

Analysis of the Laplacian on the moduli space of polarized Calabi-Yau manifolds

Zhiqin Lu and Hang Xu

ABSTRACT. In this paper, we generalize the spectrum relation in the paper *On the spectrum of the Laplacian, Math. Ann.*, 359(1-2):211–238, 2014 (by Nelia Charalambous and Zhiqin Lu) to any Hermitian manifolds. We also prove that the closure of Laplace operator $\square = \delta d$ on the moduli space of polarized Calabi-Yau manifolds is self-adjoint.

1. Introduction

Let (M, g) be a Hermitian manifold with a holomorphic vector bundle (E, h) . Suppose $\square$ is the Hodge Laplacian on smooth E -valued (p, q) forms. Though $\square$ in general is only symmetric but not self-adjoint, one can consider self-adjoint extensions of the Hodge Laplacian. One well-known self-adjoint extension is the so-called Gaffney extension $\square_G$ ([5]). In this note, we generalize the spectrum relations in [2] to the Gaffney extension on incomplete manifolds. One key ingredient for the spectrum relations is a generalized version of the Weyl's criterion.

Another well-know extension of $\square$ is the Friedrichs extension $\square_F$. $\square_G$ and $\square_F$ are in general different on incomplete manifolds. In the special case of the moduli space of polarized Calabi-Yau manifolds $\mathcal{M}$ with the Weil-Petersson metric ω_{WF} , we prove the Cauchy boundary of $\mathcal{M}$ has zero capacity, and therefore $\square_G = \square_F$ on functions. Furthermore, we also show that the Hodge Laplacian on functions with certain $\text{Dom}\square$ is essentially self-adjoint, which is a generalization of the results in [6] and [10].

Using the spectrum results we obtain on different self-adjoint extensions of the Laplacians, we study the L^2 -estimates on incomplete manifolds. The L^2 -estimate played one of the most crucial roles in several complex variables and complex geometry. The method allows us to construct a lot of holomorphic functions and holomorphic sections in various function spaces.

One of the most important applications of the L^2 -estimate is the proof of Kodaira's embedding theorem. Let L be a positive line bundle over a compact complex manifold X . Then there exists a positive integer k such that the line bundle $L^k = L \otimes \cdots \otimes L$ has a lot of (ample) holomorphic sections.

2010 *Mathematics Subject Classification.* Primary 32Q15, 32Q25; Secondary 53C55.
The first author is partially supported by the NSF grant DMS-1547878.
Part of the paper is a refinement of the thesis of the second author.

In this paper, we study the case when X is *not* a complete complex manifold. As it is well known, on an incomplete manifold, the extension of the Laplacian as a self-adjoint operator is not unique. So we need to specify the extension. Secondly, the L^2 estimates heavily depends on the spectrum gap on the bundle-valued $(0, 1)$ forms. Therefore, it is useful to generalize the results in [2] to the incomplete case.

The main result of this paper is in §5, where we re-prove the results of Masamune [10, 11]. We found a gap in his proof and we showed this gap by a counter-example in §8.

In §7 and §8, we apply our results to the moduli space of Calabi-Yau manifolds.

Acknowledgments. We are very thankful to the anonymous referees for many helpful comments.

2. Two Self-Adjoint Extensions of Hodge Laplacian

In this section, we assume (M, g) is a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h) . Consider Hodge Laplacian on E -valued (p, q) forms with compact support. As the Hodge Laplacian is symmetric but not self-adjoint, we consider the self-adjoint extensions of the Hodge Laplacian via the corresponding closed quadratic forms. By endowing the quadratic form with different domain of definition, we will get two important self-adjoint extensions, which are respectively Gaffney extension and Friedrichs extension. For more details about this section, we recommend references [9, 15].

We begin with the d-bar differential operator

$$\bar{\partial}_{p,q} : L^2(M, \Lambda^{p,q}(E)) \rightarrow L^2(M, \Lambda^{p,q+1}(E)),$$

with

$$Dom(\bar{\partial}_{p,q})$$

$$= \{\varphi \in L^2(M, \Lambda^{p,q}(E)) : \text{the distributional derivative } \bar{\partial}\varphi \in L^2(M, \Lambda^{p,q+1}(E))\}.$$

With the above domain of definition, the operator $\bar{\partial}_{p,q}$ is a densely defined closed operator. We denote the L^2 inner product on $L^2(M, \Lambda^{p,q}(E))$ as $(\cdot, \cdot)_{p,q}$. With respect to the L^2 inner product on $L^2(M, \Lambda^{p,q}(E))$ and $L^2(M, \Lambda^{p,q+1}(E))$, we have the adjoint operator of $\bar{\partial}_{p,q}$ as

$$\bar{\partial}_{p,q+1}^* : L^2(M, \Lambda^{p,q+1}(E)) \rightarrow L^2(M, \Lambda^{p,q}(E)),$$

with

$$Dom(\bar{\partial}_{p,q+1}^*) = \{\phi \in L^2(M, \Lambda^{p,q+1}(E)) : \exists \varphi \in L^2(M, \Lambda^{p,q}(E)) \text{ such that}$$

$$(\bar{\partial}u, \phi)_{p,q+1} = (u, \varphi)_{p,q} \text{ for any } u \in Dom(\bar{\partial}_{p,q})\}.$$

And in the above notation, $\bar{\partial}^*\phi$ is defined to be φ .

In the following, we will suppress the indices p, q in the operators and inner product for simplicity when there is no confusion from context.

Now let us recall Hodge Laplacian and the associated quadratic form. We use the notation $\mathcal{D}(M, \Lambda^{p,q}(E))$ to denote the set of all smooth E -valued (p, q) forms with compact support.

DEFINITION 2.1. i) Let $\square : \mathcal{D}(M, \Lambda^{p,q}(E)) \rightarrow \mathcal{D}(M, \Lambda^{p,q}(E))$ be the Hodge Laplacian defined as

$$\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

ii) Let $Q : \mathcal{D}(M, \Lambda^{p,q}(E)) \times \mathcal{D}(M, \Lambda^{p,q}(E)) \rightarrow \mathbb{C}$ be the quadratic form associated to $\square$ defined as

$$Q(\varphi, \phi) = (\bar{\partial}\varphi, \bar{\partial}\phi) + (\bar{\partial}^*\varphi, \bar{\partial}^*\phi) \text{ for any } \varphi, \phi \in \mathcal{D}(M, \Lambda^{p,q}(E)).$$

Since $\bar{\partial}, \bar{\partial}^*$ are closed operators, if we endow quadratic form Q with $\text{Dom}(Q) = \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$, then Q is closed. That means, for any sequence $\varphi_n \in \text{Dom}(Q)$, $\varphi_n \xrightarrow{L^2} \varphi$ and $Q(\varphi_m - \varphi_n, \varphi_m - \varphi_n) \rightarrow 0$ as $m, n \rightarrow \infty$, then $\varphi \in \text{Dom}(Q)$ and $Q(\varphi_n - \varphi, \varphi_n - \varphi) \rightarrow 0$.

We cite the following theorem from [14] in Chapter VIII.6.

THEOREM 2.1 ([14]). *If Q is a closed semibounded quadratic form, then Q is the quadratic form of a unique self-adjoint operator.*

By applying this theorem to our quadratic form Q with $\text{Dom}(Q) = \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \subset L^2(M, \Lambda^{p,q}(E))$, we get a self-adjoint extension of $\square$, which is called Gaffney extension and denoted as $\square_G$. The domain of $\square_G$ is

$$(2.1) \quad \begin{aligned} \text{Dom}(\square_G) &= \{\varphi \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) : \exists \eta \in L^2(M, \Lambda^{p,q}(E)) \text{ such that} \\ &\quad Q(\varphi, \phi) = (\eta, \phi) \text{ for any } \phi \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)\}. \end{aligned}$$

And in the same notation as above, $\square_G \varphi$ is defined to be η .

The following Gaffney's Theorem from [5] (See also chapter 3 in [9]) tells us that Gaffney extension can be viewed as the composition of $\bar{\partial}$ and $\bar{\partial}^*$ as follows.

THEOREM 2.2 (Gaffney).

$$(2.2) \quad \text{Dom}(\square_G) = \{\varphi \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) : \bar{\partial}\varphi \in \text{Dom}(\bar{\partial}^*) \text{ and } \bar{\partial}^*\varphi \in \text{Dom}(\bar{\partial})\}.$$

And for any $\varphi \in \text{Dom}(\square_G)$, we have

$$\square_G \varphi = \bar{\partial}\bar{\partial}^*\varphi + \bar{\partial}^*\bar{\partial}\varphi.$$

Similarly, we will introduce Friedrichs extension by endowing Q with a different domain of definition. Let's first recall the following Sobolev spaces. We denote $Q_1(\cdot, \cdot) = Q(\cdot, \cdot) + (\cdot, \cdot)$. It is not hard to see Q_1 is an inner product on $\mathcal{D}(M, \Lambda^{p,q}(E))$.

DEFINITION 2.2 (Sobolev Spaces).

$$(2.3) \quad \begin{aligned} W_0^1(M, \Lambda^{p,q}(E)) &= \text{Completion of } \mathcal{D}(M, \Lambda^{p,q}(E)) \\ &\quad \text{with respect to } Q_1 \text{ inner product,} \end{aligned}$$

$$(2.4) \quad \begin{aligned} W^1(M, \Lambda^{p,q}(E)) &= \text{Completion of } \{\varphi \in \mathcal{C}^\infty(M, \Lambda^{p,q}(E)) : Q_1(\varphi, \varphi) < \infty\} \\ &\quad \text{with respect to } Q_1 \text{ inner product.} \end{aligned}$$

REMARK 2.3. Note that φ is not necessarily in $\text{Dom}(\bar{\partial}_{p,q}^*)$ when $\varphi \in \mathcal{C}^\infty(M, \Lambda^{p,q}(E))$. So in the definition of $W^1(M, \Lambda^{p,q}(E))$, to be precise, $Q_1(\varphi, \varphi) < \infty$ means $\varphi \in L^2(M, \Lambda^{p,q}(E))$ and the point-wise differentials $\bar{\partial}\varphi, \bar{\partial}^*\varphi$ belong to $L^2(M, \Lambda^{p,q+1}(E))$ and $L^2(M, \Lambda^{p,q-1}(E))$ respectively. And one can prove $\varphi \in$

$W^1(M, \Lambda^{p,q}(E))$ if and only if $\varphi \in L^2(M, \Lambda^{p,q}(E))$ and the distributional differentials $\bar{\partial}\varphi, \bar{\partial}^*\varphi$ belong to $L^2(M, \Lambda^{p,q+1}(E))$ and $L^2(M, \Lambda^{p,q-1}(E))$ respectively.

REMARK 2.4. Note that $W_0^1 \subset \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \subset W^1$. But they are generally not equal to each other.

If we endow Q with $\text{Dom}(Q) = W_0^1$, then it becomes a closed quadratic form. By applying Theorem 2.1 again, we will get a different self-adjoint extension of Hodge Laplacian $\square$, which is called Friedrichs extension and denoted as $\square_F$. Note that $\square_F$ is generally different from $\square_G$ by Remark 2.4.

EXAMPLE 2.5. Take the Hermitian manifold $M = \Omega \subset \mathbb{C}^n$ be a bounded open set with smooth boundary. Let Hermitian vector bundle E be the trivial line bundle. Assume $u \in C^\infty(\bar{\Omega}, \Lambda^{p,q})$. Let us investigate the boundary conditions induced from $\square_G$ and $\square_F$ in this case.

