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The spectrum of continuously perturbed operators and the Laplacian on forms ☆

Nelia Charalambous [a,*], Zhiqin Lu [b]

[a] Department of Mathematics and Statistics, University of Cyprus, Nicosia, 1678, Cyprus
[b] Department of Mathematics, University of California, Irvine, Irvine, CA 92697, USA

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ABSTRACT

In this article we study the variation in the spectrum of a self-adjoint nonnegative operator on a Hilbert space under continuous perturbations of the operator. In the particular case of the Laplacian on k-forms over a complete manifold we will use this analytic result to obtain some interesting and significant properties of its spectrum. In particular, we will prove the continuous deformation of the spectrum of the Laplacian under a continuous deformation of the metric of the noncompact manifold. We will also show that the spectrum on 1-forms always contains the function spectrum on any open manifold.

1. Introduction

We consider a densely defined, self-adjoint operator H on a Hilbert space $\mathcal{H}$. The spectrum of H, $\sigma(H)$, consists of all points $\lambda \in \mathbb{C}$ for which $H - \lambda I$ fails to be invertible. When the operator is nonnegative, its spectrum is contained in $[0, \infty)$. We separate the spectral points into two subsets; the set of isolated eigenvalues of finite multiplicity which we refer to as the discrete isolated spectrum, and the essential spectrum of H, $\sigma_{\text{ess}}(H)$, which is the set of cluster points in the spectrum together with the isolated eigenvalues of infinite multiplicity. It is well known that both $\sigma(H)$ and $\sigma_{\text{ess}}(H)$ are closed sets in $\mathbb{R}$ and in $\mathbb{C}$.

An interesting problem in the study of the spectrum of self-adjoint operators, is to find the variation in the spectrum under perturbations of the operator. In the first main result of this paper, Theorem 2.4, we

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* Corresponding author.
E-mail addresses: nelia@ucy.ac.cy (N. Charalambous), zlu@uci.edu (Z. Lu).

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prove the continuous variation in the spectrum and essential spectrum of such a self-adjoint operator, under a continuous perturbation of its quadratic form. The proof of this theorem is based on a generalized Weyl criterion for computing the spectrum that the authors have previously shown in [6].

Our second goal is to study the spectrum and the essential spectrum of a particular operator, namely the Hodge Laplacian on differential forms over a complete orientable Riemannian manifold. In the compact case the spectrum of the Laplacian is a discrete set of isolated eigenvalues and Dodziuk has shown that these eigenvalues vary continuously with the metric [11]. In the noncompact case the spectrum of the Laplacian can have both pure point and essential spectral points. It is well known that the spectrum depends not only on topological invariants of the manifold, but also on its geometric ones (see [4,6] and references therein). In Theorem 3.3 we prove that the spectrum of the Laplacian on k-forms over a noncompact orientable Riemannian manifold varies continuously under continuous perturbations of the metric. Note that a C^2 perturbation of the metric results in a continuous perturbation for the Laplace operator, and a C^1 perturbation in the metric gives a continuous perturbation in the corresponding quadratic form. As a result, from the regularity point of view, Theorem 3.3 reduces the smoothness of the perturbation required for the full quadratic form of the operator to a mere continuous one. It is also worth mentioning that our proof requires the analytic tools developed in Sections 2 and 3, and is therefore not a simple extension of the argument for the compact case. Moreover, our method allows us to specify the test functions that can be used in a direct proof that a point λ belongs to the spectrum. Our result is related to the spectral continuity results due to Fukaya [19,20] and Cheeger-Colding [8] for the function spectrum and to those of Lott [26,27], and more recently Honda [22], on the form spectrum.

We will also obtain further interesting results for the k-form spectrum. We will use the generalized Weyl criterion to show that a point in the k-form spectrum of the Laplacian over a complete manifold must belong to either the $(k-1)$-form spectrum or to the $(k+1)$-form spectrum (Theorem 4.1). As an immediate consequence we have that the spectrum of the Laplacian on 1-forms always contains the spectrum of the Laplacian on functions (Corollary 4.4). We emphasize the fact that this theorem is an analytic result, which does not impose any assumptions on the curvature nor the volume growth of the manifold. It also implies that we do not always have to make stronger geometric assumptions on the manifold to compute the k-form spectrum, in comparison to the Laplacian on functions. In Section 4 we will join Theorem 4.1 with a previous result of ours to conclude that on manifolds with Ricci curvature asymptotically nonnegative in the radial direction, the essential spectrum of the Laplacian on 1-forms is $[0, \infty)$, whenever the volume of the manifold is infinite, or if the volume is finite but the volume does not decay exponentially.

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2. Continuous perturbations of the operator and its spectrum

Let $\mathcal{H}$ be a Hilbert space with two inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$. We consider two densely defined nonnegative operators H_0 and H_1 on $\mathcal{H}$ that are self-adjoint with respect to the inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ respectively. Let Q_0, Q_1 be their respective quadratic forms and denote the two norms on $\mathcal{H}$ by $\|\cdot\|_0$ and $\|\cdot\|_1$. Note that both Q_0 and Q_1 are nonnegative.

We denote the domain of the Friedrichs extension of H_0 and H_1 by $\mathfrak{Dom}(H_0)$ and $\mathfrak{Dom}(H_1)$ respectively. We assume that there exists a dense subspace $\mathcal{C} \subset \mathcal{H}$ such that $\mathcal{C}$ is contained in $\mathfrak{Dom}(H_0) \cap \mathfrak{Dom}(H_1)$ (in our applications $\mathcal{C}$ will be the space of smooth forms with compact support).

