

THE CONTINUUM LIMIT OF THE KURAMOTO MODEL ON SPARSE RANDOM GRAPHS*

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Abstract. In this paper, we study convergence of coupled dynamical systems on convergent sequences of graphs to a continuum limit. We show that the solutions of the initial value problem for the dynamical system on a convergent graph sequence tend to that for the nonlocal diffusion equation on a unit interval, as the graph size tends to infinity. We improve our earlier results in [Medvedev, The nonlinear heat equation on W-random graphs, Arch. Rational Mech. Anal., 212(3): 781–803] and extend them to a larger class of graphs, which includes directed and undirected, sparse and dense, random and deterministic graphs.

There are three main ingredients of our approach. First, we employ a flexible framework for incorporating random graphs into the models of interacting dynamical systems, which fits seamlessly with the derivation of the continuum limit. Next, we prove the averaging principle for approximating a dynamical system on a random graph by its deterministic (averaged) counterpart. The proof covers systems on sparse graphs and yields almost sure convergence on time intervals of order $\log n$, where n is the number of vertices. Finally, we prove convergence of the averaged model to the continuum limit.

The analysis of this paper covers the Kuramoto model of coupled phase oscillators on a variety of graphs including sparse Erdős-Rényi, small-world, and power law graphs.

Keywords. interacting dynamical systems; continuum limit; random graph; sparse graph; graph limit.

AMS subject classifications. 34C15; 45J05; 45L05; 74A25; 05C90; 92D25.

1. Introduction

Understanding principles of collective dynamics in large ensembles of interacting dynamical systems is a fundamental problem in nonlinear science with applications ranging from neuronal and genetic networks to power grids and the Internet. The key distinction of coupled dynamical systems considered in this paper from classical spatially extended systems such as partial differential equations or lattice dynamical systems is that the spatial domain of the former class of models is a general graph. Given an enormous variety of graphs and their complexity, analyzing dynamical systems on large and, in particular, on random graphs is a challenging problem.

In [10, 11], we initiated a study of the continuum limit of systems of coupled phase oscillators on convergent families of graphs. We used the fact that a large class of (dense) graphs, including many of those of interest in applications, can be described analytically by a measurable function on a unit square, called a graphon [7]. Roughly speaking, a graphon represents a limit of the adjacency matrix of a graph as its size tends to infinity. Using graphons, we were able to derive and justify the continuum limit for the Kuramoto model (KM) on a great variety of graphs, which led to new studies of the KM on nontrivial graphs [3, 12, 13, 15]. Importantly, the same approach can be successfully applied to justify the mean field limit for coupled dynamical systems on graphs [2, 6].

The analysis in [10, 11] did not cover the KM on sparse graphs. The progress in this direction became possible with the theory of L^p graphons used to define graph limits for sparse graphs of unbounded degree [1]. Using the insights from [1], we addressed

*Received: June 22, 2018; Accepted (in revised form): November 3, 2018. Communicated by Pierre-Emmanuel Jabin.

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the problem of the continuum limit of the KM on sparse graphs in [5]. While we were able to extend many of our techniques to the KM on a large class of sparse graphs (including power-law graphs), some of the results in [5] apply only to systems with linear diffusion. In the present work, we unify and, in the case of the KM on random graphs, significantly improve the results in [5, 10, 11]. The contribution of this paper is twofold. First, we propose a flexible framework for describing directed and undirected, sparse and dense, random and deterministic graphs to be used in interacting dynamical systems models. This framework naturally leads to continuum models approximating dynamical systems on large graphs. Second, we refine our techniques to obtain stronger results on convergence to the continuum limit, which, in addition, apply to a wider class of graphs than in [5, 11]. Even for the KM on dense graphs, our results are much stronger: we show convergence of solutions on the time intervals of order $\log n$, compared to finite intervals in [11] (see also [9]). Furthermore, in the present work, convergence is shown with probability 1 versus convergence in probability in the earlier papers [5, 11]. Finally, our results apply to the KM on sparse directed graphs, which have not been considered in [5, 11]. Taken together, the results of this paper reveal a fuller potential of our method for proving convergence of discrete problems on graphs to a continuum limit.

As in [11], the main result of this work is the proof of convergence of solutions of the initial value problems (IVPs) for the KM on graphs to the solution of that for the limiting nonlocal diffusion equation as the size of the graph tends to infinity. In its most basic version, the result may be seen as convergence of numerical discretization of a nonlocal diffusion equation. The contribution of this paper, however, is much deeper and more interesting. First, we consider dynamical problems on random graphs. This situation is not treated in classical numerical analysis. More importantly, we use minimal regularity assumptions on the limiting graphon W . The only assumption is that W is a square integrable function on a unit square subject to technical conditions (W-1), (W-2). This allows us to treat a huge class of graphs and affords great flexibility in applications. The fact that W does not require any regularity beyond integrability means, in particular, that the order, in which vertices are sampled, is irrelevant. Last but not least, the convergence problem analyzed in this work is motivated by concrete questions about the dynamics of large networks [3, 14, 16].

There are three main ingredients in our proof of convergence. First, as we commented above, we construct convergent families of graphs in the spirit of W -random graphs [8]. This description covers a broad class of graphs and fits seamlessly with the analysis of convergence of the discrete models to the continuum limit. In particular, the limit of the graph sequence, given by a measurable real-valued function W on the unit square, is used later in the derivation of the continuum model as a kernel of a nonlocal diffusion term. Many random graph models like small-world, Erdős-Rényi, and even power law graphs have relatively simple graph limits, which makes the corresponding continuum models amenable to analysis [11, 14, 15]. The key tool for dealing with the models on random graphs is the averaging principle, which justifies approximation of a coupled system on a random graph by an averaged deterministic model on a complete weighted graph. Finally, we prove convergence of the discrete averaged models to the continuum limit.

The organization of the paper is as follows. In the next section, we define convergent graph sequences that are used in the remainder of this paper and formulate the KM on random graphs. In Section 3, we state the main result about the convergence of the discrete model on graphs to the continuum model. Here, we also explain the main steps

of the proof. In Section 4, we prove the averaging principle, the first main ingredient of the proof of convergence to the continuum limit. It allows to approximate the KM on a random graph by a deterministic model via averaging over all realizations of the random graph model. The averaged model then suggests the continuum limit in the form of a nonlinear nonlocal diffusion equation. In Section 5, we establish the continuum limit for the discrete averaged models. Together with the averaging principle and some auxiliary estimates, this implies the convergence of the KM on (sparse) random graphs.