If $u \in \text{Dom}(\bar{\partial}^*)$, then

$$(\bar{\partial}\varphi, u) = (\varphi, \bar{\partial}^*u) \text{ for any } \varphi \in C^\infty(\bar{\Omega}, \Lambda^{p,q-1}).$$

Note

$$(\bar{\partial}\varphi, u) = \int_{\Omega} \bar{\partial}\varphi \wedge * \bar{u} = \int_{\partial\Omega} \varphi \wedge * \bar{u} + (-1)^{p+q} \int_{\Omega} \varphi \wedge \bar{\partial} * \bar{u} = \int_{\partial\Omega} \varphi \wedge * \bar{u} + (\varphi, \bar{\partial}^*u).$$

Here $*$ is the Hodge star operator. The second equality follows from Stokes Theorem and the last one is based on the identity $\bar{\partial}^* = - * \bar{\partial} *$. Therefore we have

$$\int_{\partial\Omega} \varphi \wedge * \bar{u} = 0 \text{ for any } \varphi \in C^\infty(\bar{\Omega}, \Lambda^{p,q-1}).$$

It implies $*u|_{\partial\Omega} = 0$ (the restriction of $*u$ to $\partial\Omega$). So by Theorem 2.2, $u \in \text{Dom}(\square_G)$ implies the boundary condition $*u|_{\partial\Omega} = 0$ and $*\bar{\partial}u|_{\partial\Omega} = 0$.

For the Friedrichs extension, $u \in \text{Dom}(\square_F)$ implies $u \in W_0^1$. Then there exists a sequence $u_j \in \mathcal{D}(M, \Lambda^{p,q})$ such that $u_n \rightarrow u$ in W_0^1 . By Weitzenböck formula, we have

$$(2.5) \quad \square u_j = - \sum_{i=1}^n \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^i} u_j = - \sum_{i=1}^n \frac{\partial}{\partial \bar{z}^i} \frac{\partial}{\partial z^i} u_j.$$

Therefore by taking the inner product with u_j ,

$$(2.6) \quad Q(u_j, u_j) = \sum_{i=1}^n \left(\frac{\partial}{\partial z^i} u_j, \frac{\partial}{\partial \bar{z}^i} u_j \right) = \sum_{i=1}^n \left(\frac{\partial}{\partial \bar{z}^i} u_j, \frac{\partial}{\partial z^i} u_j \right).$$

Then we have

$$(2.7) \quad \frac{\partial}{\partial \bar{z}^i} u_j \rightarrow \frac{\partial}{\partial \bar{z}^i} u \quad \text{and} \quad \frac{\partial}{\partial z^i} u_j \rightarrow \frac{\partial}{\partial z^i} u \quad \text{in } L^2 \text{ norms for each } i.$$

If we write $u = u_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}$, then each function $u_{I\bar{J}}$ is in the standard Sobolev space $H_0^1(\Omega)$, which implies $u_{I\bar{J}}|_{\partial\Omega} = 0$ for each multi-index I, J .

3. Spectrums of Gaffney Extension

The main goal of this section is to prove the following spectrum relations of Gaffney extension.

THEOREM 3.1. *Let (M, g) be a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h) . Consider Gaffney extension of Hodge Laplacian, $\square_{p,q} : L^2(M, \Lambda^{p,q}(E)) \rightarrow L^2(M, \Lambda^{p,q}(E))$. We have the following spectrum relations.*

$$(3.1) \quad \text{Spec}(\square_{p,q}) \cup \{0\} = \text{Spec}(\bar{\partial}\bar{\partial}_{p,q+1}^*) \cup \text{Spec}(\bar{\partial}^*\bar{\partial}_{p,q-1}) \cup \{0\}.$$

$$(3.2) \quad \text{Spec}(\square_{p,q}) \cup \{0\} = \text{Spec}(\bar{\partial}\bar{\partial}_{p,q}^*) \cup \text{Spec}(\bar{\partial}^*\bar{\partial}_{p,q}) \cup \{0\}.$$

REMARK 3.1. The above notation $\bar{\partial}\bar{\partial}_{p,q}^*$ means $\bar{\partial}_{p,q-1}\bar{\partial}_{p,q}^*$ and $\bar{\partial}^*\bar{\partial}_{p,q}$ means $\bar{\partial}_{p,q+1}^*\bar{\partial}_{p,q}$. Note that $\bar{\partial}\bar{\partial}_{p,q}^*$ and $\bar{\partial}^*\bar{\partial}_{p,q}$ are self-adjoint operators by Von Neumann's Theorem (see Chapter X in [13]) since both $\bar{\partial}_{p,q}$ and $\bar{\partial}_{p,q}^*$ are densely defined closed operators. In the following we will omit the sub-indices p, q when there is not confusion from context.

This is a generalization of results in [2], where similar spectrum relations were proved for complete Riemannian manifolds. One main tool we are going to use is the generalized Weyl criterion from [2]. The advantage of this generalized Weyl criterion is that we do not necessarily pick the test sequence from the domain of an unbounded operator. After proving it, we will mention a well known relation between Gaffney extension and L^2 estimates, which serves a preparation for later sections.

We will split the proof of Theorem 3.1 into to several Lemmas. First, we prove one containment relation of (3.1).

LEMMA 3.2. *Under the same assumption as Theorem 3.1, we have*

$$(3.3) \quad \text{Spec}(\square_{p,q}) \subset \text{Spec}(\bar{\partial}\bar{\partial}_{p,q+1}^*) \cup \text{Spec}(\bar{\partial}^*\bar{\partial}_{p,q-1}) \cup \{0\}.$$

PROOF. In this proof, we will use $\square$ to represent $\square_{p,q}$ for simplicity. Take $\lambda_0 \in \text{Spec}(\square)$ and $\lambda_0 > 0$. By Weyl's criterion, there exists a sequence $u_j \in \text{Dom}(\square)$ with $\langle u_j, u_j \rangle = 1$ such that

$$(\square - \lambda_0)u_j \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Since $\square$ is non-negative and self-adjoint, $(1 + \square)^{-1} : L^2(M, \Lambda^{p,q}(E)) \rightarrow \text{Dom}(\square_{p,q}) \subset L^2(M, \Lambda^{p,q}(E))$ is a bounded operator. By identity (2.1), we have

$$(3.4) \quad Q((1 + \square)^{-2}u_j, (1 + \square)^{-2}u_j) = (\square(1 + \square)^{-2}u_j, (1 + \square)^{-2}u_j).$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of $\square$. Then

$$(3.5) \quad (\square(1 + \square)^{-2}u_j, (1 + \square)^{-2}u_j) = \int_0^\infty \frac{\lambda}{(1 + \lambda)^4} d(P_\lambda u_j, u_j).$$

Take $C(\lambda_0) = \min_{\lambda \in [\frac{\lambda_0}{2}, \frac{3\lambda_0}{2}]} \frac{\lambda}{(1 + \lambda)^4} > 0$. Then

$$(3.6) \quad \int_0^\infty \frac{\lambda}{(1 + \lambda)^4} d(P_\lambda u_j, u_j) \geq C(\lambda_0) \int_{\frac{1}{2}\lambda_0}^{\frac{3}{2}\lambda_0} d(P_\lambda u_j, u_j) \geq C(\lambda_0) \|P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)} u_j\|^2.$$

We denote $u_j^{(1)} = P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)} u_j$ and $u_j^{(2)} = u_j - u_j^{(1)}$. By using the Projection Valued Measure again, we have

$$((\square - \lambda_0)u_j, (\square - \lambda_0)u_j) = \int_0^\infty (\lambda - \lambda_0)^2 d(P_\lambda u_j, u_j) \geq \frac{\lambda_0^2}{4} \|u_j^{(2)}\|^2.$$

Since we know $(\square - \lambda_0)u_j \rightarrow 0$ as j goes to infinity, we have

$$\|u_j^{(2)}\| \rightarrow 0 \text{ as } j \rightarrow \infty,$$

whence

$$\|u_j^{(1)}\| \rightarrow 1 \text{ as } j \rightarrow \infty.$$

Together with (3.4), (3.5) and (3.6), we have for sufficiently large j

$$(3.7) \quad \|\bar{\partial}(1 + \square)^{-2}u_j\|^2 + \|\bar{\partial}^*(1 + \square)^{-2}u_j\|^2 \geq \frac{C(\lambda_0)}{2} > 0.$$

On the other hand, we have

$$\begin{aligned} & \|(\bar{\partial}\bar{\partial}^* - \lambda_0)\bar{\partial}(1 + \square)^{-2}u_j\|^2 + \|(\bar{\partial}^*\bar{\partial} - \lambda_0)\bar{\partial}(1 + \square)^{-2}u_j\|^2 \\ &= \|\bar{\partial}(\square - \lambda_0)(1 + \square)^{-2}u_j\|^2 + \|\bar{\partial}^*(\square - \lambda_0)(1 + \square)^{-2}u_j\|^2 \\ &= (\square(1 + \square)^{-2}(\square - \lambda_0)u_j, (1 + \square)^{-2}(\square - \lambda_0)u_j) \\ &\leq \|(\square - \lambda_0)u_j\|^2. \end{aligned}$$

The first equality is because $\bar{\partial} \circ \bar{\partial} = 0$ on $\text{Dom}(\bar{\partial})$ and $\bar{\partial}^* \circ \bar{\partial}^* = 0$ on $\text{Dom}(\bar{\partial}^*)$. The second one follows from (2.1) and the commutativity of $\square$ and $(1 + \square)^{-1}$. And the last inequality follows from $\|(1 + \square)^{-1}\|_{L^2 \rightarrow L^2} \leq 1$ and $\|\square(1 + \square)^{-1}\|_{L^2 \rightarrow L^2} \leq 1$. Therefore

$$(3.8) \quad \|(\bar{\partial}\bar{\partial}^* - \lambda_0)\bar{\partial}(1 + \square)^{-2}u_j\|^2 + \|(\bar{\partial}^*\bar{\partial} - \lambda_0)\bar{\partial}(1 + \square)^{-2}u_j\|^2 \rightarrow 0.$$

Combining (3.7) and (3.8), we have $\lambda_0 \in \text{Spec}(\bar{\partial}\bar{\partial}^*_{p,q+1}) \cup \text{Spec}(\bar{\partial}^*\bar{\partial}_{p,q-1})$ by Weyl criterion. So the result follows. $\square$

Now we prove the other containment of (3.1).

LEMMA 3.3. *Under the same assumption as Theorem 3.1, we have*

$$\text{Spec}(\bar{\partial}\bar{\partial}^*_{p,q+1}) \cup \text{Spec}(\bar{\partial}^*\bar{\partial}_{p,q-1}) \subset \text{Spec}(\square_{p,q}) \cup \{0\}.$$

In order to prove this lemma, we will use one generalized Weyl criterion from [2].

THEOREM 3.2 (Charalambous-Lu). *Let H be a non-negative self-adjoint operator on Hilbert space $\mathcal{H}$. A positive real number λ_0 is contained in $\text{Spec}(H)$ if there exists a sequence $u_j \in \mathcal{H}$ such that*

- (1) *For any j , $\|u_j\| = 1$.*
- (2) *$((H - \lambda_0)(1 + H))^{-m}u_j, u_j \rightarrow 0$ for $m = 1, 2$.*

Note that compared to the classical Weyl criterion, the above theorem does not require $u_j \in \text{Dom}(H)$. We give a proof of this theorem here for the completeness.

PROOF. Note that

$$(H - \lambda_0)^2(1 + H)^{-2} = (H - \lambda_0)(1 + H)^{-1} - (\lambda_0 + 1)(H - \lambda_0)(1 + H)^{-2}.$$

The assumptions imply that

$$(3.9) \quad ((H - \lambda_0)^2(1 + H)^{-2}u_j, u_j) \rightarrow 0.$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of H . Then

$$(3.10) \quad ((H - \lambda_0)^2(1 + H)^{-2}u_j, u_j) = \int_0^\infty \frac{(\lambda - \lambda_0)^2}{(1 + \lambda)^2} d(P_\lambda u_j, u_j).$$

Define $u_j^{(1)} = P_{(\lambda_0 - \varepsilon_j, \lambda_0 + \varepsilon_j)} u_j$ and $u_j^{(2)} = u_j - u_j^{(1)}$. The constants $\varepsilon_j \in (0, \frac{\lambda_0}{2})$ are to be selected later. Note the integrand $\frac{(\lambda - \lambda_0)^2}{(1 + \lambda)^2}$ in (3.10) has the following lower bound for $\lambda \notin (\lambda_0 - \varepsilon_j, \lambda_0 + \varepsilon_j)$.

$$(3.11) \quad \frac{(\lambda - \lambda_0)^2}{(1 + \lambda)^2} \geq \min \left(\frac{\varepsilon_j^2}{(1 + \lambda_0 - \varepsilon_j)^2}, \frac{\varepsilon_j^2}{(1 + \lambda_0 + \varepsilon_j)^2} \right) \geq \frac{\varepsilon_j^2}{(1 + \frac{3}{2}\lambda_0)^2}.$$

Therefore

$$(3.12) \quad ((H - \lambda_0)^2(1 + H)^{-2} u_j, u_j) \geq \frac{\varepsilon_j^2}{(1 + \frac{3}{2}\lambda_0)^2} \|u_j^{(2)}\|^2.$$

Choose a sequence $\varepsilon_j \in (0, \frac{\lambda_0}{2})$ such that

- i) $\varepsilon_j \rightarrow 0$.
- ii) $((H - \lambda_0)^2(1 + H)^{-2} u_j, u_j) / \varepsilon_j^2 \rightarrow 0$.