Definition 2.1. We say that the operators H_0, H_1 are ε-close, if there exists a positive constant $0 < \varepsilon < 1$ such that for all $u \in \mathcal{C}$ the following two inequalities hold

$$(1-\varepsilon)\,\|u\|_0^2 \le \|u\|_1^2 \le (1+\varepsilon)\,\|u\|_0^2; \tag{1}$$

$$(1-\varepsilon)\,Q_0(u,u) \le Q_1(u,u) \le (1+\varepsilon)\,Q_0(u,u). \tag{2}$$

If H_0, H_1 are ε-close, then for any $u, v \in \mathcal{C}$

$$|(u,v)_1 - (u,v)_0| \le \varepsilon(\|u\|_0\,\|v\|_0); \tag{3}$$

$$|Q_1(u,v) - Q_0(u,v)| \le \varepsilon\,[Q_0(u,u)\,Q_0(v,v)]^{1/2}. \tag{4}$$

To prove (3) we first observe that if either $u = 0$ or $v = 0$, then clearly the inequality holds. So we will prove it when neither vanishes. We first observe that

$$\begin{aligned}|(u,v)_1 - (u,v)_0| &= \frac{1}{4}\,\big|\,[\|u+v\|_1^2 - \|u+v\|_0^2 - (\|u-v\|_1^2 - \|u-v\|_0^2)]\,\big|\\ &\le \frac{1}{4}\,\varepsilon\,[\|u+v\|_0^2 + \|u-v\|_0^2] \le \frac{1}{2}\,\varepsilon\,[\|u\|_0^2 + \|v\|_0^2].\end{aligned}$$

If in the above inequality we replace u by au and v by v/a with $a^2 = \|v\|_0/\|u\|_0$, then (3) follows immediately. Inequality (4) follows in a similar manner given that Q_0 and Q_1 are nonnegative.

We will show that the spectra of two ε-close operators also remain close. Our proof will require a comparison of their resolvent operators which is Lemma 2.3 below. The following boundedness result for the resolvent operators is a direct consequence of the spectral theorem.

Lemma 2.2. *Let H be densely defined self-adjoint nonnegative operator on a Hilbert space $\mathcal{H}$. Then for any nonnegative integer m and $\alpha > 0$, $(H+\alpha)^{-m}$ is a bounded operator on $\mathcal{H}$ with operator norm bounded by α^{-m}.*

Lemma 2.3. *Let H_0 and H_1 be two self-adjoint nonnegative operators that are ε-close on $\mathcal{H}$ as in Definition 2.1, with $0 < \varepsilon < 1/2$.*

Fix $\alpha \ge 1$. Then for all $u, v \in \mathcal{C}$

$$\big|(\,(H_1+\alpha)^{-m}u, v\,)_1 - (\,(H_0+\alpha)^{-m}u, v\,)_0\big| \le (2m+1)\,\varepsilon\,\|u\|_0\|v\|_0$$

for any nonnegative integer $m \ge 0$.

Proof. For $m = 0$, the result follows from (3).

Let $m > 0$ and assume that the lemma holds for $m-1$. Then

$$\begin{aligned}&|((H_1+\alpha)^{-m+1}u, (H_1+\alpha)^{-1}v)_1 - ((H_0+\alpha)^{-m+1}u, (H_1+\alpha)^{-1}v)_0|\\ &\qquad \le (2m-1)\,\varepsilon\|u\|_0\|v\|_0.\end{aligned}$$

Let $w = (H_0+\alpha)^{-m+1}u$, $w_1 = (H_0+\alpha)^{-1}w$, and $v_1 = (H_1+\alpha)^{-1}v$. Then

$$\begin{aligned}&|(w, v_1)_0 - (w_1, v)_0|\\ &\le |((H_0+\alpha)w_1, v_1)_0 - (w_1, (H_1+\alpha)v_1)_1| + |(w_1, v)_0 - (w_1, v)_1|\\ &\le \varepsilon\,((Q_0(w_1,w_1)\cdot Q_0(v_1,v_1))^{1/2} + \|w_1\|_0 \cdot \|v\|_0) \le 2\varepsilon\|u\|_0 \cdot \|v\|_0.\end{aligned}$$

The result follows for m after combining the above two estimates. □

For our applications, it would be convenient to consider the spectrum of any nonnegative self-adjoint operator H as a complete metric space. As is well-known, the spectrum $\sigma(H)$ is a closed subset of $[0, \infty)$, hence it is a complete metric space with distance function the one induced from $\mathbb{R}$. Let

$$\tilde{\sigma}(H) = \{-1\} \cup \sigma(H).$$

Then $(\tilde{\sigma}(H), -1)$ is a pointed complete metric space. We denote the pointed-Hausdorff distance as $d_{\mathfrak{h}}$.

Similar notions can be defined for the essential spectrum. We feel that the above definition is a convenient notion in many of our results, and we shall use it throughout this paper. However, the point -1 is just an abstract point, and in order to simplify notation, we shall use $\sigma(H)$, instead of the more complicated $(\tilde{\sigma}(H), -1)$, for the rest of the paper.

We will now describe the proximity of the spectra of two ε-close operators.

Theorem 2.4. *Let H_0, H_1 be two nonnegative operators on $\mathcal{H}$ that are ε-close as in Definition 2.1 for some $0 < \varepsilon < 1/2$. Fix $A > 0$. Then for any $\lambda \in \sigma(H_1) \cap [0, A]$*

$$\operatorname{dist}(\lambda, \sigma(H_0)) < c(A)\varepsilon^{\frac{1}{3}}$$

for a constant $c(A)$ depending only on A. In particular, we have

$$d_{\mathfrak{h}}(\sigma(H_0), \sigma(H_1)) = o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

The proof of the theorem relies on a generalized Weyl criterion for self-adjoint operators that the authors have proved in [6] (see also the proof in [7]).

