Local Properties via Color Energy Graphs and Forbidden Configurations*

Sara Fish[†] Cosmin Pohoata[‡] Adam Sheffer[§]

Abstract

The local properties problem of Erdős and Shelah generalizes many Ramsey problems and some distinct distances problems. In this work, we derive a variety of new bounds for the local properties problem and its variants, by extending the color energy technique — a variant of the additive energy technique from additive combinatorics (color energy was originally introduced by the last two authors [12]). We generalize the concept of color energy to higher color energies, and combine these with bounds on the extremal numbers of even cycles.

Let $f(n,k,\ell)$ denote the minimum number of colors required to color the edges of K_n such that every k vertices span at least ℓ colors. It can be easily shown that $f(n,k,\binom{k}{2}-\lfloor\frac{k}{2}\rfloor+2)=\Theta(n^2)$. Erdős and Gyárfás asked what happens when $\ell=\binom{k}{2}-\lfloor k/2\rfloor+1$, one away from the easy case, and derived the bound $f(n,k,\ell)=\Omega(n^{4/3})$. Our technique significantly improves this to $f(n,k,\binom{k}{2}-\lfloor k/2\rfloor+1)=\Omega(n^{2-8/k})$.

Keywords: Local properties, Ramsey theory, color energy, difference sets, forbidden configurations.
Subject classification (MSC2010): 05C35, 05C55.

1 Introduction

Erdős and Shelah [7, Section V] suggested a problem concerning local properties of a graph. Consider a complete graph $K_n=(V,E)$ and a coloring of the edges $\chi : E\to C$ (throughout the paper we assume that C is a set of colors). For positive integers k and ℓ with $\ell\le\binom{k}{2}$, assume that every induced subgraph over k vertices of V contains at least ℓ colors. We define $f(n,k,\ell)$ as the minimum size of C such that there exists a coloring $\chi : E\to C$ satisfying the above property. For example, $f(n,3,3)$ is the minimum number of colors in an edge coloring of K_n where every triangle contains three distinct colors. In this case no vertex can be adjacent to two edges of the same color, so $f(n,3,3)\ge n-1$. The problem is usually studied for fixed k,ℓ and $n\to\infty$.

One reason for studying the local properties problem is that it generalizes many Ramsey problems, and also some distinct distances problems. For example, the classical Ramsey problem is equivalent to determining $f(n,k,2)$. As another example, many works study the minimum number of distinct distances spanned by n points in $\mathbb{R}^2$ such that every k points span at least ℓ distinct distances (for a recent example, see [9]). Denoting this quantity as $\phi(n,k,\ell)$, we have

*This research project was done as part of the 2018 CUNY Combinatorics REU, supported by NSF grant DMS-1710305.

[†]California Institute of Technology, Pasadena, CA, USA *sfish@caltech.edu*. Supported by Caltech's Summer Undergraduate Research Fellowships (SURF) program.

[‡]California Institute of Technology, Pasadena, CA, USA *apohoata@caltech.edu*.

[§]Department of Mathematics, Baruch College, City University of New York, NY, USA. *adamsh@gmail.com*. Supported by NSF award DMS-1802059 and PSC-CUNY award 61666-00-49.

$\phi(n,k,\ell) \geq f(n,k,\ell)$. The problem of determining the thresholds of $f(n,k,\ell)$ has a rich history and attracted much attention (see, for example, [12]).

The *linear threshold* of the local properties problem is the smallest ℓ (for a given k) for which $f(n,k,\ell) = \Omega(n)$. Similarly, the *quadratic threshold* is the smallest ℓ for which $f(n,k,\ell) = \Omega(n^2)$. Erdős and Gyárfás [8] proved that the quadratic threshold is $\ell = \binom{k}{2} - \lfloor \frac{k}{2} \rfloor + 2$ and that the linear threshold is $\ell = \binom{k}{2} - k + 3$. Conlon, Fox, Lee, and Sudakov [5] showed that the *polynomial threshold* (the smallest ℓ for which $f(n,k,\ell) = \Omega(n^\varepsilon)$ for some $\varepsilon > 0$) is $\ell = k$. Recently, a family of additional thresholds appeared in [12]: For each integer $m \geq 2$ there exists $0 \leq c_m \leq m+1$ such that the threshold for having $f(n,k,\ell) = \Omega\left(n^{(m+1)/m}\right)$ is $\ell = \binom{k}{2} - m \cdot \lfloor \frac{k}{m+1} \rfloor + c_m$. Deriving this family of polynomial thresholds was based on introducing a new tool called the *color energy* of a graph. Color energy is a natural variant of the concept of additive energy from additive combinatorics (for example, see [15, Chapter 2]).

The following upper bound was obtained by Erdős and Gyárfás [8].

$$f\left(n,k,\ell\right) = O\left(n^{\frac{k-2}{\binom{k}{2}-\ell+1}}\right). \tag{1}$$

In the current work we derive a variety of new bounds for the local properties problem and its variants. First, recall that the quadratic threshold is $\ell = \binom{k}{2} - \lfloor \frac{k}{2} \rfloor + 2$. Erdős and Gyárfás [8] asked what happens when we move one away from the quadratic threshold. That is, they studied the case of $\ell = \binom{k}{2} - \lfloor \frac{k}{2} \rfloor + 1$, and derived the bound $f\left(n,k,\ell\right) = \Omega(n^{4/3})$. This was later improved in [12] to $f\left(n,k,\ell\right) = \Omega(n^{3/2})$. We show that $f\left(n,k,\ell\right)$ becomes arbitrarily close to n^2 as k grows.