2. The KM on graphs

Let $\Gamma_n = \langle V(\Gamma_n), E_d(\Gamma_n), A_n \rangle$ be a weighted directed graph on n nodes. $V(\Gamma_n) = [n]$ stands for the node set of Γ_n . $A_n = (a_{n,ij})$ is an $n \times n$ weight matrix. The edge set

$$E_d(\Gamma_n) = \{(i,j) \in [n]^2 : a_{n,ij} \neq 0\}.$$

An edge (i,j) is an ordered pair of nodes. We will also use $j \rightarrow i$ to denote the edge (i,j) . Loops are allowed.

We will also consider undirected weighted graphs $\Gamma_n = \langle V(\Gamma_n), E(\Gamma_n), A_n \rangle$. In this case, A_n is a symmetric matrix and the edges are unordered pairs of nodes

$$E(\Gamma_n) = \{\{i,j\} \in [n]^2 : a_{n,ij} \neq 0\}.$$

We will use $i \sim j$ as a shorthand for $\{i,j\} \in E(\Gamma_n)$.

Consider a system of coupled oscillators on a sequence of weighted (directed or undirected) graphs Γ_n

$$\dot{u}_{n,i} = f(u_{n,i}, t) + (n\alpha_n)^{-1} \sum_{j=1}^n a_{n,ij} D(u_{n,j} - u_{n,i}), \quad i \in [n], \quad (2.1)$$

$$u_{n,i}(0) = u_{n,i}^0. \quad (2.2)$$

Here, $u_{n,i} : \mathbb{R} \rightarrow \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ stands for the phase of oscillator $i \in [n]$ as a function of time. D is a 2π -periodic Lipschitz continuous function, $\text{Lip}(D) = L_D$. Without loss of generality,

$$\max_{u \in \mathbb{T}} |D(u)| = 1. \quad (2.3)$$

Function $f(u, t)$ is Lipschitz continuous in u , $\text{Lip}_u(f) = L_f$, and continuous in t . The sum on the right-hand side of (2.1) models the interaction between oscillators. Finally, unless otherwise specified $\alpha_n = 1$. The scaling factor α_n will be needed for the KM on sparse random graphs, as explained below.

Equation (2.1) generalizes the original KM by allowing nonlinearity $f(u, t)$ and sequence $\{\Gamma_n\}$ as a spatial domain. We are interested in the large n limit of (2.1), (2.2). One can expect a limiting behavior of solutions of (2.1), (2.2), only if the graph sequence $\{\Gamma_n\}$ has a well-defined asymptotic behavior in the limit as $n \rightarrow \infty$. We define the asymptotic structure of $\{\Gamma_n\}$ using function $W \in L^1(I^2)$, called a graphon. To define $\{\Gamma_n\}$, we discretize the unit interval by points $x_{n,j} = j/n$, $j \in \{0\} \cup [n]$ and denote $I_{n,i} := (x_{n,i-1}, x_{n,i}]$, $i \in [n]$. We chose the uniform mesh $\{x_{n,i}, i = 0, 1, \dots, n\}$, $n \in \mathbb{N}$, for simplicity, as this is sufficient for the applications we have in mind. In general, any dense set of points from $[0, 1]$ can be used. In particular, one could use random points sampled from the uniform distribution on $[0, 1]$, as was done in [11].

The following constructions are used to model a variety of dense and sparse, directed and undirected, random and deterministic graphs.

(DDD) Deterministic dense directed graphs $\Gamma_n = \langle V(\Gamma_n), E_d(\Gamma_n), A_n = (a_{n,ij}) \rangle$:

$$a_{n,ij} = \langle W \rangle_{I_{n,i} \times I_{n,j}} := n^2 \int_{I_{n,i} \times I_{n,j}} W(x,y) dx dy. \quad (2.4)$$

(DDU) If W is a symmetric function, the same formula defines a deterministic dense undirected graph $\Gamma_n = \langle V(\Gamma_n), E(\Gamma_n), A_n \rangle$.

(RDD) W-random graphs. Let $W: I^2 \rightarrow I$ be a nonnegative measurable function. $\Gamma_n = G_d(n, W)$ is a directed random graph on n defined as follows:

$$\mathbb{P}(j \rightarrow i) = \langle W \rangle_{I_{n,i} \times I_{n,j}}. \quad (2.5)$$

(RDU) If W is a symmetric function, define a random dense undirected random graph $\Gamma_n = G(n, W)$ as follows

$$\mathbb{P}(i \sim j) = \langle W \rangle_{I_{n,i} \times I_{n,j}}. \quad (2.6)$$

(RSD) Sparse directed W-random graph $\Gamma_n = G_d(n, W, \alpha_n)$. Here, we assume that $W \in L^1(I^2)$ is a nonnegative function and $1 \geq \alpha_n \searrow 0$ such that $n\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$. The probability of connection between two nodes is defined as follows¹

$$\mathbb{P}(j \rightarrow i) = \alpha_n \langle \tilde{W}_n \rangle_{I_{n,i} \times I_{n,j}}, \quad \tilde{W}_n(x,y) := \alpha_n^{-1} \wedge W(x,y). \quad (2.7)$$

(RSU) The undirected sparse W-random graph $\Gamma_n = G(n, W, \alpha_n)$ is defined in exactly the same way

$$\mathbb{P}(i \sim j) = \alpha_n \langle \tilde{W}_n \rangle_{I_{n,i} \times I_{n,j}}, \quad (2.8)$$

assuming that W is a symmetric nonnegative function.

In the KM (2.1) on random graphs, we assume that $a_{n,ij}$ are Bernoulli random variables with the probability of success defined by (2.5)-(2.8). For undirected graphs, we assume that $a_{n,ij} = a_{n,ji}$.

REMARK 2.1. The sequences of undirected graphs constructed above are convergent in the sense of convergence of dense graphs [7] and its generalization to sparse random graphs of unbounded degree [1]. In this paper, we will refer to any of the graph sequence constructed above as a convergent sequence of graphs. The graphon W determines the asymptotic properties of each of these graph sequences. For this reason, W is called a graph limit.