Proposition 2.5. *A nonnegative real number λ belongs to the spectrum $\sigma(H)$ if, and only if, there exists a positive constant α and a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathfrak{Dom}(H)$ such that*

(1) $\forall j \in \mathbb{N}, \quad \|\psi_j\| = 1,$
(2) $((H + \alpha)^{-m}\psi_j, (H - \lambda)\psi_j) \to 0$ *for* $m = 1, 2$.

Moreover, λ belongs to the essential spectrum $\sigma_{\mathrm{ess}}(H)$ if, and only if, in addition to the above properties

(3) $\psi_j \to 0$, *weakly as $j \to \infty$ in $\mathcal{H}$.*

Furthermore, if for some $0 < \delta < 1$,

$$|((H + \alpha)^{-m}\psi_j, (H - \lambda)\psi_j)| \leq \delta$$

for both $m = 1, 2$ and all j, then there exists a constant $c(\lambda, \alpha) > 0$, depending only on λ, α, such that

$$\operatorname{dist}(\lambda, \sigma(H)) < c(\lambda, \alpha)\, \delta^{\frac{1}{3}}.$$

For other versions of generalized Weyl criteria see [4,23].

Proof of Theorem 2.4. We start by taking a point $0 \leq \lambda \leq A$ in the spectrum of H_1 and fix $\alpha \geq 1$. By Proposition 2.5 for $m = 1, 2$ we have

$$|\,((H_1+\alpha)^{-m}\psi_j,(H_1-\lambda)\psi_j)_1\,| \le \varepsilon\|\psi_j\|_1$$

for a sequence $\{\psi_j\}$ with unit norm as $j \to \infty$. The identity

$$\begin{aligned}&((H_1+\alpha)^{-m}\psi_j,(H_1-\lambda)\psi_j)_1\\&=((H_1+\alpha)^{-m+1}\psi_j,\psi_j)_1-(\alpha+\lambda)\,(\,(H_1+\alpha)^{-m}\psi_j,\psi_j)_1\end{aligned} \tag{5}$$

together with Lemma 2.3 imply that the corresponding expression for H_0 should also tend to zero. In fact, we have

$$|((H_0+\alpha)^{-m}\psi_j,(H_0-\lambda)\psi_j)_0| \le c(A,\alpha)\varepsilon$$

for some constant $c(A,\alpha)$ depending only on A,α. By Proposition 2.5, the conclusion is true for some, possibly different, constant $c(A,\alpha)$. □

Theorem 2.4 demonstrates that whenever $H_\varepsilon \to H_0$ under the topology of ε-closeness, then $\sigma(H_\varepsilon) \to \sigma(H_0)$ as pointed metric spaces with respect to the pointed Hausdorff distance. At the same time, it implies that gaps in the spectrum of H_0, if they exist, do not vanish instantaneously.

In [32] Post studies the variation in the spectrum when a family of noncompact manifolds approaches a metric graph. To achieve this, he proves the continuous variation in the spectrum in the context of ε-close operators over Hilbert spaces that are related through quasi-unitary operators. In our context of operators induced by quadratic forms, our ε-closeness assumption for the metric (3) is equivalent to Post's assumption (A.9′) in the case when the two Hilbert spaces are the same and the operators J and J' are the identity. Moreover, our assumption (4) implies Post's (A.11′), when the operators J_1 and J_1' between the first Sobolev spaces are also the identity. See Definition A.1 in [32] for his detailed conditions.

3. The spectrum of the Laplacian on forms under continuous deformations of the metric

Let (M^n,g) be a complete n-dimensional orientable Riemannian manifold. The metric g induces a pointwise inner-product on the space of k-forms $\Lambda^k(M)$ which is denoted $\langle\cdot,\cdot\rangle$. We denote the L^2 pairing as

$$(\cdot,\cdot)=\int_M\langle\cdot,\cdot\rangle$$

and the L^2 norm as $\|\cdot\|$.

Let $L^2(\Lambda^k(M))$ denote the space of L^2 integrable k-forms. Denote by Δ_k the Laplacian on k-forms as well as its Friedrichs extension on L^2. We denote the domain of the Laplacian on k-forms by $\mathfrak{Dom}(k,\Delta)$. For the remaining of this paper, we shall write Δ instead of Δ_k for $0 \le k \le n$ whenever the order of the form is implied.

Recall that the Laplacian is given by $\Delta = d\delta + \delta d$ where d is the exterior derivative and δ is its formal adjoint with respect to the Riemannian metric. We consider the two operators that make up the Laplacian. We set

$$\mathcal{L}^1=\delta d$$

with associated quadratic form $Q^1(\omega)=(d\omega,d\omega)$ and

$$\mathcal{L}^2=d\delta$$

with associated quadratic form $Q^2(\omega)=(\delta\omega,\delta\omega)$ on k-forms.

Each one of the operators $\mathcal{L}^1, \mathcal{L}^2$ has a self-adjoint Friedrichs extension which is nonnegative. It can be easily seen that $\mathfrak{Dom}(k,\Delta) = \mathfrak{Dom}(k,\mathcal{L}^1) \cap \mathfrak{Dom}(k,\mathcal{L}^2)$. We will use the notation $\sigma(k,\Delta)$ ($\sigma_{\text{ess}}(k,\Delta)$ *resp.*) to denote the spectrum (essential spectrum *resp.*) of the Laplacian on k-forms, where $0 \leq k \leq n$, and similarly for the partial Laplace operators $\mathcal{L}^1, \mathcal{L}^2$.

The generalized Weyl Criterion, Proposition 2.5, allows to illustrate how the spectra of the above three operators are related, which is of its own interest.