Theorem 1.1. *For every $k \geq 8$, if $r = \lfloor k/4 \rfloor$ then*

$$f\left(n,k,\binom{k}{2} - \lfloor k/2 \rfloor + 1\right) = \Omega(n^{2-2/r}).$$

In particular, when k is a multiple of four, Theorem 1.1 yields the bound $\Omega(n^{2-8/k})$. From (1), we obtain

$$f\left(n,k,\binom{k}{2} - k/2 + 1\right) = O(n^{2-4/k}).$$

We prove Theorem 1.1 by extending the color energy technique from [12]. The revised energy technique can be used to obtain additional bounds. As another example, we derive the following result.

Theorem 1.2. $f\left(n,24,\binom{24}{2} - 15\right) = \Omega(n^{9/8})$.

To the best of our knowledge, none of the previous techniques lead to a non-trivial lower bound for $f\left(n,24,\binom{24}{2} - 15\right)$. From (1) we obtain $f\left(n,24,\binom{24}{2} - 15\right) = O\left(n^{11/8}\right)$.

We also introduce the following family of thresholds for $f(n,k,\ell)$, whose proofs do not rely on color energy.

Theorem 1.3. *(a) For any integers $2 \leq m \leq k/2$,*

$$f\left(n,k,\binom{k}{2} - m(k-m) + 2\right) = \Omega(n^{1/m}).$$

(b) For any integer $t \geq 3$ and $k = \binom{t+1}{2}$,

$$f\left(n,k,\binom{k}{2} - t(t-1) + 1\right) = \Omega\left(n^{1/2+1/(4t-6)}\right).$$

In the first part of Theorem 1.3, we are studying the thresholds for $f(n,k,\ell) = \Omega(n^{1/m})$ for $m \geq 2$. By (1), the threshold for $\Omega(n^{1/m})$ is at least $\ell = \binom{k}{2} - m(k-2) + 1$. Thus, Theorem 1.3(a) establishes an almost tight threshold for $\Omega(n^{1/2})$; this threshold is $\binom{k}{2} - 2k - 3 \leq \ell \leq \binom{k}{2} - 2k - 2$. When $m \geq 3$, there remains a larger gap

$$\binom{k}{2} - m(k-2) + 2 \leq \ell \leq \binom{k}{2} - m(k-m) + 1.$$

We also use our techniques to derive new bounds for the arithmetic variant of the local properties problem. In this variant we have a set A of n real numbers. We define the *difference set* of A as

$$A - A = \{a - a' : \ a, a' \in A \text{ and } a - a' > 0\}.$$

For simplicity, we ignore non-positive differences throughout. This makes no difference with the asymptotics.

Let $g(n,k,\ell)$ denote the minimum size of $A - A$ for any set A of n real numbers that satisfies the following property: Every subset $A' \subset A$ of size k satisfies $|A' - A'| \geq \ell$. Equivalently, this is the original local properties problem $f(n,k,\ell)$ when every vertex corresponds to an element of A and the color of an edge (a,a') is $|a - a'|$.

A discussion about the distinction between $f(n,k,\ell)$ and $g(n,k,\ell)$ can be found in [12]. Note that $g(n,k,\ell) \geq f(n,k,\ell)$. Using our tools we derive significantly stronger lower bounds for $g(n,k,\ell)$. For example, while the linear threshold of $f(n,k,\ell)$ is $\ell = \binom{k}{2} - k + 3$, we get that $g(n,k,\ell)$ is super-linear also when $\ell \approx \binom{k}{2}/2$.

Theorem 1.4. *For all $k > r \geq 2$,*

$$g\left(n, 2rk, \binom{2rk}{2} - \binom{2k}{2} \cdot \left[\binom{r}{2} + (r-1)\right] + 1\right) = \Omega\left(n^{\frac{r}{r-1} \cdot \frac{k-1}{k}}\right).$$

For example, by setting $r = 2$ in Theorem 1.4, we get that for every even $k \geq 4$,

$$g\left(n, k, \binom{k}{2} - 2 \cdot \binom{k/2}{2} + 1\right) = \Omega\left(n^{2 - \frac{8}{k}}\right).$$

For large k, the expression $2 \cdot \binom{k/2}{2}$ is almost half of $\binom{k}{2}$. That is, the number of allowed difference repetitions is about half of the total number of pairs. This behavior is very different than the behavior of $f(n,k,\ell)$, where the linear threshold occurs already when there are about k repetitions. As we increase r in Theorem 1.4, the number of allowed repetitions increases while the lower bound for the number of differences decreases.

Finally, we observe a simple upper bound for $g(n,k,\ell)$.

Proposition 1.5. *For every $\varepsilon > 0$, any sufficiently large c satisfies the following. For every sufficiently large integer k,*

$$g\left(n, k, c \cdot k \cdot \log^{1/4-\varepsilon} k\right) = n \cdot 2^{O(\sqrt{\log n})}.$$

Our approach. Consider a graph $G = (V,E)$ and a coloring $\chi : E \to C$. For an edge $e = (v_1, v_2) \in E$, we also write $\chi(v_1, v_2) = \chi(v_2, v_1) = \chi(e)$. We define the *color energy* of G as

$$\mathbb{E}(\chi) = \left|\left\{(v_1, v_2, v_3, v_4) \in V^4 : \ \chi(v_1, v_2) = \chi(v_3, v_4)\right\}\right|. \tag{2}$$

Color energy was introduced in [12], imitating the concept of additive energy. Studying this quantity immediately led to new bounds for $f(n, k, \ell)$. In the current work, we further push this technique in several different ways. We first show how extremal numbers of even cycles can be used to amplify uses of color energy. We use this approach to derive Theorem 1.1 in Section 3.

