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MATHEMATICAL ANALYSIS OF SURFACE PLASMON RESONANCE BY A NANO-GAP IN THE PLASMONIC METAL*

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Abstract. We develop a mathematical theory for the excitation of surface plasmon resonance on an infinitely thick metallic slab with a nano-gap defect. Using layer potential techniques, we establish the well-posedness of the underlying scattering problem. We further obtain the asymptotic expansion of the scattering solution in order to characterize the leading-order term of the surface plasmonic waves, and derive sharp estimates for both the plasmonic and nonplasmonic parts of the solution. The explicit dependence of the surface plasmon resonance on the size of the nano-gap, and the real and imaginary parts of the metal dielectric constant, are given.

Key words. integral equation, Helmholtz equation, surface plasmon resonance, subwave-length structure

AMS subject classifications. 35B34, 35C20, 35Q60

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1. Introduction.

Surface plasmonics is an emerging field in photonics which studies the coupling of optical light with collective oscillations of free electron density on a metal-dielectric interface or localized metallic nano-structures. The resonant coupling induces the so-called surface plasmon resonance, which enables localization of electromagnetic field at the subwavelength scale as well as enhancement of optical scattering and absorption. Many important applications in bio-sensing and design of novel optical devices have been proposed based on these remarkable optical properties [15, 19, 20, 24]. There are two types of surface plasmon resonance: (i) localized resonance which occurs on metal nano-structures with finite size such as nano-particles; (ii) surface plasmon polariton which occurs on an infinite metal-dielectric interface with surface waves propagating along the surface. The mathematical theory for the first type of plasmon resonance relies on the analysis of the spectrum for the Neumann–Poincaré operators. This has been investigated extensively recently (see [2, 6, 8, 10, 11, 16, 17, 18] and the references therein). We also refer the reader to [4, 5, 7] for the applications of localized surface plasmon in meta-surface and bio-sensing. Regarding to the second type of plasmon resonance, it is well known that certain defects or corrugations have to be created along the metal-dielectric interface in order to excite surface plasmon with an incident plane wave [24, 26]. However, no rigorous mathematical theory has yet been developed for the corresponding plasmon resonance.

In this paper, we investigate a setup where the defect along the metal-dielectric interface is formed by a nano-slit filled with perfect conducting materials, and we present rigorous mathematical analysis for the surface plasmon resonance induced by the small defect. The study is motivated by recent attempts to understand light

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interactions with subwavelength hole structures and the so-called extraordinary optical transmission (EOT) [15, 28]. This topic has drawn increasing interest in optics research since the report [14] by Ebbesen et al. Various resonance mechanisms can lead to the EOT phenomenon. In a series of studies [20, 21, 22, 23], we have established rigorous mathematical theories for EOT in nano-slit structures perforated in a perfectly conducting metallic slab. We also refer to [9, 12, 29, 30] for resonant scattering in several other perfect conducting or dielectric slab structures. However, as of today, a rigorous mathematical theory for resonances through plasmonic metallic nano-holes remains open. In such a scenario, both the surface plasmon mode supported along the metal-dielectric interface and the cavity resonant modes supported in the hole structure can play a role in EOT through the nano-holes. This paper is the first step toward a complete understanding of resonances for such a problem. The conceptually simplified model proposed in this paper excludes the complications induced by the interaction between the surface plasmon modes and the cavity resonant modes and hence allows us to focus on the former only. By capturing the essence of physics, the mathematical analysis provides insight into the underlying mechanism of the surface plasmon resonance. In addition, we expect that the mathematical approach developed here can be generalized to the study of surface plasmon resonance with other types of defects.

To be more specific, we consider the two-dimensional model where the medium consists of two layers that are separated by the interface $\Gamma = \{(x_1, x_2) \mid x_2 = 0\}$. The top layer is a vacuum that occupies the upper half plane $\Omega_1 := \{(x_1, x_2) \mid x_2 > 0\}$, and the bottom layer is a metal that occupies the lower half plane $\Omega_2 := \{(x_1, x_2) \mid x_2 < 0\}$. The relative permittivity ε on the x_1x_2 plane is given by

$$\varepsilon(x) = \begin{cases} 1, & x \in \Omega_1, \\ \varepsilon_m, & x \in \Omega_2, \end{cases}$$

where $\varepsilon_m = \varepsilon_m' + i\,\varepsilon_m''$ is the relative permittivity for the metal. Note that ε_m is a complex number depending on the frequency through the following Drude's model [27]:

$$\varepsilon_m(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)},$$

where ω_p is the plasmon frequency of the metal and γ is the damping coefficient. Here we are interested in the frequency range where the real part ε_m' is negative and it holds that $|\varepsilon_m'| \gg 1$ and $|\varepsilon_m'| \gg |\varepsilon_m''|$. This is true for noble plasmonic metals such as gold and silver in the optical frequency regime. More precisely, throughout the paper, we assume the following holds.

Assumption 1. *$\varepsilon_m' < 0$, $\varepsilon_m'' > 0$, $|\varepsilon_m'| \gg 1$, and $|\varepsilon_m'| \gg |\varepsilon_m''|$.*

The lower half plane is perturbed by an infinitely long and perfectly conducting nano-slit $S_\delta^\infty := \{(x_1, x_2) \mid -\delta < x_1 < \delta, -\infty < x_2 < 0\}$, whose boundary consists of three segments Γ_δ^0, Γ_v^-, and Γ_v^+, respectively as shown in Figure 1. Then the remaining parts of Ω_2 consist of two disjoint semi-infinite domains Ω_2^- and Ω_2^+. We denote the left and right segments of the metal-vacuum interface by Γ_δ^- and Γ_δ^+, respectively, with the presence of the slit.

Throughout the paper, it is assumed that the slit width is much smaller than the incident wavelength such that $\delta \ll \lambda$. For convenience of asymptotic analysis, we write

$$\varepsilon_m' = -\delta^\alpha, \quad \varepsilon_m'' = \delta^\beta,$$

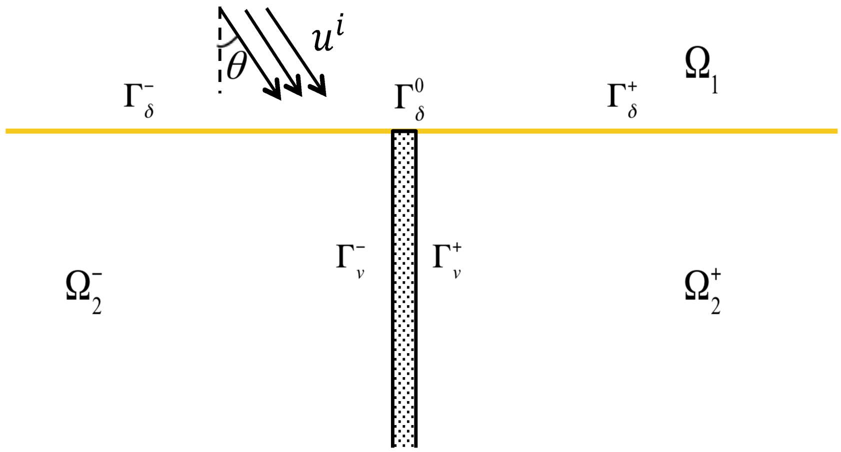

FIG. 1. *Geometry of the model. The domain of vacuum and metal is denoted by Ω_1 and Ω_2, respectively. The infinitely long and perfectly conducting slit S_δ perforated in the slab Ω_2 has a width of 2δ. The remaining part of the metal consists of two disjoint semi-infinite domains Ω_2^- and Ω_2^+. The scaling of the geometry is given by $\delta \ll \lambda$.*

where the δ-dependent variables α, β are defined by

$$\alpha(\delta) = -\frac{\ln|\varepsilon_m'|}{|\ln\delta|}, \quad \beta(\delta) = \frac{\ln\varepsilon_m''}{|\ln\delta|}.$$

In the main results, we further impose the condition that $\alpha - \beta > -6$ and $\beta < 2$ for the even case and that $\alpha - \beta > -4$ for the odd case. Note that for a fixed ε_m or frequency, in the asymptotic limit $\delta \to 0$, both α and β tend to 0. These additional conditions are satisfied automatically. Therefore, we expect that our results hold for a large frequency range that are of interest.

We consider the transverse magnetic polarization when the magnetic field is pointing along the invariant x_3 direction such that $H = (0,0,u)$. Let u^i be a plane wave incident from the above and $u^s := u - u^i$ be the scattered field. The total field u after the scattering consist of u^i and u^s in Ω_1 and u^s only in $\Omega_2^\pm$. It satisfies the following equations:

$$\begin{cases} \nabla\cdot\left(\dfrac{1}{\varepsilon(x)}\nabla u\right) + k^2 u = 0 \qquad \text{in } \Omega_1 \cup \Omega_2^+ \cup \Omega_2^-, \\ [u] = 0, \left[\dfrac{1}{\varepsilon}\dfrac{\partial u}{\partial \nu}\right] = 0 \qquad \text{on } \Gamma_\delta^- \cup \Gamma_\delta^+, \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma_\delta^0 \cup \Gamma_v^- \cup \Gamma_v^+, \end{cases} \tag{1.1}$$

where $[\cdot]$ denotes the jump of the quantity when the limit is taken along the positive and negative unit normal direction ν, respectively. Since the metallic structure is infinite in size, the usual Sommefield radiation condition does not hold for the scattered field. We will enforce the radiation condition given naturally by the associated Green's functions.

In this paper, we first formulate the boundary integral equation for the above scattering problem and establish the existence and uniqueness of the solution. The well-posedness for (1.1) with a positive ε_m' is well-known, while the case with $\varepsilon_m' < 0$ has not been studied before. Second, we aim to obtain the asymptotic expansion of the solution to the integral equation and the scattering solution in order to characterize the leading-order term of the surface plasmonic waves.

The main challenge for the investigation of the scattering problem (1.1) lies in the infinite length of the slab and the opposite permittivity values for the two media. These lead to a Fourier integral operator, whose inverse attains complex-valued poles $\xi_\pm = \pm k\sqrt{\varepsilon_m/(\varepsilon_m+1)}$ in the Fourier domain (see section 3.2 for detailed discussions). These poles are extremely close to the real axis, and they are associated with the surface plasmon polariton along the metal-vacuum interface [24]. Therefore, when solving the scattering problem, the spectral component of the current source is significantly amplified in the neighborhood the plasmonic frequency $\mathrm{Re}\,\xi_\pm$. This is the so-called surface plasmon resonance.

In order to address the above difficulties, we decompose the underlying integral operator into two frequency bands $\Delta := \{\xi \,|\, |\xi| \le 2k\}$ and $\mathbb{R}\backslash\Delta$ in the Fourier domain. The former contains the plasmonic poles while the latter doesn't. This allows for analysis of the solution amplification due to the surface plasmon. Accordingly, the scattering solution is decomposed into the plasmonic and nonplasmoinc parts. In the Fourier domain, they correspond to the spectral components of the solution in the frequency band Δ and $\mathbb{R}\backslash\Delta$, respectively. In this paper, we shall derive sharp estimates for the plasmonic and nonplasmonic parts of the solution. Their explicit dependence on the gap size δ and the permittivity value ε_m will be given in Corollaries 4.15 and 5.8.

The rest of the paper is organized as follows. The integral equation formulation for the scattering problem (1.1) is derived in section 2. In section 3, we investigate the integral operators thoroughly and present relevant estimates. The solution of the integral equation is then studied in sections 4 and 5 when the incident wave is even and odd with respect to the x_1 variable.

