

Robust Approximate Simulation for Hierarchical Control of Linear Systems under Disturbances

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Abstract—Approximate simulation, an extension of simulation relations from formal methods to continuous systems, is a powerful tool for hierarchical control of complex systems. Finding an approximate simulation relation between the full “concrete” system and a simplified “abstract” system establishes a bound on the output error between the two systems, allowing one to design a controller for the abstract system while formally certifying performance on the concrete system. However, many real-world control systems are subject to external disturbances, which are not accounted for in the standard approximate simulation framework. We present a notion of robust approximate simulation, which considers external disturbances to the concrete system. We derive output error bounds for the case of linear systems subject to two types of additive disturbances: bounded disturbances and a sequence of (unbounded) impulse disturbances. We demonstrate the need for robust approximate simulation and the effectiveness of our proposed approach with a simulated robot motion planning example.

I. INTRODUCTION

A. Motivation

Complex and high-dimensional systems are often difficult to control directly. This leads naturally to hierarchical control strategies, where a simpler (abstract) system model is used in the controller design process. One particularly useful framework for hierarchical control is *approximate simulation* [1]. An approximate simulation relation between the abstract system and the full (concrete) system certifies that the outputs of both systems can remain ϵ -close.

Approximate simulation relations give rise to control architectures like that shown in Figure 1. The interface, which maps controls from the abstract system to the concrete system, is designed to enforce ϵ -closeness of the outputs. Given such an interface, we can design a controller for the abstract system and guarantee that the concrete system’s output will remain ϵ -close. Furthermore, approximate simulation offers elegant connections to other areas of control theory, as an approximate simulation relation can be certified by finding a Lyapunov-like simulation function, which bounds the output error between the two systems.

Approximate simulation is a powerful framework for hierarchical controller design. Since it builds off of the notions of

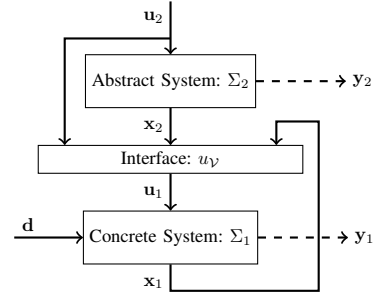


Fig. 1. Hierarchical control system architecture considered in this work. We extend the notion of approximate simulation [3] to account for disturbances d to the concrete system.

simulation and bisimulation relations from formal methods, it can be efficiently applied to discrete transition systems and hybrid systems, as well as continuous systems. Recent results suggest that approximate simulation can be used for control of complex, high-dimensional, and highly nonlinear systems such as legged robots [2]. However, the traditional notion of approximate simulation does not account for disturbances to the concrete system. This means that any guarantees regarding ϵ -closeness of the outputs may not be valid when disturbances enter the concrete system. This property is especially important when it comes to robotic and cyber-physical systems which operate in the real world, and are thus subject to a variety of disturbances.

In this work, we extend the approximate simulation framework to account for disturbances to the concrete system. We present this extension as a general notion of robust approximate simulation for continuous systems, and provide specific results for two special cases: linear systems subject to bounded additive disturbances, and linear systems subject to additive impulse disturbances.

B. Related Work

The notion of approximate simulation has its roots in the formal methods literature, where exact simulation relations are defined in terms of transition systems [4]. Such simulation and bisimulation (both systems simulate each other) relations have had a powerful impact on model checking and formal synthesis, with widespread application in software and design verification [4, 5]. Furthermore, simulation and bisimulation relations have been successfully applied to controller synthesis from temporal logic specifications [6–8]. If two systems can be shown to be (bi)similar, the formal design process can consider only the simpler model, improving computational efficiency while maintaining formal guarantees.

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More recently, there has been a growing movement to apply these impactful techniques from formal methods to (continuous) control systems. Large strides have been made in this direction through the use of approximate simulation and approximate bisimulation [9–11]. Since requiring two continuous systems to have exactly the same output may be too strict, approximate (bi)simulation relaxes this requirement to enforce only that the outputs remain ϵ -close. Beyond enabling the application of formal methods techniques to continuous systems, approximate simulation provides an elegant bridge between formal methods and classical control theory: approximate simulation can be equivalently defined in terms of a Lyapunov-like simulation function [3].