We also introduce the concept of higher color energies. The r-th color energy of a graph is a variant of $\mathbb{E}(\chi)$ that consists of $2r$-tuples instead of quadruples. We study a higher color energy by modeling it as the number of edges in a graph with about n^r vertices. We refer to such a graph as an r-th energy graph of the problem. Higher color energies and energy graphs are properly introduced in Section 2, and are then used in Section 3.

Section 4 contains our proofs of Theorem 1.3 and of Proposition 1.5, which do not rely on color energy.

Acknowledgements. We would like to thank Yufei Zhao for a discussion that led to Proposition 1.5. We would like to thank Robert Krueger and Rados Radoicic for several helpful discussions. We are grateful to the anonymous referee who helped us improve this paper.

2 Preliminaries

In this section we describe several results that are used in our proofs. We divide these results according to topics.

Extremal graph theory. *Forbidden subgraph problems* are a main topic of extremal graph theory. For a graph F, let $\text{ex}(n, F)$ denote the maximum number of edges in an F-free graph on n vertices. For $r \geq 2$, Turán proved that $\text{ex}(n, K_{r+1}) = n^2(r-1)/2r$. The Zarankiewicz problem asks about the value of $\text{ex}(n, K_{s,t})$, and is still open.

We can also consider the forbidden subgraph problem in a bipartite graph. For a bipartite graph F, let $\text{ex}(m, n, F)$ denote the maximum number of edges a bipartite graph with sides of size m and n can have without containing F as a subgraph. The following is a classical result of Kővári, Sós, and Turán (for example, see [4, Section IV.2]).

Lemma 2.1. *For integers $s, t \geq 2$, we have that* $\text{ex}(m, n, K_{s,t}) = O_{s,t}(mn^{1-1/s} + n)$.

A classical result about graphs with no cycles of a given length was originally stated by Erdős [6] without proof. For an elegant proof, see Naor and Verstraete [11].

Theorem 2.2. *For every integer $k \geq 2$, we have that* $\text{ex}(n, C_{2k}) = O(n^{1+1/k})$.

Given a graph $G = (V, E)$, a *subdivision* of G is the following bipartite graph. The first vertex set of the subdivision contains a vertex for every element of V and the second vertex set contains a vertex for every edge of E. Every edge of the subdivision is between an edge $e \in E$ and a vertex $v \in V$ such that v is an endpoint of e. Let H_t be the subdivision of K_t. The following is a recent result of Janzer [10].

Theorem 2.3. *For every integer $t \geq 3$, we have that* $\text{ex}(n, H_t) = O(n^{3/2-1/(4t-6)})$.

Our color energy technique turns a local properties problem into a problem involving a graph with about n^r vertices. We derive local properties bounds by showing that these larger graphs do not contain specific subgraphs and applying the relevant bounds on $\text{ex}(n, F)$. Better bounds on $\text{ex}(n, F)$ would immediately imply better bounds for the corresponding local properties problems.

Probabilistic method. We will require the following lemma, obtained by using straightforward probabilistic arguments (see for example [1]). We thus only provide a brief proof sketch.

Lemma 2.4. *Consider a graph $G = (V, E)$ with $|V| = n$, for a sufficiently large n.*

(a) The set V can be partitioned into disjoint sets V_1, V_2 such that $|V_1| = \lceil n/2 \rceil, |V_2| = \lfloor n/2 \rfloor$, and at least $|E|/3$ of the edges of E do not have both of their endpoints in the same V_j.

(b) For an integer $r \geq 2$, let $T \subset E^r$. Then V can be partitioned into disjoint sets $V_1, \dots, V_r$, each of size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$, such that $\Omega_r(|T|)$ of the tuples of T contains only edges having both of their endpoints in the same V_j.

Proof sketch. (a) Pick a uniform random partition of V among the set of partitions satisfying $|V_1| = \lceil n/2 \rceil$ and $|V_2| = \lfloor n/2 \rfloor$. Let X denote the number of edges of E that do not have both of their endpoints in the same V_j. Since the expected size of X is larger than $|E|/3$, there exists at least one partition for which $X > |E|/3$.

(b) Pick a uniform random partition $V_1, \dots, V_r$ of V among the set of partitions into r parts of size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. Let X denote the number of r-tuples of T that contain only edges with both their endpoints in the same V_j. It is not difficult to show that the probability of $2r$ specific vertices being in a specific V_j is at least $(4r)^{-2r}$. This can in turn be used to show that the expected size of X is larger than $|T|(4r)^{-2r}$. $\square$

Energy graphs. Consider a copy of K_n denoted as $G = (V, E)$ and a coloring $\chi : E \to C$. For an integer $r \geq 2$, we define the *r-th color energy* of the coloring as

$$\mathbb{E}_r(\chi) = \left|\left\{(a_1, a_2, \cdots, a_{2r}) \in V^{2r},\ \chi(a_1, a_2) = \chi(a_3, a_4) = \cdots = \chi(a_{2r-1}, a_{2r})\right\}\right|.$$

For a color $c \in C$, we set

$$m_c = \left|\left\{(v_1, v_2) \in V^2 :\ v_1 \neq v_2 \text{ and } \chi(v_1, v_2) = c\right\}\right|.$$