For example, we can take $\varepsilon_j = ((H - \lambda_0)^2(1 + H)^{-2} u_j, u_j)^{\frac{1}{3}}$. Therefore (3.12) implies

$$\|u_j^{(2)}\| \rightarrow 0 \text{ as } j \rightarrow \infty,$$

whence

$$(3.13) \quad \|u_j^{(1)}\| \rightarrow 1 \text{ as } j \rightarrow \infty.$$

On the other hand, as

$$\int_0^\infty \lambda^2 d(P_\lambda u_j^{(1)}, u_j^{(1)}) \leq (\lambda_0 + \varepsilon_j)^2 \|u_j\|^2 < \infty,$$

the sequence $u_j^{(1)} \in \text{Dom}(H)$. So we can apply the classical Weyl Criterion to the sequence $u_j^{(1)}$. By Projection Valued Measure again,

$$(3.14) \quad \|(H - \lambda_0)u_j^{(1)}\|^2 = \int_0^\infty (\lambda - \lambda_0)^2 d(P_\lambda u_j^{(1)}, u_j^{(1)}) \leq \varepsilon_j^2 \rightarrow 0,$$

which implies $\lambda_0 \in \text{Spec}(H)$. So the result follows. $\square$

REMARK 3.4. Note that the condition (2) in the theorem can be weakened to $((H - \lambda_0)^2(1 + H)^{-2} u_j, u_j) \rightarrow 0$ by the proof.

REMARK 3.5. The above theorem also holds for $\lambda_0 = 0$. And in fact we can also prove conditions (1) and (2) are not only sufficient but also necessary for $\lambda_0 \in \text{Spec}(H)$. More details can be found in [2].

With the generalized Weyl criterion 3.2, we are ready to prove Lemma 3.3.

PROOF. Here we prove $\text{Spec}(\bar{\partial}\bar{\partial}_{p,q+1}^*) \subset \text{Spec}(\square_{p,q}) \cup \{0\}$. The other containment $\text{Spec}(\bar{\partial}^*\bar{\partial}_{p,q-1}) \subset \text{Spec}(\square_{p,q}) \cup \{0\}$ can be proved similarly.

Take $\lambda_0 \in \text{Spec}(\bar{\partial}\bar{\partial}^*)$ and $\lambda_0 > 0$. By classical Weyl criterion, there exists a sequence $u_j \in \text{Dom}(\bar{\partial}\bar{\partial}^*)$ with $(u_j, u_j) = 1$ such that

$$(3.15) \quad ((\bar{\partial}\bar{\partial}^* - \lambda_0)u_j, (\bar{\partial}\bar{\partial}^* - \lambda_0)u_j) \rightarrow 0.$$

We will verify that the sequence $\bar{\partial}^* u_j$ satisfies conditions in Theorem 3.2. For $m = 1, 2$,

$$\begin{aligned} & ((\square - \lambda_0)(1 + \square)^{-m} \bar{\partial}^* u_j, \bar{\partial}^* u_j) \\ &= ((\square - \lambda_0)(1 + \square)^{-m} u_j, \bar{\partial} \bar{\partial}^* u_j) \\ &= ((\bar{\partial} \bar{\partial}^* - \lambda_0)(1 + \square)^{-m} u_j, \bar{\partial} \bar{\partial}^* u_j) \\ &= (\bar{\partial} \bar{\partial}^* (1 + \square)^{-m} u_j, (\bar{\partial} \bar{\partial}^* - \lambda_0) u_j). \end{aligned}$$

The first equality is because $(1 + \square)^{-1} \bar{\partial}^* = \bar{\partial}^* (1 + \square)^{-1}$ on $\text{Dom}(\bar{\partial}^*)$, which follows from Theorem 2.2. The second one follows from $\bar{\partial} \circ \bar{\partial} = 0$ on $\text{Dom}(\bar{\partial})$. The third one comes from the self-adjointness of $\bar{\partial} \bar{\partial}^*$ and straightforward calculations. Since

$$(3.16) \quad \|\bar{\partial} \bar{\partial}^* (1 + \square)^{-m} u_j\| \leq \|\square (1 + \square)^{-m} u_j\| \leq \|u_j\| = 1,$$

(3.15) implies

$$(3.17) \quad ((\square - \lambda_0)(1 + \square)^{-m} \bar{\partial}^* u_j, \bar{\partial}^* u_j) \rightarrow 0 \text{ for } m = 1, 2.$$

The other thing we need to verify is that $\|\bar{\partial}^* u_j\|$ has a positive lower bound uniformly for all j . This is from the following calculations:

$$(3.18) \quad (\bar{\partial}^* u_j, \bar{\partial}^* u_j) = ((\bar{\partial} \bar{\partial}^* - \lambda_0) u_j, u_j) + \lambda_0 \rightarrow \lambda_0 > 0.$$

Since $\|\bar{\partial}^* u_j\|$ has a uniform lower bound, we can apply Theorem 3.2 to the scaled sequence $\bar{\partial}^* u_j / \|\bar{\partial}^* u_j\|$ and the result follows immediately. $\square$

Now we are going to finish the proof of Theorem 3.1 in next lemma.

LEMMA 3.6. *Under the same assumption as Theorem 3.1, we have*

$$\text{Spec}(\square_{p,q}) \subset \text{Spec}(\bar{\partial} \bar{\partial}^*_{p,q}) \cup \text{Spec}(\bar{\partial}^* \bar{\partial}_{p,q}) \cup \{0\}.$$

PROOF. Take $\lambda_0 \in \text{Spec}(\square)$ and $\lambda_0 > 0$. Then by classical Weyl criterion, there exists a sequence $u_j \in \text{Dom}(\square)$ with $\|u_j\| = 1$ such that

$$(3.19) \quad (\square - \lambda_0) u_j \rightarrow 0.$$

We will use $\bar{\partial} \bar{\partial}^* (1 + \square)^{-2} u_j$ and $\bar{\partial}^* \bar{\partial} (1 + \square)^{-2} u_j$ as the test sequences. By the fact that $\bar{\partial} \circ \bar{\partial} = 0$ on $\text{Dom}(\bar{\partial})$ and $(1 + \square)^{-1} \square = \square (1 + \square)^{-1}$ on $\text{Dom}(\square)$, we have

$$(3.20) \quad (\bar{\partial} \bar{\partial}^* - \lambda_0) \bar{\partial} \bar{\partial}^* (1 + \square)^{-2} u_j = \bar{\partial} \bar{\partial}^* (1 + \square)^{-2} (\square - \lambda_0) u_j.$$

Since $\|\bar{\partial} \bar{\partial}^* (1 + \square)^{-2}\|_{L^2 \rightarrow L^2} \leq 1$, it implies

$$(3.21) \quad \|(\bar{\partial} \bar{\partial}^* - \lambda_0) \bar{\partial} \bar{\partial}^* (1 + \square)^{-2} u_j\| \leq \|(\square - \lambda_0) u_j\| \rightarrow 0.$$

Similarly, we also have

$$(3.22) \quad \|(\bar{\partial}^* \bar{\partial} - \lambda_0) \bar{\partial}^* \bar{\partial} (1 + \square)^{-2} u_j\| \leq \|(\square - \lambda_0) u_j\| \rightarrow 0.$$

Now we need to check either $\|\bar{\partial} \bar{\partial}^* (1 + \square)^{-2} u_j\|$ or $\|\bar{\partial}^* \bar{\partial} (1 + \square)^{-2} u_j\|$ has a positive lower bound. Note

$$(3.23) \quad \|\bar{\partial} \bar{\partial}^* (1 + \square)^{-2} u_j\|^2 + \|\bar{\partial}^* \bar{\partial} (1 + \square)^{-2} u_j\|^2 = \|\square (1 + \square)^{-2} u_j\|^2.$$

Let $\{P_\lambda\}$ be the Projection Valued Measure of $\square$. Then

$$(3.24) \quad \|\square (1 + \square)^{-2} u_j\|^2 = \int_0^\infty \frac{\lambda^2}{(1 + \lambda)^4} d(P_\lambda u_j, u_j) \geq C(\lambda_0) \|P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)} u_j\|^2.$$

Note $(\square - \lambda_0)u_j \rightarrow 0$ implies

$$(3.25) \quad \|P_{(\frac{1}{2}\lambda_0, \frac{3}{2}\lambda_0)}u_j\| \rightarrow 1.$$

Therefore for sufficiently large j ,

$$(3.26) \quad \|\bar{\partial}\bar{\partial}^*(1 + \square)^{-2}u_j\|^2 + \|\bar{\partial}^*\bar{\partial}(1 + \square)^{-2}u_j\|^2 \geq \frac{C(\lambda_0)}{2} > 0.$$

So $\lambda_0 \in \text{Spec}(\bar{\partial}\bar{\partial}^*_{p,q}) \cup \text{Spec}(\bar{\partial}^*\bar{\partial}_{p,q})$ by classical Weyl criterion and the result follows. $\square$

One direct corollary from Theorem 3.1 is the following spectrum relations of Gaffney extensions.

COROLLARY 3.7. *Under the same assumption as Theorem 3.1, we have*

$$(3.27) \quad \text{Spec}(\square_{p,q}) \subset \text{Spec}(\square_{p,q+1}) \cup \text{Spec}(\square_{p,q-1}) \cup \{0\}.$$

At the end of this section, let us recall the well known relation between the spectrum of Gaffney extension and L^2 estimates.