Lemma 3.1. *For any $0 \leq k \leq n$ the following containments hold*

$$\sigma(k,\Delta) \subset \sigma(k,\mathcal{L}^1) \cup \sigma(k,\mathcal{L}^2) \tag{6}$$

and

$$\sigma(k,\Delta) \supset \{\sigma(k,\mathcal{L}^1) \cup \sigma(k,\mathcal{L}^2)\} \setminus \{0\}. \tag{7}$$

The result is also true for the essential spectra of the operators.

Proof. We first remark that $\Delta, \mathcal{L}^1, \mathcal{L}^2$ are all closed. Therefore the forms to which we apply the Weyl Criteria can be taken to be smooth with compact support.

If $k = 0$, then $\Delta = \mathcal{L}^1$ and $\mathcal{L}^2 = 0$, and the statement is trivially true. Similarly, for $k = n$, $\Delta = \mathcal{L}^2$. As a result we only consider the case $0 < k < n$.

We begin by proving (6). We first show that 0 is always a point in $\sigma(k,\mathcal{L}^1) \cup \sigma(k,\mathcal{L}^2)$. This follows from the simple fact that for any smooth compactly supported $(k-1)$-form ω, $d\omega$ is a k-eigenform of $\mathcal{L}^1 = \delta d$ corresponding to the eigenvalue 0. Moreover, since $k \geq 1$ we can always find a sequence of compactly supported approximate $(k-1)$-forms u_j such that $\|du_j\| = 1$ on M. This implies that $0 \in \sigma(k,\mathcal{L}^1)$. As a result, if $0 \in \sigma(k,\Delta)$, then $0 \in \sigma(k,\mathcal{L}^1) \cup \sigma(k,\mathcal{L}^2)$.

We now consider $\lambda > 0$ in $\sigma(k,\Delta)$. By the classical Weyl criterion (see for example [12]), there exists a sequence of approximate eigenforms $\{\psi_j\}_{j\in\mathbb{N}}$ with $\|\psi_j\| = 1$ such that for any $0 < \varepsilon < \lambda/2$, we have

$$\|(\Delta - \lambda)\psi_j\| < \varepsilon \quad \text{as} \quad j \to \infty.$$

By the triangle inequality,

$$\|\Delta\psi_j\| \geq \frac{\lambda}{2}$$

for j large enough. Since $\|\Delta\psi_j\|^2 = \|\mathcal{L}^1\psi_j\|^2 + \|\mathcal{L}^2\psi_j\|^2$, there must exist a subsequence of j, for which either

$$\|\mathcal{L}^1\psi_j\|^2 \geq \frac{\lambda}{4} \;\text{ or }\; \|\mathcal{L}^2\psi_j\|^2 \geq \frac{\lambda}{4}$$

(note that the ψ_j are smooth with compact support).

Suppose that $\|\mathcal{L}^1\psi_j\|^2 \geq \frac{\lambda}{4}$. Observe that on smooth forms with compact support,

$$\mathcal{L}^1\mathcal{L}^2 = \mathcal{L}^2\mathcal{L}^1 = 0$$

and

$$\Delta\mathcal{L}^i = \mathcal{L}^i\Delta$$

for $i = 1, 2$. Thus for $m = 1, 2$

$$\begin{aligned} &|\,(\,(\mathcal{L}^1+1)^{-m}\mathcal{L}^1\psi_j, (\mathcal{L}^1-\lambda)\mathcal{L}^1\psi_j)\,| = |\,(\,(\mathcal{L}^1+1)^{-m}\mathcal{L}^1\psi_j, (\Delta-\lambda)\mathcal{L}^1\psi_j)\,| \\ &= |\,(\,(\mathcal{L}^1+1)^{-m}(\mathcal{L}^1)^2\psi_j, (\Delta-\lambda)\psi_j)\,| \le \|\mathcal{L}^1\psi_j\| \cdot \|(\Delta-\lambda)\psi_j\| \le \varepsilon\,\frac{4}{\lambda}\,\|\mathcal{L}^1\psi_j\|^2 \end{aligned}$$

where we have used that $\|(\mathcal{L}^1+1)^{-m}\mathcal{L}^1\| \le 1$ which can also be proved via the spectral theorem similarly to Lemma 2.2. We set $\tilde{\psi}_j = \mathcal{L}^1\psi_j/\|\mathcal{L}^1\psi_j\|$ and rescale the above inequalities. Then, for a sequence of $\varepsilon \to 0$, we can find a sequence of $\tilde{\psi}_j$ that satisfy the conditions of Proposition 2.5 with $\alpha = 1$. Therefore $\lambda \in \sigma(k, \mathcal{L}^1)$. The argument for the case $\|\mathcal{L}^2\psi_j\|^2 \ge \frac{\lambda}{4}$ is identical. We thus conclude that λ belongs either to $\sigma(k, \mathcal{L}^1)$ or to $\sigma(k, \mathcal{L}^2)$.

To prove (7) we now suppose that $\lambda > 0$ belongs to $\sigma(k, \mathcal{L}^1)$. Again by the classical Weyl criterion there exists a sequence of smooth approximate eigenforms $\{\psi_j\}_{j\in\mathbb{N}}$ with $\|\psi_j\| = 1$ such that for any $\varepsilon > 0$, $0 < \varepsilon < \lambda/2$, we have

$$\|(\mathcal{L}^1-\lambda)\psi_j\| < \varepsilon \quad \text{as} \quad j \to \infty.$$

As a result,

$$\frac{\lambda}{2} \le \|\mathcal{L}^1\psi_j\| \le 2\lambda$$

for j large enough. For $m = 1, 2$ we similarly get

$$\begin{aligned} &|\,(\,(\Delta+1)^{-m}\mathcal{L}^1\psi_j, (\Delta-\lambda)\mathcal{L}^1\psi_j)\,| = |\,(\,(\Delta+1)^{-m}\mathcal{L}^1\psi_j, (\mathcal{L}^1-\lambda)\mathcal{L}^1\psi_j)\,| \\ &= |\,(\,(\Delta+1)^{-m}\Delta\mathcal{L}^1\psi_j, (\mathcal{L}^1-\lambda)\psi_j)\,| \le \|\mathcal{L}^1\psi_j\| \cdot \|(\mathcal{L}^1-\lambda)\psi_j\| \le \varepsilon\,\frac{4}{\lambda}\,\|\mathcal{L}^1\psi_j\|^2. \end{aligned}$$

Again we set $\tilde{\psi}_j = \mathcal{L}^1\psi_j/\|\mathcal{L}^1\psi_j\|$ and rescale the above inequalities. Then for a sequence of $\varepsilon \to 0$ we can get a sequence of $\tilde{\psi}_j$ that satisfy the conditions of Proposition 2.5. Therefore, λ belongs to the spectrum of Δ.

