic solutions to the quadrupedal dynamics. As Fig. II shows, we decompose this quadruped into two bipeds, whose dynamics are on a CCS graph (according to definition 1): $\mathbf{\Gamma} \triangleq \big(\mathbf{N} = \{\mathrm{f}, \mathrm{r}\}, \mathbf{E} = \{e = (\mathrm{f}, \mathrm{r}), \bar{e} = (\mathrm{r}, \mathrm{f})\}\big)$, where f, r label the *front* and *rear* bipedal systems, correspondingly. We pick the coordinates for these two subsystems as $q_\mathrm{f} = (\xi_\mathrm{f}^\top, \theta_{\mathrm{L}_2}^\top, \theta_{\mathrm{L}_0}^\top)^\top, q_\mathrm{r} = (\xi_\mathrm{r}^\top, \theta_{\mathrm{L}_1}^\top, \theta_{\mathrm{L}_3}^\top)^\top$ with $\xi_i \in \mathbb{R}^3 \times \mathrm{SO}(3)$ and the leg joints $\theta_{L_*} \in \mathbb{R}^3$. Since all leg joints are actuated, the inputs are $u_i \in \mathcal{U} \subset \mathbb{R}^6$. The (continuous-time) dynamics of a quadruped as two coupled bipedal systems are given by a set of Differential Algebraic Equations (DAEs):

$$\mathcal{R}_{\mathcal{Q}} \triangleq \begin{cases} D_i\ddot{q}_i + H_i = J_i^\top F_i + B_i u_i + J_e^\top \lambda_e & (5) \\ J_i\ddot{q}_i + \dot{J}_i\dot{q}_i = 0 & (6) \\ \text{s.t.} \quad c_e(\xi_i, \xi_j) = \xi_i - \xi_j \equiv 0 & (7) \\ \qquad \lambda_e = -\lambda_{\bar{e}} & (8) \end{cases}$$

with $D_i(q_i) \in \mathbb{R}^{n\times n}$ the mass-inertia matrix, $H_i(q_i, \dot{q}_i) \in \mathbb{R}^n$

the drift vector, and $B_i = \begin{bmatrix}\mathbf{0}_{6\times6} & I_{6\times6}\end{bmatrix}$ the actuation matrix. The contact (holonomic) constraint $h_i(q_i) \equiv 0$ is enforced via ground reaction forces $F_i \in \mathbb{R}^3$, whose second derivative is given in (6). More details of these notations can be found in [10].Note that F_i can be eliminated by the solving (5)-(6) to have a shorter form: $D_i\ddot{q}_i + \bar{H}_i = \bar{B}_i u_i + \bar{J}_e^\top \lambda_e$. The derivation is straightforward hence omitted.

To obtain a CCS as in (1), we pick "normal form" type coordinates (see [15]), with the "output" (also known as virtual constraint [19]) that we wish to zero, given by

$$y_i(q_i, \alpha_i) = y^a(q_i) - y^d(\xi_i, \alpha_i), \tag{9}$$

where y^a, y^d are the actual and desired outputs, ξ_i represents a parameterization of time and $\alpha_i \in \mathbb{R}^{6\times6}$ are the coefficients for six 5^{th}-order Beźier polynomials that are designed by the optimization algorithm in Sec.IV. Since our goal is to find a *symmetric ambling* gait for quadrupeds, we chose $\alpha_\mathrm{r} = \mathcal{M}\alpha_\mathrm{f}$, with the matrix $\mathcal{M}$ representing a mirroring relation. It is important to note that the output coordinate here utilizes a state-feedback structure, instead of the time-based construction of [10]. We can then construct our internal states $x_i = (y_i^\top, \dot{y}_i^\top)^\top$, leaving the coupled states as $z_i = (\xi_i^\top, \dot{\xi}_i^\top)^\top$. The end result is a CCS of the form given in (1) for this mechanical system:

$$\dot{x}_i = \underbrace{\begin{bmatrix}\dot{y}_i \\ \dot{J}_{y_i}\dot{q}_i - J_{y_i}D_i^{-1}\bar{H}_i\end{bmatrix}}_{f_i(x_i,z_i)} + \underbrace{\begin{bmatrix}0 \\ J_{y_i}D_i^{-1}\bar{B}_i\end{bmatrix}}_{g_i(x_i,z_i)} u_i + \underbrace{\begin{bmatrix}0 \\ J_{y_i}D_i^{-1}\bar{J}_e^\top\end{bmatrix}}_{\breve{g}_e(x_i,z_i,z_j)} \lambda_e$$

$$\dot{z}_i = \underbrace{\begin{bmatrix}\dot{\xi}_i \\ -J_\xi D_i^{-1}\bar{H}_i\end{bmatrix}}_{p_i(x_i,z_i)} + \underbrace{\begin{bmatrix}0 \\ J_\xi D_i^{-1}\bar{B}_i\end{bmatrix}}_{q_i(x_i,z_i)} u_i + \underbrace{\begin{bmatrix}0 \\ J_\xi D_i^{-1}\bar{J}_e^\top\end{bmatrix}}_{\breve{q}_e(x_i,z_i,z_j)} \lambda_e$$

$$\text{s.t.} \quad c_e(z_i, z_j) = z_i - z_j \equiv 0, \quad \lambda_e = -\lambda_{\bar{e}},$$

where $J_{y_i} = \partial y_i(q_i)/\partial q_i$, $J_\xi = \partial\xi/\partial q = \begin{bmatrix}I_{6\times6} & \mathbf{0}_{6\times6}\end{bmatrix}$, and we suppressed the dependency on x_i, z_i for all entries.