Since every ordered pair of distinct vertices in V^2 contributes to m_c for exactly one c, we get that $\sum_{c \in C} m_c = n(n-1)$. The number of $2r$-tuples that contribute to $\mathbb{E}_r(\chi)$ and correspond to the color c is exactly m_c^r. This implies that $\mathbb{E}_r(\chi) = \sum_{c \in C} m_c^r$. By Hölder's inequality,

$$\mathbb{E}_r(\chi) = \sum_{c \in C} m_c^r \geq \frac{\left(\sum_{c \in C} m_c\right)^r}{\left(\sum_{c \in C} 1\right)^{r-1}} = \frac{n^r(n-1)^r}{|C|^{r-1}}. \tag{3}$$

Note that the "standard" color energy $\mathbb{E}(\chi)$ is the second color energy $\mathbb{E}_2(\chi)$. By (3), to obtain a lower bound for the number of colors, it suffices to derive an upper bound for $\mathbb{E}_r(\chi)$.

By Lemma 2.4(b), there exists a partition of V into r disjoint subsets $V_1, \dots, V_r$, each of size $\Theta(n)$, with the following property. When removing from E every edge that does not have both of its endpoints in the same V_j, the energy $\mathbb{E}_r(\chi)$ does not change asymptotically. (That is, after removing an edge $(u, v) \in E$, we remove from $\mathbb{E}_r(\chi)$ the contribution coming from $2r$-tuples that involve $\chi(u, v)$.) We indeed remove from G every such edge.

An *r-th energy graph* of G and χ, denoted $\hat{G}^r(\chi)$, is defined as follows. Each such graph corresponds to a different partition of V into $V_1, \dots, V_r$ of the form described in Lemma 2.4(b). The set of vertices is $V(\hat{G}^r) = V_1 \times V_2 \times \cdots \times V_r$. An edge between $(v_1, \dots, v_r), (v_1', \dots, v_r') \in V_1 \times \cdots \times V_r$ is in $E(\hat{G}^r)$ if and only if $\chi(v_1, v_1') = \chi(v_2, v_2') = \cdots = \chi(v_r, v_r')$. Note that $\mathbb{E}_r(\chi) = \Theta(|E(\hat{G}^r)|)$. Thus, to obtain a lower bound for the number of colors, we can derive an upper bound on the number of edges in an r-th energy graph of K_n.

We remove "unpopular" colors from $\hat{G}^r$, as follows. Every edge $e \in E(\hat{G}^r)$ corresponds to several edges of E that have the same color. We say that e also has this color. For every color

$c \in C$ that appears in E at most $\log n$ times, we remove from $E(\hat{G}^r)$ every edge that is associated with c. Note that every such color is associated with $O(\log^r n)$ edges of $E(\hat{G}^r)$. Since $|C| = O(n^2)$, this step removes $O(n^2 \log^r n)$ edges from the r-th energy graph. This number is too small to have an effect on any of our proofs. We refer to the resulting graph as a *pruned r-th energy graph* of G, and denote it as G^r.

3 Color energy with no cycles

In this section we derive local properties bounds by using color energy. We begin with our simplest use of color energy — the proof of Theorem 1.1. This proof demonstrates how color energy can be combined with upper bounds on $\text{ex}(n, C_{2r})$. We first recall the statement of Theorem 1.1.

Theorem 1.1. *For every $k \geq 8$, if $r = \lfloor k/4 \rfloor$ then*

$$f\left(n, k, \binom{k}{2} - \lfloor k/2 \rfloor + 1\right) = \Omega(n^{2-2/r}).$$

Proof. Consider a complete graph K_n denoted as $G = (V, E)$ and a coloring $\chi : E \to C$, such that every induced K_k contains at least $\binom{k}{2} - k/2 + 1$ colors. Consider also the color energy $\mathbb{E}_2(\chi)$, as defined and pruned in Section 2. By (3), to obtain a lower bound for $|C|$ it suffices to derive an upper bound for $\mathbb{E}_2(\chi)$. Let G^2 be a pruned second energy graph of χ. By the discussion in Section 2, it suffices to derive an upper bound on $|E(G^2)|$. The definition of G^2 relies on a partition of V into parts V_1, V_2, but we will not rely on this property in the current proof.

To bound $|E(G^2)|$, we wish to apply Theorem 2.2 on G^2. For this purpose, we assume for contradiction that G^2 contains a simple cycle γ of length $2r$. We write $\gamma = (a_1, b_1), \cdots, (a_{2r}, b_{2r})$, where the vertices $a_1, \ldots, a_{2r}, b_1, \ldots, b_{2r} \in V$ may not be distinct. Let S be the set of these vertices, so $|S| \leq 4r \leq k$.

For some intuition, we first consider the case where S consists of $2r$ distinct vertices of V. If necessary, we arbitrarily add vertices to S until $|S| = k$. For every $1 \leq j \leq 2r$, the edge between (a_j, b_j) and (a_{j+1}, b_{j+1}) implies that $\chi(a_j, a_{j+1}) = \chi(b_j, b_{j+1})$ (where $a_{2r+1} = a_1$ and $b_{2r+1} = b_1$). This in turn implies that the number of distinct colors spanned by the vertices of S is at most $\binom{k}{2} - 2r \leq \binom{k}{2} - \lfloor k/2 \rfloor$. We obtained a contradiction to the assumption that every k vertices of V span at least $\binom{k}{2} - \lfloor k/2 \rfloor + 1$ colors.

