ESSENTIAL SELF-ADJOINTNESS OF THE WAVE OPERATOR AND THE LIMITING ABSORPTION PRINCIPLE ON LORENTZIAN SCATTERING SPACES

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Abstract. We discuss the essential self-adjointness of wave operators, as well as the limiting absorption principle, in generalizations of asymptotically Minkowski settings. This is obtained via using a Fredholm framework for inverting the spectral family first, and then refining its conclusions to show dense range of $\Box - \lambda$, $\lambda \notin \mathbb{R}$, in L^2_{sc} when acting on an appropriate subdomain.

1. Introduction

In this short note we discuss the self-adjointness of the wave operator on generalizations of Minkowski space, answering a question of Jan Dereziński. More precisely, the setting is that of non-trapping sc-metrics, an extension of Lorentzian scattering metrics introduced in [2] and studied in more detail in [9], [6] and [22], with the Feynman propagator, whose role is discussed below, being particularly closely examined in the latter two papers. These are Lorentzian analogues of the Riemannian scattering metrics introduced by Melrose in [12]. The non-trapping condition is a condition on null-geodesics on M, namely they should converge to a replacement of the light cone at infinity in both the forward and backward directions. In fact, the signature of the metric makes no difference; the same conclusion is true for non-trapping semi-Riemannian metrics of any signature, as the proof goes through without any changes. Later on, we also discuss the limiting absorption principle, for which there is also a 'non-trapping at energy λ' condition which is a condition on the limiting geodesics at infinity corresponding to the spectral parameter being considered, see Section 2 for detail.

For the statement of the first result recall that for a (densely defined) unbounded operator L self-adjointness is a symmetry *plus* an invertibility (for the operator $L \pm \imath$), or equivalently surjectivity statement; essential self-adjointness amounts to a symmetry plus a dense range statement for the operator $L \pm \imath$.

Theorem 1. *Suppose (M, g) is a non-trapping Lorentzian sc-metric. Then $\Box_g$ is essentially self-adjoint on $\mathcal{C}^\infty_c(M^\circ)$.*

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As far as the author is aware, the first general mathematically precise result in this direction is that of Dereziński and Siemssen [5], see also [4], who assumed time-translation invariance, though there is a long history in the physics literature of treating the wave operator as at least a potentially self-adjoint operator. Note that this time translation invariance means that the overlap of the present paper with [5] is minimal. Our result also relates to recent/ongoing work of Nakamura and Taira, see [15], with details of the directly relevant aspects being parts of works in progress. In the work of Nakamura and Taira [15], the non-trapping condition (for the purpose of essential self-adjointness) is replaced by a positive energy condition.

The key part of proving the theorem is to show that $(\Box_g \pm \imath)u = f$ is solvable when $f \in \mathcal{C}_c^\infty(M^\circ)$. Concretely, we take

$$D = \{u \in H_{\mathrm{sc}}^{1,-1/2} \cap L^2_{\mathrm{sc}} : \Box_g u \in L^2_{\mathrm{sc}}\},$$

the weighted scattering Sobolev spaces being recalled below, and then a straightforward regularization argument shows that $\mathcal{C}_c^\infty(M^\circ)$ is dense in it and $\Box_g$ is symmetric on it. The main point is thus to show that $\mathcal{C}_c^\infty(M^\circ) \subset (\Box_g \pm \imath)D \subset L^2_{\mathrm{sc}}$ and hence $(\Box_g \pm \imath)D$ is dense in L^2_{sc}. We do so by using a Fredholm framework for inverting $\Box_g - \lambda$ on appropriate variable order Sobolev spaces discussed below; this in fact works uniformly to the real axis in λ, thus giving the limiting absorption principle. We then show additional regularity of the solution, proving Theorem 1.

The aforementioned Fredholm framework gives rise to the massive Feynman propagators via the limiting absorption principle. Gérard and Wrochna studied these in a different Fredholm setting in [8, 7], based in part on earlier work of Bär and Strohmaier [1].

We finally comment on some generalizations. Considering electromagnetic potentials A amounts to working with $(-\imath d - A)_g^*(-\imath d - A) + V$. If A, V are real and symbols of negative order (thus decaying) with values in one-forms, resp. scalars, all our results and arguments are unaffected. If A, V are real and symbols of order 0 then essential self-adjointness (including its proof) is unaffected.

Working with the wave operator on differential forms of any (or all) form degree or on tensors again does not affect the Fredholm theory; again adding to these symmetric (on $\mathcal{C}_c^\infty(M^\circ)$ with values in forms or tensors, with the L^2-inner product) first order operators with symbolic of negative order coefficients (such as the electromagnetic terms above) does not affect the Fredholm theory either. In particular, the limiting absorption principle holds in the sense of Theorem 9, i.e. a limiting resolvent exists, assuming that at $\lambda \in \mathbb{R}$ the a priori finite dimensional nullspace of $P = \Box_g - \lambda$ and P^* is trivial. (*Both* of these have to be assumed to be trivial: in Theorem 9, index 0 considerations mean that only one of these need to be assumed.) However, the argument for showing the triviality of the a priori finite dimensional nullspace of $P = \Box_g - \lambda$, $\lambda \notin \mathbb{R}$, and its adjoint in Lemma 4 would be affected since the inner product with respect to which the operator is symmetric is no longer positive definite. However, if the operator has additional structure, the nullspace may be shown to be trivial by other arguments, e.g. Wick rotations work in the case of translation invariant metrics on $\mathbb{R}^n$; the perturbation stability of the Fredholm framework implies the same conclusion on perturbations of these in the scattering category. Hence, in these cases (when the conclusion of Lemma 4 holds), the essential self-adjointness also holds in the sense that all of the above statements regarding D continue to hold.

2. Background

We now recall some background. We refer to [12] for the introduction of scattering, or sc-, structures, and to [17, 21] for another discussion which emphasizes an $\mathbb{R}^n$-based perspective localizing to asymptotic cones. Recall that on a manifold with boundary M, the space of *b-vector fields* $\mathcal{V}_{\mathrm{b}}(M)$ is the Lie algebra of smooth vector fields tangent to ∂M (indeed, this is the definition even for manifolds with corners, which will be used below for the compactified cotangent bundle), while the space of *scattering vector fields or sc-vector fields* is $\mathcal{V}_{\mathrm{sc}}(M) = x\mathcal{V}_{\mathrm{b}}(M)$, where x is any *boundary defining function*, i.e. a non-negative $\mathcal{C}^\infty$ function on M, with zero set exactly ∂M such that dx is non-zero at ∂M. Such vector fields are exactly all smooth sections of a vector bundle, ${}^{\mathrm{sc}}TM$, over M, called the *scattering tangent bundle*, which over the interior M° is naturally identified with TM°. Indeed, notice that in a local coordinate chart, in which x is one of the coordinates, and the other coordinates (coordinates on ∂M) are $y_1, \dots, y_{n-1}$, $V \in \mathcal{V}_{\mathrm{sc}}(M)$ means exactly that $V = a_0(x^2\partial_x) + \sum_{j=1}^{n-1} a_j(x\partial_{y_j})$ with a_j smooth in the chart, so $x^2\partial_x$, $x\partial_{y_1}, \dots, x\partial_{y_{n-1}}$ give a local basis of smooth sections, and thus a local basis for the fibers of the vector bundle ${}^{\mathrm{sc}}TM$. Hence, the a_j are coordinates on the fibers of ${}^{\mathrm{sc}}TM$ (locally), and thus $x, y_1, \dots, y_{n-1}, a_0, a_1, \dots, a_{n-1}$ are local coordinates on the bundle ${}^{\mathrm{sc}}TM$. There is a dual vector bundle, ${}^{\mathrm{sc}}T^*M$, called the *scattering cotangent bundle*, with local basis $\frac{dx}{x^2}, \frac{dy_1}{x}, \dots, \frac{dy_{n-1}}{x}$. A *sc-metric of signature* $(k, n-k)$ is then a smooth (in the base point p) non-degenerate symmetric bilinear map ${}^{\mathrm{sc}}T_pM \times {}^{\mathrm{sc}}T_pM \to \mathbb{R}$ of signature $(k, n-k)$. Equivalently, it is a smooth section of ${}^{\mathrm{sc}}T^*M \otimes_s {}^{\mathrm{sc}}T^*M$ (symmetric tensor product) of the appropriate signature. Then L^2_{sc} is the L^2-space of the metric density of any sc-metric (either Lorentzian or Riemannian, or of another definite signature), with all choices being equivalent in that they define the same space and equivalent norms, and $H^{s,r}_{\mathrm{sc}}$ is the corresponding weighted Sobolev space, so if $s \geq 0$ is an integer then

$$H^{s,0}_{\mathrm{sc}} = \{u \in L^2_{\mathrm{sc}}(M) : \ \forall k \leq s \ \forall V_1, \dots V_k \in \mathcal{V}_{\mathrm{sc}}(M), \ V_1 \dots V_k u \in L^2_{\mathrm{sc}}\},$$

and $H^{s,r}_{\mathrm{sc}}(M) = x^r H^{s,0}_{\mathrm{sc}}(M)$.