In a similar manner we can prove that $\sigma(k, \mathcal{L}^2) \setminus \{0\} \subset \sigma(k, \Delta)$. As a result

$$\{\sigma(k, \mathcal{L}^1) \cup \sigma(k, \mathcal{L}^2)\} \setminus \{0\} \subset \sigma(k, \Delta).$$

The case of the essential spectrum follows in a similar manner. □

Remark 3.2. Lemma 3.1 could also have been proved using a similar version of the Hodge decomposition theorem over a noncompact manifold [24]. Our method however, offers a direct proof that specifies the test functions that can be used in showing that a point λ belongs to the spectrum.

We now consider a manifold M to which we can assign two Riemannian metrics, g_0, g_1 such that (M, g_0) and (M, g_1) are smooth complete manifolds with respect to both. We say that the two metrics are ε-close if for some $0 < \varepsilon < 1$

$$(1-\varepsilon)g_0 \le g_1 \le (1+\varepsilon)g_0. \tag{8}$$

We denote by δ_i the adjoint of d on (M, g_i) for $i = 0, 1$ and the associated Laplacian operators by

$$\Delta_i = d\delta_i + \delta_i d.$$

We denote by $(\cdot,\cdot)_i$ the L^2 pairing in the g_i metric and by $\|\cdot\|_i$ the respective L^2 norm. In this section we will show that Theorem 2.4 can be extended to Δ_i. Let

$$\mathcal{L}_i^1 = \delta_i d \quad \text{and} \quad \mathcal{L}_i^2 = d\delta_i.$$

Their associated quadratic forms are given by

$$Q_i^1(\omega,\omega) = (d\omega, d\omega)_i \quad \text{and} \quad Q_i^2(\omega,\omega) = (\delta_i\omega, \delta_i\omega)_i,$$

respectively.

As mentioned in the Introduction, the notation $\sigma(k,\Delta)$, $\sigma_{\text{ess}}(k,\Delta)$ *resp.*, actually refers to the pointed complete metric space

$$(\sigma(k,\Delta)\cup\{-1\},-1), \qquad (\sigma_{\text{ess}}(k,\Delta)\cup\{-1\},-1) \quad \textit{resp.}$$

We set

$$\sigma(-1,\Delta) = \sigma(n+1,\Delta) = \sigma_{ess}(-1,\Delta) = \sigma_{ess}(n+1,\Delta) = \emptyset$$

which, according to the above convention, means that they are all the single point metric space $\{-1\}$.

We now state the main result of this section.

Theorem 3.3. *Let M^n be an orientable manifold, and let g_0, g_1 be two smooth complete Riemannian metrics on M that are ε-close for some $0<\varepsilon<1/2$.*

Fix $A>0$. Then for any $\lambda \in \sigma(k,\Delta_1)\cap[0,A]$

$$\text{dist}(\lambda, \sigma(k,\Delta_0)) < c(A,n)\,\varepsilon^{\frac{1}{3}}$$

for some constant $c(A)$ depending only on A. A similar result holds for the essential spectra of the operators. In particular,

$$d_{\mathfrak{h}}(\sigma(k,\Delta_1), \sigma(k,\Delta_0)) = o(1),$$

where $o(1)\to 0$, as $\varepsilon\to 0$.

Proof. Given that the $*$-operator is an isometry in the respective metric, for all $0\le k\le n$, we have

$$\mathfrak{Dom}(k,\mathcal{L}_i^2) = \mathfrak{Dom}(n-k,\mathcal{L}_i^1)$$

for $i=0,1$. Moreover,

$$\sigma(k,\mathcal{L}_i^2) = \sigma(n-k,\mathcal{L}_i^1).$$

The same holds true for the essential spectrum.

Since d is metric independent, and since $\mathfrak{Dom}(k,\mathcal{L}_1^1)\cap\mathfrak{Dom}(k,\mathcal{L}_0^1)\supset\mathcal{C}$, where $\mathcal{C}$ is the space of smooth forms with compact support, by the ε-closeness of the metrics we have

$$(1-\varepsilon')\,Q_0^1(\omega) \le Q_1^1(\omega) \le (1+\varepsilon')\,Q_0^1(\omega)$$

for any $\omega \in \mathcal{C}$ and for $\varepsilon' = \max\left\{(\frac{1+\varepsilon}{1-\varepsilon})^n - 1\,, 1 - (\frac{1-\varepsilon}{1+\varepsilon})^n\right\} \leq c(n)\varepsilon$ when $0 < \varepsilon < 1/2$ and where $c(n)$ is a constant that only depends on the dimension n. In other words, the operators $\mathcal{L}_1^1$ and $\mathcal{L}_0^1$ are $c(n)\varepsilon$-close. For simplicity in notation we will replace $c(n)\varepsilon$ by ε. For any $\lambda \in \sigma(k, \mathcal{L}_1^1) \cap (0, A]$, by Lemma 3.1, we have

$$\operatorname{dist}(\lambda, \sigma(k, \Delta_0)) \leq \operatorname{dist}(\lambda, \sigma(k, \mathcal{L}_0^1)) < c(A, n)\, \varepsilon^{\frac{1}{3}}.$$