III. Isolating Control Subsystems

The main idea in approaching the analysis and design of controllers for coupled control systems is to isolate subsystems that encode the behavior of the overall CCS. This section outlines the procedure for isolating the subsystems through a two-step approach: restricting systems to the *zero dynamics* manifold, and leveraging this to explicitly calculate the coupling conditions. We then can reduce the full-order CCS to a subsystem that no longer depends on the internal states of the other subsystem. We establish the main result of the paper encapsulating these constructions: solutions to the subsystem yield solutions to the full-order dynamics.

A. λ-Coupled Subsystem

Given a CCS $\mathcal{C}_\mathcal{C}$, we define the ***zero dynamics manifold*** for each subsystem $i \in \mathbf{N}$ as:

$$\mathbf{Z}_i \triangleq \{(x, z) \in \mathcal{X} \times \mathcal{Z} \mid x_i \equiv 0\}. \tag{10}$$

Thus, the zero dynamics manifold for i^{th} subsystem consists of the internal states, x_i, being zero, i.e., the system evolves only according to the coupled states z.

The key idea underlying the analysis of CCSs is to reduce the entire coupled system into the behavior of a single subsystem. This is achieved through the above constructions related to the zero dynamics. We start by designing controllers for the overall CCS on the zero dynamics of subsystem $j \in \mathbf{N}$. A controller $u_j^{\mathbf{Z},\lambda}(x_j, z)$ is said to ***render the zero dynamics manifold*** $\mathbf{Z}_j$ ***invariant*** if it satisfies:

$$0 \equiv f_j(0, z_j) + g_j(0, z_j)u_j^{\mathbf{Z},\lambda}(0, z) + \breve{g}_{\bar{e}}(0, z)\lambda_{\bar{e}} \tag{11}$$

where $u_j^{\mathbf{Z},\lambda}$ implicitly depends on $\lambda_{\bar{e}}$ for $\bar{e} = (j, i) \in \mathbf{E}$. By applying $u_j^{\mathbf{Z},\lambda}$, we obtain a ***λ-coupled control subsystem (λ-CCSub)*** for the i^{th} subsystem:

$$\mathcal{C}_i^{\mathbf{Z},\lambda} \triangleq \begin{cases} \dot{x}_i = f_i(x_i, z_i) + g_i(x_i, z_i)u_i + \breve{g}_e(x_i, z)\lambda_e \\ \dot{z}_i = p_i(x_i, z_i) + q_i(x_i, z_i)u_i + \breve{q}_e(x_i, z)\lambda_e \\ \dot{z}_j = p_j(0, z_j) + q_j(0, z_j)u_j^{\mathbf{Z},\lambda}(0, z) + \breve{q}_{\bar{e}}(0, z)\lambda_{\bar{e}} \\ \text{s.t.} \quad c_e(z) \equiv 0, \quad \lambda_e = -\lambda_{\bar{e}} \end{cases} \tag{12}$$

Thus, the i^{th} subsystem evolves according to its own dynamics and the zero dynamics of all remaining systems—all of which are coupled via the coupling inputs λ.

B. Explicit Coupling Conditions

The coupling between the control systems (1) is enforced via λ and the coupling constraints of the form (4). Similarly, even in the reduction to a subsystem (12), the coupling is still achieved through λ. We wish to generalize this so as to remove the coupling, i.e., isolate subsystems, while still preserving the overall behavior of the full system. We first define the ***coupling relation*** that allows the use of the controllers $u_j^{\mathbf{Z},\lambda}$ to eliminate the dependence on the controllers and internal states of the other subsystem.

Definition 2. For a λ-CCSub $\mathcal{C}_i^{\mathbf{Z},\lambda}$ and $i \in \mathbf{N}$, a ***coupling relation*** is a functional relationship on the coupling inputs

$$\lambda_e^{\mathbf{Z}}(x_i, z; u_i) = A_e^{\mathbf{Z}}(x_i, z)u_i + b_e^{\mathbf{Z}}(x_i, z), \tag{13}$$

that satisfies the coupling constraint (3) for all $e = (i, j) \in \mathbf{E}$.