EXAMPLE 2.1.

(1) *Sparse power law graph.* Let $0 < \beta < \gamma < 1$, $\alpha_n = n^{-\gamma}$ and

$$W(x,y) = (1 - \beta)^2 (xy)^{-\beta}. \quad (2.9)$$

Then the probability of connections in $\Gamma_n = G(n, W, \alpha_n)$ is given by

$$\mathbb{P}(i \sim j) = n^{-\gamma} \langle n^\gamma \wedge W \rangle_{I_{n,i} \times I_{n,j}}. \quad (2.10)$$

The expected degree $\mathbb{E}\deg(i) = C(\beta, \gamma, n) i^{-\beta}$ for some positive constant $C(\beta, \gamma, n)$ [5, Lemma 2.2]. Thus, this is a power law graph. On the other hand the expected edge density is $O(n^{-\beta})$. Thus, $\{\Gamma_n\}$ is a sparse sequence.

¹Throughout this paper, $a \wedge b := \min\{a, b\}$.

If (2.10) is replaced by

$$\mathbb{P}(j \rightarrow i) = n^{-\beta} \langle n^\beta \wedge W \rangle_{I_{n,i} \times I_{n,j}},$$

we obtain a sequence of sparse directed graphs with power law distribution.

(2) *Sparse Erdős-Rényi graph.* Let $\alpha_n = n^{-\gamma}$, $0 < \gamma < 1$ and $W \equiv 1$. $\Gamma_n = G(n, W, \alpha_n)$ is a graph on n nodes with the probability of edges being

$$\mathbb{P}(i \sim j) = n^{-\gamma}. \quad (2.11)$$

The expected value of the edge density in this case is $n^{-\gamma}$ and it is vanishing as $n \rightarrow \infty$. However, the expected degree $n^{1-\gamma}$ remains unbounded.

If (2.11) is replaced by

$$\mathbb{P}(j \rightarrow i) = n^{-\gamma}.$$

we obtain a sequence of sparse directed Erdős-Rényi graphs.

Let $\Gamma_n = G(n, W, \alpha_n)$ be a random sparse directed graph (cf. (RSD)). The number of directed edges pointing to $i \in [n]$ is called an in-degree of i :

$$d_{n,i}^+ = \sum_{j=1}^n \mathbf{1}_{\{j \rightarrow i\}}. \quad (2.12)$$

Similarly,

$$d_{n,i}^- = \sum_{j=1}^n \mathbf{1}_{\{i \rightarrow j\}} \quad (2.13)$$

is called an out-degree of $i \in [n]$.

From the definition of $\Gamma_n = G(n, W, \alpha_n)$, we immediately have

$$\mathbb{E} d_{n,i}^+ = \sum_{j=1}^n \alpha_n \langle \tilde{W} \rangle_{I_{n,i} \times I_{n,j}} = \alpha_n n \int_0^1 \bar{W}_n(x, y) dy, x \in I_{n,i}, \quad (2.14)$$

$$\mathbb{E} d_{n,i}^- = \sum_{j=1}^n \alpha_n \langle \tilde{W} \rangle_{I_{n,j} \times I_{n,i}} = \alpha_n n \int_0^1 \bar{W}_n(y, x) dy, x \in I_{n,i}, \quad (2.15)$$

where $\bar{W}_n = \sum_{i,j=1}^n \langle \tilde{W} \rangle_{I_{n,i} \times I_{n,j}} \mathbf{1}_{I_{n,i} \times I_{n,j}}$.

In the remainder of this paper, we will assume that $W \in L^1(I^2)$ satisfies

$$\text{esssup}_{x \in I} \int_I |W(x, y)| dy \leq W_1 \quad (\text{W-1})$$

and

$$\text{esssup}_{y \in I} \int_I |W(x, y)| dx \leq W_2. \quad (\text{W-2})$$

For the nonnegative W used in the definitions of random graphs (RDD) -(RSU) above, (W-1) and (W-2) imply

$$\sup_{n \in \mathbb{N}} \sup_{x \in I} \int_I \bar{W}_n(x, y) dy \leq W_1, \quad \sup_{n \in \mathbb{N}} \sup_{y \in I} \int_I \bar{W}_n(x, y) dx \leq W_2. \quad (2.16)$$

Indeed, for arbitrary $n \in \mathbb{N}$, $i \in [n]$ and $x \in I_{n,i}$, we have

$$\begin{aligned} \int_I \tilde{W}_n(x, y) dy &= \int_I \sum_{j \in [n]} \langle \tilde{W}_n \rangle_{I_{n,i} \times I_{n,j}} \mathbf{1}_{I_{n,j}}(y) dy \\ &= n \sum_{j \in [n]} \int_{I_{n,i} \times I_{n,j}} \tilde{W}_n dx dy \\ &\leq n \int_{I_{n,i}} \left\{ \int_I |W(x, y)| dy \right\} dx \\ &\leq \text{esssup}_{x \in I_{n,i}} \int_I |W(x, y)| dy \leq W_1. \end{aligned}$$

The second inequality in (2.16) is proved similarly.

REMARK 2.2. Conditions (2.16) apply to directed and undirected random graphs. In the undirected case, the two conditions are equivalent, since W is a symmetric function. Furthermore, by setting $\alpha_n \equiv 1$ and restricting to $W \in L^\infty(I^2)$, both conditions apply to directed and undirected dense W -random graphs. With these conventions, below it will be always assumed that conditions (2.16) hold for any of the above types of graphs.

Conditions (2.16) mean that the (in-) and out-degree of any node in Γ_n are $O(\alpha_n n)$. The uniformity here is the key. Both conditions clearly hold for all dense graphs (i.e., $W \in L^\infty(I^2)$) and many sparse graphs. For instance, sparse Erdős-Rényi and small-world graphs satisfy this condition. However, not every $\Gamma_n = G_d(n, W, \alpha_n)$ satisfies (2.16). For instance, the power law graph defined in Example 2.1 does not satisfy (2.16). At the end of the next section, we show that the KM on the power law graphs, after a suitable rescaling of the coupling term, can still be analyzed with the techniques of this paper.