THEOREM 3.3. *Let (M, g) be a Hermitian manifold with a holomorphic Hermitian vector bundle (E, h) . Assume the Gaffney extension of Hodge Laplacian $\square_{p,q+1} : L^2(M, \Lambda^{p,q+1}(E)) \rightarrow L^2(M, \Lambda^{p,q+1}(E))$ satisfies $\text{Spec}(\square_{p,q+1}) \subset [a, \infty)$ for some positive number a . Then for any $f \in \ker \bar{\partial}_{p,q+1} \subset L^2(M, \Lambda^{p,q+1}(E))$, there exists $u \in L^2(M, \Lambda^{p,q}(E))$ such that $\bar{\partial}u = f$ with the following estimate*

$$(3.28) \quad (u, u) \leq \frac{1}{a}(f, f).$$

PROOF. In the proof, we will use $\square$ to represent $\square_{p,q+1}$ for simplicity. By the condition $\text{Spec} \square \subset [a, \infty)$, we have $\square^{-1} : L^2(M, \Lambda^{p,q+1}(E)) \rightarrow \text{Dom}(\square) \subset L^2(M, \Lambda^{p,q+1}(E))$ is a bounded operator with

$$(3.29) \quad \|\square^{-1}\|_{L^2 \rightarrow L^2} \leq \frac{1}{a}.$$

Take $u = \bar{\partial}^*\square^{-1}f$ and we will verify u satisfies all the conclusions. First, since the Gaffney extension satisfies $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ by Theorem 2.2, we have

$$(3.30) \quad \bar{\partial}u = \bar{\partial}\bar{\partial}^*\square^{-1}f = f - \bar{\partial}^*\bar{\partial}\square^{-1}f.$$

Therefore $f \in \ker \bar{\partial}$ implies $\bar{\partial}^*\bar{\partial}\square^{-1}f \in \ker \bar{\partial}$. By taking the following inner product

$$(3.31) \quad 0 = (\bar{\partial}\bar{\partial}^*\bar{\partial}\square^{-1}f, \bar{\partial}\square^{-1}f) = (\bar{\partial}^*\bar{\partial}\square^{-1}f, \bar{\partial}^*\bar{\partial}\square^{-1}f),$$

we have

$$(3.32) \quad \bar{\partial}^*\bar{\partial}\square^{-1}f = 0.$$

Again by taking the following inner product with $\square^{-1}f$

$$(3.33) \quad 0 = (\bar{\partial}^*\bar{\partial}\square^{-1}f, \square^{-1}f) = (\bar{\partial}\square^{-1}f, \bar{\partial}\square^{-1}f),$$

we have

$$(3.34) \quad \bar{\partial}\square^{-1}f = 0.$$

Together with (3.30), we have

$$\bar{\partial}u = f.$$

Second, we will verify the estimate (3.28). By (3.34) and straightforward calculations, we have

$$(u, u) = (\bar{\partial}^* \square^{-1} f, \bar{\partial}^* \square^{-1} f) = (\bar{\partial} \bar{\partial}^* \square^{-1} f, \square^{-1} f) = (f, \square^{-1} f).$$

Therefore (3.29) implies the result. $\square$

REMARK 3.8. Note we cannot directly use $\bar{\partial}_{p,q+2} \square_{p,q+1}^{-1} f = \square_{p,q+2}^{-1} \bar{\partial}_{p,q+1} f = 0$ in the proof as we do not know the existence of $\square_{p,q+2}^{-1}$.

4. Spectrums of Friedrichs Extension

In this section, we assume (M, ω) is a Kähler manifold with a holomorphic Hermitian line bundle (L, h) . Since $\text{Dom}(\square_F) \subset W_0^1$, we will not get any boundary term when doing integration by parts for sections in $\text{Dom}(\square_F)$. By using the Weitzenböck formula, we will prove the following spectrum lower bound for Friedrichs extension under certain curvature conditions.

THEOREM 4.1. *Let (M, ω) be a Kähler manifold with a holomorphic Hermitian line bundle (L, h) . Consider Friedrichs extension of Hodge Laplacian, $\square_{0,q} : L^2(M, \Lambda^{0,q}(L)) \rightarrow L^2(M, \Lambda^{0,q}(L))$. If $\text{Ric}(TM) + \text{Ric}(L) \geq a\omega$ for some positive number a , then*

$$(4.1) \quad \text{Spec} \square_{0,q} \subset [aq, \infty).$$

REMARK 4.1. In this section, $\square_{0,q}$ always represents the Friedrichs extension and we will omit the subindex $\{0, q\}$ when there is no ambiguity.

PROOF. Take $\varphi \in \text{Dom}(\square_{0,q})$. As $\text{Dom}(\square_{0,q}) \subset W_0^1(M, \Lambda^{0,q}(L))$, there exists a sequence $\varphi_n \in \mathcal{D}(M, \Lambda^{0,q}(L))$ such that $\varphi_n \rightarrow \varphi$ in W_0^1 . By the Weitzenböck formula $\square_{0,q} = -\bar{\nabla} \nabla + q \text{Ric}(TM) + q \text{Ric}(L)$, we have

$$(4.2) \quad Q(\varphi_n, \varphi_n) = (\nabla \varphi_n, \nabla \varphi_n) + (q(\text{Ric}(TM) + \text{Ric}(L))\varphi_n, \varphi_n) \geq aq(\varphi_n, \varphi_n).$$

Letting $n \rightarrow \infty$, we have

$$(4.3) \quad Q(\varphi, \varphi) \geq aq(\varphi, \varphi).$$

As $Q(\varphi, \varphi) = (\square_{0,q} \varphi, \varphi)$, the result follows. $\square$

REMARK 4.2. Let $n = \dim M$. As the Weitzenböck formula for L -valued (n, q) form is $\square_{n,q} = -\bar{\nabla} \nabla + q \text{Ric}(L)$. If $\text{Ric}(L) \geq a\omega$ for some positive constant a , then the Friedrichs extension $\square_{n,q}$ satisfies $\text{Spec} \square_{n,q} \subset [aq, \infty)$.

5. Manifolds with Almost Polar Boundary

Let (M, g) be a Riemannian manifold. Similar as the Definition 2.2, we can define the Sobolev space for functions by taking the quadratic form $Q_1(\cdot, \cdot) = (\cdot, \cdot) + (d\cdot, d\cdot)$.

DEFINITION 5.1.

$$(5.1) \quad W_0^1(M) = \text{Completion of } \mathcal{D}(M) \text{ with respect to } Q_1 \text{ inner product,}$$

$$(5.2) \quad W^1(M) = \text{Completion of } \{\varphi \in C^\infty(M) : Q_1(\varphi, \varphi) < \infty\} \\ \text{with respect to } Q_1 \text{ inner product.}$$

Generally we know $W^1(M) = W_0^1(M)$ for complete Riemannian manifolds. In [10, 11], Masamune proved $W^1(M) = W_0^1(M)$ for Riemannian manifolds with almost polar boundary. We will repeat the proof here for the sake of completeness and because there is a gap in Masamune's proof.

We first introduce the definition and notations. Let d be the distance function induced by the length of piecewise curves on M . Then (M, d) is a metric space. We use $(\overline{M}_c, d)$ to denote the Cauchy completion of (M, d) . We define the Cauchy boundary $\partial_c M = \overline{M}_c - M$.

DEFINITION 5.2. We define the capacity of an open set $O \subset \overline{M}_c$ by

$$(5.3) \quad \text{cap}(O) = \inf\{Q_1(u, u) : u \in W^1(M), 0 \leq u \leq 1 \text{ and } u|_{O \cap M} = 1\}.$$

We also define the capacity of an arbitrary set $\Sigma \subset \overline{M}_c$ by

$$(5.4) \quad \text{cap}(\Sigma) = \inf\{\text{cap}(O), \Sigma \subset O, O \subset \overline{M}_c \text{ is open}\}.$$

A set Σ is said to be almost polar if $\text{cap}(\Sigma) = 0$.

REMARK 5.3. For any open set $O \subset \overline{M}_c$, $e \in W^1(M)$ is called the equilibrium potential of O if it satisfies

1. $Q_1(e, e) = \text{cap}(O)$.
2. $e|_O = 1$.
3. $0 \leq e \leq 1$.

It is known that the equilibrium potential exists for any open set $O \subset \overline{M}_c$. See [3] for more details.

Here is the main theorem we are going to prove.

THEOREM 5.1. *Let (M, g) be a Riemannian manifold. If $\text{cap}(\partial_c M) = 0$, then*

$$(5.5) \quad W^1(M) = W_0^1(M).$$

Before going to the proof, let's explain the main idea. First we show that $L^\infty(M) \cap W^1(M) \subset W^1(M)$ is dense. Then it is sufficient to consider $f \in L^\infty(M) \cap W^1(M)$. Choosing a sequence of open sets $\{V_n\}$ decreasing to $\partial_c M$, by using the equilibrium potential of V_n , say e_n , we can approximate f by $(1 - e_n)f$ whose support is contained in $M - V_n$. In the last, we want to modify the function $(1 - e_n)f$ to be compactly supported. As $(\overline{M}_c, d)$ is only a complete metric space, the closed metric ball excluding an open set containing $\partial_c M$ might not be a compact set even if $\text{cap}(\partial_c M) = 0$ (See Section 8 for more details). So we will refer to the intrinsic distance and verify that the intrinsic distance induces the same topology as d on $M - V_n$. As the closed metric ball with respect to the intrinsic distance is compact by Hopf-Rinow-Cohn-Vossen Theorem (see Theorem 2.5.28 in [1]). And we will use some cut-off function to finish the modification on support.

We begin the proof with the following lemma described above.

LEMMA 5.4. *For any Riemannian manifold (M, g) , $L^\infty(M) \cap W^1(M)$ is dense in $W^1(M)$.*

PROOF. Take $f \in W^1(M)$. Define a cut-off function $\rho \in C^\infty(\mathbb{R})$ such that

$$\rho(x) = \begin{cases} 1 & x \leq 1 \\ 0 & x \geq 2 \end{cases},$$

and

$$0 \leq \rho \leq 1, \quad -C \leq \rho' \leq 0.$$

We define $\rho_m(x) = \rho(\frac{x}{m})$ and $f_m = \rho_m(|f|)f$. Note $f_m \in L^\infty(M) \cap W^1(M)$ and we will prove $f_m \rightarrow f$ in $W^1(M)$. By dominated convergence theorem, we directly get $f_m \rightarrow f$ in $L^2(M)$.

As to df_m , we have

$$(5.6) \quad df_m - df = (\rho(\frac{|f|}{m}) - 1)df + \frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f|.$$

The first term on the right hand side converges to 0 in $L^2(M, \Lambda^1)$ as $|\rho(\frac{|f|}{m}) - 1| \leq \chi_{\{|f| \geq m\}}$. For the second term, since

$$(5.7) \quad \left| \frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f| \right| \leq 2C\chi_{\{m \leq |f| \leq 2m\}}|df|,$$

it follows that $\frac{1}{m}\rho'(\frac{|f|}{m})f \cdot d|f| \rightarrow 0$ in $L^2(M, \Lambda^1)$. So we have $f_m \rightarrow f$ in $W^1(M)$ and the result follows. $\square$

In next two lemmas, we will construct open sets containing $\partial_c M$ with smooth boundary.

LEMMA 5.5. $\partial_c M \subset \overline{M}_c$ is a closed subset.

PROOF. Since M is the complement of $\partial_c M$ in $\overline{M}_c$, it is equivalent to check that $M \subset \overline{M}_c$ is an open subset. For any $x \in M$, let i_x be the injectivity radius of (M, g) at x . Then for any $r \in (0, i_x)$, by considering the exponential map at x , we know $\overline{B_M(x, r)} = \{y \in M, d(x, y) \leq r\}$ is compact, whence complete. Therefore $B_M(x, r) = \overline{B_M(x, r)} = \{y \in \overline{M}_c, d(x, y) < r\}$ since we will not add any new point to $B_M(x, r)$ during the Cauchy completion of M . So $B_{\overline{M}_c}(x, r) \subset M$ and the result follows. $\square$

LEMMA 5.6. For any open set $U \subset \overline{M}_c$ containing $\partial_c M$, there exists an open set $V \subset \overline{M}_c$ such that $\partial_c M \subset V \subset \overline{V} \subset U$ and $\partial(\overline{M}_c \setminus V) \subset M$ is a smooth submanifold of codimension 1.

PROOF. Let U^C be the complement of U in $\overline{M}_c$. Since $\partial_c M$ and U^C are both closed in $(\overline{M}_c, d)$. By Urysohn's Lemma, there exists a function $f \in \mathcal{C}(\overline{M}_c)$ such that $0 \leq f \leq 1$, $f^{-1}(\{0\}) = \partial_c M$ and $f^{-1}(\{1\}) = U^C$. Take $S = f^{-1}([0, \frac{1}{2}))$. Then S is an open subset of $\overline{M}_c$ such that $\partial_c M \subset S \subset \overline{S} \subset U$.