The coupling relation is then summarized in the following:

Lemma 1. *For a CCS $\mathcal{C}_\mathcal{C}$, if we have*

$$\breve{Q}(x_i, z) \triangleq \begin{bmatrix} g_j(0, z_j) & -\breve{g}_{\bar{e}}(0, z) \\ J_c^{(j,i)}q_j(0, z_j) & J_c^{(i,j)}\breve{q}_e(x_i, z) - J_c^{(j,i)}\breve{q}_{\bar{e}}(0, z) \end{bmatrix}$$

invertible, there exists a controller $u_j^{\mathbf{Z}}$ that renders $\mathbf{Z}_j$ invariant and a coupling relation in (13)*, given by:*

$$\begin{bmatrix} u_j^{\mathbf{Z}}(0, z; u_i) \\ \lambda_e^{\mathbf{Z}}(x_i, z; u_i) \end{bmatrix} = \breve{Q}^{-1}\Big(\begin{bmatrix} 0 \\ -J_c^{(i,j)}q_i(x_i, z_i) \end{bmatrix} u_i + \begin{bmatrix} -f_j(0, z_j) \\ -J_c^{(i,j)}p_i(x_i, z_i) - J_c^{(j,i)}p_j(0, z_j) \end{bmatrix} \Big) \tag{14}$$

Proof. Evaluating (3) along the zero dynamics manifold $\mathbf{Z}_j$, i.e., $x_j \equiv 0$, yields: $J_c^{(i,j)}(z)(p_i(x_i, z) + q_i(x_i, z)u_i + \breve{q}_e(x_i, z)\lambda_e) + J_c^{(j,i)}(z)(p_j(0, z) + q_j(0, z)u_i + \breve{q}_{\bar{e}}(0, z)\lambda_e) = 0$ Combining this with (11) and simultaneously solving for $u_j^{\mathbf{Z}}$ and $\lambda_e^{\mathbf{Z}}$ yields the desired result. □

Recall that the controller $u_j^{\mathbf{Z},\lambda}$ that renders the zero dynamics surface invariant implicitly depends on $\lambda_{\bar{e}}$ via (11). Now with a coupling relation, the dependence of $\lambda_{\bar{e}}$ is removed, and as a result we say that $u_j^{\mathbf{Z}}$ *renders the zero dynamics manifold* $\mathbf{Z}_j$ *invariant* if:

$$0 \equiv f_j^{\mathbf{Z}}(0, z) + g_j^{\mathbf{Z}}(0, z)u_i + g_j(0, z_j)\left(u_j^{\mathbf{Z}}(0, z; u_i) - u_i\right) \tag{15}$$

where $u_j^{\mathbf{Z}}$ is now a function of u_i and

$$\begin{cases} f_j^{\mathbf{Z}}(x_j, z) & \triangleq f_j(x_j, z_j) - \breve{g}_{\bar{e}}(x_j, z) b_e^{\mathbf{Z}}(x_i, z), \\ g_j^{\mathbf{Z}}(x_j, z) & \triangleq g_j(x_j, z_j) - \breve{g}_{\bar{e}}(x_j, z) A_e^{\mathbf{Z}}(x_i, z). \end{cases} \quad (16)$$

Returning to (4), given a coupling relation we can rewrite this coupling constraint as:

$$\dot{c}_e(x_i, z) = J_c^{(i,j)}(z) \left(p_i^{\mathbf{Z}}(x_i, z) + q_i^{\mathbf{Z}}(x_i, z) u_i \right) + J_c^{(j,i)}(z) \left(p_j^{\mathbf{Z}}(x_i, z) + q_j^{\mathbf{Z}}(x_i, z) u_i \right) \equiv 0 \quad (17)$$

where for the subsystem $\mathcal{C}_i^{\mathbf{Z},\lambda}$ we have

$$\begin{cases} p_i^{\mathbf{Z}}(x_i, z) \triangleq p_i(x_i, z_i) + \breve{q}_e(x_i, z) b_e^{\mathbf{Z}}(x_i, z) \\ q_i^{\mathbf{Z}}(x_i, z) \triangleq q_i(x_i, z_i) + \breve{q}_e(x_i, z) A_e^{\mathbf{Z}}(x_i, z) \\ p_j^{\mathbf{Z}}(x_i, z) \triangleq p_j(0, z_j) + q_j(0, z_j) u_j^{\mathbf{Z}}(0, z) - \breve{q}_{\bar{e}}(0, z) b_e^{\mathbf{Z}}(x_i, z) \\ q_j^{\mathbf{Z}}(x_i, z) \triangleq -\breve{q}_{\bar{e}}(0, z) A_e^{\mathbf{Z}}(x_i, z) \end{cases} \quad (18)$$

C. Isolating Subsystems

We now arrive at the key concept for which all of the previous constructions have built — reducing a CCS to a single subsystem that can be used to give guarantees about the entire CCS. This is based on the following definition.