As an example, M could be the *radial compactification* $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$, in which a sphere $\mathbb{S}^{n-1}$ is attached as the ideal boundary of $\mathbb{R}^n$, so the compactification is diffeomorphic to a closed ball. Concretely, a neighborhood of the boundary is diffeomorphic to $[0,\epsilon)_x \times \mathbb{S}^{n-1}$, $\epsilon > 0$, whose interior, $(0,\epsilon)_x \times \mathbb{S}^{n-1}$ is identified with the subset $\{z \in \mathbb{R}^n : \ |z| > \epsilon^{-1}\}$ via the reciprocal spherical coordinate map, $(0,\epsilon) \times \mathbb{S}^{n-1} \ni (x, \omega) \mapsto x^{-1}\omega \in \mathbb{R}^n$, where the sphere is regarded as a submanifold of $\mathbb{R}^n$ to make sense of the map. Then any translation invariant metric of any signature is (i.e. can be naturally identified with) a sc-metric of the same signature. In fact, $\mathcal{C}^\infty(M)$ is then the space of *classical (one-step polyhomogeneous) symbols of order* 0 on $\mathbb{R}^n$, $\dot{\mathcal{C}}^\infty(M)$ (the space of $\mathcal{C}^\infty$ functions on M vanishing to infinite order at ∂M) is the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{V}_{\mathrm{sc}}(M)$ is spanned by the lift of translation invariant vector fields ∂_{z_j}, $j = 1, \dots, n$, over $\mathcal{C}^\infty(M)$, i.e. an element of $\mathcal{V}_{\mathrm{sc}}(M)$ is of the form $\sum a_j \partial_{z_j}$ with $a_j \in \mathcal{C}^\infty(M)$, i.e. a classical symbol of order 0, and similarly ${}^{\mathrm{sc}}T^*M \otimes_s {}^{\mathrm{sc}}T^*M$ is spanned by the lifts of $dz_i \otimes_s dz_j$ with $\mathcal{C}^\infty(M)$ coefficients. Correspondingly, L^2_{sc} is the standard Lebesgue space, $H^{s,r}_{\mathrm{sc}}$ the standard weighted Sobolev space.

More generally, a coordinate neighborhood of a point on the boundary of a manifold with boundary M can be identified with a similar coordinate neighborhood of a point on the boundary of $\overline{\mathbb{R}^n}$. Note that from the perspective of $\mathbb{R}^n$, such a neighborhood is asymptotically conic. Rather than following the above intrinsic definitions, one could transplant the notions discussed above directly from $\mathbb{R}^n$ via such an identification, *exactly* how standard notions on manifolds without boundary are defined by identifying coordinate charts with open subsets of $\mathbb{R}^n$; this is the approach taken by [21, 17].

In particular, this gives a convenient way of introducing scattering pseudodifferential operators $\Psi^{m,l}_{\mathrm{scc}}(M)$ by reducing to the case of $\overline{\mathbb{R}^n}$, or equivalently to an appropriate uniform structure in the interior, $\mathbb{R}^n$. (The notation Ψ_{scc} stands for 'scattering conormal'; [12] uses Ψ_{sc} for 'classical' scattering; classical symbols are the one-step polyhomogeneous ones.) In the present case these are simply quantizations of *(product-type) symbols* of order (m,l), $a \in S^{m,l}$, i.e. $\mathcal{C}^\infty$ functions on $\mathbb{R}^n_z \times \mathbb{R}^n_\zeta$ such that for all α, β

$$|(D_z^\alpha D_\zeta^\beta a)(z,\zeta)| \leq C_{\alpha\beta}\langle z\rangle^{l-|\alpha|}\langle\zeta\rangle^{m-|\beta|}.$$

Note that as in [17, 21], the second, decay order, uses the *opposite sign convention* than Melrose's original definition [12]; thus, the space $\Psi^{m,l}_{\mathrm{scc}}(M)$ gets bigger with increasing m, l. One also has variable order pseudodifferential symbols and operators. In this paper the relevant order is the second, decay order, which we must allow to vary, thus l is in $S^{0,0}$; for this purpose we also need to relax the *type* of the symbol estimate and allow small power (more optimally logarithmic) losses: thus the estimates for $a \in S^{m,l}_\delta$ are, for fixed $\delta \in (0, 1/2)$, which could be taken small as one wishes for our purposes,

$$|(D_z^\alpha D_\zeta^\beta a)(z,\zeta)| \leq C_{\alpha\beta}\langle z\rangle^{l(z,\zeta)-|\alpha|+\delta(|\alpha|+|\beta|)}\langle\zeta\rangle^{m-|\beta|+\delta(|\alpha|+|\beta|)}.$$

The standard concepts, such as the principal symbol, still work; in the present case it lies in $S^{m,l}_\delta / S^{m-1+2\delta,l-1+2\delta}_\delta$, and it is multiplicative, i.e. the principal symbol of a product is the product of the principal symbols, of course with the appropriate orders as usual. Since the choice of δ is usually irrelevant, we typically suppress it in the subscripts. One can thus define the ellipticity, etc., as usual. Then the elements of $\Psi^{0,0}_{\mathrm{scc}}$ are bounded operators on all weighted Sobolev spaces $H^{s,r}_{\mathrm{sc}}$, and one can define variable order Sobolev spaces (with just r variable for notational simplicity) by taking $r_0 < \inf r$, and $A \in \Psi^{s,r}_{\mathrm{scc}}$ elliptic, and saying

$$H^{s,r}_{\mathrm{sc}} = \{u \in H^{s,r_0}_{\mathrm{sc}} : \ Au \in L^2_{\mathrm{sc}}\},$$

with the norm whose square is

$$\|u\|^2_{H^{s,r}_{\mathrm{sc}}} = \|u\|^2_{H^{s,r_0}_{\mathrm{sc}}} + \|Au\|^2_{L^2_{\mathrm{sc}}};$$

see [17, 21] for details.

It is also important that in addition to principal symbols of products, we can compute principal symbols of commutators. Concretely, if $A \in \Psi^{s,r}_{\mathrm{scc}}$ and $B \in \Psi^{s',r'}_{\mathrm{scc}}$ then

$$[A,B] \in \Psi^{s+s'-1+2\delta,r+r'-1+2\delta}_{\mathrm{scc}}$$

and with a, resp. b, denoting the principal symbols of A, resp. B, its principal symbol is the Poisson bracket $\frac{1}{i}\{a,b\}$ (which, recall, arises from the symplectic structure on T^*M°). In the case of M being the radial compactification of $\mathbb{R}^n$, this is simply

$$\frac{1}{i}\{a,b\}=\frac{1}{i}\sum_{j=1}^{n}(\partial_{\zeta_j}a)(\partial_{z_j}b)-(\partial_{z_j}a)(\partial_{\zeta_j}b).$$

Writing coordinates on the fibers of the scattering cotangent bundle as $\tau,\mu_1,\ldots,\mu_{n-1}$, sc-dual to the coordinates $x,y_1,\ldots,y_{n-1}$ discussed above, i.e. sc-covectors are written as

$$\tau\,\frac{dx}{x^2}+\sum_j\mu_j\,\frac{dy_j}{x}$$

we have

$$\begin{aligned}\{a,b\}=H_ab=&x\Big((\partial_\tau a)\big((x\partial_x+\mu\cdot\partial_\mu)b\big)-\big((x\partial_x+\mu\cdot\partial_\mu)a\big)(\partial_\tau b)\\&+\sum_j\big((\partial_{\mu_j}a)(\partial_{y_j}b)-(\partial_{y_j}a)(\partial_{\mu_j}b)\big)\Big),\end{aligned}\tag{1}$$

see [12, Equation (5.24)], as follows by a change of variables computation. Here H_a is called the *Hamilton vector field* of a.