2. Integral equation formulation. Let $G_1(x,y)$ be the Green's function in the upper layer that satisfies

$$\begin{cases} \Delta G_1(x,y) + k^2 G_1(x,y) = \tilde{\delta}(x-y), & x,y \in \Omega_1, \\ \dfrac{\partial G_1(x,y)}{\partial \nu_y} = 0 \quad \text{on } \partial\Omega_1. \end{cases}$$

Then

$$G_1(x,y) = -\frac{i}{4}\left(H_0^{(1)}(k|x-y|) + H_0^{(1)}(k|x'-y|)\right),$$

where $H_0^{(1)}$ is the first kind Hankel function of order 0, and $x' = (x_1, -x_2)$. Let $\frac{\partial u^s}{\partial \nu}(y_1, 0\pm)$ denote the limit of the function from above and below the interface, respectively. From the Green's second identity, one obtains an integral equation for the scattered field u^s:

$$u^s(x) = \int_\Gamma G_1(x,y)\frac{\partial u^s}{\partial \nu}(y_1, 0+)ds_y, \quad x \in \Omega_1.$$

From the continuity of the single-layer potential and the fact that $\frac{\partial u}{\partial \nu} = 0$ on Γ_δ^0, it follows that the total field satisfies

$$u(x) = \int_{\Gamma_\delta^- \cup \Gamma_\delta^+} G_1(x,y) \frac{\partial u}{\partial \nu}(y_1, 0+)\, ds_y + 2u^i(y), \quad x \in \Gamma_\delta^- \cup \Gamma_\delta^+. \tag{2.1}$$

Let $G_{2,\pm}(x,y)$ be the Green's function in the domain $\Omega_2^\pm$ that satisfies

$$\begin{cases} \Delta G_{2,\pm}(x,y) + k^2 \varepsilon_m G_{2,\pm}(x,y) = \varepsilon_m \tilde{\delta}(x-y), & x, y \in \Omega_2^\pm, \\ \dfrac{\partial G_{2,\pm}(x,y)}{\partial \nu_y} = 0 \quad \text{on } \Gamma_v^\pm \cup \Gamma_\delta^\pm. \end{cases}$$

It is easy to check that

$$\begin{aligned} G_{2,\pm}(x,y) &= G_2^{(0)}(x_1, x_2, y_1, y_2) + G_2^{(0)}(x_1, -x_2, y_1, y_2) \\ &\quad + G_2^{(0)}(\pm 2\delta - x_1, x_2, y_1, y_2) + G_2^{(0)}(\pm 2\delta - x_1, -x_2, y_1, y_2), \end{aligned}$$

where $G_2^{(0)}(x_1, x_2, y_1, y_2) := \varepsilon_m H_0^{(1)}(k_m |x-y|)$ is the Green's function in the homogeneous medium of metal. By the Green's second identity, we obtain

$$u(x) = -\int_{\Gamma_\delta^\pm} G_{2,\pm}(x,y) \frac{1}{\varepsilon_m} \frac{\partial u}{\partial \nu}(y_1, 0-)\, ds_y, \quad x \in \Omega_2^\pm.$$

Taking the limit leads to

$$u(x) = -\int_{\Gamma_\delta^\pm} G_{2,\pm}(x,y) \frac{1}{\varepsilon_m} \frac{\partial u(y)}{\partial \nu}(y_1, 0-)\, ds_y, \quad x \in \Gamma_\delta^\pm. \tag{2.2}$$

Let us define a function $\varphi \in H^{-1/2}(\mathbb{R})$ by letting

$$\varphi(x_1) = \begin{cases} \partial_\nu u(x_1, 0), & x_1 \in (-\infty, -\delta) \cup (\delta, \infty), \\ 0, & x_1 \in (-\delta, \delta). \end{cases}$$

From the continuity conditions $\partial_\nu u(x_1, 0+) = \frac{1}{\varepsilon_m} \partial_\nu u(x_1, 0-)$ along the interfaces $\Gamma_\delta^\pm$, a combination of (2.1) and (2.2) leads to the system of integral equations

$$\begin{cases} \displaystyle\int_{-\infty}^{-\delta} \Big[G_1(x_1, 0; y_1, 0) + G_{2,-}(x_1, 0; y_1, 0) \Big] \varphi(y_1)\, dy_1 \\ \quad \displaystyle + \int_\delta^\infty G_1(x_1, 0; y_1, 0)\, \varphi(y_1)\, dy_1 + 2u^i = 0, \\ \displaystyle\int_{-\infty}^{-\delta} G_1(x_1, 0; y_1, 0)\, \varphi(y_1)\, dy_1 \\ \quad \displaystyle + \int_\delta^\infty \Big[G_1(x_1, 0; y_1, 0) + G_{2,+}(x_1, 0; y_1, 0) \Big] \varphi(y_1)\, dy_1 + 2u^i = 0, \end{cases} \tag{2.3}$$

where the first equation holds for $x_1 < -\delta$ and the second holds for $x_1 > \delta$.

Due to the symmetry of the geometry for the scattering problem, one can decompose the incident wave as a superposition of an even and an odd part and solve (2.3) for each part. Thus in what follows, we consider two cases when the incident wave u^i is even and odd respectively with respect to x_1. To this end, let us define

$$u_e^i = \frac{1}{2}(e^{i(k_1x_1+k_2x_2)} + e^{i(-k_1x_1+k_2x_2)}) = \cos(k_1x_1) \cdot e^{ik_2x_2}$$

and

$$u_o^i = \frac{1}{2i}(e^{i(k_1x_1+k_2x_2)} - e^{i(-k_1x_1+k_2x_2)}) = \sin(k_1x_1) \cdot e^{ik_2x_2}.$$

(i) $u^i = u_e^i$ such that $\varphi(x_1) = \varphi(-x_1)$. The system (2.3) reduces to the following integral equation on Γ_δ^+:

(2.4)
$$\int_\delta^\infty \Big[G_1(x_1,0;y_1,0)+G_1(x_1,0;-y_1,0)+G_{2,+}(x_1,0;y_1,0)\Big]\varphi(y_1)\,dy_1+2u^i = 0 \quad x_1 > \delta.$$

(ii) $u^i = u_o^i$ such that $\varphi(x_1) = -\varphi(-x_1)$. The system reduces to

(2.5)
$$\int_\delta^\infty \Big[G_1(x_1,0;y_1,0)-G_1(x_1,0;-y_1,0)+G_{2,+}(x_1,0;y_1,0)\Big]\varphi(y_1)\,dy_1+2u^i = 0 \quad x_1 > \delta.$$

Define the integral operators

$$K_1^e\varphi(x_1) = \int_\delta^\infty \Big[G_1(x_1,0;y_1,0) + G_1(x_1,0;-y_1,0)\Big]\,\varphi(y_1)\,dy_1, \quad x_1 > \delta, \tag{2.6}$$

$$K_1^o\varphi(x_1) = \int_\delta^\infty \Big[G_1(x_1,0;y_1,0) - G_1(x_1,0;-y_1,0)\Big]\,\varphi(y_1)\,dy_1, \quad x_1 > \delta, \tag{2.7}$$

$$K_2\varphi(x_1) = \int_\delta^\infty G_{2,+}(x_1,0;y_1,0)\,\varphi(y_1)\,dy_1, \quad x_1 > \delta. \tag{2.8}$$

We express the integral equations (2.4) and (2.5) as

$$(K_1^e + K_2)\varphi = -2u_e^i \quad \text{and} \quad (K_1^o + K_2)\varphi = -2u_o^i, \tag{2.9}$$

respectively.

3. Analysis of the integral operators.

In this section, we analyze the integral operators K_1^e, K_1^o, and K_2 via the spectral decomposition. This leads to the definition of plasmonic poles and the decomposition of the integral operator into two parts, which allows for the analysis of the operator in the frequency band with and without the poles separately.

3.1. Preliminaries.

Letting $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$ the standard fractional Sobolev space with the norm

$$\|u\|^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u defined by

$$\hat{u}(\xi) := \int_{-\infty}^{\infty} u(x_1)e^{-i\xi x_1}\,dx_1.$$

Let I be an interval in $\mathbb{R}$ and define

$$H^s(I) := \{u = U|_I \;\big|\; U \in H^s(\mathbb{R})\}.$$

Then $H^s(I)$ is a Hilbert space with the norm

$$\|u\|_{H^s(I)} = \inf\{\|U\|_{H^s(\mathbf{R})} \bigm| U \in H^s(\mathbb{R}) \text{ and } U|_I = u\}.$$

We also define

$$\tilde{H}^s(I) := \{u = U|_I \bigm| U \in H^s(\mathbb{R}) \text{ and supp } U \subset \bar{I}\}.$$

One can show that (see [1]) the space $\tilde{H}^s(I)$ is the dual of $H^{-s}(I)$ and the norm for $\tilde{H}^s(I)$ can be defined via the duality. As such $\tilde{H}^s(I)$ is also a Hilbert space. We refer to [1] for more details about the fractional Sobolev spaces. In what follows, we are mostly concerned with the case when $s = \pm\frac{1}{2}$ and $I = (\delta, \infty)$ for some $\delta \geq 0$. We denote

$$\tilde{H}^{-\frac{1}{2}}(\delta, \infty) = \tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+), \quad H^{\frac{1}{2}}(\delta, \infty) = H^{\frac{1}{2}}(\Gamma_\delta^+).$$

Remark 1. For a given function $\varphi \in \tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)$, we may associate it with a function defined over the whole real line that vanishes on $\mathbb{R}\backslash\Gamma_\delta^+$. Without stating this explicitly and with the abuse of notation, we denote the new function as φ here and in several places throughout the paper, and it is obvious that $\varphi \in H^{-\frac{1}{2}}(\mathbb{R})$.

We now define the even and odd extension operators ($\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+) \to H^{-1/2}(\mathbb{R})$) by letting

$$E\varphi(x_1) = \varphi(x_1) + \varphi(-x_1), \quad O\varphi(x_1) = \varphi(x_1) - \varphi(-x_1).$$

It is clear that $E\varphi$ and $O\varphi$ are an even and an odd function, respectively.

LEMMA 3.1. *Let $\varphi \in \tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)$ where $\delta > 0$. Then $E\varphi$ and $O\varphi$ are an even and an odd function, respectively, and it holds that*

$$\widehat{E\varphi}(\xi) = \widehat{\varphi}(\xi) + \widehat{\varphi}(-\xi), \quad \widehat{O\varphi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(-\xi).$$

Furthermore,

$$||\varphi||_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)} \lesssim ||E\varphi||_{H^{-1/2}(\mathbb{R})} \lesssim ||\varphi||_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)},$$
$$||\varphi||_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)} \lesssim ||O\varphi||_{H^{-1/2}(\mathbb{R})} \lesssim ||\varphi||_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)}.$$

We postpone the proof of the lemma to the appendix.

3.2. Spectral representation of the integral operators and surface plasmonic polaritons. It is known that the Hankel functions admit the spectral decomposition (cf. [13]):

$$-\frac{i}{4}H_0^{(1)}(k|x-y|) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2\rho_0(\xi)} e^{-\rho_0(\xi)(x_2-y_2)} e^{i\xi(x_1-y_1)} d\xi,$$
$$-\frac{i}{4}H_0^{(1)}(k_m|x-y|) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2\rho_m(\xi)} e^{-\rho_m(\xi)(x_2-y_2)} e^{i\xi(x_1-y_1)} d\xi,$$

where

$$\rho_0(\xi) = \sqrt{\xi^2 - k^2}, \quad \rho_m(\xi) = \sqrt{\xi^2 - k^2\varepsilon_m}.$$

With the above spectral decompositions, we may rewrite the operators K_1^e, K_1^o, and K_2 as follows:

$$K_1^e\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi x_1}+e^{-i\xi x_1}}{\rho_0(\xi)}\int_{\delta}^{\infty}\varphi(y_1)e^{-i\xi y_1}\,dy_1 d\xi$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\rho_0(\xi)}\left(\widehat{\varphi}(\xi)+\widehat{\varphi}(-\xi)\right)e^{i\xi x_1}\,d\xi$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\rho_0(\xi)}\widehat{E\varphi}(\xi)e^{i\xi x_1}\,d\xi; \tag{3.1}$$

$$K_1^o\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi x_1}-e^{-i\xi x_1}}{\rho_0(\xi)}\int_{\delta}^{\infty}\varphi(y_1)e^{-i\xi y_1}\,dy_1 d\xi$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\rho_0(\xi)}\left(\widehat{\varphi}(\xi)-\widehat{\varphi}(-\xi)\right)e^{i\xi x_1}\,d\xi$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\rho_0(\xi)}\widehat{O\varphi}(\xi)e^{i\xi x_1}\,d\xi; \tag{3.2}$$

$$K_2\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon_m(e^{i\xi x_1}+e^{i\xi(2\delta-x_1)})}{\rho_m(\xi)}\int_{\delta}^{\infty}\varphi(y_1)e^{-i\xi y_1}\,dy_1 d\xi$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon_m}{\rho_m(\xi)}\left(\widehat{\varphi}(\xi)+e^{-i2\delta\xi}\widehat{\varphi}(-\xi)\right)e^{i\xi x_1}\,d\xi$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon_m}{\rho_m(\xi)}e^{-i\delta\xi}\,\widehat{E\varphi_\delta}(\xi)\,e^{i\xi x_1}\,d\xi. \tag{3.3}$$

where φ_δ is defined by

$$\varphi_\delta(x_1) = \varphi(x_1+\delta).$$

If $\delta = 0$, the symbol (multiplier) associated with the operator $K_1^e + K_2$ is given by $\frac{\rho_0(\xi)\varepsilon_m+\rho_m(\xi)}{\rho_0(\xi)\rho_m(\xi)}$, which attains zeros at $\xi_\pm(k) = \pm k\sqrt{\varepsilon_m/(\varepsilon_m+1)}$. This implies that when inverting the operator $K_1^e + K_2$, the corresponding symbol will attain poles at $\xi = \xi_\pm(k)$. The poles are associated with the surface plasmon polariton, which gives rise to eigenmodes that are localized along the metal-vacuum interface [24]. Correspondingly, the spectral component of the current source function is amplified in the neighborhood of the plasmonic frequency $\xi'_\pm$ when inverting the operator, and the surface plasmon is excited. It can be calculated that

$$\xi_\pm = \xi'_\pm + i\xi''_\pm, \tag{3.4}$$

where $\xi'_\pm = k(1+O(1/|\varepsilon'_m|))$ and $\xi'_\pm = O(k\varepsilon''_m/|\varepsilon'_m|^2)$. Namely, $\xi_\pm$ lies in the vicinity of $\pm k$.