Note that $\overline{S} \setminus \partial_c M = \overline{S} \cap M$ and U^C are both closed in M . By the Smooth Urysohn's Lemma in [12], there exists a function $g \in \mathcal{C}^\infty(M)$ such that $0 \leq g \leq 1$, $g^{-1}(\{0\}) = \overline{S} \setminus \partial_c M$ and $g^{-1}(\{1\}) = U^C$. By Sard's Theorem, without loss of generality, we can assume $\frac{1}{2}$ is a regular value of g . Take $V = g^{-1}([0, \frac{1}{2})) \cup \partial_c M \subset \overline{M}_c$. Then it's easy to see $V = g^{-1}([0, \frac{1}{2})) \cup S$. Therefore V is open in $\overline{M}_c$ such that $\partial_c M \subset V \subset \overline{V} = g^{-1}([0, \frac{1}{2})) \cup \overline{S} \subset U$. The remaining part of the lemma follows from $\partial(\overline{M}_c \setminus V) = g^{-1}(\{\frac{1}{2}\})$ and $\frac{1}{2}$ is a regular value of g . $\square$

Let V be an open subset satisfying the conclusion in the above lemma. Denote $V^C = \overline{M}_c \setminus V$ as the complement of V in $\overline{M}_c$. Then $V^C = \cup_{\lambda \in \Lambda} A_\lambda$, where each A_λ is a connected component of V^C and Λ is the index set. Since V^C is locally path connected, each A_λ is both open and closed in V^C . Define the intrinsic distance function d_{A_λ} on A_λ as

DEFINITION 5.7. Define the intrinsic distance on A_λ as $d_{A_\lambda} : A_\lambda \times A_\lambda \rightarrow [0, \infty)$,

$$(5.8) \quad d_{A_\lambda}(x, y) = \inf_{l \in L_{A_\lambda}} \|l\|$$

where $L_{A_\lambda} = \{\text{all piecewise smooth curves contained in } A_\lambda \text{ from } x \text{ to } y\}$ and $\|l\|$ denotes the length of curve l .

REMARK 5.8. $d(x, y) \leq d_{A_\lambda}(x, y)$ for any $x, y \in A_\lambda$ as d is the infimum over a larger set.

In general, d and d_{A_λ} are not globally equivalent to each other on A_λ . The next lemma shows that they are locally equivalent on A_λ .

LEMMA 5.9. *For any $x \in A_\lambda$, there exists $r = r(x) > 0$ such that*

$$(5.9) \quad d_{A_\lambda}(x, y) \leq 4d(x, y) \quad \text{for any } y \in B_{A_\lambda}(x, r).$$

where $B_{A_\lambda}(x, r) = \{y \in A_\lambda, d(x, y) < r\}$.

PROOF. For any $x \in A_\lambda \subset V^C \subset M$, either x is in the interior of V^C or $x \in \partial V^C$. In the first case, take $r < i_x$ (i_x denotes the injectivity radius at x) small enough such that $B_M(x, r) \subset A_\lambda$. Then for any $y \in B_M(x, r)$, there existed a minimizing geodesic $l \subset B_M(x, r)$ such that $\|l\| = d(x, y)$. Therefore $d_{A_\lambda}(x, y) = d(x, y)$ for any $y \in B_{A_\lambda}(x, r) = B_M(x, r)$.

In the second case, i.e. $x \in \partial V^C$, take $r < i_x$. We can identify $B_{\mathbb{R}^m}(o, r)$ (w.r.t the Euclidean metric g_x) with $B_M(x, r)$ by the exponential map Exp_x at x . By shrinking r , we can assume the Riemannian metric on $B_M(x, r)$ is equivalent to the metric at x , say $\frac{1}{2}g_x \leq g \leq 2g_x$. Let $\{e_i\}_{i=1}^m$ be the standard orthonormal basis of $\mathbb{R}^m$. Up to an orthonormal linear transformation, we can assume $\{e_i\}_{i=1}^{m-1} \subset T_x(\partial V^C)$ and e_m is the normal direction of ∂V^C at x . By Lemma 5.6, possibly shrinking r again, we can assume $\partial V^C = \{(x_1, x_2, \dots, x_m) \in B(o, r), x_m = h(x_1, x_2, \dots, x_{m-1})\}$ where $h \in C^\infty(\mathbb{R}^{m-1})$ and $h(0, \dots, 0) = 0$. Since $\{e_i\}_{i=1}^{m-1}$ are tangent vectors of ∂V^C at x , $\nabla h(0, \dots, 0) = 0$. By shrinking r again, we can assume $|\nabla h| \leq 1$ in $B_{\mathbb{R}^{m-1}}(o, r)$.

For any point $y \in B_{\mathbb{R}^m}(o, r)$, consider the curve $l_1 = (ty_1, ty_2, \dots, ty_{m-1}, h(ty_1, \dots, ty_{m-1}))$ for $t \in [0, 1]$ and $l_2 = (y_1, y_2, \dots, y_{m-1}, ty_m + (1-t)h(y_1, y_2, \dots, y_{m-1}))$ for $t \in [0, 1]$. Then the concatenation $l_1 \cup l_2 \subset V^C$ is from x to y . The Euclidean length of l_1, l_2 are respectively

$$\begin{aligned} \|l_1\|_{\mathbb{R}^m} &= \int_0^1 \sqrt{y_1^2 + y_2^2 + \dots + y_{m-1}^2 + |\nabla h(ty_1, ty_2, \dots, ty_{m-1}) \cdot (y_1, y_2, \dots, y_{m-1})|^2} dt \\ &\leq 2\sqrt{y_1^2 + y_2^2 + \dots + y_{m-1}^2}, \\ \|l_2\|_{\mathbb{R}^m} &= |y_m - h(y_1, y_2, \dots, y_{m-1})| \\ &\leq |y_m| + \sqrt{y_1^2 + y_2^2 + \dots + y_{m-1}^2}. \end{aligned}$$

Therefore

$$\begin{aligned} d_{A_\lambda}(x, y) &\leq \|l_1\| + \|l_2\| \\ &\leq 2\|l_1\|_{\mathbb{R}^m} + 2\|l_2\|_{\mathbb{R}^m} \\ &\leq 4\sqrt{y_1^2 + y_2^2 + \dots + y_{m-1}^2 + y_m^2} \\ &= 4d(x, y). \end{aligned}$$

The second inequality is because $\frac{1}{2}g_x \leq g \leq 2g_x$. So the result follows. $\square$

Base on Remark 5.8 and Lemma 5.9, we have the following properties on $(A_\lambda, d_{A_\lambda})$.

PROPOSITION 5.10. $(A_\lambda, d_{A_\lambda})$ satisfies the following property.

- (a). $(A_\lambda, d_{A_\lambda})$ and (A_λ, d) have the same topology.
- (b). $(A_\lambda, d_{A_\lambda})$ is locally compact.
- (c). $(A_\lambda, d_{A_\lambda})$ is complete.

PROOF. Part (a) directly follows from Remark 5.8 and Lemma 5.9.

Now we prove part (b). Since V^C is a closed subset of (M, d) and (M, d) is locally compact, (V^C, d) is locally compact. And we know A_λ is a closed subset of (V^C, d) , therefore (A_λ, d) is locally compact. The result follows by part (a).

Last we prove part (c). Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in $(A_\lambda, d_{A_\lambda})$. By Remark 5.8, $\{x_n\}_{n=1}^\infty$ is also a Cauchy sequence in (A_λ, d) . Since A_λ is closed in (V^C, d) and V^C is closed in the complete space $(\overline{M}_c, d)$, (A_λ, d) is complete. Then there exists some $x \in A_\lambda$ such that $\lim d(x, x_n) = 0$. By Lemma 5.9, $\lim d_{A_\lambda}(x_n, x) = 0$ and therefore the result follows. $\square$

For any $x_0 \in A_\lambda$, define the function $r_{x_0} : A_\lambda \rightarrow [0, \infty)$ as $r_{x_0}(x) = d_{A_\lambda}(x_0, x)$. Then r_{x_0} has the following property.

PROPOSITION 5.11. For the function r_{x_0} defined as above, we have

$$(5.10) \quad |\nabla r|_g \leq 4.$$

PROOF. Since $|r(x) - r(y)| \leq d_{A_\lambda}(x, y)$, the result follows from Lemma 5.9. $\square$

The closed metric ball induced by d_{A_λ} is compact though it is not the case for the closed metric ball induced by d . The following lemma is essentially Hopf-Rinow-Cohn-Vossen Theorem. See Theorem 2.5.28 in [1] for more details.

LEMMA 5.12. For any $x \in A_\lambda, r > 0$, $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)}$ is compact. Here $B_{(A_\lambda, d_{A_\lambda})}(x, r)$ denotes the set $\{y \in A_\lambda, d_{A_\lambda}(x, y) < r\}$.

REMARK 5.13. By part (a) in Proposition 5.10, the closures of $B_{(A_\lambda, d_{A_\lambda})}(x, r)$ in (A_λ, d) and in (A_λ, d_λ) are the same. The compactness in (A_λ, d) and that in (A_λ, d_λ) are also the same. So there is no ambiguity in the above lemma.

PROOF. By part (b) in Proposition 5.10, the set

$$\{r > 0, \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)} \text{ is compact}\}$$

is nonempty. So we can define $r_0 = \sup\{r > 0, \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r)} \text{ is compact}\}$. Now it suffices to prove $r_0 = \infty$. Assume not. Then $r_0 \in (0, \infty)$.

First, we prove that $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is compact. Take an arbitrary $\varepsilon > 0$. For any $y \in \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$, since $d_{A_\lambda}(x, y) \leq r_0$, there exists a piecewise smooth curve $l \subset A_\lambda$ from x to y such that $\|l\| < r_0 + \varepsilon$. Reparametrize the curve l by arc length. Then the restriction $l|_{[r_0 - \varepsilon, \|l\|]}$ is a piecewise smooth curve from a point in $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}$ to y . Since

$$\|l|_{[r_0 - \varepsilon, \|l\|]}\| < 2\varepsilon, y \in B_{(A_\lambda, d_{A_\lambda})}(\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}, 2\varepsilon).$$

Therefore

$$\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)} \subset B_{(A_\lambda, d_{A_\lambda})}(\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}, 2\varepsilon).$$

Since $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 - \varepsilon)}$ is compact by the definition of r_0 , $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is totally bounded in $(A_\lambda, d_{A_\lambda})$. Therefore $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is compact by part (c) in Proposition 5.10.

Second, we prove that $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 + \delta)}$ is compact for some $\delta > 0$, which contradicts the definition of r_0 and therefore we get the result. Since $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ is also compact, together with part (b) in Proposition 5.10, $\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$ has a finite cover $\{B_{(A_\lambda, d_{A_\lambda})}(y_i, \delta_i)\}_{i=1}^N$, such that $y_i \in \overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0)}$, $\delta_i > 0$ and $\overline{B_{(A_\lambda, d_{A_\lambda})}(y_i, 2\delta_i)}$ is compact for each i . Take $\delta = \min_{1 \leq i \leq N} \delta_i$. Then

$$\overline{B_{(A_\lambda, d_{A_\lambda})}(x, r_0 + \delta)} \subset \cup_{i=1}^N \overline{B_{(A_\lambda, d_{A_\lambda})}(y_i, 2\delta_i)}$$

is compact. $\square$

Now we are ready to prove the Theorem 5.1.

PROOF. Since $\text{cap}(\partial_c M) = 0$, there exists a sequence of open sets $\{U_n\}_{n=1}^\infty$ such that $\partial_c M \subset U_n$ and $\lim \text{cap}(U_n) = 0$. For U_1 , by Lemma 5.6, there exists an open set V_1 such that $\partial_c M \subset V_1 \subset \overline{V_1} \subset U_1$ and $\partial(V_1^C)$ is a smooth submanifold. Then for $V_1 \cap U_2$, by Lemma 5.6, there exists an open set V_2 such that $\partial_c M \subset V_2 \subset \overline{V_2} \subset V_1 \cap U_2$ and $\partial(V_2^C)$ is a smooth submanifold. Inductively, we construct V_{i+1} by applying Lemma 5.6 to $V_i \cap U_{i+1}$. So we get a sequence of decreasing open sets $\{V_n\}_{n=1}^\infty$ such that $\partial_c M \subset V_n \subset \overline{V_n} \subset V_{n-1} \cap U_n$ and $\partial(V_n^C)$ is a smooth submanifold. Since in particular $V_n \subset U_n$, we have $\lim \text{cap}(V_n) = 0$.

Take $f \in W^1(M) \cap L^\infty$. It suffices to prove $f \in W_0^1(M)$.

