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Subtour Elimination Constraints Imply a Matrix-Tree Theorem SDP Constraint for the TSP

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Abstract

De Klerk, Pasechnik, and Sotirov [4] give a semidefinite programming constraint for the Traveling Salesman Problem (TSP) based on the matrix-tree Theorem. This constraint says that the aggregate weight of all spanning trees in a solution to a TSP relaxation is at least that of a cycle graph. In this note, we show that the semidefinite constraint holds for any weighted 2-edge-connected graph and, in particular, is implied by the subtour elimination constraints of the subtour elimination linear program. Hence, this semidefinite constraint is implied by a finite set of linear inequality constraints.

1 Introduction and The Matrix-Tree Theorem

The Traveling Salesman Problem (TSP) is a fundamental problem in combinatorial optimization and a canonical NP-hard problem. Efficiently computable relaxations of the TSP are used to find optimal and near-optimal TSP solutions, and recently, several relaxations based on semidefinite programs (SDPs) have been proposed (see, e.g., Cvetković, Čangalović, and Kovačević-Vujčić [2], de Klerk, Pasechnik, and Sotirov [4], and de Klerk and Sotirov [5]).

A common source of SDP constraints for the TSP is spectral graph theory: the SDP of Cvetković et al. [2] is based on algebraic connectivity, and de Klerk et al. [4] give a constraint based on Kirchoff's matrix-tree theorem. Goemans and Rendl [6] show that the constraints used in the SDP relaxation of Cvetković et al. [2] are implied by the canonical TSP relaxation, the subtour elimination linear program (see Equation (1) below for the precise definition of this linear program). In this note we show that the matrix-tree theorem constraint of de Klerk et al. [4] is also implied by the subtour elimination linear program constraints.

The matrix-tree theorem dates back to the mid-19th century (Kirchoff [10]) and connects the number of spanning trees of a graph to the Laplacian matrix of that graph. Let $G = (V, E)$ be a simple, undirected graph, and suppose each edge e has weight $x_e \geq 0$. Let X be the corresponding weighted adjacency matrix, so that X has zero diagonal and $X_{ij} = X_{ji} = x_{\{i,j\}}$. The **Laplacian** of X is the $|V| \times |V|$ matrix $L(X)$ defined entrywise as

$$L(X)_{i,j} = \begin{cases} -x_e, & \{i,j\} \in E \\ \sum_{e: e\cap i \neq \emptyset} x_e, & i = j \\ 0, & \text{else.} \end{cases}$$

Suppose that $\mathcal{T}_G$ is the set of spanning trees of G. The matrix-tree theorem is the remarkable result that any *principal minor* of $L(X)$ (i.e., the determinant of the matrix obtained by removing the ith row and column of $L(X)$ for any $1 \leq i \leq |V|$) equals $\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e$. In the case that $x_e = 1$ for every edge in G, the term $\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e$ counts the number of spanning trees of G. See Theorem VI.29 in [13], e.g., for a proof of this general version of the matrix-tree theorem.

De Klerk et al. [4] notice that any Hamiltonian cycle on n vertices has n spanning trees (delete any individual edge). They use the matrix-tree theorem to derive a constraint for SDP relaxations of the TSP saying that "the aggregate weight of spanning trees is at least n." We show that this constraint is implied by constraints in the subtour elimination linear program:

Theorem 1.1. *Let $x \in \mathbb{R}^E$ be a feasible solution to the subtour LP (1) and let G be the complete graph. Let X be the symmetric matrix where $X_{ij} = X_{ji} = x_{\{i,j\}}$ and $X_{ii} = 0$ for all i. Then X satisfies the matrix-tree theorem constraint:*

$$\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

Our results show that the matrix-tree theorem constraint (requiring linear matrix inequalities) of de Klerk et al. [4] is weaker than the subtour LP (using just linear inequalities). They can also be stated more generally: any graph G that is 2-edge-connected in a weighted sense (i.e. $\sum_{e:|e \cap S|=1} x_e \geq 2$ for every $\emptyset \subsetneq S \subsetneq V$) satisfies $\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n$. Our result follows from a theorem of Ok and Thomassen [12] that lower-bounds the number of spanning trees in an unweighted, loopless, undirected multigraph. In Section 2, we provide background on the TSP and relaxations. In Section 3 we then state the theorem from Ok and Thomassen [12] and use it to deduce Theorem 1.1.