To address the difficulties induced by the surface plasmonic poles for solving the integral equation, we decompose the operator K_1^e by treating the spectral components with and without the poles separately. To this end, let $\Delta := \{\xi \,|\, |\xi| \le 2k\}$ and χ be the corresponding characteristic function. We decompose the operator K_1^e as

$$K_1^e = K_{1,0}^e + K_{1,1}^e,$$

where

$$K_{1,0}^e\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\widehat{E\varphi}(\xi)\,e^{i\xi x_1}\,d\xi, \tag{3.5}$$

$$K_{1,1}^e\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\chi_\Delta(\xi)}{\rho_0(\xi)}\widehat{E\varphi}(\xi)\,e^{i\xi x_1}\,d\xi. \tag{3.6}$$

Similarly, the operator K_1^o is decomposed as

$$K_1^o = K_{1,0}^o + K_{1,1}^o,$$

where

$$K_{1,0}^o\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\,\widehat{O\varphi}(\xi)\,e^{i\xi x_1}\,d\xi, \tag{3.7}$$

$$K_{1,1}^o\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\chi_\Delta(\xi)}{\rho_0(\xi)}\,\widehat{O\varphi}(\xi)\,e^{i\xi x_1}\,d\xi. \tag{3.8}$$

It is straightforward that the operators $K_{1,0}^e$ and $K_{1,0}^o$ can be extended to bounded operators from $\tilde{H}^{-1/2}(\Gamma_\delta^+)$ to $H^{1/2}(\mathbb{R})$ and hence are bounded from $\tilde{H}^{-1/2}(\Gamma_\delta^+)$ to $H^{1/2}(\Gamma_\delta^+)$.

We also decompose the operator K_2 as

$$K_2 = K_{2,0} + K_{2,1},$$

where $K_{2,0}$ is the corresponding integral operator when $\delta = 0$. Namely,

$$K_{2,0}\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)}\,\widehat{E\varphi}(\xi)e^{i\xi x_1}\,d\xi, \tag{3.9}$$

$$K_{2,1}\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)}\,(e^{-i2\delta\xi}-1)\,\widehat{\varphi}(-\xi)\,e^{i\xi x_1}\,d\xi. \tag{3.10}$$

Equivalently, this gives

$$K_{2,0}\varphi(x_1) = 2\int_\delta^\infty \left(G_2^{(0)}(x_1,0;y_1,0) + G_2^{(0)}(-x_1,0;y_1,0)\right)\,\varphi(y_1)\,dy_1, \tag{3.11}$$

$$\begin{aligned} K_{2,1}\varphi(x_1) = \int_\delta^\infty \Big(&G_{2,+}(x_1,0;y_1,0) - 2G_2^{(0)}(x_1,0;y_1,0) \\ &- 2G_2^{(0)}(-x_1,0;y_1,0)\Big)\,\varphi(y_1)\,dy_1, \end{aligned} \tag{3.12}$$

where $G_2^{(0)}(x_1,x_2,y_1,y_2) = \varepsilon_m H_0^{(1)}(k_m|x-y|)$.

LEMMA 3.2. *The operator K_2 is bounded from $\tilde{H}^{-1/2}(\Gamma_\delta^+)$ to $H^{1/2}(\Gamma_\delta^+)$. Furthermore, the inverse K_2^{-1} exists and there holds*

$$||K_2^{-1}|| \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}.$$

Proof. For $\varphi \in \tilde{H}^{-1/2}(\Gamma_\delta^+)$, it is clear that $K_2\varphi$ can be extended naturally to $H^{1/2}(\mathbb{R})$. With abuse of notation, we also denote the extension as $K_2\varphi$. It is straightforward to check that K_2 bounded from $\tilde{H}^{-1/2}(\Gamma_\delta^+)$ to $H^{1/2}(\Gamma_\delta^+)$. We next show that K_2 is invertible. Indeed,

$$\begin{aligned} \langle K_2\varphi,\varphi\rangle_{L^2(\Gamma_\delta^+)} &= \langle \widehat{K_2\varphi},\widehat{\varphi}\rangle_{L^2(\mathbb{R})} \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} e^{-i\delta\xi}\,\widehat{E\varphi_\delta}(\xi)\,\overline{\widehat{\varphi}(\xi)}\,d\xi \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \widehat{E\varphi_\delta}(\xi)\,\overline{\widehat{\varphi_\delta}(\xi)}\,d\xi. \end{aligned}$$

Noting that $\widehat{E\varphi_\delta}$ is an even function and $\widehat{\varphi_\delta} - \widehat{E\varphi_\delta}$ is an odd function, we have

$$\int_{-\infty}^{\infty} \frac{1}{\rho_m(\xi)} \widehat{E\varphi_\delta}(\xi) \left(\overline{\widehat{\varphi_\delta}(\xi)} - \overline{\widehat{E\varphi_\delta}(\xi)} \right) \, d\xi = 0.$$

Consequently,

$$\langle K_2\varphi, \varphi\rangle = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \left| \widehat{E\varphi_\delta}(\xi) \right|^2 \, d\xi.$$

Therefore,

$$\begin{aligned} |\langle K_2\varphi, \varphi\rangle| &\geq \mathrm{Re} \left\{ -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \left| \widehat{E\varphi_\delta}(\xi) \right|^2 \, d\xi \right\} \\ &\geq C \int_{-\infty}^{\infty} \frac{|\varepsilon_m'|}{\sqrt{\xi^2 - k^2\varepsilon_m'}} \left| \widehat{E\varphi_\delta}(\xi) \right|^2 \, d\xi \end{aligned} \tag{3.13}$$

for some universal constant C. Here we have used the fact that $|\varepsilon_m'| \gg |\varepsilon_m''|$ in the last inequality. Note that $\varepsilon_m' < 0$, hence

$$\int_{\{|\xi|^2 > k^2|\varepsilon_m'|\}} \frac{|\varepsilon_m'|}{\sqrt{\xi^2 - k^2\varepsilon_m'}} \left| \widehat{E\varphi_\delta}(\xi) \right|^2 \, d\xi \geq \frac{|\varepsilon_m'|}{\sqrt{2}} \int_{\{|\xi|^2 > k^2|\varepsilon_m'|\}} \frac{1}{\sqrt{\xi^2+1}} \left| \widehat{E\varphi_\delta}(\xi) \right|^2 \, d\xi,$$

$$\int_{\{|\xi|^2 < k^2|\varepsilon_m'|\}} \frac{|\varepsilon_m'|}{\sqrt{\xi^2 - k^2\varepsilon_m'}} \left| \widehat{E\varphi_\delta}(\xi) \right|^2 d\xi \geq \frac{\sqrt{|\varepsilon_m'|}}{\sqrt{2}k} \int_{\{|\xi|^2 < k^2|\varepsilon_m'|\}} \frac{1}{\sqrt{\xi^2+1}} \left| \widehat{E\varphi_\delta}(\xi) \right|^2 d\xi.$$

Substituting into (3.13) yields

$$\begin{aligned} |\langle K_2\varphi, \varphi\rangle| &\geq C\sqrt{|\varepsilon_m'|} \cdot \|E\varphi_\delta\|^2_{H^{-1/2}(\mathbb{R})} \geq C\sqrt{|\varepsilon_m'|} \cdot ||\varphi_\delta||^2_{H^{-1/2}(\mathbb{R}^+)} \\ &= C\sqrt{|\varepsilon_m'|} \cdot ||\varphi||^2_{H^{-1/2}(\Gamma_\delta^+)}. \end{aligned}$$

Therefore, we obtain

$$||K_2\varphi||_{H^{1/2}(\Gamma_\delta^+)} \geq C\sqrt{|\varepsilon_m'|} \cdot ||\varphi||_{\tilde{H}^{-1/2}(\Gamma_\delta^+)}. \tag{3.14}$$

It follows that K_2 is injective and the range $\mathrm{Ran}(K_2)$ is closed.

A parallel calculation as above shows that the adjoint operator $K_2\varphi^*$ has a similar property. Especially, $K_2\varphi^*$ is injective. Thus we have

$$\mathrm{Ran}(K_2) = (\mathrm{Ker}((K_2)^*)^\perp = \{0\}^\perp = H^{1/2}(\Gamma_\delta^+). \tag{3.15}$$

From (3.14) and (3.15), we conclude that K_2 is invertible and

$$||K_2^{-1}|| \leq \frac{C}{\sqrt{|\varepsilon_m'|}}. \qquad \square$$

4. Solution of the integral equation for the even case. In this section, we develop a mathematical framework for solving the integral equation (2.9) of the even case and obtain the asymptotic expansion of the solution. In particular, we characterize the energy of the surface plasmon wave in terms of the gap size δ and the permittivity values ε_m. The study for the odd case is presented in section 5.

4.1. An overview of the methodology. Let us introduce the integral operator

$$D := K_1^e + K_2,$$

and the integral equation (2.9) reads

$$D\varphi = -2u_e^i. \tag{4.1}$$

Let $\tilde{D} : H^{-1/2}(\mathbb{R}^+) \to H^{1/2}(\mathbb{R}^+)$ be the sum of K_1^e and K_2 when the slit width $\delta = 0$. Its spectral representation is given by

$$\tilde{D}\varphi = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\rho_0(\xi)} + \frac{\varepsilon_m}{\rho_m(\xi)}\right)\widehat{E\varphi}(\xi)e^{i\xi x_1}\,d\xi, \quad x_1 > 0.$$

It follows by a direct calculation that the solution of the integral equation $\tilde{D}\varphi_{00} = -2u_e^i$ is given by

$$\varphi_{00} = R\cdot\cos(k_1x_1)\cdot\chi_{(0,+\infty)}, \tag{4.2}$$

where the coefficient

$$R = -\frac{2\rho_0(k_1)\rho_m(k_1)}{\rho_m(k_1) + \varepsilon_m\rho_0(k_1)}.$$

We view the operator D at the presence of a slit as a perturbation of $\tilde{D}$. As such let us decompose the solution of (2.9) as $\varphi = \varphi_0 + \varphi_1$, where

$$\varphi_0 = \varphi_{00}\cdot\chi_{[\delta,\infty)},$$

and φ_1 satisfies

$$D\varphi_1 = \tilde{D}\varphi_{00} - D\varphi_0, \tag{4.3}$$

Therefore, with a suitable decomposition of $\tilde{D}\varphi_{00} - D\varphi_0$ to be accomplished in section 4.4, we will need to investigate the solution of the following integral equation in order to obtain φ_1:

$$D\varphi = f, \tag{4.4}$$

where f lies in some finite energy space to be specified in section 4.3. From the spectral representation of integral operators in section 3.2, the symbol of the operator D contains plasmonic poles in the frequency band $\Delta = \{\xi \,|\, |\xi| \le 2k\}$. Consequently, the spectral component of the source function f would be amplified in the neighborhood of the plasmonic frequency $\xi_\pm'$ when inverting the operator D, and the surface plasmon resonance occurs.

Following the decomposition of the integral operators (3.7)–(3.9), we decompose the operator D as $D = D_0 + K_{1,1}^e$, where the operator $D_0 : \tilde{H}^{-1/2}(\Gamma_\delta^+) \to H^{1/2}(\Gamma_\delta^+)$ is given by

$$D_0 := K_{1,0}^e + K_2. \tag{4.5}$$

We can view D_0 as a preconditioner for the operator D. It is clear that the symbol of the operator D_0 does not contain plasmonic poles. In fact, it can be shown that D_0 is invertible.

PROPOSITION 4.1. *The operator $D_0 : \tilde{H}^{-1/2}(\Gamma_\delta^+) \to H^{1/2}(\Gamma_\delta^+)$ is invertible and there holds*

$$||D_0^{-1}|| \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}.$$

Proof. From Lemma 3.2, K_2^{-1} is invertible and we may rewrite D_0 as

$$D_0 = K_2 \cdot \left[I + (K_2)^{-1} K_{1,0}^e\right].$$

Since $||K_{1,0}^e|| \lesssim 1$ and $||K_2^{-1}|| \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}$, it follows that $I + K_2^{-1}K_{1,0}^e$ is invertible. Therefore,

$$||D_0^{-1}|| \leq \left\| \left[I + K_2^{-1}K_{1,0}^e\right]^{-1} \right\| \cdot \left\| K_2^{-1} \right\| \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}. \qquad \square$$

To analyze the operator D, we need to introduce two function spaces:

$$V_1 = \left\{ \varphi \in \tilde{H}^{-1/2}(\Gamma_\delta^+) : \int \frac{1}{|\rho_0(\xi)|} \, |\widehat{E\varphi}(\xi)|^2 \, d\xi < \infty \right\},$$
$$V_2 = \left\{ \varphi = U|_{\Gamma_\delta^+} : \int |\rho_0(\xi)| \, |\hat{U}(\xi)|^2 \, d\xi < \infty \right\}.$$

One can show that V_1 is a Hilbert space with the norm

$$\|\varphi\|_{V_1}^2 = \int \frac{1}{|\rho_0(\xi)|} \, |\widehat{E\varphi}(\xi)|^2 \, d\xi.$$

Moreover, one can show that V_2 is the dual space of V_1.