The case $\lambda = 0$ can be treated directly. If $0 \in \sigma(\Delta_1)$, we claim

$$\operatorname{dist}(0, \sigma(\Delta_0)) \leq 100\varepsilon.$$

If not, then there exists ε_0 such that $\operatorname{dist}(0, \sigma(\Delta_0)) > 100\varepsilon_0$. Let ω be a smooth compactly supported k-form such that $\|\omega\|_1 = 1$ and

$$\|\Delta_1 \omega\|_1 < \varepsilon_0.$$

Then

$$\|d\omega\|_1 \leq \sqrt{\varepsilon_0}, \quad \|\delta_1 \omega\|_1 \leq \sqrt{\varepsilon_0}.$$

Since 0 is not in the spectrum of Δ_0, then Δ_0^{-1} is well defined. We set $\eta = \Delta_0^{-1}\omega, \quad \eta_1 = d\eta$ and $\eta_2 = \delta_0 \eta$. Then

$$\|\eta\|_0 \leq \frac{1}{100\varepsilon_0}, \quad \|\eta_1\|_0 \leq \frac{1}{5\sqrt{\varepsilon_0}}, \quad \|\eta_2\|_0 \leq \frac{1}{5\sqrt{\varepsilon_0}}, \quad \text{and} \quad \|d\eta_2\|_0 \leq 2.$$

Therefore,

$$|(\omega, \delta_0 \eta_1)_0| = |(d\omega, \eta_1)_0| \leq \|d\omega\|_0 \cdot \|\eta_1\|_0 \leq 2\|d\omega\|_1 \cdot \|\eta_1\|_0 \leq \frac{2}{5}.$$

Moreover,

$$|(\omega, d\eta_2)_1| = |(\delta_1 \omega, \eta_2)_1| \leq \|\delta_1 \omega\|_1 \cdot 2\|\eta_2\|_0 \leq \frac{2}{5},$$

whereas by the ε-closeness of the metrics we have

$$|(\omega, d\eta_2)_1 - (\omega, d\eta_2)_0| \leq 2\varepsilon \|d\eta_2\|_0 \leq 4\varepsilon.$$

As a result,

$$|(\omega, d\eta_2)_0| \leq \frac{2}{5} + 4\varepsilon.$$

Observing that $\omega = \delta_0 \eta_1 + d\eta_2$ and combining the above estimates we get

$$\|\omega\|_1 = |(\omega, \delta_0 \eta_1)_0 + (\omega, d\eta_2)_0| \leq \frac{4}{5} + 4\varepsilon < \frac{9}{10}$$

which gives a contradiction for ε small enough. This completes the proof. □

The following corollary is now immediate.

Corollary 3.4. *Let M be a complete noncompact orientable manifold, and let $\{g_\varepsilon\}_{\varepsilon\in[0,1/2]}$ be a family of smooth complete Riemannian metrics on M such that*

$$(1-\varepsilon)g_0 \le g_\varepsilon \le (1+\varepsilon)g_0.$$

Then

$$d_{\mathfrak{h}}(\sigma(k,\Delta_\varepsilon),\sigma(k,\Delta_0)) = o(1),$$

where $o(1)\to 0$, as $\varepsilon\to 0$. A similar result holds for the essential spectrum of the operators.

The Corollary implies that if for a sequence of $\varepsilon_m \to 0$ there exists a sequence of points $\lambda_{\varepsilon_m} \to \lambda > 0$ with the property $\lambda_{\varepsilon_m} \in \sigma_{\text{ess}}(\Delta_{\varepsilon_m})$ for all m, then $\lambda \in \sigma_{\text{ess}}(\Delta_0)$. In other words, if $\lambda \notin \sigma_{\text{ess}}(\Delta_0)$, then there exists a $\delta > 0$ such that $\lambda \notin \sigma_{\text{ess}}(\Delta_\varepsilon)$ for all $\varepsilon < \delta$.

When M is a compact manifold the above result for the pointed Hausdorff convergence of the spectrum implies that for any given i, the i-th eigenvalues are convergent as $\varepsilon \to 0$ as was also proved in [11]. Although our proof is an adaptation of Dodziuk's argument for the compact case it is not a mere extension of it, as the continuous variation in the spectrum of the partial operators $\mathcal{L}^1, \mathcal{L}^2$ requires the analytical proofs that we have developed in Section 2.

Finally, we would like to make the following remark. As the proof of Theorem 3.3 illustrates, when two metrics are ε-close, then the spectra of the $\mathcal{L}_i^2$ operators are also close. However, given that the dual form $*\omega$ of ω is defined differently for each metric, we actually do not have that the operators $\mathcal{L}_i^2$ are ε-close whenever the metrics are close. As a result, the Laplacians themselves are also not ε-close. The ε-closeness of the $\mathcal{L}_i^2$ would in fact require the C^1 closeness of the metrics, see for example the paper by Baker and Dodziuk where they prove the convergence of eigenvalues in the case of compact manifolds with boundary when the metrics converge in the C^1 sense [2].

4. The spectrum of the Laplacian on k-forms

The following result is true on any complete orientable Riemannian manifold, without any further topological nor geometric assumptions.

Theorem 4.1. *Let (M,g) be a complete orientable Riemannian manifold. For any $0 \le k \le n$, suppose that $\lambda > 0$ belongs to $\sigma(k,\Delta)$. Then one of the following holds:*

(a) $\lambda \in \sigma(k-1,\Delta)$, *or*
(b) $\lambda \in \sigma(k+1,\Delta)$.

The same result is true for the essential spectrum.