We next consider the general case, where S might contain fewer than $2r$ vertices of V. We go one-by-one over the edges of γ. In particular, in the j-th step we consider the edge between (a_j, b_j) and (a_{j+1}, b_{j+1}) (as before, $a_{2r+1} = a_1$ and $b_{2r+1} = b_1$). At each step, we have $\chi(a_j, a_{j+1}) = \chi(b_j, b_{j+1})$ and this is either a new color repetition or a repetition we already counted in one of the previous steps. If we are in the latter case, that means that both a_j and b_j already appeared in previous edges of the cycle.

Let m mark the number of steps in which we did not find a new color repetition. In other words, there are at least $2r - m$ distinct color repetitions. In each of the m steps without a new repetition we also had two repeating vertices, so $|S| \leq 2r - 2m$. Let $c = \chi(a_1, a_2)$. We add to S the endpoints of m more edges with color c, and note that $|S| \leq 2r$. By the definition of G^2, the color c has at least $\log n$ edges associated with it, so this is always possible. If necessary, we add to S additional arbitrary vertices until it is of size k. Since the vertices of S span at most $\binom{k}{2} - 2r \leq \binom{k}{2} - \lfloor k/2 \rfloor$ colors, we again obtain a contradiction.

The above contradiction implies that G^2 does not contain a cycle of length $2r$. By Theorem

2.2, we obtain

$$\mathbb{E}_2(\chi) = O(|E(G^2)|) = O\left(\text{ex}(V(G^2), C_{2r})\right) = O\left(\left(n^2\right)^{1+1/r}\right) = O(n^{2+2/r}).$$

Combining this with (3) implies the asserted bound $|C| = \Omega(n^{2-2/r})$. $\square$

Remark. One may wonder what happens when, in the proof of Theorem 1.1, we replace Theorem 2.2 with Theorem 2.3 (that is, forbid a subdivision H_t in the energy graph instead of a cycle). Such a change leads to the bound $f\left(n, k, \binom{k}{2} - k + 2\sqrt{k} + 1\right) = \Omega\left(n^{1+1/(2\sqrt{k}-2)}\right)$. This bound is subsumed by a result of Sárközy and Selkow [14].

We now move to proofs that rely on higher color energies. That is, proofs that rely on $\mathbb{E}_r(\chi)$ with $r \geq 3$. We start with deriving a lower bound on $g(n, k, \ell)$.

Theorem 1.4. *For $k > r \geq 2$, let $\ell = \binom{2rk}{2} - \binom{2k}{2} \cdot \left[\binom{r}{2} + (r-1)\right] + 1$. Then*

$$g\left(n, 2rk, \ell\right) = \Omega\left(n^{\frac{r}{r-1} \cdot \frac{k-1}{k}}\right).$$

Proof. Let A be a set of n real numbers such that every subset $A' \subset A$ of size $2rk$ satisfies $|A' - A'| \geq \ell$. Let $G = (V, E)$ be a copy of K_n with a vertex corresponding to each element of A. We associate a color with each element of $A - A$ and color an edge (a, a') with the color associated with $|a - a'|$ (recall that we define $A - A$ as containing only positive differences). Let C be the set of colors and note that $|C| = |A - A|$. By (3), to obtain a lower bound for $|C|$ it suffices to derive an upper bound for $\mathbb{E}_r(\chi)$. Let G^r be a pruned r-th energy graph of χ. By the discussion in Section 2, it suffices to derive an upper bound on $|E(G^r)|$.

Consider an edge $((v_1, \ldots, v_r), (v'_1, \ldots, v'_r)) \in E(G^r)$. Thinking of the vertices $v_j \in V$ as their corresponding elements in A, we have $|v_1 - v'_1| = |v_2 - v'_2| = \cdots = |v_r - v'_r|$. We associate this edge with a sequence of $r-1$ symbols from $\{+, -\}$, as follows (that is, we associate the edge with an element of $\{+, -\}^{r-1}$). For every $2 \leq j \leq r$, the $(j-1)$-th element of the sequence is '+' if $v_1 - v'_1 = v_j - v'_j$, and '-' if $v_1 - v'_1 = v'_j - v_j$. That is, the associated symbol encodes how to remove the absolute values from $|v_1 - v'_1| = |v_2 - v'_2| = \cdots = |v_r - v'_r|$.

We partition G^r into 2^{r-1} graphs, as follows. Each graph contains the same set of vertices $V(G^r)$, and each graph corresponds to one of the 2^{r-1} sequences of $\{+, -\}^{r-1}$. A graph that corresponds to a specific sequence $s \in \{+, -\}^{r-1}$ contains the edges of $E(G^r)$ that are associated with s. Note that every edge of G^r corresponds to exactly one of the 2^{r-1} graphs. Thus, to obtain an upper bound for $|E(G^r)|$ it suffices to bound the number of edges in each of these graphs.

Let $H = (V(G^r), E_H)$ be one of the 2^{r-1} graphs constructed in the preceding paragraph. Assume for contradiction that H contains a cycle γ of length $2k$, and denote the vertices of γ as

$$(v_{1,1}, \ldots, v_{1,r}), (v_{2,1}, \ldots, v_{2,r}), \ldots, (v_{2k,1}, \ldots, v_{2k,r}).$$

Recall that every edge of E_H corresponds to the same symbol $s \in \{+, -\}^{r-1}$. By the definition of an edge in an energy graph, for every $1 \leq j \leq 2k$ we have

$$v_{j,1} - v_{j+1,1} = s_2(v_{j,2} - v_{j+1,2}) = \cdots = s_r(v_{j,r} - v_{j+1,r}). \tag{4}$$