THEOREM 4.2. *The operator $D : V_1 \to V_2$ is bounded and is invertible. Moreover,*

$$||D^{-1}|| \lesssim \frac{|\varepsilon_m'|}{\varepsilon_m''}.$$

Proof. First, it is straightforward that D is bounded. We need only to show that D is invertible and the inverse is also bounded. Let φ be a solution to $D\varphi = f$, we first show that

$$\|\varphi\|_{V_1} \lesssim \frac{|\varepsilon_m'|}{\varepsilon_m''} \|f\|_{V_2}.$$

Indeed,

$$\begin{aligned} \langle D\varphi, \varphi \rangle &= \langle K_1^e \varphi, \varphi \rangle + \langle K_2 \varphi, \varphi \rangle \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} \, \widehat{E\varphi}(\xi) \, \overline{\widehat{\varphi}(\xi)} \, d\xi - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \, \widehat{E\varphi_\delta}(\xi) \, \overline{\widehat{\varphi_\delta}(\xi)} \, d\xi \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} \, |\widehat{E\varphi}(\xi)|^2 \, d\xi - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \, |\widehat{E\varphi_\delta}(\xi)|^2 \, d\xi. \end{aligned}$$

Using $|\varepsilon_m''| \gg \varepsilon_m'$, one can show that

$$\text{Im} \frac{\varepsilon_m}{\rho_m(\xi)} \geq \frac{1}{3} \frac{\varepsilon_m''}{|\rho_m(\xi)|}, \quad \text{Re} \frac{\varepsilon_m}{\rho_m(\xi)} \leq 2 \frac{|\varepsilon_m'|}{|\rho_m(\xi)|}.$$

As a result,

$$|\text{Im} \, \langle D\varphi, \varphi \rangle| \geq \int_{-k}^{k} \frac{1}{2\pi |\rho_0(\xi)|} \, |\widehat{E\varphi}(\xi)|^2 \, d\xi + \frac{1}{6\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_m''}{\rho_m(\xi)} \, |\widehat{E\varphi_\delta}(\xi)|^2 \, d\xi.$$

On the other hand,

$$|\langle D\varphi, \varphi \rangle| \leq |\langle f, \varphi \rangle| \leq \|f\|_{V_2} \cdot \|\varphi\|_{V_1}.$$

We obtain

$$\int_{-\infty}^{\infty} \frac{\varepsilon_m''}{\rho_m(\xi)} \, |\widehat{E\varphi_\delta}(\xi)|^2 \, d\xi \lesssim \|f\|_{V_2} \cdot \|\varphi\|_{V_1}.$$

Therefore,

$$\begin{aligned}\int_{|\xi|\geq k} \frac{1}{|\rho_0(\xi)|} \, |\widehat{E\varphi}(\xi)|^2 \, d\xi &\leq |\mathrm{Re}\,\langle D\varphi, \varphi\rangle| + \left|\mathrm{Re} \int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \, |\widehat{E\varphi_\delta}(\xi)|^2 \, d\xi\right| \\ &\lesssim \|f\|_{V_2} \cdot \|\varphi\|_{V_1} + \frac{|\varepsilon_m'|}{\varepsilon_m''} |\mathrm{Im} \int_{-\infty}^{\infty} \frac{\varepsilon_m''}{\rho_m(\xi)} \, |\widehat{E\varphi_\delta}(\xi)|^2 \, d\xi| \\ &\lesssim \|f\|_{V_2} \cdot \|\varphi\|_{V_1} + \frac{|\varepsilon_m'|}{\varepsilon_m''} \|f\|_{V_2} \cdot \|\varphi\|_{V_1} \\ &\lesssim \frac{|\varepsilon_m'|}{\varepsilon_m''} \|f\|_{V_2} \cdot \|\varphi\|_{V_1}.\end{aligned}$$

It follows that

$$\|\varphi\|_{V_1}^2 = \int \frac{1}{|\rho_0(\xi)|} \, |\widehat{E\varphi}(\xi)|^2 \, d\xi \lesssim \frac{|\varepsilon_m'|}{\varepsilon_m''} \|f\|_{V_2} \cdot \|\varphi\|_{V_1},$$

whence

$$\|\varphi\|_{V_1} \lesssim \frac{|\varepsilon_m'|}{\varepsilon_m''} \|f\|_{V_2}.$$

We conclude that the D is injective; moreover, the range of D is closed in V_2. We next show that the range of D is dense in V_2. For this, we consider the adjoint of D, denoted by D^*, which is defined by the following identity:

$$\langle D\varphi, \psi\rangle = \langle \varphi, D^*\psi\rangle$$

where $\psi \in V_1$. A direct computation shows that

$$D^*\psi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\rho_0(\xi)} \, \widehat{E\psi}(\xi) e^{i\xi x_1} \, d\xi - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} e^{i\delta\xi} \, \widehat{E\psi}(\xi) \, e^{i\xi x_1} \, d\xi, \; x_1 > \delta.$$

Therefore, a similar argument as for the operator D shows that D^* is injective. Consequently, the range of D is dense in V_2. This combines with the fact that the range of D is also closed in V_2 yields that D is onto the space V_2. Recall that D is also injective. The open mapping theorem gives that D is invertible and the inverse is also bounded. Moreover, the inverse satisfies that desired estimate. ∎

Remark 2. The above theorem establishes the existence and uniqueness of the solution to the integral equation (4.1) when the source term is in the space V_2. It also provides an energy estimate for the solution. We would like to point out that such estimation is not sharp. In what follows, we shall derive a sharp estimate for the solution by treating the plasmonic part and the nonplasmonic part separately. Moreover, we shall characterize the magnitude of the wave field induced by surface plasmonic resonance.

Observe that the operator equation (4.4) can be rewritten as

$$\varphi + D_0^{-1} K_{1,1}^e \varphi = D_0^{-1} f. \tag{4.6}$$

We first aim to express the term $D_0^{-1}K_{1,1}^e\varphi$ in the above equation in terms of φ by solving the equation $D_0\phi = K_{1,1}^e\varphi$, which excludes plasmoinc resonances. Then we study the enhancement effect induced from the surface plasmoinc resonances by solving the whole equation (4.6). These two steps are addressed in sections 4.2 and 4.3, respectively. Finally, we summarize the solution of (4.1) in section 4.4.

4.2. Solution of $D_0\phi = K_{1,1}^e\psi$. In this section, we solve the equation

$$D_0\phi = K_{1,1}^e\psi, \tag{4.7}$$

where it holds that $\|\frac{\widehat{E\psi}}{\sqrt{|\rho_0|}}\|_{L^2(\Delta)} < \infty$. It is clear that $K_{1,1}^e\psi \in V_2$.

Using the spectral decomposition of the operators (3.1) and (3.3), let us decompose the operator D_0 as $D_0 = D_{0,0} + D_{0,1}$, where

$$D_{0,0} := (K_{1,0}^e + K_{2,0}) \quad \text{and} \quad D_{0,1} := K_{2,1}. \tag{4.8}$$

We also introduce the operator $\tilde{D}_{0,0} : H^{-1/2}(\mathbb{R}^+) \to H^{1/2}(\mathbb{R}^+)$ as follows:

$$\tilde{D}_{0,0}\varphi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon_m}{\rho_m(\xi)} + \frac{1-\chi_\Delta}{\rho_0(\xi)}\right)\widehat{E\varphi}(\xi)\, e^{i\xi x_1}\, d\xi, \quad x_1 > 0. \tag{4.9}$$

Define the function $\phi_{00,\psi}(x_1)$ over the whole real line such that its Fourier transform is given by

$$\widehat{\phi_{00,\psi}}(\xi) = \frac{\chi_\Delta \cdot \hat{\psi}(\xi)}{\rho_0(\xi)\left(\dfrac{\varepsilon_m}{\rho_m(\xi)} + \dfrac{1-\chi_\Delta}{\rho_0(\xi)}\right)} = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m}\cdot \widehat{E\psi}(\xi)\cdot\chi_\Delta. \tag{4.10}$$

LEMMA 4.3. *The following estimate holds for $\phi_{00,\psi}$:*

$$\|\widehat{\phi_{00,\psi}}\|_{L^1(\Delta)} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}\|\frac{\widehat{E\psi}}{\sqrt{|\rho_0|}}\|_{L^2(\Delta)}.$$

Moreover, $\phi_{00,\psi}$ is a smooth and even function with

$$\|\phi_{00,\psi}\|_{C^3(\mathbb{R})} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}\|\frac{\widehat{E\psi}}{\sqrt{|\rho_0|}}\|_{L^2(\Delta)}.$$

Proof. From (4.10) we see that

$$|\widehat{\phi_{00,\psi}}(\xi)| \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}\frac{1}{|\rho_0(\xi)|}\cdot\widehat{E\psi}(\xi).$$

Therefore,

$$\|\widehat{\phi_{00,\psi}}\|_{L^1(\Delta)} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}\left\|\frac{\widehat{\psi}\cdot\chi_\Delta}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}\cdot\left\|\frac{\chi_\Delta}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}\cdot\left\|\frac{\widehat{E\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}.$$

The second estimate follows immediately. ∎

Let $\phi_{00,+} = \phi_{00,\psi}\cdot\chi_{(0,\infty)}$. By observing that

$$\phi_{00,\psi}(x_1) = \phi_{00,+}(x_1) + \phi_{00,+}(-x_1) = E\phi_{00,+}(x_1) \quad \text{and} \quad \widehat{\phi_{00,\psi}}(\xi) = \widehat{E\phi_{00,+}}(\xi),$$

we have

$$\left(\frac{\varepsilon_m}{\rho_m(\xi)} + \frac{1-\chi_\Delta}{\rho_0(\xi)}\right)\widehat{E\phi_{00,+}}(\xi) = \frac{\chi_\Delta}{\rho_0(\xi)}\widehat{E\psi}(\xi).$$

Consequently, it holds that

$$\tilde{D}_{0,0}\,\phi_{00,+} = K^{e}_{1,1}\psi, \tag{4.11}$$

where we have extended $K^{e}_{1,1}\psi$ naturally to $\mathbb{R}^+$.

We decompose the solution ϕ of the operator equation (4.7) as $\phi_0 + \phi_1$, where

$$\phi_0 = \phi_{00,\psi}\cdot\chi_{[\delta,\infty)}.$$

Using (4.11), it is seen that ϕ_1 satisfies

$$D_0\phi_1 = q, \tag{4.12}$$

where

$$q(x_1) := \tilde{D}_{0,0}\,\phi_{00,+}(x_1) - [D_{0,0}\phi_0(x_1) + D_{0,1}\phi_0(x_1)] \quad \text{for } x_1 > \delta. \tag{4.13}$$

For brevity of notation, here and henceforth, we let

$$g(x_1 - y_1) := G_2^{(0)}(x_1, 0; y_1, 0) = \varepsilon_m H_0^{(1)}(k_m|x_1 - y_1|). \tag{4.14}$$

Lemma 4.4. *Let q be defined in* (4.13)*; then $q(x_1) = q_1(x_1) + q_2(x_1)$, where*

$$q_1(x_1) = -2\int_0^\infty g(x_1 + y_1 - \delta)\,(\phi_{00,\psi}(y_1+\delta) - \phi_{00,\psi}(y_1 - \delta))\,dy_1, \quad x_1 > \delta, \tag{4.15}$$

$$q_2(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\chi_\Delta}{\rho_0(\xi)}\,(\phi_{00,\psi}\cdot\chi_{(-\delta,\delta)})^\wedge(\xi)\,e^{i\xi x_1}\,d\xi, \quad x_1 > \delta. \tag{4.16}$$

Moreover, the following asymptotic expansions hold for $x_1 > \delta$:

$$q_1(x_1) = \phi''_{00,\psi}(0)\cdot q_{1,0}(x_1)\cdot\varepsilon_m\delta^3 + O(\varepsilon_m\delta^4)\cdot\|\phi_{00,\psi}\|_{C^3(\mathbb{R})}, \tag{4.17}$$

$$q_2(x_1) = \phi_{00,\psi}(0)\cdot q_{2,0}(x_1) + O\left(\delta^3\sqrt{|\ln\delta|})\cdot\|\phi_{00,\psi}\|_{C^3(\mathbb{R})}\right), \tag{4.18}$$

where

$$\|q_{1,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim 1 \quad \textit{and} \quad \|q_{2,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta\sqrt{|\ln\delta|}.$$