Hierarchical control and system equivalence have been studied in the context of control systems as well. Notions such as asymptotic model matching [12] enforce output global asymptotic stability of the joint system [13]. Since approximate simulation requires only closeness of outputs, asymptotic model matching implies approximate simulation but not vice versa [1]. Approximate simulation is most closely related to Input-to-Output Stability (IOS) [14]. Specifically, a simulation function can be viewed as an IOS Lyapunov function [15] of the joint system. The key difference between IOS and approximate simulation is that the input $\mathbf{u}_2$ to the abstract system is considered a control parameter rather than a disturbance in the framework of approximate simulation [1]. Given this emphasis, a key weakness of approximate simulation is that disturbances to the concrete system are not considered. Drawing inspiration from the robust control literature, we address this gap by proposing the notion of *robust approximate simulation*. To the best of our knowledge, this is the first work to extend approximate simulation to account for external disturbances.

The remainder of this paper is organized as follows. Section II introduces necessary background on approximate simulation, with particular attention to the case of linear systems. Section III presents our main results, which include a definition of robust approximate simulation for general continuous systems and detailed derivation of error bounds for linear systems subject to bounded and impulse disturbances. We illustrate these results with a robot motion planning example in Section IV and conclude with Section V.

II. BACKGROUND

A. Approximate Simulation

The classical notion of approximate simulation is defined in terms of two systems, Σ_1 and Σ_2 :

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1, \mathbf{u}_1) \\ \mathbf{y}_1 = g_1(\mathbf{x}_1) \end{cases}, \quad \Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2 = f_2(\mathbf{x}_2, \mathbf{u}_2) \\ \mathbf{y}_2 = g_2(\mathbf{x}_2) \end{cases}, \quad (1)$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$ are the system states, $\mathbf{u}_i \in \mathbb{R}^{p_i}$ are the control inputs, and $\mathbf{y}_i \in \mathbb{R}^m$ are the system outputs. Note that the states may be different sizes but the outputs must be the same size. Without loss of generality, we consider Σ_1 to be a more complex “concrete” model and Σ_2 to be the simpler “abstract” model. This typically means that $n_2 < n_1$.

Approximate simulation is defined in terms of a Lyapunov-like simulation function $\mathcal{V}$ and an interface function $u_{\mathcal{V}}$:

Definition 1 (Girard and Pappas [3]). *Consider two systems of the form (1). Let $\mathcal{V} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^+$ be a smooth function and $u_{\mathcal{V}} : \mathbb{R}^{p_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{p_1}$ be a continuous function. $\mathcal{V}$ is a simulation function of Σ_2 by Σ_1 and $u_{\mathcal{V}}$ is an associated interface if there exists a class- κ function¹ γ such that for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,*

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) \geq \|g_1(\mathbf{x}_1) - g_2(\mathbf{x}_2)\|, \quad (2)$$

and for all $\mathbf{u}_2 \in \mathbb{R}^{p_2}$ satisfying $\gamma(\|\mathbf{u}_2\|) < \mathcal{V}(\mathbf{x}_1, \mathbf{x}_2)$,

$$\frac{\partial \mathcal{V}}{\partial \mathbf{x}_2} f_2(\mathbf{x}_2, \mathbf{u}_2) + \frac{\partial \mathcal{V}}{\partial \mathbf{x}_1} f_1(\mathbf{x}_1, u_{\mathcal{V}}(\mathbf{u}_2, \mathbf{x}_1, \mathbf{x}_2)) < 0. \quad (3)$$

Definition 2 (Girard and Pappas [3]). Σ_1 *approximately simulates* Σ_2 if there exists a simulation function $\mathcal{V}$ of Σ_2 by Σ_1 .

The conditions (2-3) essentially state that the simulation function bounds the output error, and as long as the input to the abstract system is not too large, the simulation function will be decreasing.