2 Preliminaries

The Traveling Salesman Problem can be stated as follows. Let $G = K_n$ be the complete graph on $V = [n] := \{1, 2, ..., n\}$. For each $e = \{i, j\} \in G$, associate an edge cost c_e (interpreted as the cost of traveling from vertex i to vertex j or vice versa). The TSP is to find a minimum-cost tour on G visiting every vertex exactly once, i.e., to find a minimum-cost Hamiltonian cycle on K_n with respect to the edge costs.

The prototypical TSP relaxation is the the subtour elimination linear program (also referred to as the Dantzig-Fulkerson-Johnson relaxation [3] and the Held-Karp bound [9], and which we will refer to as the **subtour LP**). For $S \subset V$, denote the set of edges with exactly one endpoint in S by $\delta(S) := \{e = \{i, j\} : |\{i, j\} \cap S| = 1\}$ and let $\delta(v) := \delta(\{v\})$. For $F \subset E$, let $x(F)$ denote the sum of x over those edges in F: $x(F) = \sum_{e \in F} x_e$. The subtour LP is:

$$\begin{array}{lll} \text{min} & \sum_{e \in E} c_e x_e & \\ \text{subject to} & x(\delta(v)) = 2, & v = 1, \ldots, n \\ & x(\delta(S)) \geq 2, & S \subset V : S \neq \emptyset, S \neq V \\ & 0 \leq x_e \leq 1, & e \in E. \end{array} \tag{1}$$

The subtour LP is a **relaxation** of the TSP because 1) every Hamiltonian cycle has a corresponding feasible solution to the subtour LP, and 2) the value of the subtour LP for such a feasible solution equals the cost of the corresponding Hamiltonian cycle.

Significant recent research has gone into developing relaxations instead based on semidefinite programs (SDPs). See, e.g., Cvetković, Čangalović, and Kovačević-Vujčić [2] (who both introduce an SDP relaxation based on algebraic connectivity), de Klerk, Pasechnik, and Sotirov [4] (who introduce an SDP relaxation based on the theory of association schemes and give the matrix-tree theorem-based SDP constraint), and de Klerk and Sotirov [5] (who use symmetry reduction to strengthen the SDP of de Klerk et al. [4]). Various results have characterized the performance of these SDPs (Goemans and Rendl [6], Gutekunst and Williamson [7], and Gutekunst and Williamson [8]).

The TSP SDP relaxations generally have some symmetric matrix variable X that can be interpreted as a weighted adjacency matrix. There are typically constraints enforcing that $X_{ii} = 0$, and since X is symmetric, $X_{ij} = X_{ji}$ can be thought of as the weight x_e on edge $e = \{i, j\}$. In a feasible solution to the SDP relaxation taking on integral values, constraints force X to be the weighted adjacency matrix of a Hamiltonian cycle. There are generally constraints that directly enforce that X is 2-regular in a weighted sense: every row of X sums to 2, in analogy to the subtour LP constraints that $x(\delta(v)) = 2$. If G is the graph with edge weights x_e, the constraints imply that the corresponding Laplacian matrix to X is $L(X) = 2I - X$. Throughout we treat X as the weighted adjacency matrix of a complete graph $G = K_n$ where edges can have weight zero.[1]

Let A_{-i} denote the matrix obtained by deleting the ith row and column of A. Note that $2I - X$ is positive semidefinite, so $(2I - X)_{-i}$ is positive semidefinite for all i. De Klerk et al. [4]'s observation that a Hamiltonian cycle has n spanning trees, together with the aforementioned matrix-tree theorem, allows them to introduce the SDP constraint

$$\det\left((2I - X)_{-1}\right) \geq n. \tag{2}$$

Since the set $\{Z \succeq 0 : \det(Z) \geq n\}$ can be expressed as a linear matrix inequality (see, e.g., Section 3.2 of Nemirovskii [11]), the constraint in Equation (2) can be written as a linear matrix inequality for use in TSP SDP relaxations. De Klerk et al. [4] note that this constraint strengthens a semidefinite programming relaxation of the TSP from Cvetković, Čangalović, and Kovačević-Vujčić [2]. We refer to Equation (2) as the "matrix-tree theorem constraint."

3 The Matrix-Tree Theorem Constraint

To prove our main result, we will use the following result from Ok and Thomassen [12] which relates edge-connectivity to spanning trees. An unweighted, undirected, loopless multigraph $G = (V, E)$ is **k-edge-connected** if G is still connected after the removal of any $k - 1$ edges.