The dynamical model on random graphs can be generalized in the following way. The nonnegativity assumption on W can be dropped. Indeed, writing $W = W^+ - W^-$, assume that positive and negative parts of W , W^+ and W^- , satisfy (W-1), (W-2). Then one can define graphs on n nodes, Γ_n^+ and Γ_n^- , whose edge sets are defined using the graphons W^+ and W^- respectively. Thus, the original model can be rewritten as

$$\dot{u}_{n,i} = f(u_{n,i}, t) + (n\alpha_n)^{-1} \left(\sum_{j=1}^n a_{n,ij}^+ D(u_{n,j} - u_{n,i}) - \sum_{k=1}^n a_{n,ik}^- D(u_{n,k} - u_{n,i}) \right), \quad i \in [n], \quad (2.17)$$

where $(a_{n,ij}^+)$ and $(a_{n,ij}^-)$ are weighted adjacency matrices of Γ_n^+ and Γ_n^- . The derivation and analysis of the continuum limit for (2.1) with nonnegative W translates verbatim for (2.17). To simplify presentation, we restrict to the case of nonnegative W .

3. The main result

Having defined the discrete model (2.1), (2.2), we now present the main result of this work. Our goal is to describe the limiting behavior of the coupled system as $n \rightarrow \infty$. Specifically, we are going to compare the solutions of the discrete model (2.1), (2.2) for large n with the solution of the IVP for the continuum model

$$\partial_t u(t, x) = f(u, t) + \int_I W(x, y) D(u(t, y) - u(t, x)) dy, \quad x \in I, \quad (3.1)$$

$$u(0, x) = g(x). \quad (3.2)$$

For simplicity of exposition, we state our main result for the KM on a sparse directed graph. Clearly, the statement of the theorem translates easily to the KM on sparse undirected graphs, as well as both directed and undirected dense random graphs, as explained in Remark 2.2.

Below, we use the bold font to denote vector-valued functions. In particular, $\mathbf{u}(t)$ stands for the map $t \mapsto u(t, \cdot) \in L^2(I)$. Further, given the solution of the IVP for the KM (2.1), (2.2) $u_n(t) = (u_{n,1}(t), u_{n,2}(t), \dots, u_{n,n}(t))$, we define

$$u_n(t, x) = \sum_{i=1}^n u_{n,i}(t) \mathbf{1}_{I_{n,i}}(x), \quad (3.3)$$

where $\mathbf{1}_{I_{n,i}}$ stands for the characteristic function of $I_{n,i}$, $i \in [n]$. The corresponding vector-valued function is denoted by $\mathbf{u}_n(t)$.

THEOREM 3.1. *Let $u_n(t) = (u_{n,1}(t), u_{n,2}(t), \dots, u_{n,n}(t))$ be the solution of the IVP for the KM (2.1), (2.2) on $\Gamma_n = G_d(n, W, n^{-\gamma})$, $0 < \gamma < 0.5$, with nonnegative $W \in L^2(I^2)$ satisfying (W-1) and (W-2) and subject to the initial condition (2.2) with*

$$u_{n,i}^0 = \langle g \rangle_{I_{n,i}}, \quad i \in [n],$$

and $g \in L^2(I)$.

Then for any $T > 0$,

$$\lim_{n \rightarrow \infty} \|\mathbf{u}_n - \mathbf{u}\|_{C(0, T; L^2(I))} = 0 \quad a.s.,$$

where $\mathbf{u}(t)$ is the solution of the IVP for the continuum limit (3.1), (3.2) and $\mathbf{u}_n(t)$ is defined by (3.3).

Theorem 3.1 establishes convergence of the discrete models on graphs to the continuum limit under the minimal assumptions on W . We only ask that the graphon $W \in L^2(I^2)$ satisfies technical conditions (W-1) and (W-2). This allows us to treat the KM on a variety of graphs in a uniform fashion. In particular, Theorem 3.1 contains as special cases convergence of the KM on dense deterministic and random graphs analyzed in [10, 11], as well as convergence of the KM on sparse graphs considered in [5]. In the case of the KM on random graphs, in this paper the convergence is proved in the almost sure sense compared to the convergence in probability in [5, 11]. In addition, the setting of this paper includes the KM on directed graphs, while the even symmetry of W was used in certain arguments in [5]. All in all, the main result of this paper shows convergence of the KM to the continuum limit in the stronger sense and for a more general class of graphs than in the previous work on this subject.

The proof of Theorem 3.1 follows the scheme developed in [11]. The first step of the proof is estimating proximity between the solution on the IVP (2.1), (2.2) and that for the averaged equation:

$$\dot{v}_{n,i} = f(v_{n,i}, t) + n^{-1} \sum_{j=1}^n \bar{W}_{n,ij} D(v_{n,j} - v_{n,i}), \quad i \in [n], \quad (3.4)$$

$$v_{n,i}(0) = u_{n,i}^0. \quad (3.5)$$

Here, we replaced $a_{n,ij}, i, j \in [n]$, with their expected values. In Theorem 4.1 below, we prove that the solutions of the original and averaged models with probability 1 become closer and closer in the appropriate norm for increasing values of n . On the other hand,

(3.4) has the form of a Cartesian discretization of the continuum limit (3.1). Thus, the second step in the proof is to show that the averaged model approximates the nonlocal Equation (3.1).

The new challenges in implementing this plan are twofold. On the one hand, we significantly relaxed the assumptions on W , compared to $W \in L^\infty(I^2)$ used in [10, 11]. On the other hand, we consider the nonlinear interaction function D compared to the linear diffusion in [5]. To overcome these problems we refined our techniques, in particular, the use of the concentration inequalities in the proof of Theorem 4.1 to obtain finer estimates on the solutions of the averaged model. While the overall approach remains the same as in [11], the analysis in the present paper reflects a better understanding of the method of the proof of convergence to the continuum limit, and fuller reveals its potential.

4. Averaging

For KM on random graphs, the key step in the derivation of the continuum limit is the averaging procedure, when a stochastic model is approximated by a deterministic (averaged) system. In this section, we focus on the justification of the averaging.

Throughout this section, we consider the KM on random graphs (cf. **RDD**, **RDU**, **RSD**, **RSU**). Without loss of generality, we consider the KM on a random sparse directed graph $\Gamma_n = G_d(n, W, \alpha_n)$, as it represents the most general case².