If Σ_1 approximately simulates Σ_2 , we can bound the output error of the two systems as follows:

Theorem 1 (Girard and Pappas [1]). *Consider two systems of the form (1). Let $\mathcal{V}$ be a simulation function of Σ_2 by Σ_1 and $u_{\mathcal{V}}$ be an associated interface. Let $\mathbf{u}_2(t)$ be an admissible input of Σ_2 with associated state and output trajectories $\mathbf{x}_2(t)$ and $\mathbf{y}_2(t)$. Let $\mathbf{x}_1(t)$ be a state trajectory of Σ_1 satisfying*

$$\dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1, u_{\mathcal{V}}(\mathbf{u}_2, \mathbf{x}_1, \mathbf{x}_2))$$

and $\mathbf{y}_1(t)$ be the associated output trajectory. Then

$$\|\mathbf{y}_1(t) - \mathbf{y}_2(t)\| \leq \epsilon,$$

where

$$\epsilon = \max \{ \mathcal{V}(\mathbf{x}_1(0), \mathbf{x}_2(0)), \gamma(\|\mathbf{u}_2\|_{\infty}) \}. \quad (4)$$

B. Approximate Simulation for Linear Systems

Finding a simulation function and an interface for two arbitrary systems is a difficult and open problem, though some promising results with sum-of-squares programming do exist [9, 16]. For linear systems, however, there are well-defined conditions for the existence of a simulation function [1], which we summarize below.

Consider the case when both the concrete and the abstract systems are linear, i.e.,

$$\Sigma_i : \begin{cases} \dot{\mathbf{x}}_i = A_i \mathbf{x}_i + B_i \mathbf{u}_i \\ \mathbf{y}_i = C_i \mathbf{x}_i \end{cases}, \quad i = \{1, 2\}. \quad (5)$$

In this case, there are strong results regarding the form of a simulation function. First note the following Lemma:

¹A function $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a class- κ function if it is continuous, strictly increasing, and $\gamma(0) = 0$.

Lemma 1 (Girard and Pappas [10]). *If Σ_1 is stabilizable with feedback gain K , i.e. $(A_1 + B_1K)$ is Hurwitz, then there exists a positive definite symmetric matrix M and positive scalar constant λ such that the following hold:*

$$M \geq C_1^T C_1, \quad (6)$$

$$(A_1 + B_1K)^T M + M(A_1 + B_1K) \leq -2\lambda M. \quad (7)$$

Such an M can be used to show exponential convergence of $\mathbf{y}_1$ to zero with rate λ under the feedback $\mathbf{u}_1 = K\mathbf{x}_1$. Note that M and K can be computed jointly using semi-definite programming [1]: letting $\bar{K} = KM^{-1}$ and $\bar{M} = M^{-1}$, we have the equivalent linear matrix inequality conditions

$$\begin{bmatrix} \bar{M} & \bar{M}C_1^T \\ C_1\bar{M} & I \end{bmatrix} \geq 0, \quad (8)$$

$$\bar{M}A_1^T + A_1\bar{M} + \bar{K}^T B_1^T + B_1\bar{K} \leq -2\lambda\bar{M}. \quad (9)$$

We can now state the following Theorem:

Theorem 2 (Girard and Pappas [1]). *Consider two systems of the form (5). Assume that Σ_1 is stabilizable with feedback gain K and that there exist matrices $P \in \mathbb{R}^{n_1 \times n_2}$ and $Q \in \mathbb{R}^{m_1 \times n_2}$ such that the following conditions hold:*

$$PA_2 = A_1P + B_1Q, \quad (10)$$

$$C_2 = C_1P. \quad (11)$$

Then a simulation function of Σ_2 by Σ_1 is given by

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) = \sqrt{(\mathbf{x}_1 - P\mathbf{x}_2)^T M (\mathbf{x}_1 - P\mathbf{x}_2)}, \quad (12)$$

an associated interface is

$$u_V = R\mathbf{u}_2 + Q\mathbf{x}_2 + K(\mathbf{x}_1 - P\mathbf{x}_2), \quad (13)$$

and the class- κ function γ is given by

$$\gamma(\nu) = \frac{\|\sqrt{M}(B_1R - PB_2)\|}{\lambda} \nu, \quad (14)$$

where R is an arbitrary matrix of proper dimensions and M, λ satisfy (6-7).

The matrix R acts as a “feedforward” mapping from $\mathbf{u}_2$ to $\mathbf{u}_1$. While the simulation relation holds for any R of proper dimensions, choosing R to minimize (14) is a logical choice, as this tightens the error bound (4).