Theorem 3.1 (Theorem 1 in Ok and Thomassen [12]). *Let G be an weighted, loopless, undirected multigraph that is k-edge-connected. Then G has at least $n\left(\frac{k}{2}\right)^{n-1}$ spanning trees.*

[1] Any spanning tree T containing an edge of weight zero has $\prod_{e\in T} x_e = 0$ and doesn't contribute to $\sum_{T\in\mathcal{T}_G} \prod_{e\in T} x_e \geq n$. We can let $G = K_n$ without loss of generality, as any other graph can be extended to the complete graph by placing a weight of zero on all missing edges; the weighted adjacency matrix, Laplacian, and aggregate spanning tree weight $\sum_{T\in\mathcal{T}_G} \prod_{e\in T} x_e \geq n$ will not change.

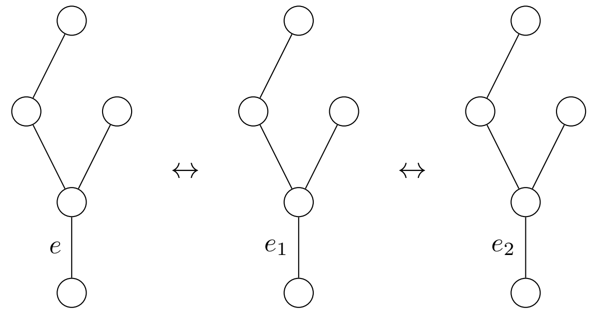

Figure 1: A sample tree instantiation in $\mathcal{T}_G^e, \mathcal{T}_{G'}^1$, and $\mathcal{T}_{G'}^2$

We first use it to prove the following:

Proposition 3.2. *Let $G = (V, E)$ be a weighted simple graph with rational edge weights given by $x \in \mathbb{R}^E$. If x is an extreme point of the subtour LP (1), then*

$$\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

Theorem 1.1 will then follow as an immediate consequence.

To prove Proposition 3.2, we start with a symmetric, simple weighted graph $G = (V, E)$ with edge weights given by $x \in \mathbb{R}^E$. Because x is rational, we will be able to scale x so that $Rx \in \mathbb{Z}^E$. Then we let $G' = (V, E')$ be an undirected, loopless, unweighted multigraph with Rx_e copies of edge e. Moreover, if $x(\delta(S)) \geq 2$ then $Rx(\delta(S)) \geq 2R$ so that G' will be $2R$-edge-connected. We can then appeal to Theorem 3.1, find a large number of spanning trees, and find corresponding spanning trees in G.

We first verify that the aggregate weight of spanning trees in G' (as an unweighted multigraph with Rx_e copies of edge e) matches the aggregate weight of spanning trees in G (as a weighted simple graph where edge e has weight Rx_e). To do so, we apply the following lemma iteratively.

Lemma 3.3. *Let G be a weighted loopless multigraph. Let $e = \{u, v\} \in G$ and let G' be obtained from G by splitting e into two copies $e_1 = e_2 = \{u, v\}$ and assigning nonnegative weights x' to the edges in G' so that $x_e = x'_{e_1} + x'_{e_2}$ (and $x_f = x'_f$ for all other edges f in G). Then*

$$\sum_{T \in \mathcal{T}_G} \prod_{f \in T} x_f = \sum_{T \in \mathcal{T}_{G'}} \prod_{f \in T} x'_f.$$

In the proof, we use $\sqcup$ to denote a partition: $S = A \sqcup B$ means $S = A \cup B$ and $A \cap B = \emptyset$. We also use $\backslash$ for set-minus, so that $S \backslash A = \{x \in S : x \notin A\}$.

Proof. This result follows by partitioning $\mathcal{T}_{G'}$. No $T \in \mathcal{T}_{G'}$ can contain both e_1 and e_2 so we write

$$\mathcal{T}_{G'} = \mathcal{T}_{G'}^0 \sqcup \mathcal{T}_{G'}^1 \sqcup \mathcal{T}_{G'}^2$$

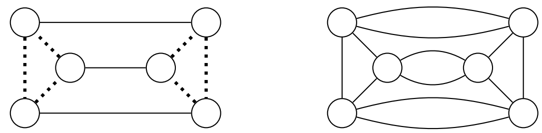

Figure 2: The left shows a simple, weighted graph G where dashed edges have weight 1/2 and full edges have weigh 1. In this case $R = 2$ and the right shows the corresponding unweighted multigraph G'.

where $\mathcal{T}^i_{G'}$ consists of those spanning trees including edge i for $i = 1, 2$ and $\mathcal{T}^0_{G'}$ consists of those trees including neither e_1 nor e_2. We analogously partition

$$\mathcal{T}_G = \mathcal{T}^0_G \sqcup \mathcal{T}^e_G,$$

where $\mathcal{T}^0_G$ consists of spanning trees not using e and $\mathcal{T}^e_G$ consists of spanning trees using e.