For convenience, we rewrite the KM on $\Gamma_n = G_d(n, W, \alpha_n)$:

$$\dot{u}_{n,i} = f(u_{n,i}, t) + \frac{1}{\alpha_n n} \sum_{j=1}^n a_{n,ij} D(u_{n,j} - u_{n,i}), \quad i \in [n], \quad (4.1)$$

Taking the expected value of the right-hand side of (4.1) on Γ_n

$$\mathbb{E} a_{n,ij} = \mathbb{P}(j \rightarrow i) = \alpha_n \langle \tilde{W}_n \rangle_{I_{n,i} \times I_{n,j}},$$

we arrive at the following averaged model

$$\dot{v}_{n,i} = f(v_{n,i}, t) + \frac{1}{n} \sum_{j=1}^n \bar{W}_{n,ij} D(v_{n,j} - v_{n,i}), \quad i \in [n], \quad (4.2)$$

where $\bar{W}_{n,ij} := \langle \tilde{W}_n \rangle_{I_{n,i} \times I_{n,j}}$.

To compare the solutions of the IVPs for the original and the averaged KMs, we adopt the discrete L^2 -norm:

$$\|u_n - v_n\|_{2,n} = \left(n^{-1} \sum_{j=1}^n (u_{n,j} - v_{n,j})^2 \right)^{1/2}. \quad (4.3)$$

THEOREM 4.1. *Let nonnegative $W \in L^1(I^2)$ satisfy (W-1), (W-2), and $\alpha_n = n^{-\gamma}$, $\gamma \in (0, 0.5)$, and*

$$L = L_f + L_D \left(2 + \frac{3}{2} W_1 + \frac{1}{2} W_2 \right) + \frac{1}{2}. \quad (4.4)$$

²For the KM on an undirected graph, assume, in addition, that W is symmetric. In the dense case, restrict to $0 \leq W \leq 1$ and set $\alpha_n \equiv 1$.

Then for solutions of the original and averaged Equations (4.1) and (4.2) subject to the same initial conditions and any $T \leq C \ln n$, $0 \leq C < (1-2\gamma)L^{-1}$, we have

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|u_n - v_n\|_{2,n} = 0 \quad \text{almost surely (a.s.)}. \quad (4.5)$$

Proof. Recall that $f(u, t)$ and D are Lipschitz continuous functions in u with Lipschitz constants L_f and L_D respectively. In addition, $f(u, t)$ is a continuous function of t and $D(u)$ is 2π -periodic function satisfying (2.3).

Further, $a_{n,ij}$, are Bernoulli random variables

$$\mathbb{P}(a_{n,ij} = 1) = \alpha_n \bar{W}_{n,ij}. \quad (4.6)$$

Denote $\psi_{n,i} := v_{n,i} - u_{n,i}$. By subtracting (4.1) from (4.2), multiplying the result by $n^{-1}\psi_{n,i}$, and summing over $i \in [n]$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\psi_n\|_{2,n}^2 &= n^{-1} \underbrace{\sum_{i=1}^n (f(v_{n,i}, t) - f(u_{n,i}, t)) \psi_{n,i}}_{I_1} \\ &\quad + n^{-2} \alpha_n^{-1} \underbrace{\sum_{i,j=1}^n (\alpha_n W_{n,ij} - a_{n,ij}) D(v_{n,j} - v_{n,i}) \psi_{n,i}}_{I_2} \\ &\quad + n^{-2} \alpha_n^{-1} \underbrace{\sum_{i,j=1}^n a_{n,ij} [D(v_{n,j} - v_{n,i}) - D(u_{n,j} - u_{n,i})] \psi_{n,i}}_{I_3} \\ &=: I_1 + I_2 + I_3, \end{aligned} \quad (4.7)$$

where $\|\cdot\|_{2,n}^2$ is the discrete L^2 -norm (cf. (4.3)).

Using Lipschitz continuity of f in u , we have

$$|I_1| \leq L_f \|\psi_n\|_{2,n}^2. \quad (4.8)$$

Using Lipschitz continuity of D and the triangle inequality, we have

$$\begin{aligned} |I_3| &\leq L_D n^{-2} \alpha_n^{-1} \sum_{i,j=1}^n a_{n,ij} (|\psi_{n,i}| + |\psi_{n,j}|) |\psi_{n,i}| \\ &\leq L_D n^{-2} \alpha_n^{-1} \left(\frac{3}{2} \sum_{i,j=1}^n a_{n,ij} \psi_{n,i}^2 + \frac{1}{2} \sum_{i,j=1}^n a_{n,ij} \psi_{n,j}^2 \right). \end{aligned} \quad (4.9)$$

Choose $0 < \delta < 1 - 2\gamma$ and denote

$$A_{n,i} = \left\{ S_{n,i} \geq \alpha_n \sum_{j=1}^n \bar{W}_{n,ij} + n^{\frac{1+\delta}{2}} \right\}, \quad n \in \mathbb{N}, i \in [n], \quad (4.10)$$

$$A_n = \bigcup_{i=1}^n A_{n,i}, \quad n \in \mathbb{N}, \quad (4.11)$$

where

$$S_{n,i} = \sum_{j=1}^n a_{n,ij}, \quad i \in [n]. \quad (4.12)$$

Noting $0 \leq a_{n,ij} \leq 1$, $\mathbb{E} S_{n,i} = \alpha_n \sum_{j=1}^n \bar{W}_{n,ij}$, we apply Chernoff-Hoeffding inequality³ to bound

$$\mathbb{P}(A_{n,i}) \leq e^{-2n^\delta}. \quad (4.13)$$

By the union bound,

$$\mathbb{P}(A_n) \leq n e^{-2n^\delta}. \quad (4.14)$$

By Borel-Cantelli lemma, with probability 1 there exists $N_\delta \in \mathbb{N}$ such that

$$S_{n,i} < \alpha_n \sum_{j=1}^n \bar{W}_{n,ij} + n^{\frac{1+\delta}{2}}, \quad (4.15)$$

for all $n \geq N_\delta$ and $i \in [n]$. Below, we restrict to the subset of probability 1, where (4.15) holds.