Definition 3. For a CCS $\mathcal{C}_{\mathcal{C}}$, and $i \neq j \in \mathbf{N}$, assume a coupling relation $\lambda_e^{\mathbf{Z}}$ such that there exist $u_j^{\mathbf{Z}}$ rendering the zero dynamics manifold $\mathbf{Z}_j$ invariant. Then the i^{th} ***control subsystem (CSub)*** associated with the CCS $\mathcal{C}_{\mathcal{C}}$ is given by:

$$\mathcal{C}_i^{\mathbf{Z}} \triangleq \begin{cases} \dot{x}_i = f_i^{\mathbf{Z}}(x_i, z) + g_i^{\mathbf{Z}}(x_i, z) u_i \\ \dot{z}_i = p_i^{\mathbf{Z}}(x_i, z) + q_i^{\mathbf{Z}}(x_i, z) u_i \\ \dot{z}_j = p_j^{\mathbf{Z}}(x_i, z) + q_j^{\mathbf{Z}}(x_i, z) u_i \end{cases} \quad (19)$$

where $f_i^{\mathbf{Z}}(x_i, z) \triangleq f_i(x_i, z_i) + \breve{g}_e(x_i, z) b_e^{\mathbf{Z}}(x_i, z)$, $g_i^{\mathbf{Z}}(x_i, z) \triangleq g_i(x_i, z_i) + \breve{g}_e(x_i, z) A_e^{\mathbf{Z}}(x_i, z)$, and $p_i^{\mathbf{Z}}, q_i^{\mathbf{Z}}$, $p_j^{\mathbf{Z}}, q_j^{\mathbf{Z}}$ are given in (18). Furthermore, when a feedback controller $u_i(x_i, z)$ is applied to $\mathcal{C}_i^{\mathbf{Z}}$, the result is a dynamical system, denoted by $\mathcal{D}_i^{\mathbf{Z}}$.

Note that the coupling constraint (17) was not explicitly stated in the CSub $\mathcal{C}_i^{\mathbf{Z}}$. This was because it was solved for via the coupling relation $\lambda_e^{\mathbf{Z}}$. That is, the system naturally evolves on the ***constraint manifold***: $\mathbf{C} \triangleq \{(x, z) \in \mathcal{X} \times \mathcal{Z} : c_e(z) \equiv 0, \ \forall\, e \in \mathbf{E}\}$. This is made formal in the following result. Additionally, it will be seen that solutions to the i^{th} subsystem, denoted by $(x_i(t), z(t), \lambda(t))$, can be used to construct solutions to the full-order CCS. Before formally stating the ultimate result of this paper, we need some notation. Let $(x_i, z) \in \mathcal{X}_i \times \mathcal{Z}$ and consider the canonical embedding $\iota : \mathcal{X}_i \times \mathcal{Z} \hookrightarrow \mathcal{X} \times \mathcal{Z}$ given by $\iota(x_i, z) = (x, z)$, where $x = \{x_i, x_j\}$ and $x_j = 0$.

Theorem 1. *Let $\mathcal{C}_{\mathcal{C}}$ be a CCS, and for the j^{th} system assume there exist $u_j^{\mathbf{Z}}$ that render the zero dynamics manifold $\mathbf{Z}_j$ invariant. Let $\mathcal{C}_i^{\mathbf{Z}}$ be the corresponding λ-CCSub for the i^{th} subsystem. Given a feedback controller $u_i(x_i, z)$ for the CSub with corresponding dynamical subsystem $\mathcal{D}_i^{\mathbf{Z}}$ with solution $(x_i(t), z(t))$ for $t \in I \subset \mathbb{R}$. If*

$$\iota(x_i(0), z(0)) \in \mathbf{C} \quad \Rightarrow \quad \iota(x_i(t), z(t)) \in \mathbf{C} \quad \forall\, t \in I \subset \mathbb{R}$$

then $(\iota(x_i(t), z(t)), \lambda^{\mathbf{Z}}(t))$ with

$$\lambda^{\mathbf{Z}}(t) = \left\{ \lambda_e^{\mathbf{Z}}\big(x_i(t), z(t); u_i(x_i(t), z(t))\big) \right\}_{e \in \mathbf{E}}$$

is a solution to $\mathcal{D}_{\mathcal{C}}$, the CDS obtained by applying $u_i, u_j^{\mathbf{Z}}$.

Proof. The condition that $(x(0), z(0)) \in \mathbf{C}$ is equivalent to $c_e(z(0)) = 0$. Concretely, $c_e(z_i(0), z_j(0)) = 0$. Since $\lambda_e^{\mathbf{Z}}$ is a coupling relation, it satisfies (4) and more explicitly (17); therefore, and being explicit about the arguments, $\dot{c}_e(x(t), z(t)) = 0$ for all $t \in \mathbf{I}$ and all $e \in \mathbf{E}$. It follows that $c_e(z(t)) = 0$ for all $t \in \mathbf{I}$ and $e \in \mathbf{E}$.