For any $1 \leq j_1 < j_2 \leq 2k$, summing (4) for $j_1 \leq j < j_2$ yields

$$v_{j_1,1} - v_{j_2,1} = s_2(v_{j_1,2} - v_{j_2,2}) = \cdots = s_r(v_{j_1,r} - v_{j_2,r}). \tag{5}$$

By the above, the vertices of the cycle γ form a clique K_{2k} in G^r. We next claim that the $2kr$ vertices $v_{j,\ell} \in V$ used to define the vertices of γ are all distinct. By the definition of $V_1, \ldots, V_r$, if $j \neq j'$ then $v_{j,\ell}$ and $v_{j',\ell'}$ must correspond to different elements of V. Assume for contradiction that $v_{j,\ell} = v_{j,\ell'}$ for some $\ell \neq \ell'$. By (5) with $j_1 = \ell$ and $j_2 = \ell'$, we obtain that $(v_{\ell,1}, \ldots, v_{\ell,r}) = (v_{\ell',1}, \ldots, v_{\ell',r})$. This contradicts γ being a simple cycle, which implies that the $2kr$ vertices $v_{j,\ell} \in V$ are indeed distinct.

Consider the set S consisting of the $2kr$ vertices $v_{j,\ell} \in V$ used to define the vertices of γ. By the preceding paragraph $|S| = 2kr$. By (4), for each of the $\binom{2k}{2}$ choices for j_1 and j_2 we have $r-1$ distinct color repetitions. Consider one such repetition $v_{j_1,\ell} - v_{j_2,\ell} = v_{j_1,\ell'} - v_{j_2,\ell'}$ with $\ell \neq \ell'$ and note that it leads to a second repetition $v_{j_1,\ell} - v_{j_1,\ell'} = v_{j_2,\ell} - v_{j_2,\ell'}$ (if instead we start with $v_{j_1,\ell} - v_{j_2,\ell} = v_{j_2,\ell'} - v_{j_1,\ell'}$ then we have the second repetition $v_{j_1,\ell} - v_{j_2,\ell'} = v_{j_2,\ell} - v_{j_1,\ell'}$). In addition to the $r-1$ repetitions from the edge definition, we obtain $\binom{r}{2}$ repetitions this way. Thus, for each of the $\binom{2k}{2}$ choices for j_1 and j_2 we actually have $r - 1 + \binom{r}{2}$ distinct color repetitions. This contradicts the local property assumption, so H does not contain a cycle of length $2k$.

Since H does not contain a cycle of length $2k$, Theorem 2.2 implies

$$|E_H| \leq \mathrm{ex}(|V(G^r)|, C_{2k}) = O\left(|V(G^r)|^{1+1/k}\right) = O(n^{r+r/k}).$$

Recall that $E(G^r)$ is partitioned into 2^{r-1} subsets, each satisfying the above upper bound for $|E_H|$. We thus have

$$\mathbb{E}_r(\chi) = \Theta\left(|E(G^r)|\right) = O(n^{r+r/k}).$$

Combining this upper bound for $\mathbb{E}_r(\chi)$ with (3) gives

$$|A - A| = |C| = \Omega\left(n^{\frac{r}{r-1}\cdot\frac{k-1}{k}}\right).$$

□

We also rely on higher color energy to prove the following result.

Theorem 1.2. $f\left(n, 24, \binom{24}{2} - 15\right) = \Omega(n^{9/8})$.

Proof. Let $G = (V, E)$ be a copy of K_n and let $\chi : E \to C$, such that every copy of K_{24} in G has at least $\binom{24}{2} - 15$ distinct colors. Let G^3 be a pruned third energy graph of χ.

Assume for contradiction that there exists a vertex $v \in V$ adjacent to at least 17 edges of color $c \in C$. Let S be a set consisting of v, of 17 vertices that form with v an edge of color c, and of six arbitrary additional vertices of V. Then S is a set of 24 vertices of V that span at most $\binom{24}{2} - 16$ colors of C. This contradicts the local property, so no vertex of V can be adjacent to 17 edges of the same color.

Pruning. We perform two additional steps of pruning $E(G^3)$. First, for every $1 \leq j \leq 3$ we partition V_j into two disjoint sets V_j' and V_j'' and discard from $E(G^3)$ every edge that has both of its endpoints containing a vertex from V_j' or both of its endpoints containing a vertex from V_j''. By imitating the proof of Lemma 2.4(a), we get that this can be done without asymptotically changing the size of $E(G^3)$.

In our second pruning step, we throw from $E(G^3)$ edges until G^3 satisfies the following property: For every $v \in V(G^3)$, no two neighbors of v have the same value in one of their three coordinates. In particular, we repeatedly choose an edge $e \in E(G^3)$ that will remain in the graph and then remove every edge $e' \in E(G^3)$ that violates the above condition together with e'. Figure 1 depicts

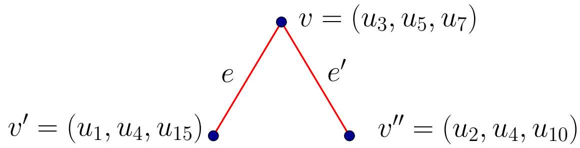

Figure 1: After deciding to keep e, we need to remove e' because of the common coordinate u_4.

a situation where after deciding to keep $e = (v, v')$, we need to remove $e' = (v, v'')$, since v and v' have the common coordinate $u_4 \in V$. We now show that this process does not asymptotically decrease the size of $E(G^3)$.