With (4.15) in hand, we return to bounding the right-hand side of (4.9)

$$\begin{aligned} n^{-2} \alpha_n^{-1} \sum_{i,j=1}^n a_{n,ij} \psi_{n,i}^2 &\leq n^{-1} \sum_{i=1}^n \left[n^{\frac{-1+\delta}{2}} \alpha_n^{-1} + n^{-1} \sum_{j=1}^n \bar{W}_{n,ij} \right] \psi_{n,i}^2 \\ &\leq (1 + W_1) \|\psi\|_{n,2}^2, \end{aligned} \quad (4.16)$$

where we used $n^{\frac{-1+\delta}{2}} \alpha_n^{-1} = n^{\frac{-1+2\gamma+\delta}{2}} \leq 1$ and the definition of W_1 (cf. (2.16)).

Similarly,

$$n^{-2} \alpha_n^{-1} \sum_{i,j=1}^n a_{n,ij} \psi_{n,j}^2 \leq (1 + W_2) \|\psi\|_{n,2}^2. \quad (4.17)$$

By plugging (4.16) and (4.17) into (4.9), we have

$$|I_3| \leq L_D \left(2 + \frac{3}{2} W_1 + \frac{1}{2} W_2 \right) \|\psi\|_{n,2}^2. \quad (4.18)$$

It remains to bound I_2 . To this end, we will need the following definitions:

$$\begin{aligned} Z_{n,i}(t) &= n^{-1} \sum_{j=1}^n b_{n,ij}(t) \eta_{n,ij}, \\ b_{n,ij}(t) &= D(v_{n,j}(t) - v_{n,i}(t)), \end{aligned}$$

³ Here and below, we are using

$$\mathbb{P} \left(\sum_{i=1}^N X_i \geq \sum_{i=1}^N \mathbb{E} X_i + t \right) \leq e^{\frac{-2t^2}{\sum_{i=1}^N (b_i - a_i)^2}}, \text{ and } \mathbb{P} \left(\left| \sum_{i=1}^N X_i - \sum_{i=1}^N \mathbb{E} X_i \right| \geq t \right) \leq 2e^{\frac{-2t^2}{\sum_{i=1}^N (b_i - a_i)^2}},$$

which hold for collectively independent random variables $a_i \leq X_i \leq b_i$, $i \in [N]$ [4].

$$\eta_{n,ij} = a_{n,ij} - \alpha_n \bar{W}_{n,ij},$$

and $Z_n = (Z_{n,1}, Z_{n,2}, \dots, Z_{n,n})$. With these definitions in hand, we estimate I_2 as follows:

$$|I_2| = |n^{-1} \alpha_n^{-1} \sum_{i=1}^n Z_{n,i} \psi_{n,i}| \leq 2^{-1} \alpha_n^{-2} \|Z_n\|_{2,n}^2 + 2^{-1} \|\psi_n\|_{2,n}^2. \quad (4.19)$$

The combination of (4.7), (4.8), (4.18) and (4.19) yields

$$\frac{d}{dt} \|\psi_n(t)\|_{2,n}^2 \leq L \|\psi_n(t)\|_{2,n}^2 + \frac{1}{\alpha_n^2} \|Z_n(t)\|_{2,n}^2, \quad (4.20)$$

where L is defined in (4.4).

Using the Grönwall's inequality, we have

$$\|\psi_n(t)\|_{2,n}^2 \leq \alpha_n^{-2} e^{Lt} \int_0^\tau e^{-Ls} \|Z_n(s)\|_{2,n}^2 ds.$$

and

$$\sup_{t \in [0, T]} \|\psi_n(t)\|_{2,n}^2 \leq \alpha_n^{-2} e^{LT} \int_0^\infty e^{-Ls} \|Z_n(s)\|_{2,n}^2 ds. \quad (4.21)$$

Our next goal is to estimate $\int_0^\infty e^{-Ls} \|Z_n(s)\|_{2,n}^2 ds$. To this end, note

$$\mathbb{E} \eta_{n,ij} = \mathbb{E} (a_{n,ij} - \alpha_n \bar{W}_{n,ij}) = 0, \quad (4.22)$$

$$\mathbb{E} \eta_{n,ij}^2 = \mathbb{E} (a_{n,ij} - \alpha_n \bar{W}_{n,ij})^2 = \alpha_n \bar{W}_{n,ij} - (\alpha_n \bar{W}_{n,ij})^2 \leq 1. \quad (4.23)$$

Further,

$$\int_0^\infty e^{-Ls} Z_{n,i}(s)^2 ds = n^{-2} \sum_{k,l=1}^n c_{n,ikl} \eta_{n,ik} \eta_{n,il}, \quad (4.24)$$

where

$$c_{n,ikl} = \int_0^\infty e^{-Ls} b_{n,ik}(s) b_{n,il}(s) ds \quad \text{and} \quad |c_{n,ikl}| \leq L^{-1}. \quad (4.25)$$

Further, from (4.24) and (4.25), we have

$$\int_0^\infty e^{-Ls} \|Z_n(s)\|_{2,n}^2 ds = n^{-3} \sum_{i,k,l=1}^n c_{n,ikl} \eta_{n,ik} \eta_{n,il}. \quad (4.26)$$

Our final goal is to bound the sum on the right-hand side of (4.26). To this end, we write

$$\sum_{i,k,l=1}^n c_{n,ikl} \eta_{n,ik} \eta_{n,il} = \sum_{i,k=1}^n c_{n,ikk} \eta_{n,ik}^2 + 2 \sum_{i=1}^n \sum_{1 \leq l < k \leq n} c_{n,ikl} \eta_{n,ik} \eta_{n,il}. \quad (4.27)$$