III. MAIN RESULTS

A. Robust Approximate Simulation

Consider the systems

$$\Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1, \mathbf{u}_1, \mathbf{d}) \\ \mathbf{y}_1 = g_1(\mathbf{x}_1) \end{cases}, \quad \Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2 = f_2(\mathbf{x}_2, \mathbf{u}_2) \\ \mathbf{y}_2 = g_2(\mathbf{x}_2) \end{cases}, \quad (15)$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$ are the system states, $\mathbf{u}_i \in \mathbb{R}^{p_i}$ are the control inputs, $\mathbf{y}_i \in \mathbb{R}^{m_i}$ are the system outputs, and $\mathbf{d} \in \mathbb{R}^q$ is an unknown disturbance.

We define robust approximate simulation in terms of a Lyapunov-like robust simulation function as follows:

Definition 3 (Robust Simulation Function). *Consider two systems of the form (15). Let $\mathcal{V} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^+$ be a*

smooth function and $u_V : \mathbb{R}^{p_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{p_1}$ be a continuous function. $\mathcal{V}$ is a robust simulation function of Σ_2 by Σ_1 and u_V is an associated interface if there exists class- κ functions γ_1, γ_2 such that for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) \geq \|g_1(\mathbf{x}_1) - g_2(\mathbf{x}_2)\| \quad (16)$$

and for all $\mathbf{u}_2 \in \mathbb{R}^{p_2}$ satisfying $\gamma_1(\|\mathbf{d}\|) + \gamma_2(\|\mathbf{u}_2\|) < \mathcal{V}(\mathbf{x}_1, \mathbf{x}_2)$,

$$\frac{\partial \mathcal{V}}{\partial \mathbf{x}_2} f_2(\mathbf{x}_2, \mathbf{u}_2) + \frac{\partial \mathcal{V}}{\partial \mathbf{x}_1} f_1(\mathbf{x}_1, u_V(\mathbf{u}_2, \mathbf{x}_1, \mathbf{x}_2), \mathbf{d}) < 0. \quad (17)$$

Definition 4 (Robust Approximate Simulation). *Σ_1 robustly approximately simulates Σ_2 if there exists a robust simulation function $\mathcal{V}$ of Σ_2 by Σ_1 .*

Note that Definition 3 is a direct generalization of the typical approximate simulation notion: taking $\mathbf{d} = \mathbf{0}$, we recover System (1) and Definition 1. The primary difference between robust approximate simulation and traditional approximate simulation is the conditions under which $\mathcal{V}$ decreases along a trajectory. This suggests that for many cases, a simulation function may also be a robust simulation function, though the resulting error bounds would be different.

As with conventional approximate simulation, finding a robust simulation function and an interface for two general systems is very difficult. In the following subsections, we consider the special cases of linear systems under bounded and impulse disturbances. For each case, we show that the conventional approximate simulation function is also a robust simulation function, and derive the associated error bounds. Proofs are available in the extended online version [17].

B. Linear Systems under Bounded Disturbances

Consider the following special case of System (15):

$$\begin{aligned} \Sigma_1 : \begin{cases} \dot{\mathbf{x}}_1 = A_1\mathbf{x}_1 + B_1\mathbf{u}_1 + B_d\mathbf{d} \\ \mathbf{y}_1 = C_1\mathbf{x}_1 \end{cases}, \\ \Sigma_2 : \begin{cases} \dot{\mathbf{x}}_2 = A_2\mathbf{x}_2 + B_2\mathbf{u}_2 \\ \mathbf{y}_2 = C_2\mathbf{x}_2 \end{cases}, \end{aligned} \quad (18)$$

where A_i, B_i, C_i , and B_d are matrices of proper dimension, and $\mathbf{d}(t) \in \mathbb{R}^q$ is an external disturbance signal that is unknown but bounded in the sense that $\|\mathbf{d}\|_\infty < d_{max}$. We assume that B_d , the mapping from disturbances to the system, is known.