The trees in $\mathcal{T}^1_{G'}$, $\mathcal{T}^2_{G'}$, and $\mathcal{T}^e_G$ all use exactly one edge $\{u, v\}$ (and other than using exactly one of e_1, e_2, or e as $\{u, v\}$, use the exact same other edges). Hence if $T \in \mathcal{T}^1_{G'}$, then $(T \backslash e_1) \cup e_2 \in \mathcal{T}^2_{G'}$ and $(T \backslash e_1) \cup e \in \mathcal{T}^e_G$. This process gives a one-to-one correspondence between trees in $\mathcal{T}^1_{G'}$, $\mathcal{T}^2_{G'}$, and $\mathcal{T}^e_G$; see Figure 1. Analogously, $\mathcal{T}^0_{G'} = \mathcal{T}^0_G$. Hence:

$$\begin{aligned}
\sum_{T \in \mathcal{T}_{G'}} \prod_{f \in T} x'_f &= \sum_{T \in \mathcal{T}^0_{G'}} \prod_{f \in T} x'_f + \sum_{T \in \mathcal{T}^1_{G'}} \prod_{f \in T} x'_f + \sum_{T \in \mathcal{T}^2_{G'}} \prod_{f \in T} x'_f \\
&= \sum_{T \in \mathcal{T}^0_G} \prod_{f \in T} x_f + (x'_{e_1} + x'_{e_2}) \sum_{T \in \mathcal{T}^1_{G'}} \prod_{f \in T, f \neq e_1} x'_f \\
&= \sum_{T \in \mathcal{T}^0_G} \prod_{f \in T} x_f + x_e \sum_{T \in \mathcal{T}^1_G} \prod_{f \in T, f \neq e} x_f \\
&= \sum_{T \in \mathcal{T}^0_G} \prod_{f \in T} x_f + \sum_{T \in \mathcal{T}^1_G} \prod_{f \in T} x_f \\
&= \sum_{T \in \mathcal{T}_G} \prod_{f \in T} x_f.
\end{aligned}$$

□

We now prove our main theorem in the special case of subtour LP extreme points.

Proof (of Proposition 3.2). Let x be a feasible extreme point of the subtour LP. Then $x(\delta(S)) \geq 2$ for all S with $1 \leq |S| \leq |V| - 1$ and, moreover, it is well-known that x is rational.[2]

We first convert G into a loopless, unweighted multigraph. To do so, suppose that $x_{e_i} = \frac{s_i}{r_i}$ in lowest terms. Let

$$R = \mathcal{LCM}(r_1, ..., r_m).$$

[2]Extreme points occur where a certain number of constraints hold with equality. Cramer's rule, e.g., shows that if the constraints of a linear program have rational coefficients, then every extreme point is rational. This is the case for the subtour LP.

Let G' denote the graph G with weights $x'_e = Rx_e$ for all $e \in G$ and let $X' = RX$. Then we make two observations: $x'_e \in \mathbb{Z}$ for all e, and properties of deteriminants imply

$$\det\left(\left(L(X')_{-1}\right)\right) = R^{n-1}\det\left(\left(L(X)_{-1}\right)\right). \tag{3}$$

To compute $\det\left(\left(L(X')_{-1}\right)\right)$ we appeal to the matrix-tree theorem; by Lemma 3.3 it is equivalent (in terms of the aggregate weight of spanning trees) to view G' as a loopless unweighted multigraph where there are x'_e copies of edge e (so that there are $s_i \frac{R}{r_i} \in \mathbb{Z}$ copies of edge e_i, each of weight 1). See Figure 2.

Note that $x(\delta(S)) \geq 2$ implies that $x'(\delta(S)) \geq 2R$. Thus G' is $2R$-edge-connected and by Theorem 3.1, G' has at least nR^{n-1} spanning trees; since every edge of G' has weight 1, the matrix-tree theorem states

$$\det\left(\left(L(X')_{-1}\right)\right) = \sum_{T \in \mathcal{T}_{G'}} \prod_{f \in T} x'_f \geq nR^{n-1}.$$

Combining with Equation 3 we have:

$$\begin{aligned} R^{n-1}\det\left(\left(L(X)_{-1}\right)\right) &= \det\left(\left(L(X')_{-1}\right)\right) \\ &\geq nR^{n-1}. \end{aligned}$$

That is,

$$\det\left(\left(L(X)_{-1}\right)\right) \geq n$$

and the matrix-tree theorem implies

$$\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

□

We can now show that the matrix-tree theorem constraint (2) holds for any feasible point of the subtour LP. We restate our main theorem in slightly more detail.