The fact that $\big(\iota(x_i(t), z(t)), \lambda^{\mathbf{Z}}(t)\big)$ is a solution to $\mathcal{D}_{\mathcal{C}}$ assuming that $(x_i(t), z(t))$ is a solution to $\mathcal{D}_i^{\mathbf{Z}}$ follows trivially from the fact that the zero dynamics $\mathbf{Z}_j$ are invariant, i.e., $\iota\big(x_i(t), z(t)\big) \in \mathbf{Z}_j, \ \forall\, t \in \mathbf{I}$. □

Periodic Orbits. In the context of quadrupedal dynamics, we will be interested in generating periodic solutions, i.e., walking. A solution to a CDS $\mathcal{D}_{\mathcal{C}}$ is ***periodic*** of period $T > 0$ if for some initial condition $(x(0), z(0), \lambda(0))$:

$$\big(x(t+T), z(t+T), \lambda(t+T)\big) = (x(t), z(t), \lambda(t))$$

with the resulting periodic orbit: $\mathcal{O} = \{(x(t), z(t)) \in \mathcal{X} \times \mathcal{Z} \mid 0 \leq t \leq T\}$. As a result of Theorem 1, periodic orbits in a subsystem correspond to the periodic orbits in the full-order dynamics.

Corollary 1. *Under the conditions of Theorem 1, assume that $(x_i(t), z(t))$ is a periodic solution to $\mathcal{D}_i^{\mathbf{Z}}$ with period $T > 0$ and corresponding orbit $\mathcal{O}_i = \{(x_i(t), z(t)) \in \mathcal{X}_i \times \mathcal{Z} \mid 0 \leq t \leq T\}$. Then $(\iota(x(t), z(t), \lambda^{\mathbf{Z}}(t))$ is a periodic solution to the CDS with period $T > 0$ and corresponding periodic orbit $\mathcal{O} = \iota(\mathcal{O}_i)$.*

Application to quadrupeds. For the quadrupedal dynamics $\mathcal{R}_{\mathcal{Q}}$, since the output (9) has (vector) relative degree 2 with respect to u_i (see [19]), we can explicitly design the controller $u_j^{\mathbf{Z},\lambda}$ that renders $\mathbf{Z}_j$ invariant:

$$u_j^{\mathbf{Z},\lambda} = (J_{y_i} D_j^{-1} \bar{B}_j)^{-1} \big(J_{y_j} D_j^{-1} \bar{H}_j - \dot{J}_{y_j} \dot{q}_j - J_{y_i} D_j^{-1} \bar{J}_e^\top \lambda_e \big),$$

as given by Lemma 1. Hence, this controller satisfies (11) and renders a λ-coupled CSub, as in (12).

For robotic systems, we take these ideas one step further to obtain "bipeds" that are the isolated subsystems associated with quadrupeds. Operating on the invariant zero dynamics manifold $\mathbf{Z}_j$ yields $y_j(q_j, \alpha_j) \equiv 0$, hence

$$\theta_a \equiv H_a^{-1} y^d(\xi_j, \alpha_j) \text{ and } q_j^{\mathbf{Z}}(\xi_j) \equiv \big(\xi_j^\top, (H_a^{-1} y^d(\xi_j, \alpha_j))^\top\big)^\top$$
$$\Rightarrow \quad \ddot{q}_j^{\mathbf{Z}}(\xi_j, \dot{\xi}_j, \ddot{\xi}_j) = J_{\mathrm{z}}(\xi_j) \ddot{\xi}_j + \dot{J}_{\mathrm{z}}(\xi_j, \dot{\xi}_j) \dot{\xi}_j.$$

where $J_{\mathrm{z}} = \partial q_j^{\mathbf{Z}}(\xi_j) / \partial \xi_j$. In another word, if $u_j^{\mathbf{Z},\lambda}$ exists and is applied to j^{th} subsystem, the j^{th} bipedal dynamics given by in (5)-(6) are equivalent to:

$$\begin{cases} D_j \ddot{q}_j^{\mathbf{Z}}(\xi_j, \dot{\xi}_j, \ddot{\xi}_j) + H_j = J_j^\top F_j + B_j u_j^{\mathbf{Z}} + J_e^\top \lambda_e & (20) \\ J_j \ddot{q}_j^{\mathbf{Z}}(\xi_j, \dot{\xi}_j, \ddot{\xi}_j) + \dot{J}_j \dot{q}_j^{\mathbf{Z}}(\xi_j, \dot{\xi}_j) = 0 & (21) \end{cases}$$

where for simplicity we have suppressed the dependencies of $D_j(q_j(\xi_j))$, $J_j(q_j(\xi_j))$ and $H_j(q_j(\xi_j), \dot{q}_j(\xi_j, \dot{\xi}_j))$. We then leverage a specific structure of rigid-body dynamics when using the floating base convention: $B_j u_j + J_{\bar{e}}^\top \lambda_e = (\lambda_{\bar{e}}^\top, u_j^\top)^\top$. Utilizing this, (21) and the first 6 rows of (20) yield the following "bipedal" dynamics:

$$\mathcal{R}_{\mathcal{B}}^{\mathbf{Z}_j} \triangleq \begin{cases} D_j^{\mathbf{Z}} \ddot{\xi}_j + H_j^{\mathbf{Z}} = \hat{J}_j^\top F_j + \lambda_e \\ J_j^{\mathbf{Z}} \ddot{\xi}_j + w_j^{\mathbf{Z}} = 0 \end{cases} \quad (22)$$