Taking inspiration from [1, Theorem 2], we can establish the following analogue to Theorem 2:

Theorem 3. *Consider two systems of the form (18). Assume that Σ_1 is stabilizable with feedback gain K and that there exist matrices P and Q such that the following conditions hold:*

$$PA_2 = A_1P + B_1Q, \quad (19)$$

$$C_2 = C_1P. \quad (20)$$

Then a robust simulation function of Σ_2 by Σ_1 is given by

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) = \sqrt{(\mathbf{x}_1 - P\mathbf{x}_2)^T M (\mathbf{x}_1 - P\mathbf{x}_2)}, \quad (21)$$

an associated interface is

$$u_V = R\mathbf{u}_2 + Q\mathbf{x}_2 + K(\mathbf{x}_1 - P\mathbf{x}_2), \quad (22)$$

the class- κ function γ_1 is given by

$$\gamma_1(\nu) = \frac{\|\sqrt{M}B_d\|}{\lambda}\nu, \quad (23)$$

and the class- κ function γ_2 is given by

$$\gamma_2(\nu) = \frac{\|\sqrt{M}(B_1R - PB_2)\|}{\lambda}\nu, \quad (24)$$

where R is an arbitrary matrix of proper dimensions and M, λ are such that (6-7) hold.

Furthermore, we can establish a modified error bound for this case of bounded disturbances:

Theorem 4. Consider two systems of the form (18). Let $\mathcal{V}$ be a robust simulation function of Σ_2 by Σ_1 and u_V be an associated interface. Let $\mathbf{u}_2(t)$ be an admissible input of Σ_2 with associated state and output trajectories $\mathbf{x}_2(t)$ and $\mathbf{y}_2(t)$. Let $\mathbf{x}_1(t)$ be a state trajectory of Σ_1 satisfying

$$\dot{\mathbf{x}}_1 = A_1\mathbf{x}_1 + B_1u_V + B_d\mathbf{d}$$

and $\mathbf{y}_1(t)$ be the associated output trajectory. Then

$$\|\mathbf{y}_1(t) - \mathbf{y}_2(t)\| \leq \max \{ \mathcal{V}(\mathbf{x}_1(0), \mathbf{x}_2(0)), \gamma_1(\|\mathbf{d}\|_\infty) + \gamma_2(\|\mathbf{u}_2\|_\infty) \}. \quad (25)$$

C. Linear Systems under Impulse Disturbances

Here we consider the case of linear systems under unbounded disturbances that take the form of impulses. We are inspired to consider this type of disturbance model by recent research applying approximate simulation to legged robot locomotion [2]. In legged locomotion, disturbances from foot impacts with the ground result in infinite-magnitude disturbances over infinitesimally small time periods, and thus cannot be described by the bounded disturbance model described above.

Consider the following special case of System (15):

$$\begin{aligned} \Sigma_1 : & \begin{cases} \dot{\mathbf{x}}_1 = A_1\mathbf{x}_1 + B_1\mathbf{u}_1 + B_d\mathbf{d} \\ \mathbf{y}_1 = C_1\mathbf{x}_1 \end{cases}, \\ \Sigma_2 : & \begin{cases} \dot{\mathbf{x}}_2 = A_2\mathbf{x}_2 + B_2\mathbf{u}_2 \\ \mathbf{y}_2 = C_2\mathbf{x}_2 \end{cases}, \end{aligned} \quad (26)$$

where $\mathbf{d}(t)$ is a sequence of unit impulses at times $\mathbb{T}$, i.e.,

$$\mathbf{d}(t) = \sum_{t_i \in \mathbb{T}} \delta(t - t_i), \quad (27)$$

where $\delta(t)$ is the Dirac delta function. $B_d(t) \in \mathbb{R}^n$ is a mapping from the impulse disturbances $\mathbf{d}(t)$ to the system, which we assume is unknown and possibly time varying, but bounded in the sense that $\|B_d\| < b_{max}$. We make the further assumption that the impulses $t_i \in \mathbb{T}$ are separated by at least a minimum dwell time $t_{dwell} > 0$.

Remark 1. Even though the disturbances we consider with this system model are unbounded, Definition 3 is flexible

enough to handle impulse disturbances. This is because at the instant of an impulse, $\mathbf{d}(t)$ is infinite magnitude, and therefore the condition (17) is not enforced.

First, we establish a straightforward analogue to Theorem 2 for the case of the impulse sequence model.