Both sums on the right-hand side of (4.27) are formed of independent bounded random variables. By Chernoff-Hoeffding inequality, for an arbitrary $\delta > 0$, we have

$$\mathbb{P} \left(\sum_{i,k=1}^n c_{n,ikk} \eta_{n,ik}^2 \geq \sum_{i,k=1}^n c_{n,ikk} \mathbb{E} \eta_{n,ik}^2 + n^2 \right) \leq e^{-n^2 L^2}, \quad (4.28)$$

$$\mathbb{P}\left(\left|\sum_{i=1}^n \sum_{1 \leq l < k \leq n} c_{n,ikl} \eta_{n,ik} \eta_{n,il}\right| \geq n^{\frac{3}{2} + \delta}\right) \leq 2e^{-n^{2\delta}L^2}, \quad (4.29)$$

where we used the bound on $c_{n,ikl}$ (see (4.25)). By Borel-Cantelli lemma, we now have

$$\sum_{i,k=1}^n c_{n,ikk} \eta_{n,ik}^2 \leq \sum_{i,k=1}^n c_{n,ikk} \mathbb{E} \eta_{n,ik}^2 + n^2 < (L^{-1} + 1)n^2, \quad (4.30)$$

$$\left| \sum_{i=1}^n \sum_{1 \leq l < k \leq n} c_{n,ikl} \eta_{n,ik} \eta_{n,il} \right| < n^{\frac{3}{2} + \delta}, \quad (4.31)$$

for sufficiently large n a.s.. Plugging in these bounds into (4.27) and (4.26), we obtain

$$\alpha_n^{-2} \int_0^\infty e^{-Ls} \|Z_n(s)\|_{2,n}^2 ds \leq \alpha_n^{-2} \left((L^{-1} + 1)n^{-1} + 2n^{\frac{-3}{2} + \delta} \right) \leq C_1 \alpha_n^{-2} n^{-1} \quad (4.32)$$

for some $C_1 \geq 0$ a. s..

Using (4.32), from (4.21) we have

$$\sup_{t \in [0, T]} \|\psi_n(t)\|_{2,n}^2 \leq C_1 e^{LT} \alpha_n^{-2} n^{-1}. \quad (4.33)$$

For $\alpha_n = n^{-\gamma}$, $0 < \gamma < \frac{1}{2}$ the right-hand side of (4.33) tends to zero on the time interval with $T \leq C \ln n$ for any $0 < C < \frac{1-2\gamma}{L}$. \square

If we restrict to finite time intervals then (4.33) yields the rate of convergence estimate.

COROLLARY 4.1. *For fixed $T > 0$, we have*

$$\lim_{n \rightarrow \infty} n^{\frac{1}{2} - \gamma - \delta} \sup_{t \in [0, T]} \|\psi_n(t)\|_{2,n} = 0 \quad a.s., \quad (4.34)$$

where $0 < \delta < \frac{1}{2} - \gamma$ is arbitrary.

REMARK 4.1. Theorem 4.1 and Corollary 4.1 clearly apply to the KM on undirected sparse graphs. Furthermore, by setting $\gamma = 0$, these results translate to the KM on dense W-random graphs.

REMARK 4.2. As we pointed out earlier, not every sparse random graph defined in (RSD, RSU) meets (2.16). However, the averaging can still be justified for the KM on such graphs if the original model is suitably rescaled. For simplicity, we explain the new scaling for the KM on undirected graphs.

Let $\Gamma_n = G(n, W, \alpha_n)$, where $W \in L^1(I^2)$ is a symmetric nonnegative function and $\alpha_n \searrow 0$, $\alpha_n n \rightarrow \infty$ as before. Consider

$$\dot{u}_{n,i} = f(u_{n,i}, t) + d_{n,i}^{-1} \sum_{j=1}^n a_{n,ij} D(u_{n,j} - u_{n,i}), \quad i \in [n], \quad (4.35)$$

where $d_{n,i} := d_{n,i}^+ = d_{n,i}^-$ is a degree of node $i \in [n]$. We claim that the conclusion of Theorem 4.1 holds for the rescaled model (4.35) for any nonnegative symmetric $W \in L^1(I^2)$. Indeed, the averaged system in this case takes the following form

$$\dot{v}_{n,i} = f(v_{n,i}, t) + n^{-1} \sum_{j=1}^n U_{n,ij} D(v_{n,j} - v_{n,i}), \quad i \in [n], \quad (4.36)$$

where

$$U_{n,ij} = \frac{\bar{W}_{n,ij}}{n^{-1} \sum_{k=1}^n \bar{W}_{n,ki}}, \quad (i,j) \in [n]^2. \quad (4.37)$$

Using $\bar{W}_{n,ij} = \bar{W}_{n,ji}$ and (4.37), we have

$$n^{-1} \sum_{k=1}^n U_{n,kj} = n^{-1} \sum_{k=1}^n U_{n,ik} = 1 \quad \forall i, j \in [n].$$

Thus, the bounds in (4.16) and (4.17) hold with $W_1 = W_2 = 1$. The rest of the proof remains unchanged.

5. The continuum limit

We now turn to the IVP for the continuum model (3.1), (3.2). Throughout this section, we assume that $W \in L^2(I^2)$ is subject to (W-1) and (W-2). With these assumptions, the proof of Theorem 3.1 of [5] can be easily modified to yield the existence of a unique classical solution $\mathbf{u} \in C^1(\mathbb{R}, L^2(I))$ of the IVP (3.1), (3.2). We continue to use the bold font to denote vector-valued functions, e.g., $\mathbf{u}(t) := u(t, \cdot) \in L^2(I)$.

The goal of this section is to show that the solutions of the averaged discrete problems (3.4), (3.5) converge to that of the continuous problem (3.1), (3.2). To this end, it is convenient to recast the former problems as

$$\partial_t v_n(t, x) = f(v_n(t, x), t) + \int_I \bar{W}_n(x, y) D(v_n(t, y) - v_n(t, x)) dy, \quad (5.1)$$

$$v_n(0, x) = g_n(x), \quad (5.2)$$

where

$$g_n(x) = \sum_{i=1}^n g_{n,i} \mathbf{1}_{I_{n,i}}(x), \quad g_{n,i} = \langle g \rangle_{I_{n,i}} := n \int_{I_{n,i}} g(x) dx, \quad (5.3)$$

$$\bar{W}_n(x, y) = \sum_{i,j=1}^n \bar{W}_{n,ij} \mathbf{1}_{I_{n,i} \times I_{n,j}}(x, y). \quad (5.4)$$

The two IVPs (3.4), (3.5) and (5.1), (5.2) are equivalent. Indeed, take $v_{n,i}(t)$, $i \in [n]$, the solution of (3.4), (3.5), and form

$$v_n(t, x) = \sum_{i=1}^n v_{n,i}(t) \mathbf{1}_{I_{n,i}}(x). \quad (5.5)$$

It is straightforward to check that $v_n(t, x)$ solves (5.1), (5.2). Conversely, the time dependent coefficients of the step functions $\mathbf{1}_{I_{n,i}}$, $i \in [n]$, in the representation (5.5) of the solution of (5.1), (5.2) yield the solution of the discrete problem (3.4), (3.5).