First, we approximate f by functions with support in some V_n^C . Let e_n be the equilibrium potential (see Remark 5.3) of V_n , i.e. e_n satisfies

- $e_n \in W$ and $Q_1(e_n, e_n) = \text{cap}(V_n)$.
- $e_n|_{V_n} = 1$.
- $0 \leq e_n \leq 1$.

Since $\|e_n\|_W = \text{cap}(V_n) \rightarrow 0$, we can assume $e_n \rightarrow 0$ a.e. by passing to a subsequence. Let $f_n = (1 - e_{n-1})f$. Then $f_n \rightarrow f$ in $W^1(M)$ and $\text{supp}(f_n) \subset V_{n-1}^C \subset \overline{V_n}^C \subset \text{interior}(V_n^C)$.

Secondly, we approximate each f_n with $\text{supp}(f_n) \subset \text{interior}(V_n^C)$ by functions with compact support. From now on, we fix f_n and V_n^C . For economy we suppress the index n . Write V^C into the disjoint union of connected component, $V^C = \cup_{\lambda \in \Lambda} A_\lambda$. Since $f \in W^1(M)$ and $\{A_\lambda\}_{\lambda \in \Lambda}$ is pairly disjoint, f vanishes on all but countably many A_λ , say $\{A_{\lambda_j}\}_{j=1}^\infty$. Denote $g_j = f\chi_{A_{\lambda_j}}$ where $\chi_{A_{\lambda_j}}$ is the characteristic function of A_{λ_j} . Note $g_j \in W^1(M)$ and $\nabla g_j = (\nabla f)\chi_{A_{\lambda_j}}$ by the fact that $\partial A_{\lambda_j} \subset \partial(V^C)$ and f vanishes close to $\partial(V^C)$ as $\text{supp } f \subset \text{interior}(V^C)$. Then $f = \sum_{j=1}^\infty g_j$ and $\|f\|_W^2 = \sum_{j=1}^\infty \|g_j\|_W^2$. Therefore for any $\varepsilon > 0$, there exists $N > 0$ such that $\|f - \sum_{j=1}^N g_j\|_W < \varepsilon$.

Now it suffices to approximate each g_j be compact supported function. Take $x_j \in A_{\lambda_j}$ and define $r_j : A_{\lambda_j} \rightarrow [0, \infty)$ as $r_j(x) = d_{A_{\lambda_j}}(x_j, x)$. Then $|\nabla r_j|_g \leq 4$ by Proposition 5.11. Let $\varphi \in C^\infty(R)$ satisfy the following conditions:

- φ is a decreasing function and $0 \leq \varphi \leq 1$.

- $\varphi|_{(-\infty, 0]} = 1$ and $\varphi|_{[1, \infty)} = 0$.
- $|\varphi'| \leq C$ and C is a fixed constant.

Define $\varphi_k(x) = \varphi(\frac{x}{k})$. Then $\varphi_k \circ r_j \rightarrow 1$ a.e. on A_{λ_j} as $k \rightarrow \infty$ and $|\nabla(\varphi_k \circ r_j)|_g \leq \frac{4C}{k}$. Therefore we have $(\varphi_k \circ r_j)g_j \rightarrow g_j$ in $W^1(M)$. And $\text{supp}((\varphi_k \circ r_j)g_j) \subset \text{supp}(\varphi_k \circ r_j) \subset \overline{B_{(A_{\lambda_j}, d_{A_{\lambda_j}})}(x_j, 2k)}$, which is compact by Lemma 5.12. So the result follows. $\square$

6. Moduli Space of Polarized Calabi-Yau Manifolds

Let (M, L) be a Calabi-Yau manifold polarized by a positive line bundle L . That is, M is a compact Kähler manifold with a Ricci flat Kähler metric ω and the metric ω is contained in the first Chern class of L . Let $\mathcal{M}$ be the moduli space of Calabi-Yau manifolds polarized by a fixed positive line bundle L . In [16], Viehweg proved the moduli space $\mathcal{M}$ is a quasi-projective variety. Take $\overline{\mathcal{M}}$ as the compactification of $\mathcal{M}$. With the classical result of Hironaka, by resolution of singularities, we can choose $\overline{\mathcal{M}}$ in such a way that the divisor $Y = \overline{\mathcal{M}} \setminus \mathcal{M}$ is a divisor of normal crossings. After passing to a finite cover, we may assume $\mathcal{M}$ and $\overline{\mathcal{M}}$ are smooth manifolds (see Lemma 4.1 in [8]). From now on, we will work on this quasi-projective Kähler manifold $(\mathcal{M}, \omega_{WP})$ with the compactification $\overline{\mathcal{M}}$ as a compact Kähler manifold.

Here is the main theorem we are going to prove in this section.

THEOREM 6.1. *The moduli space of polarized Calabi-Yau manifolds $(\mathcal{M}, \omega_{WP})$ has almost polar Cauchy boundary, i.e. $\text{cap}(\partial_c \mathcal{M}) = 0$.*

REMARK 6.1. In general, the Cauchy completion $\overline{\mathcal{M}}_c$ is not necessarily identical to the compactification $\overline{\mathcal{M}}$.

It is well-known that there is a complete Kähler metric on $\mathcal{M}$ such that it is asymptotical to the Poincaré metric near infinity. We call it Poincaré metric and denote it by ω_P (See Lemme 3.1 in [8]). The key ingredient to prove Theorem 6.1 is the following lemma in [7].

LEMMA 6.2. *For any $\varepsilon > 0$ small enough, there is a smooth real valued function $\rho_\varepsilon \in \mathcal{D}(\mathcal{M})$ such that*

- $0 \leq \rho_\varepsilon \leq 1$;
- There is a constant C , independent of ε , such that $-C\omega_P \leq \sqrt{-1}\partial\bar{\partial}\rho_\varepsilon \leq C\omega_P$;
- In a neighborhood of Y , $\rho_\varepsilon = 0$ and $\rho_\varepsilon(x) = 1$ if the Euclidean distance of $x \in M$ to Y is greater than 2ε .

PROOF. As $Y \subset \overline{\mathcal{M}}$ is a divisor of normal crossings, by [8] (see Lemma 4.1), we can find a finite cover $\{U_\alpha\}_{\alpha=1}^t$ of $\overline{\mathcal{M}}$ such that $Y \subset \cup_{\alpha=1}^s U_\alpha$ and $U_{s+1} \cup \dots \cup U_t \cap Y = \emptyset$. Furthermore, we can assume that $U_\alpha - Y = (\Delta^*)^{a_\alpha} \times (\Delta)^{b_\alpha}$ with the coordinates $(s_1^\alpha, \dots, s_{a_\alpha}^\alpha, w_1^\alpha, \dots, w_{b_\alpha}^\alpha)$ for any $1 \leq \alpha \leq s$, where Δ^* and Δ are respectively the punctured unit disk and the unit disk in $\mathbb{C}$. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth decreasing function such that $0 \leq \eta \leq 1$ and

$$\eta = \begin{cases} 1 & x \leq 0 \\ 0 & x \geq 1 \end{cases}.$$

Let

$$\eta_\varepsilon(z) = \begin{cases} 1 & |z| \leq e^{-\frac{1}{\varepsilon}} \\ \eta\left(\frac{(\log \frac{1}{|z|})^{-1} - \varepsilon}{\varepsilon}\right) & e^{-\frac{1}{\varepsilon}} \leq |z| \leq e^{-\frac{1}{2\varepsilon}} \\ 0 & |z| \geq e^{-\frac{1}{2\varepsilon}} \end{cases}.$$

And let

$$\eta_\varepsilon^\alpha(s_1^\alpha, \dots, s_{a_\alpha}^\alpha) = \prod_{j=1}^{a_\alpha} (1 - \eta_\varepsilon(s_j^\alpha)).$$

Then define the function

$$\rho_\varepsilon = \sum_{\alpha=1}^s \psi_\alpha \eta_\varepsilon^\alpha + \sum_{\alpha=s+1}^t \psi_\alpha,$$

where $\{\psi_\alpha\}$ is a partition of unity subordinated to $\{U_\alpha\}$.

Then $0 \leq \rho_\varepsilon \leq 1$. By a straightforward calculation, we have

$$\begin{aligned} \bar{\partial} \eta_\varepsilon &= \frac{1}{2\varepsilon} \eta' \frac{d\bar{z}}{\bar{z}(\log \frac{1}{|z|})^2}, \\ \partial \bar{\partial} \eta_\varepsilon &= \frac{1}{4\varepsilon^2} \eta'' \frac{dz \wedge d\bar{z}}{|z|^2 (\log \frac{1}{|z|})^4} + \frac{1}{2\varepsilon} \eta' \frac{dz \wedge d\bar{z}}{|z|^2 (\log \frac{1}{|z|})^3}. \end{aligned}$$

Note that $\eta' = 0$ unless $\varepsilon \leq (\log \frac{1}{|z|})^{-1} \leq 2\varepsilon$. Therefore

$$|\bar{\partial} \eta_\varepsilon| \leq C \left| \frac{d\bar{z}}{|z| \log \frac{1}{|z|}} \right|, \quad |\partial \bar{\partial} \eta_\varepsilon| \leq C \left| \frac{dz \wedge d\bar{z}}{|z|^2 (\log \frac{1}{|z|})^2} \right|,$$

where C is a constant independent of ε . Therefore we obtain part (b) as ψ_α are fixed smooth functions on $\overline{\mathcal{M}}$.

Let $x \in \mathcal{M}$. When x is sufficiently close to Y , $\psi_\alpha = 0$ for any $\alpha \geq s+1$ and $\eta_\varepsilon^\alpha = 0$ for any $\alpha \leq s$. Therefore $\rho_\varepsilon = 0$ in a neighborhood of Y . If the distance of x to Y is at least 2ε , then there is a constant $C > 0$ such that $|s_j^\alpha| \geq C\varepsilon$ for any $1 \leq j \leq a_\alpha$ and $1 \leq \alpha \leq s$. Since $\varepsilon e^{\frac{1}{2\varepsilon}} \rightarrow \infty$ as $\varepsilon \rightarrow 0$, when ε is small enough we have $\rho_\varepsilon(x) = \sum \psi_\alpha = 1$. $\square$

Now we are ready to prove Theorem 6.1.