with $D_j^{\mathbf{Z}} = \hat{D}_j J_{\mathrm{z}}$, $H_j^{\mathbf{Z}} = \hat{D}_j \dot{J}_{\mathrm{z}} \dot{\xi}_j + \hat{H}_j$, $J_j^{\mathbf{Z}} = J_j J_{\mathrm{z}}$, and $w_j^{\mathbf{Z}} = J_j \dot{J}_{\mathrm{z}} \dot{\xi}_j + \dot{J}_j J_{\mathrm{z}} \dot{\xi}_i$. Here, we denote $\hat{\square}$ as the first 6 rows (block) of the a variable. Hence, $\mathcal{R}_{\mathcal{B}}^{\mathbf{Z}_j}$ represents the dynamics of a subsystem j on $\mathbf{Z}_j$, i.e., (22) evolves according to (11) where F_j can be uniquely determined.

IV. Coupled System Optimization

With the previous construction of coupled control systems, we present a general optimization framework to solve for the solution to the i^{th} CSub in (19) associated with the CCS, while synthesising the controllers that render forward invariance of the zero dynamics manifolds. The approach we will take is a *locally direct collocation* based optimization method [8], which has been widely applied to finding solutions to dynamical systems such as [13]. We now pose the previous formulations as a set of constraints to represent the controlled dynamics of $\mathcal{C}_i^{\mathbf{Z}}$. Along this process, the problem formulation of our target application — the control of quadrupedal walking, will be used as an example to illustrate this method.

Optimization setup. We first discretized the time horizon $t \in [0, T]$ evenly to obtain the grid indices $\kappa = 0, 1, ...\mathrm{K}$, i.e., $t^\kappa = T\kappa/\mathrm{K}$. We define the *decision variable* associated with the i^{th} control subsystem $\mathcal{C}_i^{\mathbf{Z}}$ as:

$$\mathbf{X} \triangleq \{\vartheta^\kappa\}_{\kappa=0,1,\ldots K}, \quad \vartheta^\kappa \triangleq \{x_i^\kappa, \dot{x}_i^\kappa, z_i^\kappa, \dot{z}_i^\kappa, z_j^\kappa, \dot{z}_j^\kappa, u_i^\kappa, u_j^{\mathbf{Z},\kappa}\}$$

Note that we abbreviated the dependency on time t as $\Box^\kappa \triangleq \Box(t^\kappa)$ for notational simplicity.

Recall that given a coupling relation, we have associated zero dynamics invariance conditions given by (15). We will enforce these conditions in the optimization to ensure that $u_j^{\mathbf{Z},\kappa}$ renders $\mathbf{Z}_j$ invariant as:

$$F_{\text{zero}}(\vartheta^\kappa) \triangleq f_j^{\mathbf{Z}}(0, z^\kappa) + g_j^{\mathbf{Z}}(0, z^\kappa)u_i^\kappa + g_j(0, z_j^\kappa)\left(u_j^{\mathbf{Z},\kappa} - u_i^\kappa\right),$$

where $f_j^{\mathbf{Z}}$ and $g_j^{\mathbf{Z}}$ are given as in (16).

Next, following from the constructions in Sec.III-C, we define constraints corresponding to the dynamics of the i^{th} control subsystem $\mathcal{C}_i^{\mathbf{Z}}$ (as obtained from the coupling relation). Denote $\chi^\kappa = (x_i^\kappa, z_i^\kappa, z_j^\kappa)$ and

$$F(\chi^\kappa, u_i^\kappa) \triangleq \begin{cases} f_i^{\mathbf{Z}}(x_i^\kappa, z^\kappa) + g_i^{\mathbf{Z}}(x_i^\kappa, z^\kappa)u_i^\kappa \\ p_i^{\mathbf{Z}}(x_i^\kappa, z^\kappa) + q_i^{\mathbf{Z}}(x_i^\kappa, z^\kappa)u_i^\kappa \\ p_j^{\mathbf{Z}}(x_i^\kappa, z^\kappa) + q_j^{\mathbf{Z}}(x_i, z)u_i^\kappa \end{cases}$$

to obtain the *dynamic constraints* as

$$F_{\text{dyn}}(\vartheta^\kappa) \triangleq \dot{\chi}^\kappa - F(\chi^\kappa, u_i^\kappa) = 0, \tag{C.2}$$

which is an equality constraint imposed on the κ^{th} node to enforce all of the states and controllers satisfy the dynamics in (19). Further, to guarantee that those local solutions satisfying (C.2) stay on the same vector flow, i.e., belong to one unique solution, we employ an implicit stage-3 Runge-Kutta method for formulating this objective as an equality constraint. Concretely, we use Hermite interpolation to compute the interpolated value of χ_c^κ and its slope $\dot{\chi}_c^\kappa$ (see equation (30) of [8]) at the center of the subinterval $[t^\kappa, t^{\kappa+1}]$. Then the ***collocation constraints*** are formed as:

$$d(\chi^\kappa, \chi^{\kappa+1}, u_i^\kappa) \triangleq \dot{\chi}_c^\kappa - F(\chi_c^\kappa, u_i^\kappa) = 0 \tag{C.3}$$