The following theorem establishes the continuum limit for the IVP for the averaged Equation (5.1), (5.2).

THEOREM 5.1. *Let $W \in L^2(I^2)$ satisfy (W-1), (W-2) and $g \in L^2(I)$. Recall that $f(u, t)$ and $D(u)$ are Lipschitz continuous functions in u . In addition, $f(u, t)$ is a continuous function of t . Then for any given $T > 0$*

$$\lim_{n \rightarrow \infty} \|\mathbf{v}_n - \mathbf{u}\|_{C(0, T; L^2(I))} = 0,$$

where $\mathbf{u}(t)$ is the solution of the IVP for the continuum limit (3.1), (3.2) and $\mathbf{v}_n(t) = v_n(t, \cdot)$ is the solution of the IVP (5.1), (5.2).

REMARK 5.1. Theorem 5.1 combined with Theorem 4.1 implies Theorem 3.1, and, thus, provides a rigorous justification for the continuum limit of the KM on sparse graphs.

Proof. Denote the difference between the solutions of the original IVP (3.1), (3.2) and the averaged IVP (5.1), (5.2)

$$w_n(t, x) = u(t, x) - v_n(t, x). \quad (5.6)$$

By subtracting (5.1) from (3.1), multiplying the resultant equation by w_n , and integrating over I , we obtain

$$\begin{aligned} \int_I \partial_t w_n(t, x) w_n(t, x) dx &= \int_I (f(u(t, x), t) - f(v_n(t, x), t)) w_n(t, x) dx \\ &+ \int_I \int_I W(x, y) [D(u(t, y) - u(t, x)) - D(v_n(t, y) - v_n(t, x))] w_n(t, x) dy dx \\ &+ \int_I \int_I (W(x, y) - W_n(x, y)) D(v_n(t, y) - v_n(t, x)) w_n(t, x) dy dx. \end{aligned} \quad (5.7)$$

By Lipschitz continuity of $f(u, t)$ in u , we have

$$\left| \int_I (f(u(t, x), t) - f(v_n(t, x), t)) w_n(t, x) dx \right| \leq L_f \|w_n(t, \cdot)\|^2. \quad (5.8)$$

Here and below, $\|\cdot\|$ stands for the norm in $L^2(I)$ or in $L^2(I^2)$ depending on the context.

Using the bound for D (2.3) and Young's inequality, we obtain

$$\begin{aligned} &\left| \int_I \int_I (W(x, y) - W_n(x, y)) D(v_n(t, y) - v_n(t, x)) w_n(t, x) dy dx \right| \\ &\leq \frac{1}{2} \|W - W_n\|^2 + \frac{1}{2} \|w_n(t, \cdot)\|^2. \end{aligned} \quad (5.9)$$

Finally, using Lipschitz continuity of D and Young's inequality, we estimate

$$\begin{aligned} &\left| \int_I \int_I W(x, y) [D(u(t, y) - u(t, x)) - D(v_n(t, y) - v_n(t, x))] w_n(t, x) dy dx \right| \\ &\leq L_D \int_I \int_I W(x, y) (|w_n(t, y)| + |w_n(t, x)|) |w_n(t, x)| dy dx \\ &\leq L_D \int_I \int_I W(x, y) \left(\frac{1}{2} |w_n(t, y)|^2 + \frac{3}{2} |w_n(t, x)|^2 \right) dy dx \\ &\leq \frac{3L_D}{2} \int_I \int_I W(x, y) |w_n(t, x)|^2 dy dx + \frac{L_D}{2} \int_I \int_I W(x, y) |w_n(t, y)|^2 dy dx \\ &\leq 2^{-1} L_D (3W_1 + W_2) \|w_n(t, \cdot)\|^2, \end{aligned} \quad (5.10)$$

where we also used Fubini's theorem, (W-1) and (W-2) to get the bound in the last line.

By combining (5.7)-(5.10), we arrive at

$$\frac{d}{dt} \|w_n(t, \cdot)\|^2 \leq L \|w_n(t, \cdot)\|^2 + \|W_n - W\|^2, \quad (5.11)$$

where $L = 2L_f + 1 + L_D(3W_1 + W_2)$.

By Grönwall's inequality, we have

$$\sup_{t \in [0, T]} \|w_n(t, \cdot)\| \leq e^{LT/2} (\|g - g_n\| + \|W_n - W\|).$$

It remains to show that $\|W_n - W\| \rightarrow 0$ as $n \rightarrow \infty$. To this end, let P_n denote the L^2 -projection from $L^2(I^2)$ onto the subspace of $L^2(I^2)$ generated by step functions $\{\mathbf{1}_{I_{n,i} \times I_{n,j}}, i, j \in [n]\}$. Recall that $\bar{W}_n = P_n \tilde{W}_n$.

Thus, by the triangle inequality, we have

$$\|W - \bar{W}_n\| \leq \|W - \tilde{W}_n\| + \|\tilde{W}_n - P_n \tilde{W}_n\|. \quad (5.12)$$

Using the triangle inequality again, we have

$$\begin{aligned} \|\tilde{W}_n - P_n \tilde{W}_n\| &\leq \|\tilde{W}_n - W\| + \|W - P_n W\| + \|P_n(W - \tilde{W}_n)\| \\ &\leq 2\|\tilde{W}_n - W\| + \|W - P_n W\|. \end{aligned} \quad (5.13)$$

From the definition of \tilde{W}_n , $\tilde{W}_n \rightarrow W$ a.e. on I^2 . By the dominated convergence theorem $\|\tilde{W}_n - W\| \rightarrow 0$. Further, $\|W - P_n W\| \rightarrow 0$ as $n \rightarrow \infty$ by the L^2 -Martingale Convergence Theorem (cf. [5, Lemma 5.3]). Thus, from (5.12) and (5.13), we have $\|W - \bar{W}_n\| \rightarrow 0$. This concludes the proof. \square

Acknowledgements. The author thanks Gideon Simpson for insightful comments, which helped to improve the manuscript. This work was supported in part by the NSF grant DMS 1715161.

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