Theorem 5. Consider two systems of the form (26). Assume that Σ_1 is stabilizable with feedback gain K and that there exist matrices P and Q such that the following conditions hold:

$$PA_2 = A_1P + B_1Q, \quad (28)$$

$$C_2 = C_1P. \quad (29)$$

Then a robust simulation function of Σ_2 by Σ_1 is given by

$$\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) = \sqrt{(\mathbf{x}_1 - P\mathbf{x}_2)^T M (\mathbf{x}_1 - P\mathbf{x}_2)}, \quad (30)$$

an associated interface is

$$u_V = R\mathbf{u}_2 + Q\mathbf{x}_2 + K(\mathbf{x}_1 - P\mathbf{x}_2), \quad (31)$$

the class- κ function γ_1 is given by

$$\gamma_1(\nu) = \nu \quad (32)$$

and the class- κ function γ_2 is given by

$$\gamma_2(\nu) = \frac{\|\sqrt{M}(B_1R - PB_2)\|}{\lambda}\nu, \quad (33)$$

where R is an arbitrary matrix of proper dimensions and M, λ are such that (6-7) hold.

We can now establish an error bound analogous to that of Theorem 1. To do so, first consider the case of a single impulse disturbance at time t_i . For this case, we establish a relaxed upper bound on the output error as follows:

Proposition 1. Consider two systems of the form (26). Let $\mathcal{V}$ be a robust simulation function of Σ_2 by Σ_1 and u_V be an associated interface. Let $\mathbf{u}_2(t)$ be a smooth admissible input of Σ_2 with associated state and output trajectories $\mathbf{x}_2(t)$ and $\mathbf{y}_2(t)$. Let $\mathbf{x}_1(t)$ be a state trajectory of Σ_1 satisfying

$$\dot{\mathbf{x}}_1 = A_1\mathbf{x}_1 + B_1u_V + B_d\mathbf{d}$$

and $\mathbf{y}_1(t)$ be the associated output trajectory. Assume that the disturbance signal $\mathbf{d}(t) = \delta(t - t_i)$ is a single impulse at time t_i . Then

$$\|\mathbf{y}_1(t) - \mathbf{y}_2(t)\| \leq \max \{ \mathcal{V}(\mathbf{x}_1(0), \mathbf{x}_2(0)), \gamma(\|\mathbf{u}_2\|_\infty) \} + b_{max}\sqrt{\lambda_{max}}, \quad (34)$$

where λ_{max} is the maximum eigenvalue of M and b_{max} is the upper bound on $\|B_d\|$.

To extend this result to the case of a sequence of impulse disturbances, we recall from the proof of [3, Theorem 2] that the decay rate of the simulation function is bounded by

$$\frac{d\mathcal{V}}{dt} \leq \lambda(-\mathcal{V}(\mathbf{x}_1, \mathbf{x}_2) + \gamma(\|\mathbf{u}_2\|)).$$

This minimum decay rate allows us to establish a dwell time t_{dwell} such that as long as the impulse disturbances are

separated by at least t_{dwell} , the simulation function decays enough between impulses that the resulting error bound is the same as that of a single impulse. This is stated formally as follows:

Proposition 2. Consider two systems of the form (26). Let $\mathcal{V}$ be a robust simulation function of Σ_2 by Σ_1 and u_V be an associated interface. Let $\mathbf{u}_2(t)$ be a smooth admissible input of Σ_2 with associated state and output trajectories $\mathbf{x}_2(t)$ and $\mathbf{y}_2(t)$. Let $\mathbf{x}_1(t)$ be a state trajectory of Σ_1 satisfying

$$\dot{\mathbf{x}}_1 = A_1 \mathbf{x}_1 + B_1 u_V + B_d d$$

and $\mathbf{y}_1(t)$ be the associated output trajectory. Assume that $t_{dwell} \geq 1/\lambda$, where λ is defined as per Lemma 1. Then

$$\|\mathbf{y}_1(t) - \mathbf{y}_2(t)\| \leq \max \{ \mathcal{V}(\mathbf{x}_1(0), \mathbf{x}_2(0)), \gamma(\|\mathbf{u}_2\|_\infty) \} + b_{max} \sqrt{\lambda_{max}}, \quad (35)$$

where λ_{max} is the maximum eigenvalue of M and b_{max} is the upper bound on $\|B_d\|$.