Physical Constraints & Periodic Constraints. A set of inequality constraints (*path constraints*) $p(\vartheta^\kappa) \geq 0$ are used to enforce conditions along the time horizon. For robotics, these are widely applied as obstacle avoidance conditions, and some feasibility conditions for the dynamical system, representing real-world physics. In our application — the walking dynamics of quadrupeds, the inequality constraints are used to define the friction cone condition and maximum ground clearance of the swing foot to be higher than 8 cm.

In addition, a set of equality constraints are imposed on the decision variables at $t = 0, T$ to "connect" the initial and final condition: $b(\chi^0, \chi^{\mathrm{K}}) = 0$, so that the optimal solution to the optimization is a periodic solution to the dynamical system. Particularly, the dynamics of quadrupedal locomotion include both continuous and discrete dynamics, forming a *hybrid control system*. To find a periodic solution (ambling motion), we have the periodic constraint as:

$$b(q_i^0, \dot{q}_i^0, q_i^{\mathrm{K}}, \dot{q}_i^{\mathrm{K}}) = \begin{bmatrix} \Delta(q_i^{\mathrm{K}})\,\dot{q}_i^{\mathrm{K}} - \dot{q}_i^0 \\ q_i^{\mathrm{K}} - q_i^0 \end{bmatrix} = 0 \tag{C.6}$$

where $\Delta(\cdot)$ represents the plastic impact dynamics that maps the pre-impact velocity $\dot{q}_i^{\mathrm{K}}$ to its post-impact term.

Optimization problem. To find the periodic solution to dynamical system (19), we now parse this coupled controlling problem of the isolated i^{th} subsystem as:

$$\begin{aligned}
\underset{\mathbf{X}}{\text{argmin}} \quad & \Phi(\mathbf{X}) && && \text{(NLP)}\\
\text{s.t.} \quad & F_{\text{zero}}(\vartheta^\kappa) = 0 && \kappa = 0, 1\ldots, \mathrm{K} && \text{(C.1)}\\
& F_{\text{dyn}}(\vartheta^\kappa) = 0 && \kappa = 0, 1\ldots, \mathrm{K} && \text{(C.2)}\\
& d(\chi^\kappa, \chi^{\kappa+1}, u_i^\kappa) = 0 && \kappa = 0, 1\ldots, \mathrm{K}-1 && \text{(C.3)}\\
& \vartheta^\kappa \in \mathcal{X} \times \mathcal{Z} \times \mathcal{U} && \kappa = 0, 1\ldots, \mathrm{K} && \text{(C.4)}\\
& p(\vartheta^\kappa) \geq 0 && \kappa = 0, 1\ldots, \mathrm{K} && \text{(C.5)}\\
& b(\mathbf{X}) = 0 && && \text{(C.6)}
\end{aligned}$$

where $\Phi(\cdot) \in \mathbb{R}$ is the cost function. Here, we pick the cost function as the acceleration of the torso orientation to yield a less energetic motion for the ease of experiments. (C.4) defines the upper and lower bounds of the decision variables, i.e., that they live in the admissible space of values. In the application of walking, this was used to define the feasible configuration space and the actuator torque less than 40N·m. The other constraints are as stated as above.

Solutions. As a result, the optimization (NLP) can simultaneously produce trajectories (solutions) of the states $\{x_i(t), z(t)\}$, $u_j^{\mathbf{Z}}(t)$ that renders the zero dynamics manifold $\mathbf{Z}_j$ invariant and the open-loop controller $u_i(t)$, $\forall t \in [0, T]$ for which these solutions are defined. Note that one can also enforce the dynamics $\dot{x}_i^\kappa + \varepsilon x_i^\kappa = 0$ with $\varepsilon > 0$ to guarantee the convergence attribute of the i^{th} isolating subsystem, in which case the controller $u_i(x_i, z)$ is equivalently an input-output feedback linearization controller. Per Theorem 1, given $u_j^{\mathbf{Z}}$ that renders invariant $\mathbf{Z}_j$ and the feedback controller $u_i(x_i, z)$, we can compute $\lambda^{\mathbf{Z}}(t)$ using (13), hence $(\iota(x_i(t), z(t)), \lambda^{\mathbf{Z}}(t))$ is a solution to the original CDS. Further, by imposing the periodic condition on the solution's boundary condition, the optimization produced a periodic solution to period T to the CCS. Therefore, according to Corollary 1, $(\iota(x_i(t), z(t), \lambda^{\mathbf{Z}}(t))$ is a periodic solution to the CDS with period T.