PROOF. Take the function ρ_ε constructed in Lemma 6.2. As $\rho_\varepsilon \in \mathcal{D}(\mathcal{M})$ and $0 \leq \rho_\varepsilon \leq 1$, we have

$$(6.1) \quad \text{cap}(\partial_c \mathcal{M}) \leq \int_{\mathcal{M}} |1 - \rho_\varepsilon|^2 \frac{\omega_{WP}^n}{n!} + \int_{\mathcal{M}} |d(1 - \rho_\varepsilon)|^2 \frac{\omega_{WP}^n}{n!}, \text{ for any } \varepsilon > 0.$$

Since $\rho_\varepsilon \rightarrow 1$ pointwise on $\mathcal{M}$ and the volume of Weil-Petersson metric is finite by Theorem 1.1 in [8],

$$(6.2) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}} |1 - \rho_\varepsilon|^2 \frac{\omega_{WP}^n}{n!} = 0.$$

It suffices to prove that $\int_{\mathcal{M}} |d\rho|^2 \rightarrow 0$. Note

$$\begin{aligned} \int_{\mathcal{M}} |d\rho_\varepsilon|^2 \omega_{WP}^n &= 2 \int_{\mathcal{M}} |\bar{\partial}\rho_\varepsilon|^2 \omega_{WP}^n = 2n \int_{\mathcal{M}} \sqrt{-1} \bar{\partial}\rho_\varepsilon \wedge \bar{\partial}\rho_\varepsilon \wedge \omega_{WP}^{n-1} \\ &= -2n \int_{\mathcal{M}} \sqrt{-1} \rho_\varepsilon \bar{\partial}\bar{\partial}\rho_\varepsilon \wedge \omega_{WP}^{n-1}. \end{aligned}$$

Since $-C\omega_P \leq \sqrt{-1}\bar{\partial}\bar{\partial}\rho_\varepsilon \leq C\omega_P$ and $\omega_{WP} \leq C\omega_P$ (see Proposition 3.1 in [8]), we have

$$\int_{\mathcal{M}} |\bar{\partial}\rho_\varepsilon|^2 \omega_{WP}^n \leq C \int_{\text{supp}(\bar{\partial}\rho_\varepsilon)} \omega_P^n.$$

Use the same cover of $\{U_\alpha\}_{\alpha=1}^t$ of $\overline{\mathcal{M}}$ as in Lemma 6.2. Then $Y \subset \cup_{\alpha=1}^s U_\alpha$, $U_{s+1} \cup \dots \cup U_t \cap Y = \emptyset$ and $U_\alpha - Y = (\Delta^*)^{a_\alpha} \times (\Delta)^{b_\alpha}$ with the coordinates $(s_1^\alpha, \dots, s_{a_\alpha}^\alpha, w_1^\alpha, \dots, w_{b_\alpha}^\alpha)$ for any $1 \leq \alpha \leq s$. When ε is small enough, we can assume that $\text{supp}(\bar{\partial}\rho_\varepsilon) \cap U_\alpha \subset \{|s_j^\alpha| \leq \frac{1}{2}, |w_j^\alpha| \leq \frac{1}{2}\}$ for any $1 \leq \alpha \leq s$. Since in $U_\alpha - Y$ for any $1 \leq \alpha \leq s$, the Poincaré metric ω_P is asymptotic to

$$(6.3) \quad \frac{\sqrt{-1}}{2} \left(\sum_{j=1}^{a_\alpha} \frac{ds_j^\alpha \wedge d\bar{s}_j^\alpha}{|s_j^\alpha|^2 (\log \frac{1}{|s_j^\alpha|})^2} + \sum_{j=1}^{b_\alpha} dw_j^\alpha \wedge d\bar{w}_j^\alpha \right),$$

we have

$$(6.4) \quad \int_{\text{supp}(\bar{\partial}\rho_\varepsilon)} \omega_P^n \leq C \sum_{\alpha=1}^s \prod_{j=1}^{a_\alpha} \int_{e^{-\frac{1}{2\varepsilon}}}^{e^{-\frac{1}{\varepsilon}}} \frac{1}{|s_j^\alpha| (\log \frac{1}{|s_j^\alpha|})^2} d|s_j^\alpha| \prod_{j=1}^{b_\alpha} \int_0^{\frac{1}{2}} |w_j^\alpha| d|w_j^\alpha| \leq C\varepsilon.$$

So we have

$$(6.5) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M}} |d\rho_\varepsilon|^2 \omega_{WP}^n = 0$$

and the result follows. $\square$

7. Self-Adjointness of the Laplacian on Moduli Space

In this section, we will consider the self-adjointness of Laplacian on $(\mathcal{M}, \omega_{WP})$. Let us consider the differential operators d and δ defined on C^1 functions and C^1 forms on $\mathcal{M}$ respectively. We define the domain $\text{Dom}(d)$ of d to be the set of C^1 functions f defined on $\mathcal{M}$ such that both f and df are in L^2 . Similarly, we define the domain $\text{Dom}(\delta)$ of δ to be the set of C^1 1-forms w such that both w and δw are in L^2 . We then define the Laplacian Δ with respect to ω_{WP} by Δ with $\text{Dom}(\Delta)$ given by the set of C^2 functions f such that $f \in \text{Dom}(d)$ and $df \in \text{Dom}(\delta)$. In this section, we will prove the closure $\overline{\Delta}$ of Δ is self-adjoint.

THEOREM 7.1. *On $(\mathcal{M}, \omega_{WP})$, the closure $\overline{\Delta}$ of Laplacian on functions is self-adjoint.*

It is proved in [6] that $\overline{\Delta}$ is self-adjoint on $M \setminus \Sigma_M$ when M is an algebraic variety with the induced Fubini-Study metric and Σ_M is the singular set at least of real codimension 2. Here our result is different as we are considering the Weil-Petersson metric.

PROOF. By the theorem of Gaffney in [4], in order to show $\overline{\Delta}$ is self-adjoint, it is sufficient to prove

$$(7.1) \quad (df, w) = (f, \delta w)$$

for any $f \in \text{Dom}(d)$ and $w \in \text{Dom}(\delta)$. By Theorem 6.1 and 5.1, we have $W^1(\mathcal{M}) = W_0^1(\mathcal{M})$. Since $\text{Dom}(d) \subset W^1(\mathcal{M})$, there exists a sequence $f_n \in \mathcal{D}(\mathcal{M})$ such that $f_n \rightarrow f$ in $W^1(\mathcal{M})$. As each f_n has compact support, by integration by parts, we have

$$(7.2) \quad (df_n, w) = (f_n, \delta w).$$

The result follows by taking $n \rightarrow \infty$. $\square$

8. An Example

Let (M, g) be a Riemannian manifold. A closed metric ball in $(\overline{M}_c, d)$ excluding an open set containing $\partial_c M$ might not be compact even if $\text{cap}(\partial_c M) = 0$. In this section, we will give a concrete example.

Consider the Riemannian manifold (M, g) as follows. $M = \mathbb{R}^3$ and in terms of the cylindrical coordinates (r, θ, z) ,

$$(8.1) \quad g = e^{2z}(dr^2 + f^2(r)d\theta^2 + dz^2).$$

Here the function $f \in C^\infty([0, \infty))$ satisfies the following properties:

- $f(r) = r$ for $r \in [0, \frac{1}{2}]$.
- f is increasing on $[0, 1]$ and $f(1) = 1$.
- f is decreasing on $[1, \infty)$.
- $f(r) = e^{-r}$ for $r \in [2, \infty)$.

For any piecewise smooth curve $l : [a, b] \rightarrow M$, we denote the length of l by $\|l\|$, i.e.

$$(8.2) \quad \|l\| = \int_a^b e^{z(t)} \sqrt{\dot{r}^2(t) + f^2(r(t))\dot{\theta}^2(t) + \dot{z}^2(t)} dt$$

And define the distance function d as

$$d(p, q) = \inf_{l \in L} \|l\|,$$

where $L = \{\text{all piecewise smooth curves from } p \text{ to } q\}$. Then we know (M, d) is a metric space.

LEMMA 8.1. *For any $P_1, P_2 \in M$, denote the coordinate of P_i as (r_i, θ_i, z_i) for $i = 1, 2$. Then*

$$(8.3) \quad d(P_1, P_2) \leq e^{z_1} + e^{z_2}.$$

PROOF. For any $t_0 < \min(z_1, z_2)$. Define the following three smooth curves.

- $l_1 : (r_1, \theta_1, t)$ for $t \in [t_0, z_1]$ oriented from z_1 to t_0 .
- $l_2 : (r_1 + (r_2 - r_1)t, \theta_1 + (\theta_2 - \theta_1)t, t_0)$ for $t \in [0, 1]$.
- $l_3 : (r_2, \theta_2, t)$ for $t \in [t_0, z_2]$.

Then $l_1 \cup l_2 \cup l_3$ is a piecewise smooth curve connecting P_1 and P_2 . We can calculate the length of these curves straightforwardly.

$$\begin{aligned} \|l_1\| &= \int_{t_0}^{z_1} e^t dt = e^{z_1} - e^{t_0}, \\ \|l_3\| &= \int_{t_0}^{z_2} e^t dt = e^{z_2} - e^{t_0}, \\ \|l_2\| &= \int_0^1 e^{t_0} \sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2 f^2(r_1 + (r_2 - r_1)t)} dt \\ &\leq e^{t_0} \sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2}. \end{aligned}$$

Therefore

$$d(P_1, P_2) \leq e^{z_1} + e^{z_2} - 2e^{t_0} + e^{t_0} \sqrt{(r_2 - r_1)^2 + (\theta_2 - \theta_1)^2}.$$

Taking $t_0 \rightarrow -\infty$, the result follows. $\square$

Define $H_I = \mathbb{R}^2 \times I = \{(r, \theta, z) : z \in I\}$ for any $I \subset \mathbb{R}$. And we will use $\text{diam } S$ to denote the diameter of set $S \subset M$.

COROLLARY 8.2. $\text{diam } H_{(-\infty, 0]} \leq 2$.

PROOF. For any $P_1, P_2 \in H_{(-\infty, 0]}$, we have $d(P_1, P_2) \leq e^{z_1} + e^{z_2} \leq 2$. $\square$

LEMMA 8.3. For any $P_1, P_2 \in M$,

$$(8.4) \quad d(P_1, P_2) \geq |e^{z_1} - e^{z_2}|.$$

PROOF. For any piecewise smooth curve $l : [0, 1] \rightarrow M$ from P_1 to P_2 , we have

$$\begin{aligned} \|l\| &= \int_0^1 e^{z(t)} \sqrt{\dot{r}^2(t) + f^2(r(t)) \dot{\theta}^2(t) + \dot{z}^2(t)} dt \\ &\geq \int_0^1 e^z |\dot{z}(t)| dt \\ &\geq |e^{z_1} - e^{z_2}|. \end{aligned}$$

$\square$

Note that the metric space (M, d) is not complete. $\{(0, 0, -n)\}_{n=1}^\infty$ is a Cauchy sequence since $d((0, 0, -m), (0, 0, -n)) \leq e^{-m} + e^{-n}$. But it is not convergent in M .

THEOREM 8.1. Let $\overline{M}$ be the completion of M with respect to metric d . Then $\overline{M} = M \cup \{\infty\}$ where $\{\infty\}$ is defined as the Cauchy sequence $\{(0, 0, -n)\}_{n=1}^\infty$.

We want show that for any Cauchy sequence $\{P_n\}_{n=1}^\infty$, either it is convergent in M or it is equivalent to the Cauchy sequence $\{(0, 0, -n)\}_{n=1}^\infty$. We split the proof into following lemmas.

LEMMA 8.4. Let $\{P_n\}_{n=1}^\infty$ be a Cauchy sequence in M and denote $P_n = (r_n, \theta_n, z_n)$. Then $\{z_n\}_{n=1}^\infty$ is either convergent in $\mathbb{R}$ or $\lim_{n \rightarrow \infty} z_n = -\infty$.

PROOF. By inequality (8.4), we have $d(P_m, P_n) \geq |e^{z_m} - e^{z_n}|$. Therefore $\{e^{z_n}\}$ is a Cauchy sequence in $\mathbb{R}$. So the result follows. $\square$

LEMMA 8.5. Let $\{P_n\}_{n=1}^\infty$ be a Cauchy sequence in M and denote $P_n = (r_n, \theta_n, z_n)$. If $\{z_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$, then $\{r_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$.