Proof. From the definition (3.12),

$$D_{0,1}\phi_0(x_1) = \int_\delta^\infty \Big(G_{2,+}(x_1, 0; y_1, 0) - 2G_2^{(0)}(x_1, 0; y_1, 0) - 2G_2^{(0)}(-x_1, 0; y_1, 0)\Big)\,\phi_0(y_1)\,dy_1.$$

Therefore,

$$
\begin{aligned}
(4.19)\quad D_{0,1}\phi_0(x_1) &= 2\int_{\delta}^{\infty} \left(g(x_1+y_1-2\delta) - g(x_1+y_1)\right)\, \phi_0(y_1)\, dy_1 \\
&= 2\int_{-\delta}^{\infty} g(x_1+y_1)\, \phi_0(y_1+2\delta)\, dy_1 - 2\int_{\delta}^{\infty} g(x_1+y_1)\, \phi_0(y_1)\, dy_1 \\
&= 2\int_{-\delta}^{\infty} g(x_1+y_1)\, (\phi_{00,\psi}(y_1+2\delta) - \phi_{00,\psi}(y_1))\, dy_1 \\
&\quad + 2\int_{-\delta}^{\delta} g(x_1+y_1)\, \phi_{00,\psi}(y_1)\, dy_1 \\
&= 2\int_{0}^{\infty} g(x_1+y_1-\delta)\, (\phi_{00,\psi}(y_1+\delta) - \phi_{00,\psi}(y_1-\delta))\, dy_1 \\
&\quad + 2\int_{-\delta}^{\delta} g(x_1+y_1)\, \phi_{00,\psi}(y_1)\, dy_1.
\end{aligned}
$$

On the other hand, from (4.8) and (4.9), it follows that for $x_1 > \delta$,

$$
\begin{aligned}
(4.20)\qquad & \tilde{D}_{0,0}\, \phi_{00,+}(x_1) - D_{0,0}\phi_0(x_1) \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\varepsilon_m}{\rho_m(\xi)} + \frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\right) (\phi_{00,\psi}\cdot\chi_{(-\delta,\delta)})^\wedge(\xi)\, e^{i\xi x_1}\, d\xi.
\end{aligned}
$$

Note that

$$
(4.21)\quad -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)}\, (\phi_{00,\psi}\cdot\chi_{(-\delta,\delta)})^\wedge(\xi)\, e^{i\xi x_1}\, d\xi = 2\int_{-\delta}^{\delta} g(x_1+y_1)\, \phi_{00,\psi}(y_1)\, dy_1,
$$

then (4.15) and (4.16) follows by combining (4.19)–(4.21).

We now derive the asymptotics for q_1 and q_2. Note that

$$
\phi_{00,\psi}(y_1+\delta) - \phi_{00,\psi}(y_1-\delta) = 2\delta\, \phi''_{00,\psi}(0) y_1 + R_1(y_1)y_1^3 + R_2(y_1)y_1^2\delta + R_3(y_1)y_1\delta^2,
$$

where R_1, R_2, R_3 are smooth functions such that

$$
\|R_j\|_{C^1(\mathbb{R})} \lesssim \|\phi_{00,\psi}\|_{C^3(\mathbb{R})}, \quad j = 1,2,3.
$$

Correspondingly, we decompose q_1 as

$$
q_1 =: q_{1,0} + q_{1,1} + q_{1,2} + q_{1,3},
$$

where $q_{1,j}$ is the integral of the above density.

Setting $x_1' = (x-\delta)/\delta$, $y_1' = y_1/\delta$, and $k' = k_m\delta$, then it follows that $k' = O(1)$ and

$$
\begin{aligned}
\tilde{q}_{1,0}(x_1') &:= q_{1,0}(\delta x_1' + \delta) = 2\delta^3\, \phi''_{00,\psi}(0)\cdot\varepsilon_m\cdot \int_0^\infty H_0^{(1)}(ik'|x_1'+y_1'|)\, y_1'\, dy_1', \\
\tilde{q}_{1,1}(x_1') &:= q_{1,1}(\delta x_1' + \delta) = \delta^4\, \cdot\varepsilon_m\cdot \int_0^\infty H_0^{(1)}(ik'|x_1'+y_1'|) R_1(\delta y_1')\, (y_1')^3\, dy_1', \\
\tilde{q}_{1,2}(x_1') &:= q_{1,2}(\delta x_1' + \delta) = \delta^4\, \cdot\varepsilon_m\cdot \int_0^\infty H_0^{(1)}(ik'|x_1'+y_1'|) R_2(\delta y_1')\, (y_1')^2\, dy_1', \\
\tilde{q}_{1,3}(x_1') &:= q_{1,3}(\delta x_1' + \delta) = \delta^4\, \cdot\varepsilon_m\cdot \int_0^\infty H_0^{(1)}(ik'|x_1'+y_1'|) R_3(\delta y_1')\, y_1'\, dy_1'.
\end{aligned}
$$

Since $H_0^{(1)}(ik'|y_1'|)$ decays exponentially, we can show that

$$\left\| \int_0^\infty H_0^{(1)}(ik'|x_1'+y_1'|)\, y_1'\, dy_1' \right\|_{H^{1/2}(0,\infty)} = O(1),$$
$$\left\| \int_0^\infty H_0^{(1)}(ik'|x_1'+y_1'|) R_1(\delta y_1')\, (y_1')^3\, dy_1' \right\|_{H^{1/2}(0,\infty)} \lesssim \|R_1\|_{C^1(\mathbb{R})},$$
$$\left\| \int_0^\infty H_0^{(1)}(ik'|x_1'+y_1'|) R_2(\delta y_1')\, (y_1')^2\, dy_1' \right\|_{H^{1/2}(0,\infty)} \lesssim \|R_2\|_{C^1(\mathbb{R})},$$
$$\left\| \int_0^\infty H_0^{(1)}(ik'|x_1'+y_1'|) R_2(\delta y_1')\, (y_1')^2\, dy_1' \right\|_{H^{1/2}(0,\infty)} \lesssim \|R_3\|_{C^1(\mathbb{R})}.$$

By the translation and scaling invariance of $||\cdot||_{H^{1/2}}$ norm, we deduce that the integral $q_1 \in H^{1/2}(\Gamma_\delta^+)$. Furthermore,

$$q_1(x_1) = \phi''_{00,\psi}(0)\, \varepsilon_m \delta^3 \cdot q_{1,0}(x_1) + O(\varepsilon_m \delta^4) \cdot \|\phi_{00,\psi}\|_{C^3(\mathbb{R})} \quad \text{in } H^{1/2}(\Gamma_\delta^+), \tag{4.22}$$

where $\|q_{1,0}\|_{H^{1/2}(\Gamma_\delta^+)} = O(1)$.

We extend q_2 naturally to the whole real line and still denote it as q_2. Applying the Taylor expansion, we see that

$$q_2(x_1) = \phi_{00,\psi}(0) \cdot (1 + O(\delta^2)) \cdot q_{2,0}(x_1), \tag{4.23}$$

where

$$q_{2,0} := \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\, \widehat{\chi_{(-\delta,\delta)}}(\xi)\, e^{i\xi x_1}\, d\xi.$$

It follows that

$$\begin{aligned} ||q_{2,0}||^2_{H^{1/2}(\mathbb{R})} &\lesssim \int_{-\infty}^\infty \frac{1}{\sqrt{1+|\xi|}} \frac{\sin^2(\delta\xi)}{\xi^2}\, d\xi \\ &= \delta^2 \cdot \int_{-\infty}^\infty \frac{1}{\sqrt{\delta^2+\xi^2}} \frac{\sin^2 \xi}{\xi^2}\, d\xi \\ &\le C\delta^2 |\ln \delta|. \end{aligned} \tag{4.24}$$

The proof is complete by combining (4.23) and (4.24). ∎

From the above discussions, we can obtain the expansion of the solution for the operator equation (4.7). In particular, by virtue of (4.12), Proposition 4.1, and Lemma 4.3–4.4, we arrive at the following conclusion.

THEOREM 4.5. *Let ϕ be the solution of the equation $D_0\phi = K_{1,1}^e \psi$. Let $\phi_{00,\psi}$ be defined by*

$$\hat{\phi}_{00,\psi}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m} \cdot \widehat{E\psi}(\xi) \cdot \chi_\Delta(\xi).$$

Then $\phi = \phi_0 + \phi_1$, where

$$\phi_0 = \phi_{00,\psi} \cdot \chi_{(\delta,\infty)},$$
$$\phi_1 = \delta^{3+\alpha} \cdot \phi''_{00,\psi}(0) \cdot D_0^{-1} q_{1,0} + \phi_{00,\psi}(0) \cdot D_0^{-1} q_{2,0} + D_0^{-1} q_h.$$

In addition,

$$\|\phi_{00,\psi}\|_{C^3(\mathbb{R})} \lesssim \delta^{-\frac{\alpha}{2}} \cdot \left\| \frac{\widehat{E\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)},$$

$$\|q_{1,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim 1, \quad \|q_{2,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta\sqrt{|\ln\delta|},$$

$$\|q_h\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \left(\delta^{4+\alpha} + \delta^3\sqrt{|\ln\delta|}\right)\|\phi_{00,\psi}\|_{C^3(\mathbb{R})}$$

$$\lesssim \left(\delta^{4+\frac{\alpha}{2}} + \delta^{3-\alpha/2}\sqrt{|\ln\delta|}\right)\left\| \frac{\widehat{E\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)},$$

$$\|\phi_1\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \left(\delta^3 + \delta^{1-\alpha}\sqrt{|\ln\delta|}\right)\left\| \frac{\widehat{E\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.$$

4.3. Solution of $\mathbf{D}\boldsymbol{\varphi} = \boldsymbol{f}$ and excitation of surface plasmon. Following (4.6), we rewrite the operator equation $D\varphi = f$ as

$$\varphi + D_0^{-1}K_{1,1}^e\varphi = D_0^{-1}f. \tag{4.25}$$

By Theorem 4.2, we see that the solution $\varphi \in V_1$ and it holds that

$$\left\| \frac{\widehat{E\varphi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)} \lesssim \|\varphi\|_{V_1} \lesssim \frac{|\varepsilon_m'|}{\varepsilon_m''}\|f\|_{V_2} = O(\delta^{\alpha-\beta})\|f\|_{V_2}. \tag{4.26}$$

From Theorem 4.5, it follows that $D_0^{-1}K_{1,1}^e\varphi = \phi_{00}\chi_{(\delta,\infty)} + \phi_1$, where ϕ_{00} is defined by

$$\widehat{\phi_{00}}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m} \cdot \widehat{E\varphi}(\xi) \cdot \chi_\Delta(\xi)$$

and

$$\phi_1 = \delta^{3+\alpha} \cdot \phi_{00}''(0) \cdot D_0^{-1}q_{1,0} + \phi_{00}(0) \cdot D_0^{-1}q_{2,0} + D_0^{-1}q_h.$$

By virtue of Lemma 4.3 and (4.26), we have the following estimate.

LEMMA 4.6. *The following estimate holds for ϕ_{00}:*

$$\|\widehat{\phi_{00}}\|_{L^1(\mathbb{R})} \lesssim \frac{|\varepsilon_m'|^{\frac{1}{2}}}{\varepsilon_m''}\|f\|_{V_2}.$$

Moreover, ϕ_{00} is a smooth and even function with

$$\|\phi_{00}\|_{C^3(\mathbb{R})} \lesssim \frac{|\varepsilon_m'|^{\frac{1}{2}}}{\varepsilon_m''}\|f\|_{V_2}.$$

Substituting the expansion for $D_0^{-1}K_{1,1}^e\varphi$ into (4.25), we obtain

$$\varphi + \phi_0 + \delta^{3+\alpha} \cdot \phi_{00}''(0) \cdot D_0^{-1}q_{1,0} + \phi_{00}(0) \cdot D_0^{-1}q_{2,0} + D_0^{-1}q_h = D_0^{-1}f. \tag{4.27}$$

Extending evenly over the whole real line yields

$$E\varphi + \phi_{00}(1-\chi_{(-\delta,\delta)}) + \delta^{3+\alpha} \cdot \phi_{00}''(0) \cdot ED_0^{-1}q_{1,0}$$
$$+ \phi_{00}(0) \cdot ED_0^{-1}q_{2,0} + ED_0^{-1}q_h = E(D_0^{-1}f).$$

This leads to the following equation in the Fourier domain:

$$\widehat{E\varphi}(\xi) + \widehat{\phi_{00}}(\xi) + \hat{Q}(\xi) = (ED_0^{-1}f)^\wedge(\xi), \tag{4.28}$$

where

$$\hat{Q}(\xi) := -(\phi_{00}\chi_{(-\delta,\delta)})^\wedge(\xi) + \delta^{3+\alpha}\cdot\phi_{00}''(0)\cdot(ED_0^{-1}q_{1,0})^\wedge(\xi) + \phi_{00}(0)\cdot(ED_0^{-1}q_{2,0})^\wedge(\xi) + (ED_0^{-1}q_h)^\wedge(\xi).$$