Application to quadrupeds. When posing the control problem of quadrupeds, we leverage the subsystems representing the front and rear bipeds: $\mathcal{R}_{\mathcal{B}}^{\mathbf{Z}_{\mathrm{f}}}$ and $\mathcal{R}_{\mathcal{B}}^{\mathbf{Z}_{\mathrm{r}}}$, as given in (22). Note that these subsystems are still coupled through λ—while this could be explicitly solved for via Lemma 1, we keep it implicit due to the complexity of inverting the mass-inertia

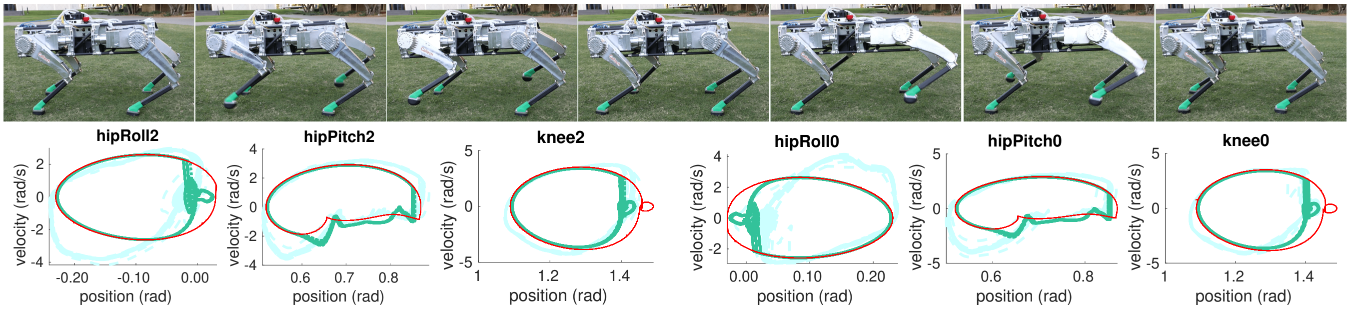

Fig. 3. *Top:* Snapshots showing a full step of the ambling gait in an outdoor lawn. *Bottom:* The periodic trajectory produced by optimization (NLP) (in red) vs. the experimental tracking data (in cyan) vs. RaiSim simulation data (in green) in the form of phase portrait (limit cycle) using 18 seconds' data.

matrix for this particular robotic application. The i^{th} subsystem yield (C.1), (C.2) and (C.3) for (NLP). Specifically for all of the grid indices $\kappa = 0, 1, ...5$, we have the decision variables: $\vartheta^\kappa = \{q_{\text{f}}^\kappa, \dot{q}_{\text{f}}^\kappa, \xi_{\text{r}}^\kappa, \dot{\xi}_{\text{r}}^\kappa, u_{\text{f}}^\kappa, F_{\text{f}}^\kappa, F_{\text{r}}^\kappa, \alpha_{\text{f}}, \lambda_e^\kappa\}$. Finally, the optimization converges to a periodic solution to the isolated bipedal system, which can then be composed to obtain the ambling motion of the quadruped (shown in Fig. 3) according to Theorem 1. We report that the optimization took 17.6s and 295 iterations of searching, which is over 58% faster than the previous full-model based approach in [9]. The computational complexity is mitigated mainly due to the dimension reduction of the state space which is enabled by the representation of the quadrupedal dynamics as bipedal subsystems. For validation purposes, both simulations in a physics engine – RaiSim – and hardware experiments were conducted with a unified, time-based PD approximation of input-output linearizing controllers to track the the desired outputs (represented by $\alpha_{\text{f}}, \alpha_{\text{r}} = \mathcal{M}\alpha_{\text{f}}$):

$$u_i(q_i, \dot{q}_i, t) = -k_p\big(\dot{y}^a(q_i) - y_t^d(t, \alpha_i)\big) - k_d\big(y^a(q_i) - \dot{y}_t^d(t, \alpha_i)\big)$$

with k_p, k_d the PD gains. The result is successful ambling in simulation, and experimentally walking on flat and outdoor uneven terrains (see video [1]). See Fig. 3 for walking tiles and the tracking performance. Remark that the averaged absolute torque inputs are 11.16 N·m, which are well within the hardware limits.

V. Conclusion

As inspired by robotic systems, this paper presented a new formulation of coupled control systems: control systems that are connected via coupling relations and coupling inputs. We demonstrated how these systems can be reduced to a single subsystem that encodes the behavior of the full-order coupled system; this was achieved through leveraging zero dynamics and coupling relations. The main result of this paper was that solutions to these isolated subsystems are solutions to the full-order systems. Building on this, we constructed a nonlinear optimization problem on only a given subsystem that yields periodic orbits for the full-order dynamics. Finally, the application of these ideas were considered for coupled control systems from which a specific example includes quadrupeds. This was demonstrated through experiments on hardware. The general formulation of the CCS problem allows for a wide variety of applications, such as coupling bipedal locomotion to get walking on sloped terrain, stair climbing, and trotting behaviors for multi-legged systems.

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