Finally, we need to discuss microlocalization. Since there are two different important behaviors, in the case of $T^*\mathbb{R}^n$ these being $|z|\to\infty$ and $|\zeta|\to\infty$, it is *even more useful to compactify phase space* than in the usual microlocal analysis setting, where using homogeneity in dilations of the fibers of the cotangent bundle is an effective substitute. In the case of $T^*\mathbb{R}^n$, this compactified phase space is

$$\overline{{}^{\mathrm{sc}}T^*\mathbb{R}^n}=\overline{\mathbb{R}^n_z}\times\overline{\mathbb{R}^n_\zeta}.$$

i.e. we compactify the position and the momentum space separately using the above radial compactification. Thus $\overline{{}^{\mathrm{sc}}T^*\mathbb{R}^n}$ is the product of two closed balls, and hence it is a manifold with corners. The two boundary hypersurfaces are $\overline{\mathbb{R}^n}\times\partial\overline{\mathbb{R}^n}$, which is *'fiber infinity'*, where standard microlocal analysis takes place, and $\partial\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$, i.e. *'base infinity'*; these intersect in the corner $\partial\overline{\mathbb{R}^n}\times\partial\overline{\mathbb{R}^n}$. The locus of microlocalization is then

$$\partial\overline{{}^{\mathrm{sc}}T^*\mathbb{R}^n}=\partial\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}\cup\overline{\mathbb{R}^n}\times\partial\overline{\mathbb{R}^n};$$

thus the *elliptic set*, the *characteristic set* and the *wave front set* are subsets of this.

These notions immediately extend to general manifolds via the local coordinate identifications. The general compactified phase space is the fiber-radial compactification $\overline{{}^{\mathrm{sc}}T^*}M$ of ${}^{\mathrm{sc}}T^*M$; the locus of microlocalization is its boundary

$$\partial\overline{{}^{\mathrm{sc}}T^*}M=\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M\cup{}^{\mathrm{sc}}S^*M,$$

where ${}^{\mathrm{sc}}S^*M$ is fiber infinity, i.e. the boundary of the fiber compactification. One can take x as a boundary defining function of base infinity $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M$ and (with the local coordinate notation from above) $\rho_\infty=\langle(\tau,\mu)\rangle^{-1}=(|(\tau,\mu)|^2+1)^{-1/2}$ as a defining function of fiber infinity, where $|.|$ is the length with respect to any Riemannian sc-metric. Relating to the above discussion of variable order spaces, one may have e.g. the decay order as smooth function on $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M$; in order to match with the previous definition one extends it to a smooth function on all of $\overline{{}^{\mathrm{sc}}T^*}M$, with all potential extensions resulting in exactly the same Sobolev space as can be seen immediately from the definition.

Note that in view of the vanishing factor x on the right hand side of (1), as well as the one order lower than that of a homogeneity of

$$H_a = x\Big((\partial_\tau a)(x\partial_x + \mu\cdot\partial_\mu) - \big((x\partial_x + \mu\cdot\partial_\mu)a\big)\partial_\tau + \sum_j \big((\partial_{\mu_j} a)\partial_{y_j} - (\partial_{y_j} a)\partial_{\mu_j}\big)\Big), \tag{2}$$

one may want to rescale H_a, factoring out this vanishing. In order to obtain a well-behaved, namely smooth and at least potentially non-degenerate, vector field on the compactification, one may consider

$$H_{a,s,r} = x^{r-1}\rho_\infty^{s-1} H_a.$$

If a is classical this becomes a smooth vector field tangent to the boundary of the compactified space $\overline{{}^{\mathrm{sc}}T^*}M$, and thus defines a flow on it (the *Hamilton flow*); for general a the vector field is conormal of order $(0,0)$ as a vector field tangent to the boundary, i.e. is in $S^{0,0}\mathcal{V}_{\mathrm{b}}(\overline{{}^{\mathrm{sc}}T^*}M)$. Note that the defining functions we factored out are defined only up to a smooth positive multiple (smooth on the compact space, thus this means bounded from above and below by positive constants in particular), and hence the rescaled vector field is only so defined, but such a change merely reparameterizes the flow, which is irrelevant for considerations below. Radial points of H_a are then points on the boundary of $\overline{{}^{\mathrm{sc}}T^*}M$ at which the $H_{a,s,r}$ vanishes (as a vector in $T\overline{{}^{\mathrm{sc}}T^*}M$), thus are critical points of the flow. Note that at such points $H_{a,s,r}$ need *not* vanish as a b-vector field, so e.g. it may have a non-trivial $x\partial_x$ component (or the analogous statement at fiber infinity); indeed this non-vanishing is what makes the radial point non-degenerate: in analytic estimates $x\partial_x$ hitting the weight x^m of a commutant is what can give a contribution of a definite sign.

After these differentiable manifold structure type discussion in the sc-category, we briefly discuss the geometry, namely metrics. For the purposes of the present paper what we need is that the metric g is a Lorentzian (or more general pseudo-Riemannian) sc-metric for which the Hamilton flow has a source/sink structure. Thus, we need that there are submanifolds of $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M$ which are transversal to ${}^{\mathrm{sc}}S^*_{\partial M}M$ and are normal sources (meaning normally to the submanifold they are sources) L_-, resp. sinks L_+, of the Hamilton flow of the dual metric function, G (the principal symbol of $\Box_g$); here the source/sink is understood in a non-degenerate sense for the Hamilton vector field $H_{G,2,0}$. Notice that this means that $\rho_\infty\partial_\tau G > 0$ at sources and $\rho_\infty\partial_\tau G < 0$ at sinks (in the sense of bounded away from 0) by (2). The definition of a *non-trapping sc-metric* g is then that g is a sc-metric (of some signature) such that all integral curves of H_G inside the characteristic set $\{G = 0\}$ at ${}^{\mathrm{sc}}S^*M$ (i.e. fiber infinity), except those contained in $L_\pm$, tend to L_+ (indeed, necessarily to $L_+ \cap {}^{\mathrm{sc}}S^*M$) in the forward and L_- in the backward direction. We call a sc-metric *non-trapping at energy* λ if it is non-trapping in the sense above, and in addition, all integral curves of H_G inside the λ characteristic set $\{G = \lambda\}$ at $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M$ (i.e. base infinity), except those contained in $L_\pm$, tend to L_+ in the forward and L_- in the backward direction.

The class of non-trapping Lorentzian sc-metrics is a much larger class of metrics than that of Lorentzian scattering metrics introduced in [2, Section 3.2]. Indeed the latter class also demands that at fiber infinity $L_\pm$ be halves of the (scattering) conormal bundles of submanifolds $S_\pm$ of ∂M, locally given by $v = 0$ within $x = 0$, at which the metric has a certain model form, generalizing that of the Minkowski metric on the radial compactification. This implies the source/sink structure at sc-fiber infinity, see [2, Section 3.6], which is in the b-setting at fiber infinity for a conformal multiple of the operator, but by homogeneity considerations the results are completely analogous for our sc-setting. Concretely, the model form of g is

$$v\frac{dx^2}{x^4} - \Big(\frac{dx}{x^2}\otimes\frac{\alpha}{x} + \frac{\alpha}{x}\otimes\frac{dx}{x^2}\Big) - \frac{\tilde g}{x^2},$$

with $\alpha = \frac{1}{2}\,dv + O(v) + O(x)$ a smooth one-form on M near $v = x = 0$, and $\tilde g$ a smooth symmetric 2-cotensor on M which is positive definite on the joint annihilator of dx and dv, and $L_\pm$ at sc-fiber infinity are given by the boundary of the span of $\frac{dv}{x}$ at $v = 0, x = 0$. Indeed, G at the span of $\frac{dv}{x}$ at $v = 0, x = 0$ is, modulo quadratically vanishing terms,

$$-4v\gamma^2 - 4\tau\gamma,$$

where we write sc-one-forms as

$$\tau\,\frac{dx}{x^2} + \frac{\gamma\,dv + \eta\,dw}{x},$$

w local coordinates on $S_\pm$, cf. [2, Equation (3.18)]; the radial sets are thus $\tau = 0$, $\eta = 0$, $x = 0$, $v = 0$, and, taking $\rho_\infty = |\gamma|^{-1}$ locally,

$$\begin{aligned}
&x^{-1}|\gamma|^{-1}H_G\\
&= |\gamma|^{-1}\Big(- 4\gamma(x\partial_x + \gamma\partial_\gamma + \eta\partial_\eta) + (8v\gamma^2 + 4\tau\gamma)\partial_\tau - (8v\gamma + 4\tau)\partial_v + 4\gamma^2\partial_\gamma\Big)\\
&= -4\,\mathrm{sgn}(\gamma)x\partial_x
\end{aligned}$$

there as a b-vector field, so $\gamma > 0$ is the sink, $\gamma < 0$ the source. (But note that only the part at fiber infinity is in the characteristic set for $\lambda \neq 0$!) In the class of Lorentzian scattering metrics one also demands global properties, such as that $\{v > 0,\ x = 0\}$ has two connected components $C_\pm$ ('spherical caps' in Minkowski space, and though they may not be spherical in any sense, we continue calling them so in general) with $S_\pm$ as their respective boundaries, while $\{v < 0,\ x = 0\} = C_0$ has two boundary components $S_\pm$.