From the Taylor expansion,

$$(\phi_{00}\chi_{(-\delta,\delta)})^\wedge(\xi) = (\phi_{00}(0)+O(\delta^2))\cdot\hat{\chi}_{(-\delta,\delta)}(\xi) = \phi_{00}(0)\cdot\hat{q}_{2,1}(\xi) + \hat{q}_{2,2}(\xi),$$

where $\hat{q}_{2,1}(\xi) = \frac{\sin\delta\xi}{\xi}$ and $q_{2,2}$ satisfy the estimate

$$\|\hat{q}_{2,1}(\xi)\|_{L^2(\Delta)} \lesssim \delta, \quad \left\|\frac{1}{\sqrt{1+|\xi|}}\hat{q}_{2,1}(\xi)\right\|_{L^2(\mathbb{R})} \lesssim \delta\sqrt{|\ln\delta|},$$
$$\|\hat{q}_{2,2}\|_{L^2(\Delta)} \lesssim \|\phi_{00}\|_{C^3(\mathbb{R})}\cdot\delta^3, \quad \left\|\frac{1}{\sqrt{1+|\xi|}}\hat{q}_{2,2}\right\|_{L^2(\mathbb{R})} \lesssim \|\phi_{00}\|_{C^3(\mathbb{R})}\cdot\delta^3\sqrt{|\ln\delta|}.$$

Correspondingly, we express $\hat{Q}(\xi)$ as

$$\hat{Q}(\xi) = \delta^{3+\alpha}\cdot\phi_{00}''(0)\cdot(ED_0^{-1}q_{1,0})^\wedge(\xi) + \phi_{00}(0)\cdot\left[(ED_0^{-1}q_{2,0})^\wedge(\xi) + \hat{q}_{2,1}(\xi)\right] + (ED_0^{-1}q_h + q_{2,2})^\wedge(\xi).$$

Lemma 4.7. *The following estimate holds for $\hat{Q}$:*

$$\left\|\hat{Q}\right\|_{L^2(\Delta)} \lesssim (|\phi_{00}(0)| + |\phi_{00}''(0)|)\cdot\left(\delta^{3+\alpha/2}+\delta\right) + \left(\delta^{4+\alpha/2}+\delta^3\right)\cdot\|\phi_{00}\|_{C^3(\mathbb{R})},$$
$$\left\|\frac{\hat{Q}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R})} \lesssim (|\phi_{00}(0)| + |\phi_{00}''(0)|)\cdot\left(\delta^{3+\alpha/2}+\delta\sqrt{|\ln\delta|}\right) + \left(\delta^{4+\frac{\alpha}{2}}+\delta^3\sqrt{|\ln\delta|}\right)\cdot\|\phi_{00}\|_{C^3(\mathbb{R})}.$$

In light of the formulas (4.10) and (4.28), the following equation holds for $\xi\in\Delta$:

$$\widehat{\phi_{00}}(\xi)\left(\frac{\rho_0\varepsilon_m}{\rho_m}+1\right) + \hat{Q}(\xi) = (ED_0^{-1}f)^\wedge(\xi).$$

Hence we can express the Fourier transform of $\tilde{\phi}_0$ as

$$\widehat{\phi_{00}}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m+\rho_m(\xi)}\cdot\left[(ED_0^{-1}f)^\wedge(\xi) - \hat{Q}(\xi)\right], \quad \xi\in\Delta. \tag{4.29}$$

On other hand, note that

$$\phi_{00}(0) = \int_\Delta \widehat{\phi_{00}}(\xi)d\xi, \quad \phi_{00}''(0) = -\int_\Delta \xi^2\widehat{\phi_{00}}(\xi)d\xi.$$

Substituting (4.29) into the above two formulas yields a linear system for $\phi_{00}(0)$ and $\phi_{00}''(0)$:

$$\Pi\begin{bmatrix}\phi_{00}(0)\\ \phi_{00}''(0)\end{bmatrix} = b, \tag{4.30}$$

where

$$\Pi = \left[\begin{array}{cc} 1 + A_1(ED_0^{-1}q_{2,0}) + A_1(q_{2,1}) & \delta^{3+\alpha} A_1(ED_0^{-1}q_{1,0}) \\ -A_2(ED_0^{-1}q_{2,0}) - A_2(q_{2,1}) & 1 - \delta^{3+\alpha} A_2(ED_0^{-1}q_{1,0}) \end{array} \right],$$

$$b = \left[\begin{array}{c} A_1(ED_0^{-1}f) - A_1(ED_0^{-1}q_h) - A_1(q_{2,2}) \\ -A_2(ED_0^{-1}f) + A_2(ED_0^{-1}q_h) + A_2(q_{2,2})) \end{array} \right].$$

In addition, the functionals A_1 and A_2 are defined as

$$A_1(\varphi) := \int_\Delta \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \hat{\varphi}(\xi)\, d\xi,$$
$$A_2(\varphi) := \int_\Delta \frac{\xi^2 \cdot \rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \hat{\varphi}(\xi)\, d\xi.$$

We first solve the above linear system to obtain $\phi_{00}(0)$ and $\phi_{00}''(0)$, which will lead to the estimation for $\hat{Q}(\xi)$ in (4.29). To this end, we study the entries in the matrix Π and the vector b. This is given in what follows.

LEMMA 4.8. *The following inequalities hold:*

$$\left\| \frac{\rho_m}{\rho_0\varepsilon_m + \rho_m} \right\|_{L^2(\Delta)} \lesssim \left(\frac{1}{\sqrt{|\varepsilon_m'|}} \ln|\varepsilon_m'| + \frac{1}{\sqrt{|\varepsilon_m''|}} \right), \tag{4.31}$$

$$\left\| \frac{\rho_m}{\sqrt{|\rho_0|}\,(\rho_0\varepsilon_m + \rho_m)} \right\|_{L^2(\Delta)} \lesssim |\varepsilon_m'|^{\frac{1}{4}} \left(\frac{1}{\sqrt{|\varepsilon_m'|}} + \frac{1}{\sqrt{\varepsilon_m''}} \right), \tag{4.32}$$

$$\left\| \frac{\rho_0\rho_m}{(\rho_0\varepsilon_m + \rho_m)} \right\|_{L^2(\Delta)} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}} \left(1 + \frac{1}{\sqrt{\varepsilon_m''}} \right). \tag{4.33}$$

In addition, if $\hat{\varphi} \in L^2(\Delta)$, then

$$|A_j(\varphi)| \lesssim \left(\frac{1}{\sqrt{|\varepsilon_m'|}} \ln|\varepsilon_m'| + \frac{1}{\sqrt{|\varepsilon_m''|}} \right) \|\hat{\varphi}\|_{L^2(\Delta)}, \quad j = 1, 2.$$

Proof. See Appendix B. ◻

LEMMA 4.9. *The following expansions hold for Π and b:*

$$\Pi = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2} \right) \left[\begin{array}{cc} O(\delta) & O\left(\delta^{3+\alpha/2}\right) \\ O(\delta) & O\left(\delta^{3+\alpha/2}\right) \end{array} \right],$$

$$b = \left[\begin{array}{c} A_1(ED_0^{-1}f) \\ -A_2(ED_0^{-1}f) \end{array} \right] + O\left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2} \right) \left(\delta^{4+\alpha/2} + \delta^3\sqrt{|\ln\delta|} \right) \|\widehat{\phi_{00}}\|_{L^1(\mathbb{R})}.$$

Moreover, assume that $-6 < \alpha - \beta < 0$; then $\phi_{00}(0)$ and $\phi_{00}''(0)$ admit the following estimate:

$$\phi_{00}(0) = A_1(ED_0^{-1}f) \cdot (1 + o(1)) + O\left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2} \right) \delta^3 \sqrt{|\ln\delta|} \|\widehat{\phi_{00}}\|_{L^1(\mathbb{R})},$$
$$\phi_{00}''(0) = A_2(ED_0^{-1}f) \cdot (1 + o(1)) + O\left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2} \right) \delta^3 \sqrt{|\ln\delta|} \|\widehat{\phi_{00}}\|_{L^1(\mathbb{R})}.$$

Proof. From the estimation in Theorem 4.5 and Lemma 4.8, we have

$$\begin{aligned}
|A_1(ED_0^{-1})q_h| &\leq \left\| \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \right\|_{L^2(\Delta)} \cdot \|(D_0^{-1}q_h)^\wedge(\xi)\|_{L^2(\Delta)} \\
&\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{-\alpha/2}\|q_h\|_{H^{1/2}(\Gamma_\delta^+)} \\
&\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\left(\delta^{4+\alpha} + \delta^3\sqrt{|\ln\delta|}\right)\|\phi_{00}\|_{C^3(\mathbb{R})}, \\
|A_1(q_{2,2})| &\leq \left\| \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \right\|_{L^2(\Delta)} \cdot \|\hat{q}_{2,2}\|_{L^2(\Delta)} \\
&\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^3\sqrt{|\ln\delta|}\|\phi_{00}\|_{C^3(\mathbb{R})}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|A_2(ED_0^{-1})q_h| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\left(\delta^{4+\frac{1}{2}\alpha} + \delta^3\sqrt{|\ln\delta|}\right)\|\phi_{00}\|_{C^3(\mathbb{R})}, \\
|A_1(ED_0^{-1}q_{1,0})| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{-\alpha/2}, \\
|A_2(ED_0^{-1}q_{1,0})| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{-\alpha/2}, \\
|A_1(ED_0^{-1}q_{2,0})| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{1-\alpha/2}\sqrt{|\ln\delta|}, \\
|A_2(ED_0^{-1}q_{2,0})| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{1-\alpha/2}\sqrt{|\ln\delta|}, \\
|A_1(q_{2,1})| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta, \\
|A_2(q_{2,1})| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta, \\
|A_1(q_{2,2})| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^3\sqrt{|\ln\delta|}\|\phi_{00}\|_{C^3(\mathbb{R})}, \\
|A_2(q_{2,2})| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^3\sqrt{|\ln\delta|}\|\phi_{00}\|_{C^3(\mathbb{R})}.
\end{aligned}$$

Finally, using the estimate

$$\|\phi_{00}\|_{C^3(\mathbb{R})} \lesssim \|\widehat{\phi_{00}}\|_{L^1(\mathbb{R})}$$

we obtain the desired estimate for $\phi_{00}(0)$ and $\phi_{00}''(0)$. ∎

Now, we are ready to discuss the solution of the operator equation $D\varphi = f$. We distinguish two types of source function f:

(i) $f \in H^{1/2}(\Gamma_\delta^+)$.

(ii) $f = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\chi_\Delta(\xi)}{\rho_0(\xi)}\,\hat{\psi}(\xi)\,e^{i\xi x_1}\,d\xi$, where $\hat{\psi}(\xi)$ is even and it holds that $\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\|_{L^2(\Delta)} < \infty$.

The main results are given in Theorems 4.10 and 4.12, respectively, for the above two cases. For each case, we shall establish the estimation for the energy of the solution in the frequency bands Δ and $\mathbb{R}\backslash\Delta$, respectively. Note that the surface plasmonic pole lies in Δ, hence the energy in this frequency band corresponds to excitation of surface plasmon given the source f, while the energy in the frequency band $\mathbb{R}\backslash\Delta$ corresponds to the nonplasmonic part of the solution. More precisely, the former is given in (4.34)–(4.35) and the latter is given in (4.36), case (i) ((4.37)–(4.38) and (4.39), respectively for case (ii)).

Remark 3. In the following theorems, the estimation of the excited surface plasmon wave is established in terms of the L^1-norm in the Fourier space. Such an estimation leads to the L^∞-norm of the excited surface plasmon wave, which shall be used in our subsequent studies of the interaction between surface plasmon resonance and Fabry–Perot resonance. We point out that the standard L^2-norm estimate or energy estimate for the excited surface plasmon can also be derived from our analysis. However, such a bound is significantly larger than that of the L^1-norm.