Proof. Let $\lambda > 0$ and $\lambda \in \sigma(k,\Delta)$. By the classical Weyl criterion we know that for each $\varepsilon > 0$, there exists an approximate eigenfunction $\omega_\varepsilon \in \mathfrak{Dom}(k,\Delta)$ such that $\|\omega_\varepsilon\| = 1$,

$$\|(\Delta-\lambda)\omega_\varepsilon\| \le \varepsilon. \tag{9}$$

As M is complete, we can in fact assume that the ω_ε are smooth and compactly supported. Choosing $\varepsilon < \lambda/2$, the triangle inequality gives

$$(\Delta\omega_\varepsilon,\omega_\varepsilon) \ge \frac{1}{2}\lambda. \tag{10}$$

Thus we have

$$\|d\omega_\varepsilon\|^2 + \|\delta\omega_\varepsilon\|^2 = (\Delta\omega_\varepsilon, \omega_\varepsilon) \geq \frac{1}{2}\,\lambda.$$

This estimate implies that either

$$\|d\omega_\varepsilon\|^2 \geq \frac{\lambda}{4} \quad \text{or} \quad \|\delta\omega_\varepsilon\|^2 \geq \frac{\lambda}{4}.$$

We first consider the case $\|d\omega_\varepsilon\|^2 \geq \frac{\lambda}{4}$. For simplicity, we denote $\omega_\varepsilon = \omega$. For any integer m, $(\Delta+1)^{-m} d\omega = d(\Delta+1)^{-m}\omega$ and $(\Delta+1)^{-m}\delta\omega = \delta(\Delta+1)^{-m}\omega$. For $m = 1, 2$ we compute

$$\begin{aligned} |\,((\Delta+1)^{-m}\, d\omega,\ (\Delta-\lambda)\, d\omega)\,| &= |\,((\Delta+1)^{-m}\,\delta d\omega, (\Delta-\lambda)\omega)\,| \\ &\leq \varepsilon\|\,((\Delta+1)^{-m}\,\delta d\omega\| \end{aligned} \tag{11}$$

by (9). At the same time, the commutativity properties of the resolvent and integration by parts give

$$\|(\Delta+1)^{-m}\,\delta d\omega\|^2 + \|(\Delta+1)^{-m}\, d\delta\omega\|^2 = \|(\Delta+1)^{-m}\,\Delta\omega\|^2 \leq \|\Delta\omega\|^2 \leq (\varepsilon+\lambda)^2,$$

where we have used Lemma 2.2 and assumption (9). Combining this with (11) we get

$$|\,((\Delta+1)^{-m}\, d\omega,\ (\Delta-\lambda)\, d\omega)\,| \leq \varepsilon(\varepsilon+\lambda) \leq \varepsilon(1+\lambda)\|\omega\|^2 \leq \frac{4\varepsilon(1+\lambda)}{\lambda}\|d\omega\|^2$$

by our assumption.

If we consider instead the case $\|d\omega_\varepsilon\|^2 \geq \frac{\lambda}{4}$, we similarly get

$$|\,((\Delta+1)^{-m}\,\delta\omega,\ (\Delta-\lambda)\,\delta\omega)\,| \leq \frac{4\varepsilon(1+\lambda)}{\lambda}\|\delta\omega\|^2.$$

By Proposition 2.5, λ must therefore belong to either $\sigma(k-1, \Delta)$ or $\sigma(k+1, \Delta)$.

The case for the essential spectrum follows similarly. □

Remark 4.2. Over a compact orientable manifold the k-form spectrum is discrete, and each element of the spectrum is an eigenvalue. To each eigenvalue λ corresponds a smooth form ω such that

$$\Delta\omega - \lambda\omega = 0.$$

It is easy to check that

$$\Delta d\omega - \lambda d\omega = 0, \quad \Delta\delta\omega - \lambda\delta\omega = 0.$$

Therefore if $\lambda \neq 0$, at least one of $d\omega$ and $\delta\omega$ should not be zero, and hence the conclusion of Theorem 4.1 is trivially true. On the other hand, it seems that in order to prove the result in the complete noncompact case, we need to make full use of our new Weyl criterion.

Remark 4.3. Gromov and Shubin proved in [21] that over any Riemannian manifold, including the noncomplete case, we have the following Hodge decomposition theorem:

$$L^2(M) = \ker\Delta \oplus \operatorname{Im} d \oplus \operatorname{Im}\delta.$$

However, the completeness assumption in Theorem 4.1 is essential as there exists a counterexample in the incomplete case. See Lu-Xu [29] for details as well as [10] for further results on the compact case.

Corollary 4.4. *The spectrum of the Laplacian on 1-forms contains the spectrum of the Laplacian on functions except possibly for the point* $\lambda = 0$*.*

Corollary 4.5. *The essential spectrum of the Laplacian on 1-forms is* $[0, \infty)$ *whenever the essential spectrum of the Laplacian on functions is* $[0, \infty)$*.*

Corollary 4.4 is an immediate consequence of Theorem 4.1. Corollary 4.5 follows from Corollary 4.4 and the fact that the essential spectrum is a closed set. Combined with Theorem 1.3 of our article [4] on the function spectrum these corollaries will give that the essential spectrum on 1-forms is $[0, \infty)$ on a significantly larger class of manifolds.

We recall the following definitions from [4]. Let p be a fixed point in M. The cut locus $\mathrm{Cut}(p)$ is a set of measure zero in M, and the manifold can be written as the disjoint union $M = \Omega \cup \mathrm{Cut}(p)$, where Ω is star-shaped with respect to p. That is, if $x \in \Omega$, then the geodesic line segment $\overline{px} \subset \Omega$. $\partial r = \partial/\partial r$ is well defined on Ω.