The 'non-trapping at energy λ' condition played no role in [2], since that paper considered the wave equation ($\lambda = 0$) and rescaled the wave operator to a b-operator, for which the non-trapping condition is exactly the above sc-non-trapping condition; this rescaling is possible and the non-trapping claim holds since the dual metric G is necessarily a homogeneous quadratic polynomial on the fibers of ${}^{\mathrm{sc}}T^*M$, unlike $G-\lambda$ for non-zero λ. However, for the particular example of Minkowski space, it is straightforward to check that the source-sink structure and the non-trapping condition hold for any non-zero real λ, with the characteristic set connected for $\lambda < 0$ and having two connected components when $\lambda > 0$. Indeed, using Euclidean variables (z, ζ), ζ is constant along the Hamilton flow, while z moves along a straight line in the direction of the tangent vector $G(\zeta)$, so the radial set at ${}^{\mathrm{sc}}T^*_{\partial M}M$ (with ∂M considered as the 'sphere at infinity' identified with the unit *Euclidean* sphere) is where the direction $\frac{z}{|z|}$ (with $|z|$ the *Euclidean* metric length) of z is that of $\pm G(\zeta)$; note that the characteristic set is where $G(\zeta, \zeta) = \lambda$, with the intersection of the two giving an $(n-1)$-dimensional submanifold in ${}^{\mathrm{sc}}T^*_{\partial M}M$ with $\zeta \in (\mathbb{R}^n)^*$ satisfying $G(\zeta, \zeta) = \lambda$ a coordinate along it, and the source is where z and $G(\zeta)$ are anti-aligned, the sink is where they are aligned. Since the propagation estimates are stable under perturbations, see [16, Section 2.7], they also hold for sc-metrics that are suitably close (in C^∞) to the Minkowski metric.

3. Fredholm theory and essential self-adjointness

In our proof of the main theorem we focus on the Lorentzian case of signature $(1, n-1)$ for the sake of being definite in terminology; the general pseudo-Riemannian case barely differs, except that the *only* reasonable problems are the Feynman and anti-Feynman problems, but they are also *the only ones that matter* below. In all of our discussions below we assume that the metrics are *non-trapping sc-metrics.*

In order to get started, we first recall that (assuming M is connected – otherwise the statement is for each connected component) for $\lambda > 0$ the Klein-Gordon operator $P = \Box_g - \lambda \in \Psi_{\mathrm{sc}}^{2,0}$ has four Fredholm problems, see [20], as well as [21, Section 5.4.8] for a detailed discussion, corresponding to the characteristic set having two connected components. (This follows from the characteristic set $\{\zeta :\ G(\zeta,\zeta) = \lambda\}$, $G = g^{-1}$, having two connected components fiberwise: this fiberwise characteristic set is the two-sheeted hyperboloid.) Indeed, in each connected component of the characteristic set one can choose the direction in which one propagates estimates for P, and then for P^* using dual spaces one propagates the estimates in the opposite direction, resulting in 2^2 possibilities. Concretely, these are the retarded, advanced, Feynman and anti-Feynman, Fredholm problems; the direction of propagation is encoded by the use of appropriate weighted Sobolev type spaces. Concretely, these are based on variable order Sobolev spaces $H_{\mathrm{sc}}^{s,r}$, where s is constant, r variable, a function on $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M$, monotone along the rescaled Hamilton flow, and satisfies the inequalities $r > -1/2$ at the radial points from which estimates are propagated, $r < -1/2$ at the radial points to which estimates are propagated, and

$$\mathcal{Y}^{s-1,r+1} = H_{\mathrm{sc}}^{s-1,r+1},\ \mathcal{X}^{s,r} = \{u \in H_{\mathrm{sc}}^{s,r} :\ (\Box_g - \lambda)u \in H_{\mathrm{sc}}^{s-1,r+1}\}, \tag{3}$$

with

$$P = \Box_g - \lambda : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r+1}$$

Fredholm. Here the threshold value $-1/2$ for the Sobolev order arises from the radial point estimates for *formally self-adjoint* P; in general, for an operator $P \in \Psi_{\mathrm{sc}}^{m,l}$ with $P = P^*$ with the same characteristic set geometry as ours, the threshold order is $\frac{l-1}{2}$, see [21, Proposition 5.26] (where the geometry, namely the role of m, l are reversed). Note also that the target space, $\mathcal{Y}^{s-1,r+1}$ loses one order, i.e. requires one extra order of regularity, in both senses relative to what one would expect given a domain $H_{\mathrm{sc}}^{s,r}$ and an *elliptic* element of $\Psi_{\mathrm{sc}}^{2,0}$ (namely $H_{\mathrm{sc}}^{s-2,r}$) necessitating the modification of $H_{\mathrm{sc}}^{s,r}$ to arrive at $\mathcal{X}^{s,r}$. Our P is not elliptic of course, and this is the real principal type loss of one order relative to elliptic estimates, going back in this manner to Hörmander's theorem on propagation of singularities [10].

To fix terminology we make the definition:

Definition 2. The function spaces in (3) with s constant, r a function on $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M$, monotone along the rescaled Hamilton flow, are called

(i) Feynman, if $r > -1/2$ at the radial sources L_- and $r < -1/2$ at the radial sinks L_+,
(ii) anti-Feynman, if $r > -1/2$ at the radial sinks L_+ and $r < -1/2$ at the radial sources L_-.

Corresponding to the above discussion, the Fredholmness of P on these spaces relies on the non-trapping nature of the bicharacteristic flow within the characteristic set, which has two parts: the part at fiber infinity, in ${}^{\mathrm{sc}}S^*M$, which is independent of λ, and the part at spatial infinity, $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M$, which does depend on λ. In particular, these non-trapping conditions are perturbation stable, and hold for the Minkowski (as well as translation invariant pseudo-Riemannian on $\mathbb{R}^n$) metrics. Note that the choice of s, r with r satisfying the above constraints is irrelevant; and the nullspace automatically lies in the intersection of all these spaces; thus in particular for elements u of the nullspace of P, $\mathrm{WF}_{\mathrm{sc}}(u)$ is a subset of the radial points towards which the estimates are propagated.

If $\lambda < 0$ (and $n = \dim M \geq 3$), then due to the behavior of the characteristic set at base infinity, the characteristic set has only one connected component (the characteristic set $\{\zeta : \ G(\zeta, \zeta) = \lambda\}$ has one connected component fiberwise: this fiberwise characteristic set is the one-sheeted hyperboloid.), and correspondingly only two of these problems remain: Feynman and anti-Feynman problems. Note that the Cauchy problem (or retarded/advances problems) is still solvable, but the solution will typically grow exponentially, thus it does not exist as far as polynomially weighted Sobolev spaces (our world in this paper) are concerned. (For $\lambda = 0$ we still have the four problems, but in weighted b-Sobolev spaces, see [6], which we do not discuss here.)

Now, if λ is made complex, of course the usual principal (and even the subprincipal!) symbol are not affected, and correspondingly estimates at fiber infinity, ${}^{\mathrm{sc}}S^*M$, are unchanged over M°. However, the estimates at $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}M$ become more delicate.

Namely, in this region one has a non-real principal symbol (since λ is part of it). Thus, by the usual propagation estimates (these are the ones used for *'complex absorption'*) one can propagate estimates in the *forward* direction along the $H_{\mathrm{Re}\,p}$ flow when $\mathrm{Im}\,\lambda \geq 0$, and in the *backward* direction when $\mathrm{Im}\,\lambda \leq 0$. (See [21, Section 5.4.5] and [17], as well as the usual microlocal analysis version in [16, Section 2.5].) Of course, the operator is elliptic at *finite* points (not at ${}^{\mathrm{sc}}S^*M$) of $\overline{{}^{\mathrm{sc}}T^*}_{\partial M}$ when $\mathrm{Im}\,\lambda > 0$; the point is that the estimates work at the corner (fiber infinity at ∂M), and they work *uniformly* in λ even as $\mathrm{Im}\,\lambda \to 0$. This propagation works for any s, r (s a priori relevant only when one is at fiber infinity at ∂M), including variable r, when r is monotone decreasing in the direction in which the estimates are propagated.