THEOREM 4.10. *Under Assumption* 1, *further assume that* $\alpha-\beta > -6$ *and* $\beta < 2$. *If* $f \in H^{1/2}(\Gamma_\delta^+)$, *the following holds for the solution of* $D\varphi = f$*:*

$$\left\| \frac{\widehat{E\varphi}}{\rho_0} \right\|_{L^1(\Delta)} \lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right) \|f\|_{H^{1/2}(\Gamma_\delta^+)}, \tag{4.34}$$

$$\left\| \widehat{E\varphi} \right\|_{L^1(\Delta)} \lesssim \left(\delta^{-\alpha/2} + \delta^{-(\alpha+\beta)/2}\right) \|f\|_{H^{1/2}(\Gamma_\delta^+)}, \tag{4.35}$$

$$\left\| \frac{\widehat{E\varphi}}{\sqrt{1+|\xi|}} \right\|_{L^2(\mathbb{R}\backslash\Delta)} \lesssim \delta^{-\alpha/2} \|f\|_{H^{1/2}(\Gamma_\delta^+)}. \tag{4.36}$$

Proof. First, using Lemma 4.8 and Proposition 4.1, we obtain

$$\begin{aligned} |A_1(ED_0^{-1}f)| &\lesssim \delta^{-\alpha/2}\left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right) \|f\|_{H^{1/2}(\Gamma_\delta^+)}, \\ |A_2(ED_0^{-1}f)| &\lesssim \delta^{-\alpha/2}\left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right) \|f\|_{H^{1/2}(\Gamma_\delta^+)}. \end{aligned}$$

On the other hand, from Lemma 4.6, we have $\|\widehat{\phi_{00}}\|_{L^1(\mathbb{R})} \lesssim \delta^{\alpha/2-\beta}\|f\|_{V_2} \lesssim \delta^{\alpha/2-\beta}\|f\|_{H^{1/2}(\Gamma_\delta^+)}$. Therefore, using Lemma 4.9, it follows that

$$\begin{aligned} |\phi_{00}(0)| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{-\alpha/2}\left(1 + \delta^{3+\alpha-\beta}\sqrt{|\ln\delta|}\right)\|f\|_{H^{1/2}(\Gamma_\delta^+)} \\ &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{-\alpha/2}\|f\|_{H^{1/2}(\Gamma_\delta^+)}, \\ |\phi_{00}''(0)| &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{-\alpha/2}\|f\|_{H^{1/2}(\Gamma_\delta^+)}. \end{aligned}$$

By Lemma 4.7, we have

$$\begin{aligned} \left\|\hat{Q}\right\|_{L^2(\Delta)} \lesssim{}& \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{-\alpha/2}\left(\delta^{3+\alpha/2} + \delta\right)\|f\|_{H^{1/2}(\Gamma_\delta^+)} \\ &+ \left(\delta^{4+\frac{\alpha}{2}} + \delta^3\right)\left\|\widehat{\phi_{00}}\right\|_{L^1(\mathbb{R})}. \end{aligned}$$

From the formula (4.29), the Cauchy–Schwarz inequality leads to an updated estimate for ϕ_{00}:

$$\begin{aligned}\left\|\widehat{\phi_{00}}\right\|_{L^1(\mathbb{R})} &\lesssim \left\|\frac{\rho_m}{\rho_0\varepsilon_m+\rho_m}\right\|_{L^2(\Delta)} \cdot \left\|\widehat{ED_0^{-1}}f+\hat{Q}\right\|_{L^2(\Delta)} \\ &\le \left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\cdot\left(\left\|\widehat{ED_0^{-1}}f\right\|_{L^2(\Delta)}+\left\|\hat{Q}\right\|_{L^2(\Delta)}\right) \\ &\lesssim \left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\left[\delta^{-\alpha/2}+\left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\delta^{-\alpha/2}\right. \\ &\quad \left.\left(\delta^{3+\alpha/2}+\delta\right)\right]\|f\|_{H^{1/2}(\Gamma_\delta^+)} \\ &\quad + \left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\left(\delta^{4+\alpha/2}+\delta^3\right)\left\|\widehat{\phi_{00}}\right\|_{L^1(\mathbb{R})}.\end{aligned}$$

Since $\alpha-\beta>-6$, the $\widehat{\phi_{00}}$ term on the right-hand side can be absorbed by the left-hand side; with the additional condition $\beta<2$, one can further derive that

$$\left\|\widehat{\phi_{00}}\right\|_{L^1(\mathbb{R})} \lesssim \delta^{-\alpha/2}\left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\|f\|_{H^{1/2}(\Gamma_\delta^+)}.$$

This also implies the improved estimates for ϕ_{00} and Q:

$$\|\phi_{00}\|_{C^3(\mathbb{R})} \lesssim \delta^{-\alpha/2}\left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\|f\|_{H^{1/2}(\Gamma_\delta^+)}$$

and

$$\begin{aligned}&\left\|\frac{\hat{Q}(\xi)}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R})} \\ &\quad\lesssim \delta^{-\alpha/2}\left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\left(\delta^{3+\alpha/2}+\delta^{1-\alpha/2}\sqrt{|\ln\delta|}\right)\|f\|_{H^{1/2}(\Gamma_\delta^+)}.\end{aligned}$$

Now, it follows (4.10) from that

$$\widehat{E\varphi}(\xi)=\frac{\rho_0(\xi)\varepsilon_m}{\rho_m(\xi)}\cdot\widehat{\phi_{00}}(\xi) \quad \text{for } \xi\in\Delta.$$

Using the estimate for $\widehat{\phi_{00}}(\xi)$ above, we obtain

$$\left\|\frac{\widehat{E\varphi}}{\rho_0}\right\|_{L^1(\Delta)} \lesssim \sqrt{|\varepsilon_m'|}\left\|\widehat{\phi_{00}}\right\|_{L^1(\mathbb{R})} \lesssim \left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\|f\|_{H^{1/2}(\Gamma_\delta^+)}.$$

On the other hand, in view of (4.29) and Lemma 4.8, we also have

$$\begin{aligned}\left\|\rho_0\widehat{\phi_{00}}\right\|_{L^1(\mathbb{R})} &\lesssim \left\|\frac{\rho_0\rho_m}{\rho_0\varepsilon_m+\rho_m}\right\|_{L^2(\Delta)}\cdot\left\|\widehat{ED_0^{-1}}f+\hat{Q}\right\|_{L^2(\Delta)} \\ &\le \left(\delta^{-\alpha/2}+\delta^{-(\alpha+\beta)/2}\right)\cdot\left(\left\|\widehat{ED_0^{-1}}f\right\|_{L^2(\Delta)}+\left\|\hat{Q}\right\|_{L^2(\Delta)}\right) \\ &\lesssim \delta^{-\alpha/2}\left(\delta^{-\alpha/2}+\delta^{-(\alpha+\beta)/2}\right)\|f\|_{H^{1/2}(\Gamma_\delta^+)}.\end{aligned}$$

Therefore,

$$\left\|\widehat{E\varphi}\right\|_{L^1(\Delta)} \lesssim \sqrt{|\varepsilon_m'|}\left\|\rho_0(\xi)\widehat{\phi_{00}}(\xi)\right\|_{L^1(\mathbb{R})} \lesssim \left(\delta^{-\alpha/2}+\delta^{-(\alpha+\beta)/2}\right)\|f\|_{H^{1/2}(\Gamma_\delta^+)}.$$

Finally, note that the support of $\widehat{\phi_{00}}(\xi)$ belongs to Δ, and the formula (4.28) leads to

$$\widehat{E\varphi}(\xi) = (ED_0^{-1}f)^\wedge(\xi) - \hat{Q}(\xi) \quad \text{for } \xi \notin \Delta.$$

We obtain

$$\begin{aligned}\left\| \frac{\widehat{E\varphi}}{\sqrt{1+|\xi|}} \right\|_{L^2(\mathbb{R}\setminus\Delta)} &\lesssim ||D_0^{-1}||\, \|f\|_{H^{1/2}(\Gamma_\delta^+)} + \left\| \frac{1}{\sqrt{1+|\xi|}} \hat{Q}(\xi) \right\|_{L^2(\mathbb{R})} \\ &\lesssim \delta^{-\alpha/2} \|f\|_{H^{1/2}(\Gamma_\delta^+)}.\end{aligned}$$

∎

For a source function f that takes the form in (ii), the solution of $D_0\phi = f$ can be expressed in the following lemma.

LEMMA 4.11. *Let $f \in V_2$ be in the form of* (ii)*; then $D_0^{-1}f$ has the expansion*

$$\widehat{ED_0^{-1}}f(\xi) = \frac{\rho_m(\xi)}{\varepsilon_m} \cdot \frac{\hat{\psi(\xi)}}{\rho_0} \cdot \chi_\Delta(\xi) + \hat{R}(\xi),$$

where the low-order term R satisfies the estimate

$$\left\| \frac{\hat{R}(\xi)}{\sqrt{1+|\xi|}} \right\|_{L^2(\mathbb{R})} \lesssim \left(\delta^3 + \delta^{1-\alpha/2}\sqrt{|\ln\delta|} \right) \cdot \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.$$

Moreover,

$$\left\| \widehat{ED_0^{-1}}f \cdot \sqrt{|\rho_0|} \right\|_{L^2(\Delta)} \lesssim \delta^{-\alpha/2} \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)},$$

$$\left\| \frac{\widehat{ED_0^{-1}}f}{\sqrt{1+|\xi|}} \right\|_{L^2(\mathbb{R}\setminus\Delta)} \lesssim \left(\delta^3 + \delta^{1-\alpha/2}\sqrt{|\ln\delta|} \right) \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)},$$

$$|A_j(ED_0^{-1}f)| \lesssim \left(\delta^{-\alpha/2} + \delta^{-\beta/2} \right) \cdot \delta^{-\alpha/4} \cdot \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}, \quad j = 1, 2.$$

Proof. From a parallel argument as in Theorem 4.5, we obtain

$$D_0^{-1}f = \phi_{00,\psi} \cdot \chi_{(\delta,\infty)} + \phi_{1,f}.$$

Using the Fourier transform of $\phi_{00,\psi}$, it follows that

$$\begin{aligned}\widehat{ED_0^{-1}}f(\xi) &= \widehat{\phi_{00,\psi}} + (\phi_{00,\psi}\chi_{(-\delta,\delta)})^\wedge(\xi) + \widehat{E\phi_{1,f}} \\ &= \frac{\rho_m(\xi)}{\varepsilon_m} \cdot \frac{\hat{\psi}}{\rho_0} \cdot \chi_\Delta(\xi) + (\phi_{00,\psi}(0) + O(\delta^2)) \cdot \hat{\chi}_{(-\delta,\delta)}(\xi) + \widehat{E\phi_{1,f}}(\xi) \\ &=: \frac{\rho_m(\xi)}{\varepsilon_m} \cdot \frac{\hat{\psi}}{\rho_0} \cdot \chi_\Delta(\xi) + \hat{R}(\xi).\end{aligned}$$

By Lemma 4.3 and Theorem 4.5, we have

$$\left\| \frac{\hat{R}(\xi)}{\sqrt{1+|\xi|}} \right\|_{L^2(\mathbb{R})} \lesssim \left(\delta^3 + \delta^{1-\alpha/2}\sqrt{|\ln\delta|} \right) \cdot \left\| \frac{\hat{\psi}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)}.$$

Therefore,

$$\left\|\widehat{ED_0^{-1}f}\cdot\sqrt{|\rho_0|}\right\|_{L^2(\Delta)}\lesssim\delta^{-\alpha/2}\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)},$$

$$\left\|\frac{\widehat{ED_0^{-1}f}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\setminus\Delta)}\lesssim\left(\delta^3+\delta^{1-\alpha/2}\sqrt{|\ln\delta|}\right)\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}.$$

Hence the estimate for $ED_0^{-1}f$ holds. Finally, using the estimate (4.32) in Lemma 4.8, and applying the Cauchy–Schwarz inequality, we arrive at

$$|A_1(ED_0^{-1}f)|\lesssim\left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\cdot\delta^{-\alpha/4}\cdot\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}. \qquad \square$$

THEOREM 4.12. *Under Assumption* 1, *further assume that* $\alpha-\beta>-6$ *and* $\beta<2$. *If* f *is given as in* (ii), *the following holds for the solution of* $D\varphi=f$*:*

$$\widehat{E\varphi}(\xi)\chi_\Delta(\xi)=\frac{\rho_m(\xi)\hat{\psi}}{\rho_0(\xi)\varepsilon_m+\rho_m(\xi)}+\frac{\rho_0(\xi)\varepsilon_m}{\rho_0(\xi)\varepsilon_m+\rho_m(\xi)}\cdot\left[\hat{R}(\xi)-\hat{Q}(\xi)\right],$$

where the low-order terms R *and* Q *have the estimate*

$$\|\hat{R}(\xi)\|_{L^2(\Delta)}+\|\hat{Q}(\xi)\|_{L^2(\Delta)}\lesssim\left(\delta^{3+\frac{1}{4}\alpha}+\delta^{1-3\alpha/4}\sqrt{|\ln\delta|}\right)\cdot\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}.$$

Moreover,

$$\left\|\frac{\widehat{E\varphi}}{\rho_0}\right\|_{L^1(\Delta)}\lesssim\left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\delta^{\alpha/4}\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}, \tag{4.37}$$

$$\left\|\widehat{E\varphi}(\xi)\right\|_{L^1(\Delta)}\lesssim\left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\delta^{-\alpha/4}\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}, \tag{4.38}$$

$$\begin{aligned}&\left\|\frac{\widehat{E\varphi}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\setminus\Delta)}\\&\lesssim\left(\delta^3+\delta^{3+\frac{\alpha}{4}-\frac{\beta}{2}}+\delta^{1-\frac{\alpha}{4}-\frac{\beta}{2}}\sqrt{|\ln\delta|}+\delta^{1-\alpha/2}\sqrt{|\ln\delta|}\right)\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}.\end{aligned} \tag{4.39}$$