PROOF. Let $z_0 = \lim z_n$. By dropping finitely many beginning terms, we can assume $z_n \in [z_0 - 1, z_0 + 1]$. Let $\delta = \delta(z_0) = e^{z_0-1} - e^{z_0-2}$. Since $\{P_n\}$ is Cauchy, by dropping more beginning terms, we can assume further that $d(P_m, P_n) < \frac{\delta}{3}$ for any $m, n \in \mathbb{Z}^+$. By the definition of metric d , there exists a piecewise smooth curve $l_{mn} : [0, 1] \rightarrow M$ from P_m to P_n such that $\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n)$. We claim

$$(8.5) \quad \min_{t \in [0, 1]} z(t) > z_0 - 2.$$

Assume not. Take $t = t_0 \in [0, 1]$ be the first time such that $z(t) = z_0 - 2$, which implies that $z(t) \geq z_0 - 2$ for $t \in [0, t_0]$. Then

$$\|l_{mn}\| \geq \int_0^{t_0} e^{z(t)} |\dot{z}(t)| dt \geq e^{z_m} - e^{z(t_0)} \geq e^{z_0-1} - e^{z_0-2} = \delta.$$

However, according to our assumption on l_{mn} , we have

$$(8.6) \quad \|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n) < \frac{\delta}{2},$$

which is a contradiction and therefore the claim follows. Thus we have

$$\frac{3}{2}d(P_m, P_n) \geq l_{mn} \geq \int_0^1 e^{z(t)} |\dot{r}(t)| dt \geq e^{z_0-2} |r_m - r_n|.$$

Therefore $\{r_n\}$ is a Cauchy sequence in $\mathbb{R}$. □

LEMMA 8.6. *Let $\{P_n\}_{n=1}^\infty$ be a Cauchy sequence in M and denote $P_n = (r_n, \theta_n, z_n)$. If $\{z_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ and $\lim r_n > 0$, then $\{\theta_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$.*

PROOF. Let $z_0 = \lim z_n$ and $r_0 = \lim r_n$. By dropping finitely many beginning terms, we can assume that $z_n \in [z_0 - 1, z_0 + 1]$ and $r_n \in [\frac{1}{2}r_0, \frac{3}{2}r_0]$ for any $n \in \mathbb{Z}$. Define $\delta(z_0) = e^{z_0-1} - e^{z_0-2}$ and $\delta(r_0, z_0) = \frac{1}{4}r_0 e^{z_0-2}$. And take $\delta = \min\{\delta(z_0), \delta(r_0, z_0)\}$. By dropping more beginning terms, we can assume further $d(P_m, P_n) < \frac{\delta}{3}$ for any $m, n \in \mathbb{Z}$. Again we take a piecewise smooth curve $l_{mn} : [0, 1] \rightarrow M$ from P_m to P_n such that $\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n)$. By the proof in Lemma 8.5, we have $\min z(t) \geq z_0 - 2$. Here we claim

$$(8.7) \quad r(t) \in [\frac{1}{4}r_0, \frac{7}{4}r_0] \text{ for any } t \in [0, 1].$$

Assume not. Then let $t = t_0$ be the first time such that $r(t_0) = \frac{1}{4}r_0$ or $\frac{7}{4}r_0$. Then

$$\|l_{mn}\| \geq \int_0^{t_0} e^{z(t)} |\dot{r}(t)| dt \geq e^{z_0-2} |r(t_0) - r_m| \geq \frac{1}{4}r_0 e^{z_0-2} = \delta(r_0, z_0).$$

But we also have

$$\|l_{mn}\| \leq \frac{3}{2}d(P_m, P_n) < \frac{\delta}{2},$$

which is a contradiction. So the claim follows. Therefore

$$\begin{aligned} \frac{3}{2}d(P_m, P_n) &\geq \|l_{mn}\| \geq \int_0^1 e^{z(t)} f(r(t)) |\dot{\theta}(t)| dt \\ &\geq e^{z_0-2} \min\{f(\frac{1}{4}r_0), f(\frac{7}{4}r_0)\} |\theta_m - \theta_n|. \end{aligned}$$

It follows that $\{\theta_n\}$ is a Cauchy sequence. □

LEMMA 8.7. *Let $P_n = (r_n, \theta_n, z_n)$ be a sequence in M . If $r_n \rightarrow r_0, \theta_n \rightarrow r_0, z_n \rightarrow z_0$ in $\mathbb{R}$, then P_n converges to $P_0 = (r_0, \theta_0, z_0)$ with respect to metric d .*

PROOF. Since $z_n \rightarrow z_0$ in $\mathbb{R}$. By dropping finitely many beginning terms, we can assume $z_n \in [z_0 - 1, z_0 + 1]$. Define a smooth curve from P_0 to P_n as $l(t) = (r_0 + (r_n - r_0)t, \theta_0 + (\theta_n - \theta_0)t, z_0 + (z_n - z_0)t)$. Then

$$\begin{aligned} d(P_0, P_n) &\leq \|l\| = \int_0^1 e^{z(t)} \sqrt{(r_n - r_0)^2 + f^2(r_0 + (r_n - r_0)t)(\theta_n - \theta_0)^2 + (z_n - z_0)^2} dt \\ &\leq e^{z_0+1} \sqrt{(r_n - r_0)^2 + (\theta_n - \theta_0)^2 + (z_n - z_0)^2} \end{aligned}$$

So the result follows. $\square$

Now we are ready to prove Theorem 8.1.

PROOF. Let $\{P_n\}_{n=1}^\infty$ be a Cauchy sequence in M . By Lemma 8.4, we have either $\lim z_n = -\infty$ or $\lim z_n = z_0$ for some $z_0 \in \mathbb{R}$. In the first case, we have

$$d(P_n, (0, 0, -n)) \leq e^{z_n} + e^{-n} \rightarrow 0.$$

Therefore Cauchy sequence $\{P_n\}$ and $\{(0, 0, -n)\}$ are equivalent to each other.

In the second case that $z_0 = \lim z_n \in \mathbb{R}$, we can assume $z_n \in [z_0 - 1, z_0 + 1]$ for any $n \in \mathbb{Z}^+$. By Lemma 8.5, we know that $\{r_n\}$ is a Cauchy sequence in $\mathbb{R}$. Let $r_0 = \lim r_n$. We have two sub-cases, either $r_0 = 0$ or $r_0 > 0$. When $r_0 = 0$, take a smooth curve l from $(0, 0, z_0)$ to P_n as $l(t) = (r_n t, \theta_n t, z_0 + (z_n - z_0)t)$. Then

$$\begin{aligned} d((0, 0, z_0), P_n) &\leq \|l\| = \int_0^1 e^{z(t)} \sqrt{r_n^2 + f^2(r_n t) \theta_n^2 + (z_n - z_0)^2} dt \\ &\leq e^{z_0+1} \int_0^1 \sqrt{r_n^2 + 4\pi^2 f^2(r_n t) + (z_n - z_0)^2} dt \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore $P_n \rightarrow (0, 0, z_0)$ in M .

In the second sub-case that $r_0 > 0$, by Lemma 8.6, we have that $\lim \theta_n = \theta_0$ for some $\theta_0 \in \mathbb{R}$. Then by Lemma 8.7, we have that P_n converges to $P_0 = (r_0, \theta_0, z_0)$ in M . So the result follows. $\square$

THEOREM 8.2. *The capacity of $\partial_c M = \{\infty\} \subset \overline{M}_c$ is zero.*

PROOF. Define a decreasing function $\varphi \in C^\infty(\mathbb{R})$ such that

$$\varphi(z) = \begin{cases} 1 & z \leq 0 \\ 0 & z \geq 1. \end{cases}$$

For any $a \in \mathbb{R}$, define $\varphi_a \in C^\infty(M)$ as $\varphi_a(P) = \varphi(z - a)$ for any $P = (r, \theta, z) \in M$. Then $\varphi = 1$ on $H_{(-\infty, a)} = B(\infty, e^a)$ and $\varphi = 0$ outside $H_{(-\infty, a+1)} =$

$B(\infty, e^{a+1})$. Then

$$\begin{aligned} \int_M \varphi_a^2 dV_g &\leq \int_{H_{(-\infty, a+1)}} dV_g \\ &= \int_{-\infty}^{a+1} \int_0^{2\pi} \int_0^\infty e^{3z} f(r) dr d\theta dz \\ &= 2\pi e^{3a+3} \int_0^\infty f(r) dr \\ &\rightarrow 0, \quad \text{as } a \rightarrow -\infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_M |\nabla \varphi_a|_g^2 dV_g &= \int_{H_{(a, a+1)}} |\varphi'(z-a)|^2 e^{-2z} dV_g \\ &= \int_a^{a+1} \int_0^{2\pi} \int_0^\infty |\varphi'(z-a)|^2 e^z f(r) dr d\theta dz \\ &\leq 2\pi(e^{a+1} - e^a) \sup_{\mathbb{R}} |\varphi'| \int_0^\infty f(r) dr \\ &\rightarrow 0, \quad \text{as } a \rightarrow -\infty. \end{aligned}$$

Therefore the result follows. $\square$

PROPOSITION 8.8. *Let $o = (0, 0, 0)$. Then $\overline{B(o, 2)} \setminus B(\infty, e^{-1})$ is not compact in M .*

PROOF. By Corollary 8.2, we have

$$\overline{B(o, 2)} - B(\infty, e^{-1}) \supset H_{(-\infty, 0]} - H_{(-\infty, -1)} = H_{(-1, 0]}.$$

Consider the sequence $P_n = (n, 0, 0)$ in $H_{(-1, 0]}$. We claim

$$(8.8) \quad d(P_m, P_n) \geq \min(e^{-1}, 1 - e^{-1}) \quad \text{for any } m \neq n.$$

Let $l : [0, 1] \rightarrow M$ be an arbitrary smooth curve from P_m to P_n . Then either $l \subset H_{(-1, +\infty)}$ or l will hit the plane $z = -1$. In the first case, we have

$$\|l\| \geq \int_0^1 e^{z(t)} |\dot{r}(t)| dt \geq e^{-1} |r_m - r_n| \geq e^{-1}$$

In the second case, take $t = t_0$ be the first time l hit the plane $z = -1$. Then

$$\|l\| \geq \int_0^{t_0} e^{z(t)} |\dot{z}(t)| dt \geq e^{z(0)} - e^{z(t_0)} = 1 - e^{-1}.$$

Combining these two cases, we have $\|l\| \geq \min(e^{-1}, 1 - e^{-1})$ for any piecewise smooth curve from P_m to P_n . So the claim follows. Therefore, there is no convergent subsequence of $\{P_n\}$ and thus the result follows. $\square$

References

- [1] Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR1835418
- [2] Nelia Charalambous and Zhiqin Lu, *On the spectrum of the Laplacian*, Math. Ann. **359** (2014), no. 1-2, 211–238, DOI 10.1007/s00208-013-1000-8. MR3201899

- [3] Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda, *Dirichlet forms and symmetric Markov processes*, Second revised and extended edition, De Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 2011. MR2778606
- [4] Matthew P. Gaffney, *The harmonic operator for exterior differential forms*, Proc. Nat. Acad. Sci. U. S. A. **37** (1951), 48–50, DOI 10.1073/pnas.37.1.48. MR0048138
- [5] Matthew P. Gaffney, *Hilbert space methods in the theory of harmonic integrals*, Trans. Amer. Math. Soc. **78** (1955), 426–444, DOI 10.2307/1993072. MR0068888
- [6] Peter Li and Gang Tian, *On the heat kernel of the Bergmann metric on algebraic varieties*, J. Amer. Math. Soc. **8** (1995), no. 4, 857–877, DOI 10.2307/2152831. MR1320155
- [7] Zhiqin Lu and Michael R. Douglas, *Gauss-Bonnet-Chern theorem on moduli space*, Math. Ann. **357** (2013), no. 2, 469–511, DOI 10.1007/s00208-013-0907-4. MR3096515
- [8] Zhiqin Lu and Xiaofeng Sun, *On the Weil-Petersson volume and the first Chern class of the moduli space of Calabi-Yau manifolds*, Comm. Math. Phys. **261** (2006), no. 2, 297–322, DOI 10.1007/s00220-005-1441-3. MR2191883
- [9] Xiaonan Ma and George Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007. MR2339952
- [10] Jun Masamune, *Essential self-adjointness of Laplacians on Riemannian manifolds with fractal boundary*, Comm. Partial Differential Equations **24** (1999), no. 3-4, 749–757, DOI 10.1080/03605309908821442. MR1683058
- [11] Jun Masamune, *Analysis of the Laplacian of an incomplete manifold with almost polar boundary*, Rend. Mat. Appl. (7) **25** (2005), no. 1, 109–126. MR2142127
- [12] Peter Petersen, *Manifold theory*, Available online at <http://www.math.ucla.edu/~petersen/manifolds.pdf>.
- [13] Michael Reed and Barry Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR0493420
- [14] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I*, 2nd ed., Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980. Functional analysis. MR751959
- [15] Emil J. Straube, *Lectures on the L^2 -Sobolev theory of the $\bar{\partial}$ -Neumann problem*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2010. MR2603659
- [16] Eckart Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995. MR1368632

DEPARTMENT OF MATHEMATICS, UC IRVINE, IRVINE, CALIFORNIA 92617
 Email address: zlu@uci.edu

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218
 Email address: hxu@math.jhu.edu