Notice that corresponding to the ellipticity at finite points, for these estimates, as well as the ones below, the only relevant non-trapping condition is the basic one concerning bicharacteristics at fiber infinity, ${}^{\mathrm{sc}}S^*M$, i.e. the 'non-trapping at energy λ' condition is only relevant when one wants to let λ to the real axis, as we do below for the limiting absorption principle (but not for the essential self-adjointness discussion), and then the relevant condition is non-trapping at *the limiting energy* λ.

Most importantly though one needs to get radial point estimates, however. For real $\lambda \neq 0$ these are two estimates, see [21, Section 5.4.7] and [17], as well as the usual microlocal analysis version in [16, Section 2.4]. The first, high 'regularity' (where here this means decay), which gives a 'free' estimate at the radial point, is of the form

$$\|Qu\|_{H^{s,r}_{\mathrm{sc}}} \leq C(\|Q_1 Pu\|_{H^{s-1,r+1}_{\mathrm{sc}}} + \|Q_1 u\|_{H^{s',r'}_{\mathrm{sc}}} + \|u\|_{H^{M,N}_{\mathrm{sc}}})$$

when $r > r' > -1/2$, s, s', M, N arbitrary (one usually considers s', M, N very large and negative; these are background error terms with relatively compact properties), Q elliptic at the radial set, with wave front set in a small neighborhood, Q_1 elliptic on $\mathrm{WF}'(Q)$ and bicharacteristics from all points in the intersection of $\mathrm{WF}'(Q)$ and the characteristic set tend to L in the appropriate (forward/backward) direction depending on the sink/source nature, remaining in $\mathrm{Ell}(Q_1)$. The second, low 'regularity' (where again here this means decay), which allows one to propagate estimates into the radial point from a punctured neighborhood, is of the form

$$\|Qu\|_{H^{s,r}_{\mathrm{sc}}} \leq C(\|Q_2 u\|_{H^{s,r}_{\mathrm{sc}}} + \|Q_1 Pu\|_{H^{s-1,r+1}_{\mathrm{sc}}} + \|Q_1 u\|_{H^{s',r'}_{\mathrm{sc}}} + \|u\|_{H^{M,N}_{\mathrm{sc}}})$$

when $r < -1/2$, s, s', r', M, N arbitrary (one considers s', r', M, N very large and negative), Q elliptic at the radial set, with wave front set in a small neighborhood, Q_1 elliptic on $\mathrm{WF}'(Q)$ and bicharacteristics from all points in $\mathrm{WF}'(Q)$ intersected with the characteristic set which are not in L, tend to L in the appropriate (forward/backward) direction depending on the sink/source nature, remaining in $\mathrm{Ell}(Q_1)$, and intersect $\mathrm{Ell}(Q_2)$ at some point in the opposite direction along the flow, still remaining in $\mathrm{Ell}(Q_1)$.

Now allowing λ complex, say $\mathrm{Im}\,\lambda \geq 0$, one can only propagate estimates in the forward direction along the $H_{\mathrm{Re}\,p}$-flow, and correspondingly one obtains the high regularity estimates only at the sources, the low regularity ones at the sinks (with sources and sinks reversed for $\mathrm{Im}\,\lambda \leq 0$). These estimates in fact become *stronger* than the ones above, cf. the complex absorption arguments in [21, Section 5.4.5] and [17], namely one can in addition control a term $\|Qu\|_{H^{s-1/2,r+1/2}_{\mathrm{sc}}}$, i.e. one that is stronger in the sense of decay (though not differentiability) than that on Qu above. This results from an extra term $\langle \check{A}^* \,\mathrm{Im}\,\lambda \check{A} u, u\rangle = \mathrm{Im}\,\lambda \|\check{A}u\|^2$ in the estimate, in addition to the commutator terms, $\langle [\check{A}^*\check{A}, \Box]u, u\rangle$, with $\check{A} \in \Psi^{m'/2, l'/2}_{\mathrm{scc}}$, $s = (m'+1)/2$, $r = (l'-1)/2$. Thus the estimates are

$$\|Qu\|_{H^{s,r}_{\mathrm{sc}}} + \mathrm{Im}\,\lambda \|Qu\|_{H^{s-1/2,r+1/2}_{\mathrm{sc}}} \leq C(\|Q_1 Pu\|_{H^{s-1,r+1}_{\mathrm{sc}}} + \|Q_1 u\|_{H^{s',r'}_{\mathrm{sc}}} + \|u\|_{H^{M,N}_{\mathrm{sc}}}) \tag{4}$$

$r > r' > -1/2$, s, s', M, N arbitrary, and

$$\begin{aligned} &\|Qu\|_{H^{s,r}_{\mathrm{sc}}} + \mathrm{Im}\,\lambda \|Qu\|_{H^{s-1/2,r+1/2}_{\mathrm{sc}}} \\ &\qquad \leq C(\|Q_2 u\|_{H^{s,r}_{\mathrm{sc}}} + \|Q_1 Pu\|_{H^{s-1,r+1}_{\mathrm{sc}}} + \|Q_1 u\|_{H^{s',r'}_{\mathrm{sc}}} + \|u\|_{H^{M,N}_{\mathrm{sc}}}) \end{aligned} \tag{5}$$

when $r < -1/2$, s, s', r', M, N arbitrary.

Now, taking r with $-1/2 < r$ at the sources, $-1/2 > r > -1$ at the sinks, monotone along the flow, $s > 1/2$, this in particular gives:

Proposition 3. *(See* [21, Section 5.4.8] *for the real λ version.) Suppose $\lambda \neq 0$. Then for s, r as above corresponding to either the Feynman spaces (*$\mathrm{Im}\,\lambda \geq 0$*) or anti-Feynman spaces (*$\mathrm{Im}\,\lambda \leq 0$*), the operator*

$$P : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r+1}$$

is Fredholm, with

$$\mathcal{Y}^{s-1,r+1} = H^{s-1,r+1}_{\mathrm{sc}},\ \mathcal{X}^{s,r} = \{u \in H^{s,r}_{\mathrm{sc}} :\ Pu \in H^{s-1,r+1}_{\mathrm{sc}}\}.$$

One can interpret the estimates (4)-(5), as well as the analogous real principal type estimates in the characteristic set between the radial points as additional regularity estimates giving that in fact

$$\mathcal{X}^{s,r} = \{u \in H_{\mathrm{sc}}^{s,r} \cap H_{\mathrm{sc}}^{s-1/2,r+1/2} :\ Pu \in H_{\mathrm{sc}}^{s-1,r+1}\}. \tag{6}$$

In particular, this lets one solve $Pu = f$, $f \in \dot{\mathcal{C}}^\infty(M)$, up to finite dimensional obstacles, namely one gets that the solution u (which exists in the complement of a finite dimensional subspace) is almost in L^2, namely $u \in H_{\mathrm{sc}}^{0,-\epsilon}$ for all $\epsilon > 0$. Indeed, we have that if $Pu = f$, $f \in \dot{\mathcal{C}}^\infty(M)$, with $u \in \mathcal{X}^{s,r}$ as above, then $u \in H_{\mathrm{sc}}^{\tilde{s}-1/2,\tilde{r}+1/2}$ for all $\tilde{s}$ and for all $\tilde{r} < -1/2$, thus in $H_{\mathrm{sc}}^{\infty,-\epsilon}$ for all $\epsilon > 0$.

This is not quite sufficient, however, since we want to conclude $u \in L^2_{\mathrm{sc}}$, and also that there are no finite codimension issues (i.e. we have invertibility and not just Fredholmness) so, for $\operatorname{Im}\lambda > 0$, one needs to do a borderline estimate, with $r = -1/2$ at the sink (everywhere else one is in L^2 already), which corresponds to $l' = 0$. Note that such an estimate cannot work when $\operatorname{Im}\lambda = 0$, and thus cannot be uniform in $\operatorname{Im}\lambda$ when $\operatorname{Im}\lambda > 0$. The key point is that in this case the commutator $[\check{A}^*\check{A}, \Box]$ will have principal symbol at ${}^{\mathrm{sc}}S^*M$ for which the normally main term (arising from the weight) vanishes at L.