Interestingly, the minimum dwell time of $1/\lambda$ does not depend on the magnitude of the disturbances B_d but only on λ , which is essentially the decay rate of the simulation function. A larger disturbance magnitude does increase the error bound, however, through the parameter $b_{max} \geq \|B_d\|$.

Finally, we consider the case of arbitrary dwell times, which may be shorter than $1/\lambda$, and derive an associated error bound:

Theorem 6. Consider two systems of the form (26). Let $\mathcal{V}$ be a robust simulation function of Σ_2 by Σ_1 and u_V be an associated interface. Let $\mathbf{u}_2(t)$ be a smooth admissible input of Σ_2 with associated state and output trajectories $\mathbf{x}_2(t)$ and $\mathbf{y}_2(t)$. Let $\mathbf{x}_1(t)$ be a state trajectory of Σ_1 satisfying

$$\dot{\mathbf{x}}_1 = A_1 \mathbf{x}_1 + B_1 u_V + B_d d$$

and $\mathbf{y}_1(t)$ be the associated output trajectory. Assume that $t_{dwell} > 0$. Then

$$\|\mathbf{y}_1(t) - \mathbf{y}_2(t)\| \leq \max \{ \mathcal{V}(0), \gamma(\|\mathbf{u}_2\|_\infty) \} + \max \left\{ 1, \frac{1}{t_{dwell}\lambda} \right\} b_{max} \sqrt{\lambda_{max}}, \quad (36)$$

where λ_{max} is the maximum eigenvalue of M , b_{max} is the upper bound on $\|B_d\|$, and λ is defined by Lemma 1.

IV. EXAMPLE

To illustrate the importance of accounting for disturbances when using approximate simulation, we present a variation of the example presented in [1]. In this example, shown in Figure 2, a robot must navigate a narrow passageway before reaching a goal region (yellow). The concrete system is a robot with triple integrator dynamics while the abstract system is a single integrator. The output of both systems represents the position of the robot in the plane.

Following [1, Section 5], we found an interface that enforces approximate simulation and output error bound (4) $\epsilon = 0.2258$. The abstract system is fully actuated, making it

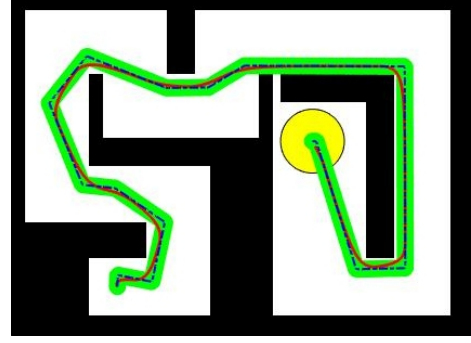


Fig. 2. A robot navigates a passageway to reach a goal (yellow circle) without any disturbances. A classical approximate simulation relation guarantees that the concrete system (red solid line) will stay close (green shaded region) to the abstract system trajectory (blue dashed lines).

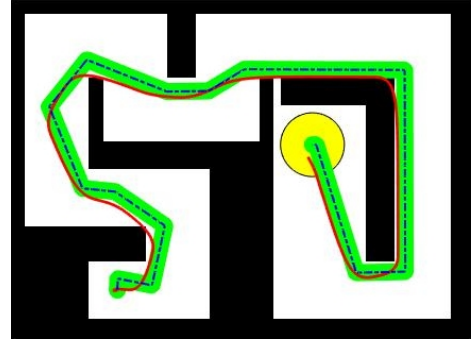


Fig. 3. When a bounded disturbance is applied to the concrete system, the classical approximate simulation relation breaks down. The robot leaves the safe region around the planned trajectory, occasionally colliding with obstacles.

easy to find a trajectory that reaches the goal and stays ϵ away from all obstacles. In this example, we used the probabilistic roadmap strategy [18] to find such a trajectory. This is shown in Figure 2 by the blue dashed lines. The green shaded region represents the area in which the concrete system is guaranteed to remain. The associated concrete system trajectory (red solid line) tracks the abstract trajectory effectively, staying within the safe region and eventually arriving at the goal. Note that the output error bound ϵ is fairly tight.