Theorem (Theorem 1.1, restated)**.** *Let $x \in \mathbb{R}^E$ be a feasible solution to the subtour LP (1) and let G be the complete graph. Let X be the symmetric matrix where $X_{ij} = X_{ji} = x_{\{i,j\}}$ and $X_{ii} = 0$ for all i. Then X satisfies the matrix-tree theorem constraint:*

$$det\left((2I - X)_{-1}\right) \geq n.$$

Equivalently,

$$\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

Proof. The subtour LP is bounded, so that every feasible x for the subtour LP can be written as a convex combination of extreme points to the subtour LP. For any extreme point of the subtour LP y, let Y be the matrix where $Y_{ij} = Y_{ji} = y_{\{i,j\}}$ and $Y_{ii} = 0$. Feasibility of the subtour LP means $y(\delta(i)) = 2$ for all $i \in V$, so the associated Laplacian is $2I - Y$. By Proposition 3.2 and the matrix-tree theorem, $\det\left((2I - Y)_{-1}\right) \geq n$.

We now show that any convex combination of two extreme points of the subtour LP also satisfies the matrix-tree theorem constraint; extending to general convex combinations is left as an exercise. Note that the determinant is well-known to be log concave on symmetric positive definite matrixes (see, e.g., section 3.1 of Boyd and Vandenberghe [1]) so that $\det(tA+(1-t)B) \geq \det(A)^t \det(B)^{1-t}$ for $0 \leq t \leq 1$ if $A, B \succ 0$.

Consider two extreme points of the subtour LP, with weighted adjacency matrices A and B. Denote their graph Laplacians as $L(A) = 2I - A$ and $L(B) = 2I - B$ respectively. For a graph with weighted adjacency matrix X, all principal subminors of $L(X)$ are nonnegative so that all principal subminors of $L(X)_{-1}$ are as well: these are just the principal subminors of $L(X)$ that include row/column 1 being removed. This implies that $L(X)_{-1} \succeq 0$. By Proposition 3.2, $\det\left((L(A)_{-1})\right), \det\left((L(B)_{-1})\right) \geq n$ so that zero cannot be an eigenvalue of $(L(A)_{-1})$ or $(L(N)_{-1})$ and so both are positive definite. By log-concavity,

$$\begin{aligned}
&\det(t\left(L(A)_{-1}\right) + (1-t)\left(L(B)_{-1}\right)) \\
&\quad \geq \left(\det\left(L(A)_{-1}\right)\right)^t \left(\det\left(L(B)_{-1}\right)\right)^{1-t} \\
&\quad \geq n^t n^{1-t} \\
&\quad = n.
\end{aligned}$$

Hence, $tA + (1-t)B$ satisfies the matrix-tree-theorem constraint (2). □

Remark 3.4. *Note that the proof of Theorem 1.1 for any x such that $x(\delta(S)) \geq 2$ for each $S \subset V$ with $1 \leq |S| \leq |V| - 1$. Hence, any x with $x(\delta(S)) \geq 2$ for all such S and corresponding weighted adjacency matrix X satisfies*

$$det\left((L(X)_{-1})\right) = \sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

However, it need not be the case that that rows of sum to X, so possibly $L(X) \neq 2I - X$.

4 Conclusion

Theorem 1.1 has several implications. Goemans and Rendl [6] show that the subtour LP is stronger than a TSP SDP relaxation of Cvetković et al. [2] in the following sense: Any feasible solution for the subtour LP corresponds to a feasible solution of the same cost for the SDP. Hence, on any given instance, the optimal value of the subtour LP is at least as close to the cost of a TSP solution as the optimal value of the SDP. Theorem 1.1 gives a comparable weakness result for the matrix-tree theorem constraint (2). Moreover, it implies that Goemans and Rendl [6]'s result extends to the case where the matrix-tree theorem constraint (2) is added to the SDP of Cvetković et al. [2]. More generally, our results show that matrix semidefinite inequalities used to impose the matrix-tree theorem are implied by a set of linear inequalities.

Acknowledgments

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