It suffices for us to consider $m' = l' = 0$, in which case the situation is very simple: we will take $\check{A}$ to be microlocally the identity near the sinks, i.e. to have $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - \check{A})$ disjoint from the sink. (Such microlocalizers play an important role in the proof of asymptotic completeness in the N-body setting; a partially microlocal version is the work of Sigal and Soffer [14] and Yafaev [23], see [19] for a discussion.)

Since $\mathrm{WF}_{\mathrm{sc}}(u)$ is in the sink when $f \in \dot{\mathcal{C}}^\infty(M)$ by the propagation estimates (first regularity at the radial source, propagated along the Hamilton flow to the complement of the sink), for $u \in H_{\mathrm{sc}}^{s',r'}$ (with e.g. $r' = r + 1/2$ from above, so < 0 but close to 0 allowed) the pairing $\langle u, [\check{A}^*\check{A}, \Box]u\rangle$ makes sense if $2r' - l' + 1 \geq 0$ and $2s' - m' - 1 \geq 0$, which holds with $l' = m' = 0$ if $r' < 0$ is close to 0 and $s' = 1$, say. Furthermore, a regularized version, with the regularization being used to expand the commutator, remains bounded: for this one uses an additional regularizer factor $(1 + \epsilon x^{-1})^{-\delta'}$ with $\delta' = -2r' > 0$ so that for $\epsilon > 0$ even $\langle u, \Box\check{A}^*\check{A}u\rangle$ makes sense, cf. $\phi_t(\rho_0^{-1})$ of [21, Equation (5.61)] for the regularizer choice. The regularizer gives the correct sign in the commutator as it behaves exactly the same way as if one had a more decaying weight, i.e. as if $l' < 0$, since

$$\begin{aligned} H_{G,2,0}x^{-2l'}(1+\epsilon x^{-1})^{-\delta'} &= \rho_\infty x^{-1} H_G x^{-2l'}(1+\epsilon x^{-1})^{-\delta'} \\ &= \Big(-2l' + \delta' \frac{\epsilon x^{-1}}{1+\epsilon x^{-1}}\Big) x^{-2l'}(1+\epsilon x^{-1})^{-\delta'}(\rho_\infty x^{-2} H_G x), \end{aligned}$$

with both terms in the big parentheses being ≥ 0 when $l' \leq 0$. Then the $\operatorname{Im}\lambda$ term gives an estimate for $\|\check{A}u\|^2$, which is an estimate for u in $H_{\mathrm{sc}}^{m'/2,l'/2} = L^2_{\mathrm{sc}}$ as desired. In particular, if $u \in \mathcal{X}^{s,r}$ and $Pu \in \dot{\mathcal{C}}^\infty(M)$ then $u \in L^2_{\mathrm{sc}}$.

Lemma 4. *Suppose that* $\operatorname{Im}\lambda > 0$, *and consider the Feynman Fredholm problem* $P : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r}$. *Then* $\mathrm{Ker}P$ *and* $\mathrm{Ker}P^*$ *(on the dual space) are trivial.*

Analogous statements hold for $\operatorname{Im}\lambda < 0$ *for the anti-Feynman Fredholm problem.*

Proof. We have already seen that elements of $\mathrm{Ker}P$ and $\mathrm{Ker}P^*$ lie in L^2_{sc}. Thus, formally the lemma follows from

$$0 = \langle Pu, u\rangle - \langle u, Pu\rangle = \langle (P - P^*)u, u\rangle = -2\imath\langle \operatorname{Im}\lambda u, u\rangle = -2\imath \operatorname{Im}\lambda \|u\|^2,$$

but the issue is that $\langle P^*u, u\rangle$ does not actually make sense a priori due to the too weak a priori differentiability of u when the unweighted spaces are used (all we know is that $u \in L^2_{\mathrm{sc}}$, so $P^* u \in H^{-2,0}_{\mathrm{sc}}$ only, unless we use $P - P^* \in \Psi^{0,0}_{\mathrm{scc}}$, but even then we need to justify the integration by parts (because the adjoint a priori puts us in dual spaces)!), so we need to have a more careful, if standard, regularization argument.

Namely, we take $\Lambda_t \in \Psi^{-\infty,0}_{\mathrm{scc}}$, $t \in [0,1]$ such that the family is uniformly bounded in $\Psi^{0,0}_{\mathrm{scc}}$ and converges to Id in $\Psi^{\epsilon,0}_{\mathrm{scc}}$, $\epsilon > 0$, as $t \to 0$, and thus strongly on L^2_{sc}. Then we have for $t > 0$, if $u \in L^2_{\mathrm{sc}}$ and $Pu \in L^2_{\mathrm{sc}}$,

$$\begin{aligned}
0 = \langle Pu, u\rangle - \langle u, Pu\rangle &= \lim_{t\to 0}\big(\langle \Lambda_t Pu, u\rangle - \langle \Lambda_t u, Pu\rangle\big)\\
&= \lim_{t\to 0}\big(\langle \Lambda_t Pu, u\rangle - \langle P^*\Lambda_t u, u\rangle\big)\\
&= \lim_{t\to 0}\big(\langle \Lambda_t Pu, u\rangle - \langle \Lambda_t P^* u, u\rangle - \langle [P^*,\Lambda_t] u, u\rangle\big)\\
&= \lim_{t\to 0}\big(\langle \Lambda_t Pu, u\rangle - \langle \Lambda_t (P + 2\imath \operatorname{Im}\lambda) u, u\rangle - \langle [P^*,\Lambda_t] u, u\rangle\big)\\
&= -2\imath \operatorname{Im}\lambda \langle u, u\rangle - \lim_{t\to 0}\langle [P^*,\Lambda_t] u, u\rangle.
\end{aligned}$$

Now, $P^* \in \Psi^{2,0}_{\mathrm{scc}}$, so $[P^*,\Lambda_t]$ is uniformly bounded in $\Psi^{1,-1}_{\mathrm{scc}}$, and it converges to $[P^*, I] = 0$ in $\Psi^{1+\epsilon,-1+\epsilon}_{\mathrm{scc}}$ for $\epsilon > 0$, thus strongly as a bounded operator $H^{1/2,-1/2}_{\mathrm{sc}} \to H^{-1/2,1/2}_{\mathrm{sc}}$. Correspondingly, if in addition $u \in H^{1/2,-1/2}_{\mathrm{sc}}$, then the last term vanishes, and we conclude that $u = 0$. But we have seen that in the Feynman spaces this holds, namely $u \in H^{\infty,-\epsilon}_{\mathrm{sc}}$ for all $\epsilon > 0$, so we conclude that $\|u\|^2 = 0$ and thus $u = 0$ as well.

The analogous argument also holds for P^* on the anti-Feynman space, which proves that P^* is also injective. □

Corollary 5. *Suppose* $\operatorname{Im}\lambda \neq 0$. *The operator* $P : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r+1}$ *is indeed invertible (not just Fredholm) and moreover we have for* $f \in \dot{\mathcal{C}}^\infty(M)$ *that* $u = P^{-1} f \in L^2_{\mathrm{sc}}$ *as well.*

We take

$$D = \{u \in H^{1,-1/2}_{\mathrm{sc}} \cap L^2_{\mathrm{sc}} : \Box_g u \in L^2_{\mathrm{sc}}\}.$$

In fact, more generally consider

$$D^{s',r'} = \{u \in H^{s',r'}_{\mathrm{sc}} \cap L^2_{\mathrm{sc}} : \Box_g u \in L^2_{\mathrm{sc}}\}, \qquad s' \in [1,2],\ r' \in [-1/2, 0),$$

so $D^{s',r'} \subset D = D^{1,-1/2}$ for all $s' \in [1,2]$, $r' \in [-1/2,0)$. Any choice of s', r' in this range would suffice for the proof of Theorem 1, with D being perhaps the simplest, but the other spaces give some slightly more precise intermediate results.

One of the reasons behind the particular ranges of s', r' is the following lemma, which implies that for $f \in \dot{\mathcal{C}}^\infty(M)$, $P^{-1} f$ lies in $D^{s',r'}$:

Lemma 6. *For either sign of* $\operatorname{Im}\lambda$, *and corresponding choices of* s, r *(only* r *depends on the sign of* $\operatorname{Im}\lambda$*) as above with the slightly stronger requirements* $s \geq 1/2 + s'$, *while* $r > -1/2 + r'$, *with* $r < -1/2$ *at the low regularity radial points (so the sinks if* $\operatorname{Im}\lambda > 0$*), we have* $\mathcal{X}^{s,r} \cap L^2_{\mathrm{sc}} \subset D^{s',r'}$.