Definition 4.6. Let M be a complete noncompact Riemannian manifold. Let p be a fixed point in M and define $r(x)$ to be the radial function with respect to p. We say that the radial Ricci curvature of M is asymptotically nonnegative with respect to p if there exists a continuous function $\delta(r)$ on $\mathbb{R}^+$ such that

(i) $\lim_{r\to\infty} \delta(r) = 0$;
(ii) $\delta(r) > 0$, and
(iii) $\mathrm{Ric}(\partial r, \partial r) \geq -(n-1)\delta(r)$ on Ω.

We remark that manifolds that satisfy the condition above have subexponential volume growth at p, but need not have uniformly subexponential volume growth as defined in [34]. In other words, the L^p independence result for the spectrum of the Laplacian on 1-forms need not hold [3]. Using Corollary 4.4 and Theorem 1.3 of [4] we obtain

Theorem 4.7. *Let* M *be a complete noncompact orientable Riemannian manifold. Suppose that, with respect to a fixed point* p*, the radial Ricci curvature is asymptotically nonnegative in the sense of Definition 4.6. If the volume of the manifold is finite we additionally assume that its volume does not decay exponentially at* p*.*

Then the essential spectrum of the Laplacian on 1-forms is $[0, \infty)$*.*

By the Poincaré duality, it is easy to see that

$$\sigma(k, \Delta) = \sigma(n-k, \Delta)$$

for any $0 \leq k \leq n$. Therefore, everything we stated for 1-forms will also be true for $n-1$ forms.

In general, the computation of the k-form spectrum of the Laplacian is a significantly more difficult task compared to the function case, due to the increased complexity in obtaining and controlling approximate eigenforms. Previous attempts to compute the form and function spectrum using the classical Weyl criterion or an L^p independence result required very strong assumptions on the decay of the curvature to show that the spectrum is $[0, \infty)$ (see for example [3,13,15–18,30,34,35]). For example, the only cases where the k-form spectrum had previously been computed to be $[0, \infty)$ were either manifolds in warped product form, or manifolds whose normal bundle map over the soul is a diffeomorphism. In addition, a strong decay assumption on both the radial and nonradial components of curvature had to be made [3,18]. Theorem 4.7 demonstrates the strength of the analytic result Theorem 4.1 as it allows us to compute the 1-form and $n-1$-form spectra under the most general conditions thus far possible.

On the other hand, even though the 1-form spectrum essentially contains the function spectrum, there is no monotonicity for the k-form spectrum with respect to k in general. In the case of hyperbolic space we have *unimodality*, which means that the spectrum is increasing for $k \le n/2$ and decreasing for other k:

Example 4.8. The essential spectrum of the Laplacian on forms over hyperbolic space $\mathbb{H}^{N+1}$ is given by

$$\sigma_{\mathrm{ess}}(k, \Delta) = \sigma_{\mathrm{ess}}(N+1-k, \Delta) = [\,(\tfrac{N}{2} - k)^2, \infty\,)$$

for $0 \le k \le \frac{N}{2}$, and whenever N is odd

$$\sigma_{\mathrm{ess}}(\tfrac{N+1}{2}, \Delta) = \{0\} \cup [\,\tfrac{1}{4}, \infty\,).$$

A proof of this result can be found in Donnelly [14]. Mazzeo and Phillips show in [24] that the same result is true over quotients of hyperbolic space, $\mathbb{H}^{N+1}/\Gamma$, that are geometrically finite and have infinite volume.

However, one cannot expect a unimodality result on every manifold as we can see from the following example:

Example 4.9. Consider the product manifold $M^4 = F^3 \times \mathbb{R}$, where F^3 is the compact flat three-manifold constructed by Hantzsche and Wendt in 1935 with first Betti number zero (see [9] for a family of manifolds of any dimension $n \ge 3$ with the same property). Note that M is a flat noncompact manifold. By Theorem 4.1 and Lemma 4.10

$$\sigma_{\mathrm{ess}}(k, \Delta) = [0, \infty) \qquad \text{for} \quad k = 0, 1, 3, 4.$$

However, since there do not exist any harmonic 1-forms nor harmonic 2-forms on F then

$$\sigma_{\mathrm{ess}}(2, \Delta) = [a, \infty) \qquad \text{for some} \quad a > 0.$$

In other words, its essential spectrum is smaller in half-dimension. Note that this does not contradict the result of Theorem 4.1.

Define

$$\alpha(B, s, n, k) = \begin{cases} 0 & s > n/2, \\ 0 & s \le n/2 \text{ and } 0 \le k \le s, n-s \le k \le n, \\ \min\{\lambda_B(l) \mid 1 \le l \le k\}, & s+1 \le k \le n-s-1. \end{cases}$$

Lemma 4.10. *Let $M^n = B^{n-s} \times \mathbb{R}^s$ as above. Let $0 \le k \le n$. Then $\sigma_{\mathrm{pt}}(k, \Delta, M) = \emptyset$, and*

$$\sigma(k, \Delta, M) = \sigma_{\mathrm{ess}}(k, \Delta, M) = [\alpha(B, s, n, k), \infty).$$

The above simple example of a product space gives us a large family of manifolds where the spectrum of the Laplacian on forms is a connected interval. In [5,6] we give even larger classes of manifolds where the form essential spectrum is a connected set. In general however, the essential spectrum can have gaps (see for example [1] where they prove the existence of an arbitrary number of gaps in the essential spectrum over cyclic coverings). There are various known examples where gaps occur for the essential spectrum of the Laplacian on functions [25,28,31,33]. Corollary 4.4 tells us that the essential spectrum on 1-forms over these latter manifolds could be larger, but it is not known whether it would also have gaps. It would be quite interesting to find sufficient conditions on the geometry of the manifold so that its essential spectrum is a connected subset of the real line.

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