We now consider planning with this same approximate simulation relation under bounded disturbances. Specifically, we choose $B_d = [-0.2 \ -0.2 \ 0 \ 0 \ 0 \ 0]^T$, $d(t) = 1$. This represents a constant disturbance pushing the robot down and to the left. This is analogous to what would happen if there was a steady gust of wind pushing the robot when it was deployed in the real world.

First, we naively (and improperly) apply the classical approximate simulation relation to this case. The resulting trajectories are shown in Figure 3. The robot is unable to stay ϵ -close to the planned abstract system trajectory, leading to several collisions with the walls of the passageway.

This motivates the use of a robust approximate simulation relation. Following Theorems 3 and 4, we find the correct error bound (25) of $\epsilon = 0.6767$. Using this robust approximate simulation relation to plan an abstract system trajectory, we obtain the more conservative results shown in Figure 4. The

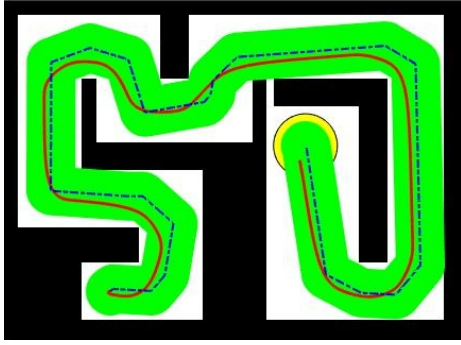


Fig. 4. Using the robust simulation relation and the correct output error bound for bounded disturbances (25) results in a more conservative plan for the abstract system and allows the robot to reach the goal.

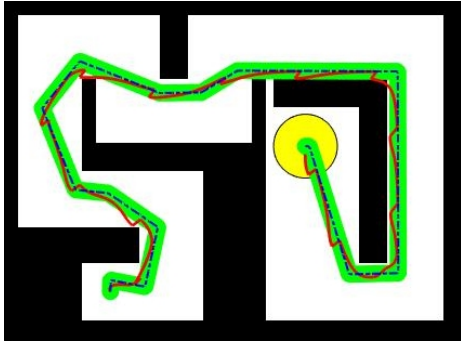


Fig. 5. When the concrete system experiences unbounded impulse disturbances, the classical approximate simulation relation breaks down. The robot leaves the safe region around the planned trajectory, occasionally colliding with obstacles.

robot is able to stay within these relaxed error bounds despite the disturbances, and successfully reaches the goal. We can also see that the associated error bound is reasonably tight.

Finally, we consider the case of unbounded (impulse) disturbances. We use the same disturbance mapping B_d as above, and consider impulses occurring every 2.5s. This corresponds to the case of the robot experiencing a push at regular intervals. As in the case of bounded disturbances, improperly applying a classical approximate simulation relation results in the robot leaving the safe region and colliding with obstacles (Figure 5). Following Theorem 6, we find a relaxed error bound of $\epsilon = 0.678$. Replanning with this revised margin leads to the safe plan shown in Figure 6. The robot stays within the safe region, avoids collisions, and reaches the goal.

V. CONCLUSION

We proposed a notion of robust approximate simulation as a generalization of approximate simulation. This framework can be used for formally correct hierarchical control in the case when the concrete system is subject to external disturbances. We provided detailed results, including the associated error bounds, for linear systems subject to bounded disturbances and impulse disturbances. Future work will focus on generalizing these results to finite-energy disturbances, nonlinear systems, and general transition systems.

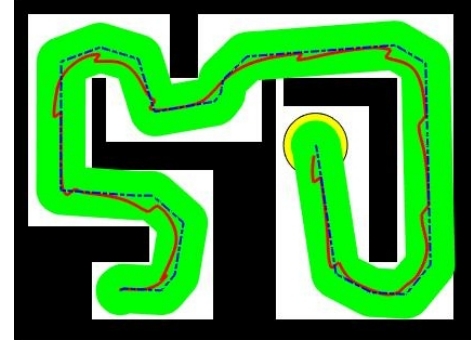


Fig. 6. Using the robust simulation relation and the correct output error bound for impulse disturbances (36) results in a more conservative plan for the abstract system and allows the robot to reach the goal.

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