Remark 7. The smaller the space $D^{s',r'}$ (i.e. the bigger s', r' are), the stronger the conclusion, though at the cost of stronger hypotheses on s, r (which however do not really matter for our purposes, the only important fact is that there *exist* compatible s, r). However, as we cannot take $r' = 0$, we do not have an 'optimal' choice of order.

Note that the existence of the function r requires the upper bound on r', namely $r' < 0$; the first sentence of the proof below also requires $r' \geq -1/2$.

Proof. First, if $u \in \mathcal{X}^{s,r} \cap L^2_{\mathrm{sc}}$ then $(\Box_g - \lambda)u \in H^{s-1,r+1}_{\mathrm{sc}} \subset L^2_{\mathrm{sc}}$ (as $r > -1/2 + r' \geq -1$ and $s \geq 3/2$) implies $\Box_g u \in L^2_{\mathrm{sc}}$. Next, by (6), $u \in H^{s-1/2,r+1/2}_{\mathrm{sc}}$. Since r takes values in $(-1/2 + r', \infty)$, so $r + 1/2$ in (r', ∞), and since $s' \leq s - 1/2$, we have $H^{s-1/2,r+1/2}_{\mathrm{sc}} \subset H^{s',r'}_{\mathrm{sc}}$. This proves the lemma. $\Box$

We have the following basic analytic facts about $D^{s',r'}$; this lemma requires lower bounds on s', r'.

Lemma 8. *The space $D^{s',r'}$ is a Hilbert space with $\dot{\mathcal{C}}^\infty(M)$ dense in it and $\Box_g$ symmetric on it.*

Proof. First, $D^{s',r'}$ is a Hilbert space in the standard manner with squared norm

$$\|u\|^2_{D^{s',r'}} = \|u\|^2_{H^{s',r'}_{\mathrm{sc}}} + \|u\|^2_{L^2_{\mathrm{sc}}} + \|\Box_g u\|^2_{L^2_{\mathrm{sc}}},$$

and the inclusion $D^{s',r'} \subset D$ is continuous.

Moreover, $\dot{\mathcal{C}}^\infty(M)$ is dense in $D^{s',r'}$ since using $\tilde{\Lambda}_t \in \Psi^{-\infty,-\infty}_{\mathrm{scc}}$ uniformly bounded in $\Psi^{0,0}_{\mathrm{scc}}$, converging to Id in $\Psi^{\epsilon,\epsilon}_{\mathrm{scc}}$ for all $\epsilon > 0$, we have $[\Box_g, \tilde{\Lambda}_t]$ uniformly bounded in $\Psi^{1,-1}_{\mathrm{scc}}$, converging to 0 in $\Psi^{1+\epsilon,-1+\epsilon}_{\mathrm{scc}}$, thus strongly as a map $H^{1,-1}_{\mathrm{sc}} \to H^{0,0}_{\mathrm{sc}} = L^2_{\mathrm{sc}}$. Hence $\dot{\mathcal{C}}^\infty(M) \ni \tilde{\Lambda}_t u \to u$ in L^2_{sc}, as well as in $H^{s',r'}_{\mathrm{sc}}$ and $\Box_g \tilde{\Lambda}_t u = \tilde{\Lambda}_t \Box_g u + [\Box_g, \tilde{\Lambda}_t]u \to \Box_g u$ in L^2_{sc} as $D^{s',r'} \subset H^{1,-1}_{\mathrm{sc}}$. See [13, Appendix A] for a more general discussion on spaces like $D^{s',r'}$; in the present context [21, Section 4] would be the relevant setting, but the present statement is not proved there, though the proof of [13, Lemma A.3] applies, mutatis mutandis.

Furthermore, $\Box_g$ is symmetric on D since

$$\begin{aligned}
\langle \Box_g u, u\rangle - \langle u, \Box_g u\rangle &= \lim_{t\to 0} \langle \tilde{\Lambda}_t \Box_g u, u\rangle - \langle \tilde{\Lambda}_t u, \Box_g u\rangle \\
&= \lim_{t\to 0} \langle \tilde{\Lambda}_t \Box_g u, u\rangle - \langle \Box_g \tilde{\Lambda}_t u, u\rangle \\
&= \lim_{t\to 0} \langle \tilde{\Lambda}_t \Box_g u, u\rangle - \langle \tilde{\Lambda}_t \Box_g u, u\rangle - \langle [\Box_g, \tilde{\Lambda}_t]u, u\rangle \\
&= -\lim_{t\to 0} \langle [\Box_g, \tilde{\Lambda}_t]u, u\rangle.
\end{aligned}$$

Indeed, as noted above $[\Box_g, \tilde{\Lambda}_t]$ is uniformly bounded in $\Psi^{1,-1}_{\mathrm{scc}}$, converging to 0 in $\Psi^{1+\epsilon,-1+\epsilon}_{\mathrm{scc}}$, thus strongly as a map $H^{1/2,-1/2}_{\mathrm{sc}} \to H^{-1/2,1/2}_{\mathrm{sc}}$, so for $u \in D \subset H^{1,-1/2}_{\mathrm{sc}} \subset H^{1/2,-1/2}_{\mathrm{sc}}$ the right hand side tends to 0 and we have the desired conclusion of symmetry. This immediately implies the general s', r' case since $D^{s',r'} \subset D$. $\Box$

Proof of Theorem 1. In the proof given in the next paragraph, we can replace D by $D^{s',r'}$, and the proof remains valid without any other changes.

We have that $\Box_g : D \to L^2_{\mathrm{sc}}$ is a continuous map, $\mathcal{C}^\infty_c(M^\circ)$ is dense in D (by virtue of $\dot{\mathcal{C}}^\infty(M)$ being so), and $\Box_g$ is a symmetric operator. In order to prove that $\Box_g$ is essentially self-adjoint, it suffices to prove that for $\lambda \notin \mathbb{R}$, $\Box_g - \lambda$ has a dense range in L^2_{sc}. But $\dot{\mathcal{C}}^\infty(M)$ is dense in L^2_{sc}, so it suffices to show that for $\operatorname{Im}\lambda \neq 0$ and $f \in \dot{\mathcal{C}}^\infty(M)$ there exists $u \in D$ such that $(\Box_g - \lambda)u = f$. But choosing s, r as in Lemma 6, it follows from Corollary 5 that under these conditions there exists $u \in \mathcal{X}^{s,r}$ such that $(\Box_g - \lambda)u = f$, and moreover $u \in L^2_{\mathrm{sc}}$, so as $\mathcal{X}^{s,r} \cap L^2_{\mathrm{sc}} \subset D$

by Lemma 6, the desired conclusion follows. This proves that $\Box_g$ is essentially selfadjoint on $\mathcal{C}_c^\infty(M^\circ)$, namely proves Theorem 1. $\Box$

4. The limiting absorption principle

The limiting absorption principle is an immediate consequence of our discussion. Namely, under the assumption of g being non-trapping at energy λ for the limiting λ (or interval of λ's, if one wishes), the estimates for $\Box_g - \lambda$ on the Feynman spaces are uniform in $\operatorname{Im}\lambda \geq 0$, and similarly on the anti-Feynman spaces in $\operatorname{Im}\lambda \leq 0$; and indeed, for $\lambda \in \mathbb{R}$, $\Box_g - \lambda$ is Fredholm on either one of these spaces. Furthermore, when $\operatorname{Im}\lambda \neq 0$, the operator is invertible, thus index 0, and this is stable under perturbations (even of the kind we discussed), cf. [16, Section 2.7], which also discusses continuity in the weak operator topology. In particular, the limit *is* the (anti-)Feynman propagator, up to finite dimensional nullspace issues on the limiting space. Thus,

Theorem 9. *Suppose $\lambda \in \mathbb{R} \setminus \{0\}$, g is a non-trapping sc-metric which is non-trapping at energy λ and $\Box_g - \lambda$ has trivial nullspace on either the Feynman or the anti-Feynman function spaces. Then $\lim_{\epsilon\to 0}(\Box_g - (\lambda \pm \imath\epsilon))^{-1}$ exist in the weak operator topology on the Feynman (+), resp. anti-Feynman (−) function spaces, and* is *the Feynman, resp. anti-Feynman propagator, i.e. the inverse of $\Box_g - \lambda$ on the appropriate function spaces.*

Remark 10. The argument of [22, Proposition 3.1] applies with minor notational changes (corresponding to the b-setting employed there and the sc-setting employed here) to prove that the primed wave front set of the Schwartz kernel of $(\Box_g - (\lambda \pm \imath 0))^{-1}$ is in the backward/forward flowout of the diagonal of the cotangent bundle T^*M° over the interior of M.