Consider an edge $e = (v, v')$ that we decided to keep in $E(G^3)$. Write $v = (u_1, u_2, u_3)$ and $v' = (u'_1, u'_2, u'_3)$, where $u_1, u_2, u_3, u'_1, u'_2, u'_3 \in V$. Let $c = \chi(u_1, u'_1) = \chi(u_2, u'_2) = \chi(u_3, u'_3)$. Assume for contradiction that v has 17 neighbors of the form $(u'_1, u'_2, *)$, where the $*$ could be any vertex of V_3. Since each of these 17 neighbors has a different vertex replacing the $*$, and each such vertex $w \in V_3$ must satisfy $\chi(u_3, w) = c$, we get that u_3 is adjacent to 17 edges of color c in G. This contradicts the above, so v has at most 16 neighbors of the form $(u'_1, u'_2, *)$.

We now assume that v has more than 16^2 neighbors of the form $(u'_1, *, *)$, where the $*$ symbols represent any $w_2 \in V_2$ and $w_3 \in V_3$, respectively. By the preceding paragraph, after fixing w_2 there are at most 16 options for w_3. Thus, there are at least 17 distinct values for w_2. As in the preceding paragraph, this implies that u_2 is adjacent to at least 17 edges of color c. This contradiction implies that v has at most 16^2 neighbors of the form $(u'_1, *, *)$. Similarly, v has at most 16^2 neighbors of the form $(*, u'_2, *)$ and at most 16^2 neighbors of the form $(*, *, u'_3)$. Therefore, after keeping an edge in $E(G^3)$ we remove at most $3 \cdot 16^2$ edges adjacent to v. Symmetrically, we remove at most $3 \cdot 16^2$ edges adjacent to v'. We conclude that the second pruning step does not change the asymptotic size of $E(G^3)$.

Cycles in the energy graph. Assume for contradiction that G^3 contains a cycle γ of length eight. We denote the vertices of γ as $(a_1, b_1, c_1), \cdots, (a_8, b_8, c_8)$. We create a set of vertices $S \subset V$ in eight steps, where during the j-th step we add a_j, b_j, c_j to S (it is possible that some of these vertices were already placed in S in a previous step). We now show that at each step we can add at most three vertices to S and obtain at least two new color repetitions in the subgraph induced by S. When beginning the j-th step, if at least two of the vertices a_j, b_j, c_j are not already in S then $\chi(a_j, a_{j+1}) = \chi(b_j, b_{j+1}) = \chi(c_j, c_{j+1})$ yields two new color repetitions. As usual, we set $a_9 = a_1, b_9 = b_1$, and $c_9 = c_1$.

At the beginning of the first step all three vertices a_1, b_1, c_1 are new, since S is empty. Recalling the partitioning of each of V_1, V_2, V_3 into two disjoint sets, we note that a_2, b_2, c_2 are all new, since they cannot be identical to vertices from the first step. Recalling also that the neighbors of a vertex of $V(G^3)$ cannot have any identical coordinates, we get that $a_3, b_3, c_3, a_4, b_4, c_4$ are also all new. That is, in the first four steps we place 12 vertices in S and have eight distinct color repetitions.

If at step j exactly one of a_j, b_j, c_j is not already in S, then the edge involving this new vertex yields one new color repetition. Since we only add to S one new vertex from a_j, b_j, c_j, we are allowed to add two additional vertices. We take an arbitrary edge of color $\chi(a_1, a_2)$ that is not already in the subgraph induced by S, and add both of the endpoints of this edge to S. Such an edge is always available by the assumption that every color appears at least $\log n$ times (see the above definition of an r-th energy graph). Note that in this case we indeed added at most three new vertices to S and at least two new color repetitions to the subgraph induced by S.

In the fifth step, at least one of the vertices a_5, b_5, c_5 is not in S yet. Indeed, since each of the sets V_1, V_2, V_3 is bipartite, these vertices cannot be equivalent to vertices from steps with even indices. Since neighbors with common coordinates are not allowed, $a_5 \neq a_3, b_5 \neq b_3$, and $c_5 \neq c_3$. Finally, it is impossible to have $(a_1, b_1, c_1) = (a_5, b_5, c_5)$ since then we would have a vertex of $V(G^3)$ repeating twice in γ, implying that γ is not a simple cycle. A similar argument shows that there is at least one new vertex of V also in the sixth, seventh, and eighth steps. This concludes the construction of S.

If after the above process S still has fewer than 24 vertices, we keep adding arbitrary vertices until $|S| = 24$. By the above, the subgraph induced by S has at least 16 color repetitions. In other words, this subgraph contains at most $\binom{24}{2} - 16$ distinct colors. This contradicts the local property, so $G(E^3)$ cannot contain a cycle of length eight. Theorem 2.2 implies

$$\mathbb{E}_3(\chi) = \Theta\left(|E(G^3)|\right) = O\left(\mathrm{ex}(|V(E^3)|, C_8)\right) = O\left(|V(E^3)|^{5/4}\right) = O(n^{15/4}).$$

Combining this with (3) (when $r = 3$) yields $|C| = \Omega(n^{9/8})$, as asserted. □

4 Proofs with no color energy

This section contains proofs where we do not rely on color energy. We repeat each result before proving it.