Of course, it is still a question whether the nullspace of $\Box_g - \lambda$ is trivial; the set of λ for which it is, is necessarily open by stability. Again, this stability is true even for the relatively drastic kind of perturbations we have which change the domain space (since the space depends strongly on λ via the condition $(\Box_g - \lambda)u \in H_{\mathrm{sc}}^{s-1,r+1}$); the point is that the relevant estimates *on fixed spaces*, with *fixed relatively compact error terms* are perturbation stable. Interestingly, cf. the discussion of [20, Section 4], adopting arguments of Isozaki [11] from N-body scattering as done in [18, Proof of Proposition 17.8], which are valid after minor modification in this setting as we discuss below, any element of the nullspace of $\Box_g - \lambda$ in either the Feynman or the anti-Feynman spaces in fact lies in $\dot{\mathcal{C}}^\infty(M)$:

Proposition 11. *On both the Feynman and anti-Feynman function spaces the nullspace of $P = \Box_g - \lambda$ is a subspace of $\dot{\mathcal{C}}^\infty(M)$.*

Proof. The arguments following [11] rely on using the commutant

$$\chi_\epsilon(x) = \epsilon^{-2r-1}\int_0^{x/\epsilon} \phi(s)^2 s^{-2r-2}\,ds,$$

where $\phi \in \mathcal{C}^\infty(\mathbb{R})$ is such that $\phi = 0$ on $(-\infty, 1]$, 1 on $[2, \infty)$, and where $r \in (-1/2, 0)$. Notice that χ_ϵ is supported in M° (namely in $x \geq \epsilon$) and

$$[x^2\partial_x, \chi_\epsilon] = x^2\partial_x(\chi_\epsilon) = x^{-2r}\phi(x/\epsilon)^2.$$

Thus, while the family $\{\chi_\epsilon :\ \epsilon \in (0,1)\}$ (considered as a family of multiplication operators) is not (uniformly) bounded in any symbol space, as (using $s' = \epsilon s$) for any l

$$\sup x^l \chi_\epsilon \geq \chi_\epsilon(1) \geq \int_{2\epsilon}^1 \phi(s'/\epsilon)^2 (s')^{-2r-2}\, ds' = \frac{1}{2r+1}((2\epsilon)^{-2r-1} - 1) \to \infty$$

as $\epsilon \to 0$ since $-2r-1 < -2(-1/2) - 1 = 0$, its commutator with $x^2\partial_x$ is bounded in symbols of order $2r$. This gives, by (1), that the principal symbol of $\imath[P, \chi_\epsilon]$ is, with p denoting the principal symbol of P,

$$H_p \chi_\epsilon = (\partial_\tau p) x^2 \partial_x(\chi_\epsilon);$$

note that the $x^2\partial_x$ component of H_p is exactly $\partial_\tau p$, so at a source, resp. sink, manifold of the boundary, $\partial_\tau p < 0$, resp. $\partial_\tau p > 0$, i.e. at such a manifold this commutator has a definite sign. While the lower order terms, which here means just the 0th order terms as P is a differential operator, involve further derivatives of χ_ϵ, they *only* involve at least first derivatives of χ_ϵ and thus the lower order terms will also be bounded in symbols of order $0, 2r-1$. Thus,

$$\imath[\Box_g - \lambda, \chi_\epsilon] = \pm\phi(x/\epsilon)(B^*B + E)\phi(x/\epsilon) + F_\epsilon, \tag{7}$$

where $B \in \Psi_{\mathrm{scc}}^{1/2,r}$, with principal symbol elliptic at the sources/sinks (depending on the choice of $\pm$, with $+$ for sinks), $E \in \Psi_{\mathrm{scc}}^{1/2,r}$ having disjoint wave front set from these, and $\{F_\epsilon :\ \epsilon \in (0,1)\}$ is uniformly bounded in $\Psi_{\mathrm{scc}}^{0,2r-1}$.

Now consider $u \in \mathrm{Ker} P$ on $\mathcal{X}^{s',r'}$, where s' may be taken arbitrarily high and r' arbitrarily high except in a neighborhood of the source/sink in accordance with the sign in $\pm$ above, where $r' \in (-1, -1/2)$ ('arbitrarily high' is in the sense that the nullspace is *independent of such choices*). Then $\langle \phi(\cdot/\epsilon)E\phi(\cdot/\epsilon)u, u\rangle$ remains bounded as on $\mathrm{WF}'_{\mathrm{sc}}(E)$, u is microlocally in $H_{\mathrm{sc}}^{\infty,\infty} = \dot{\mathcal{C}}^\infty(M)$, while $\langle F_\epsilon u, u\rangle$ also remains bounded since $u \in H_{\mathrm{sc}}^{\infty,-1/2-\delta'}$ for all $\delta' > 0$ and $2r - 1 < -1$, so one can choose $\delta' > 0$ with $2(-1/2 - \delta') - (2r-1) > 0$. On the other hand, for $\epsilon > 0$,

$$\langle \imath[\Box_g - \lambda, \chi_\epsilon]u, u\rangle = \langle \imath\chi_\epsilon u, (\Box_g - \lambda)u\rangle - \langle \imath(\Box_g - \lambda)u, \chi_\epsilon u\rangle = 0$$

since χ_ϵ is compactly supported in M°, so the integration by parts is justified. Correspondingly, one deduces that $B\phi(\cdot/\epsilon)u$ is uniformly bounded in L^2_{sc}, and thus by the standard weak-* convergence argument $Bu \in L^2_{\mathrm{sc}}$, proving that even at the source/sink where we did not have a priori knowledge of membership of u in a subspace of $H_{\mathrm{sc}}^{\infty,-1/2}$, in fact, $u \in H_{\mathrm{sc}}^{\infty,r}$ for all $r \in (0, -1/2)$. Then the standard radial point estimate, see [17, 21], implies that in fact u is microlocally in $H_{\mathrm{sc}}^{\infty,\infty}$ even there; in combination with the other a priori knowledge, we conclude that $u \in \dot{\mathcal{C}}^\infty(M)$. □

Remark 12. Notice that this argument used crucially that $Pu = 0$; since χ_ϵ is not uniformly bounded on any weighted Sobolev space, $\langle \chi_\epsilon u, (\Box_g - \lambda)u\rangle$, $\langle (\Box_g - \lambda)u, \chi_\epsilon u\rangle$ would not remain bounded as $\epsilon \to 0$ otherwise even if, say, $Pu \in \dot{\mathcal{C}}^\infty(M)$.

Also notice that the argument crucially relies that we are taking either the Feynman or the anti-Feynman space, so the points at which we do not have a priori decay are either all sources or all sinks, thus there is a single definite sign in (7), arising for the common source, or sink, nature of them. For other Fredholm problems, the elements of the nullspace are not necessarily in the 'trivial space', $\dot{\mathcal{C}}^\infty(M)$.

Thus, the absence of embedded eigenvalues depends on a unique continuation argument at infinity, namely that the rapid decay (infinite order vanishing) at ∂M of an element of Ker$P$ implies its vanishing nearby.

In the case of non-trapping Lorentzian scattering metrics (possibly long range), as in [2, 3], if one assumes that the there is a boundary defining function ρ of M such that, say, near the past 'spherical cap' $\overline{C_-}$, $\frac{d\rho}{\rho^2}$, is timelike (which for instance is true on perturbations of Minkowski space), for $\lambda > 0$ energy estimates imply that, being an element of $\dot{\mathcal{C}}^\infty(M)$, an element of this nullspace vanishes identically at first near $\overline{C_-}$, and then the non-trapping condition implying global hyperbolicity, see [9, Section 5] in this setting for this implication, vanishes globally, so the nullspace is indeed trivial.

An analogous conclusion holds by a Wick rotation argument, see [6], for the Minkowski metric, as well as pseudo-Riemannian translation invariant metrics, and again the perturbation stability implies that the conclusion also holds for their perturbations in the sc-category.

We finally remark that the $\lambda = 0$ Fredholm problem was studied in [6]; one can also discuss the limiting absorption principle there, under somewhat stronger conditions than we needed here, but we defer it to future work.

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