Theorem 1.3. *(a) For any integers $2 \leq m \leq k/2$,*

$$f\left(n, k, \binom{k}{2} - m(k-m) + 2\right) = \Omega(n^{1/m}).$$

(b) For any integer $t \geq 3$ and $k = \binom{t+1}{2}$,

$$f\left(n, k, \binom{k}{2} - t(t-1) + 1\right) = \Omega\left(n^{1/2 + 1/(4t-6)}\right).$$

Proof. (a) Consider a copy of $K_n = (V, E)$ and a coloring $\chi : E \to C$, such that every copy of K_k in the graph contains at least $\binom{k}{2} - m(k-m) + 2$ colors. Assume for contradiction that there exists a color $c \in C$ such that $\Omega(n^{2-1/m})$ edges $e \in E$ satisfy $\chi(e) = c$ (with a sufficiently large constant in the $O(\cdot)$-notation). Let $E' \subset E$ be the set of edges with color c.

By Lemma 2.4(a), we can partition the vertices of V into two disjoint sets V_1, V_2 each of size $\Theta(n)$ such that $\Theta(|E'|)$ of the edges of E' have one endpoint in V_1 and one in V_2. Let $E^* \subset E'$ denote the set of edges with one endpoint in each set. Then $G = (V_1, V_2, E^*)$ is a bipartite graph with $\Theta(n)$ vertices in each part and $\Omega\left(n^{2-1/m}\right)$ edges. Since we started with a sufficiently large constant in the $\Omega(\cdot)$-notation, by Lemma 2.1 we get that G contains a copy of $K_{m,k-m}$.

From the preceding paragraph, we have that the original colored K_n contains a copy of $K_{m,k-m}$ with all of its edges having color c. This is a set of k vertices with at most $\binom{k}{2} - m(k-m) + 1$ distinct colors. Since this contradicts the local property of the coloring, we conclude that every color of C appears $O\left(n^{2-1/m}\right)$ times in E. This immediately implies that $|C| = \Omega\left(n^{1/m}\right)$.

(b) The proof is obtained by repeating the proof of part (a), while replacing the use of Lemma 2.1 with Theorem 2.3. That is, we assume for contradiction that there exists a color $c \in C$ such that $\Omega\left(n^{3/2-1/(4t-6)}\right)$ edges $e \in E$ satisfy $\chi(e) = c$. We then obtain a contradiction by showing that the colored K_n contains a copy of H_t with all of its edges having the color c. A copy of H_t consists of $\binom{t+1}{2}$ vertices and $t(t-1)$ edges. □

For the following result, see for example [13, Theorem 9.1 of Chapter 2] combined with [3].

Theorem 4.1. *Let $\varepsilon > 0$ and let A be a set of n elements that contains no 3-term arithmetic progression. Then $|A - A| = \Omega\left(n \cdot \log^{1/4-\varepsilon} n\right)$.*

We are now ready to prove our upper bound for $g(n, k, \ell)$.

Proposition 1.5. *For every $\varepsilon > 0$, any sufficiently large c satisfies the following. For every sufficiently large integer k,*

$$g\left(n, k, c \cdot k \cdot \log^{1/4-\varepsilon} k\right) = n2^{O(\sqrt{\log n})}.$$

Proof. Behrend [2] proved that there exists a set A of n positive integers such that $|A - A| = n \cdot 2^{O(\sqrt{\log n})}$ and no three elements of A form an arithmetic progression. Let $B \subseteq A$ satisfy $|B| = k$. Since B does not contain a 3-term arithmetic progression and k is sufficiently large, Theorem 4.1 gives

$$|B - B| = \Omega(k \log^{1/4-\varepsilon} k) \leq c \cdot k \cdot \log^{1/4-\varepsilon} k,$$

where the last transition holds for sufficiently large c. This construction immediately implies the asserted bound. $\square$

References

[1] N. Alon, J. H. Spencer, *The Probabilistic Method*, Wiley–Interscience, 2004.

[2] F. A. Behrend, On sets of integers which contain no three terms in arithmetic progression, *Proc. Natl. Acad. Sci. U.S.A.* **32** (1946), 331–332.

[3] T. F. Bloom, A quantitative improvement for Roth's theorem on arithmetic progressions, *J. London Math. Soc.* **93** (2016), 643–663.

[4] B. Bollobás, *Modern graph theory*, Graduate texts in Mathematics, Springer–Verlag, 1998.

[5] D. Conlon, J. Fox, C. Lee, B. Sudakov, The Erdős–Gyárfás problem on generalized Ramsey numbers, *Proc. London Math. Soc.*, **110** (2014), 1–18.

[6] P. Erdős, Extremal problems in graph theory, *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)* , 29–36.

[7] P. Erdős, Problems and results on finite and infinite graphs, *Recent advances in graph theory* (1974), 183–192.

[8] P. Erdős and A. Gyárfás, A variant of the classical Ramsey problem, *Combinatorica* **17** (1997), 459–467.

[9] J. Fox, J. Pach, and A. Suk, More distinct distances under local conditions *Combinatorica* **38** (2018), 501–509.

[10] O. Janzer, Improved bounds for the extremal number of subdivisions, *Electron. J. Combin.* **26** (2019), P3.3.

[11] A. Naor and J. Verstraete, A Note on Bipartite Graphs without 2k-Cycles, *Comb. Probab. Comput.* **14** (2005), 845–849.

[12] C. Pohoata and A. Sheffer, Local Properties in Colored Graphs, Distinct Distances and Difference Sets, *Combinatorica* **39** (2019), 705–714.

[13] I. Z. Rusza, Sumsets and structure, *Combinatorial number theory and additive group theory* (2009), 87–210.

[14] G. Sárközy and S. Selkow, On edge colorings with at least q colors in every subset of p vertices, *Electron. J. Combin.* **8** (2001), R9.

[15] T. Tao and V. H. Vu, *Additive combinatorics*, Cambridge University Press, 2006.