Proof. The proof is similar to that of Theorem 4.10. First, by Lemmas 4.9 and 4.11, we can show that

$$|\phi_{00}(0)|\lesssim\left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\cdot\delta^{-\alpha/4}\cdot\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)},\quad|\phi_{00}''(0)|,$$

$$\lesssim\left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\cdot\delta^{-\alpha/4}\cdot\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}.$$

On the other hand, recall from (4.29) that

$$\widehat{\phi_{00}}(\xi)=\frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m+\rho_m(\xi)}\cdot\left[(ED_0^{-1}f)^\wedge(\xi)-Q(\xi)\right]\quad\text{for }\xi\in\Delta. \tag{4.40}$$

Applying Lemmas 4.7 and 4.11 yields

$$\begin{aligned}\left\|\widehat{\phi_{00}}\right\|_{L^1(\mathbb{R})} &\lesssim \left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\delta^{\alpha/4}\cdot\left\|\widehat{ED_0^{-1}f}\cdot\sqrt{|\rho_0|}\right\|_{L^2(\Delta)} \\ &\quad + \left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\left\|\hat{Q}\right\|_{L^2(\Delta)} \\ &\lesssim \left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\delta^{-\alpha/4}\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)} \\ &\quad + \left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\delta^{-\alpha/4}\left(\delta^{3+\alpha/2}+\delta\right)\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)} \\ &\quad + \left(\delta^{-\alpha/2}|\ln\delta|+\delta^{-\beta/2}\right)\left(\delta^{4+\alpha/2}+\delta^3\right)\left\|\widehat{\phi_{00}}(\xi)\right\|_{L^1(\mathbb{R})}.\end{aligned}$$

Under the assumption $\alpha-\beta>-6$, the $\widehat{\phi_{00}}(\xi)$ term on the right-hand side above can be absorbed by the left-hand side; with the help of the additional assumption $\beta<2$, one can derive the estimate

$$\left\|\widehat{\phi_{00}}\right\|_{L^1(\mathbb{R})} \lesssim \left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\cdot\delta^{-\alpha/4}\cdot\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)},$$

which further implies the improved estimate for Q (using Lemmas 4.7):

$$\left\|\hat{Q}\right\|_{L^2(\Delta)} \lesssim \left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\cdot\left(\delta^{3+\alpha/4}+\delta^{1-\alpha/4}\right)\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}, \tag{4.41}$$

$$\begin{aligned}&\left\|\frac{\hat{Q}(\xi)}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R})} \\ &\quad\lesssim \left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\cdot\left(\delta^{3+\alpha/4}+\delta^{1-\alpha/4}\sqrt{|\ln\delta|}\right)\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}.\end{aligned} \tag{4.42}$$

Next, recall that

$$\widehat{E\varphi}(\xi)\chi_\Delta = \frac{\rho_0(\xi)\varepsilon_m}{\rho_m(\xi)}\cdot\widehat{\phi_{00}}(\xi).$$

By (4.40) and Lemma 4.11, we have for $\xi\in\Delta$,

$$\begin{aligned}\widehat{E\varphi}(\xi) &= \frac{\rho_0(\xi)\varepsilon_m}{\rho_0(\xi)\varepsilon_m+\rho_m(\xi)}\cdot\left[(ED_0^{-1}f)^\wedge(\xi)-\hat{Q}(\xi)\right] \\ &= \frac{\rho_m(\xi)\hat{\psi}}{\rho_0(\xi)\varepsilon_m+\rho_m(\xi)}+\frac{\rho_0(\xi)\varepsilon_m}{\rho_0(\xi)\varepsilon_m+\rho_m(\xi)}\cdot\left[\hat{R}(\xi)-\hat{Q}(\xi)\right]\end{aligned}$$

with R, Q satisfying the desired estimate. This also implies that

$$\left\|\frac{\widehat{E\varphi}}{\rho_0}\right\|_{L^1(\Delta)} \lesssim \sqrt{|\varepsilon_m|}\|\hat{\phi}_{00}\|_{L^1(\Delta)} \lesssim \left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\cdot\delta^{\alpha/4}\cdot\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}$$

and

$$\left\|\widehat{E\varphi}\right\|_{L^1(\Delta)} \lesssim \left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\cdot\delta^{-\alpha/4}\cdot\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)},$$

Finally, from (4.28) we see that

$$(1-\chi_\Delta)\widehat{E\varphi}(\xi) = (ED_0^{-1}f)^\wedge(\xi) - \hat{Q}(\xi).$$

Applying Lemma 4.11 and the estimation (4.42), we obtain

$$\begin{aligned}\left\|\frac{\widehat{E\varphi}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\backslash\Delta)} &\lesssim \left(\delta^3+\delta^{1-\alpha/2}\sqrt{|\ln\delta|}\right)\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}\\ &\quad+\left(\delta^{-\alpha/2}+\delta^{-\beta/2}\right)\left(\delta^{3+\alpha/4}+\delta^{1-\alpha/4}\sqrt{|\ln\delta|}\right)\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}\\ &\lesssim \Big(\delta^3+\delta^{3+\alpha/4-\beta/2}\\ &\qquad+\delta^{1-\alpha/4-\beta/2}\sqrt{|\ln\delta|}+\delta^{1-\alpha/2}\sqrt{|\ln\delta|}\Big)\left\|\frac{\hat{\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}. \qquad \square\end{aligned}$$

Remark 4. From Theorems 4.10 and 4.12, we see that the amplitude of the excited surface plasmon in the frequency band Δ depends mainly on α and β, or the real part ε_m' and imaginary part ε_m'' of the metal relative permittivity. To have strong surface plasmon excitation, one needs to have small ε_m'' and $|\varepsilon_m'|$.

Remark 5. The approach in this paper relies on the assumption that $|\varepsilon_m'| \gg 1$, for which the skin depth of the metal is much smaller than the wavelength. It does not apply to the case when $|\varepsilon_m'| = O(1)$. We expect some other interesting phenomenon to occur in such scenario. We leave it as an open problem for future investigation.

4.4. Solution of the operator equation (4.1). Now we apply the mathematical framework developed in the previous section to study the solution of the operator equation (4.1). From the discussions in section 4.1, the solution of the operator equation (4.1) can be decomposed as $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 = \varphi_{00}\cdot\chi_{(\delta,\infty)}$ and φ_{00} is given by (4.2). In addition, φ_1 satisfies the operator equation

$$D\varphi_1 = p, \quad \text{where} \quad p := \tilde{D}\varphi_{00} - D\varphi_0. \tag{4.43}$$

LEMMA 4.13. *Let p be defined in* (4.43)*; then $p = p_1 + p_2 + p_3$, and the following asymptotic expansions hold for $x_1 > \delta$:*

$$\begin{aligned}p_1(x_1) &= \varphi_{00}''(0)\cdot p_{1,0}(x_1)\cdot\varepsilon_m\delta^3 + O(\delta^{4+\alpha/2}) \quad \text{in } H^{1/2}(\Gamma_\delta^+),\\ p_2(x_1) &= \varphi_{00}(0)\cdot p_{2,0}(x_1) + O\left(\delta^{3-\alpha/2}\sqrt{|\ln\delta|}\right) \quad \text{in } H^{1/2}(\Gamma_\delta^+),\\ p_3(x_1) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\chi_\Delta(\xi)}{\rho_0(\xi)}\left(\varphi_{00}\cdot\chi_{(-\delta,\delta)}\right)^\wedge(\xi)\, e^{i\xi x_1}\, d\xi,\end{aligned}$$

where

$$\|p_{1,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim 1 \quad \text{and} \quad \|p_{2,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta\sqrt{|\ln\delta|}.$$

Moreover,

$$\|p_1\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta^{3+\alpha/2}, \quad \|p_2\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta^{1-\alpha/2}\sqrt{|\ln\delta|}.$$

Proof. We first note from the explicit expression of φ_{00} that

$$\|\varphi_{00}\|_{C^3(\mathbb{R})} \lesssim \frac{1}{\sqrt{|\varepsilon'_m|}} = \delta^{-\alpha/2}.$$

Recall that $D = K_1^e + K_2 = K_1^e + K_{2,0} + K_{2,1}$, and thus

$$p = \tilde{D}\varphi_{00} - D\varphi_0 = \left[\tilde{D}\varphi_{00} - (K_1^e + K_{2,0})\varphi_0\right] - K_{2,1}\varphi_0.$$

More explicitly,

$$K_{2,1}\varphi_0 = \int_\delta^\infty \left(G_{2,+}(x_1,0;y_1,0) - G_2^{(0)}(x_1,0;y_1,0) - G_2^{(0)}(-x_1,0;y_1,0)\right)\varphi_0(y_1)\,dy_1,$$

$$\tilde{D}\varphi_{00} - (K_1^e + K_{2,0})\varphi_0 = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\varepsilon_m}{\rho_m(\xi)} + \frac{1}{\rho_0(\xi)}\right)\cdot(\varphi_{00}\cdot\chi_{(-\delta,\delta)})^\wedge(\xi)\,e^{i\xi x_1}\,d\xi.$$

By a parallel calculation in Lemma 4.4, we can decompose p as $p = p_1 + p_2 + p_3$, where

$$p_1(x_1) = -2\int_0^\infty g(x_1 + y_1 - \delta)\,(\varphi_{00}(y_1+\delta) - \varphi_{00}(y_1-\delta))\,dy_1, \quad x_1 > \delta,$$

$$p_2(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\,(\varphi_{00}\cdot\chi_{(-\delta,\delta)})^\wedge(\xi)\,e^{i\xi x_1}\,d\xi, \quad x_1 > \delta,$$

$$p_3(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\chi_\Delta(\xi)}{\rho_0(\xi)}\,(\varphi_{00}\cdot\chi_{(-\delta,\delta)})^\wedge(\xi)\,e^{i\xi x_1}\,d\xi \quad x_1 > \delta.$$

The same argument as in Lemma 4.4 leads to the assertion. ∎

THEOREM 4.14. *Under Assumption* 1, *further assume that* $\alpha - \beta > -6$ *and* $\beta < 2$. *Let* φ_{00} *be defined in* (4.2). *Then the solution of* (4.1) *admits the decomposition* $\varphi = \varphi_0 + \varphi_1$, *where*

$$\varphi_0 = \varphi_{00}\cdot\chi_{(\delta,\infty)} \quad \textit{and} \quad \varphi_1 = D^{-1}p_1 + D^{-1}p_2 + D^{-1}p_3.$$

In addition,

$$\left\|\frac{\widehat{E\varphi_1}}{\rho_0}\right\|_{L^1(\Delta)} \lesssim \delta^{3+\frac{\alpha-\beta}{2}} + \delta^3|\ln\delta| + \delta^{1-\frac{3}{4}\alpha} + \delta^{1-\frac{\alpha}{4}-\frac{\beta}{2}} \tag{4.44}$$

and

$$\left\|\frac{\widehat{E\varphi_1}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\setminus\Delta)} \lesssim \delta^3 + \delta^{4-\frac{\alpha}{4}-\frac{\beta}{2}} + \delta^{1-\alpha}\sqrt{|\ln\delta|} + \delta^{2-\frac{3\alpha}{4}-\frac{\beta}{2}}\sqrt{|\ln\delta|}. \tag{4.45}$$

Proof. Based on the decomposition of the source function p in Lemma 4.13, we write the solution of (4.43) as $\varphi_1 = \varphi_1^{(1)} + \varphi_1^{(2)}$, where

$$D\varphi_1^{(1)} = p_1 + p_2 \quad \text{and} \quad D\varphi_1^{(2)} = p_3.$$

We apply Theorem 4.10 and Lemma 4.13 for the equation $D\varphi_1^{(1)} = p_1 + p_2$ to obtain the following estimates:

$$\left\| \frac{\widehat{E\varphi_1^{(1)}}}{\rho_0} \right\|_{L^1(\Delta)} \lesssim \left(\delta^{-\alpha/2} |\ln \delta| + \delta^{-\beta/2}\right) \left(\delta^{3+\alpha/2} + \delta^{1-\alpha/2}\sqrt{|\ln \delta|}\right),$$

$$\left\| \frac{\widehat{E\varphi_1^{(1)}}}{\sqrt{1+|\xi|}} \right\|_{L^2(\mathbb{R}\backslash\Delta)} \lesssim \delta^{-\alpha/2} \left(\delta^{3+\alpha/2} + \delta^{1-\alpha/2}\sqrt{|\ln \delta|}\right) = \delta^3 + \delta^{1-\alpha}\sqrt{|\ln \delta|}.$$

On the other hand, applying Theorem 4.12 to the equation $D\varphi_1^{(2)} = p_3$ together with the estimate

$$\left\| \frac{\varphi_{00} \cdot \widehat{\chi_{(-\delta,\delta)})}}{\sqrt{|\rho_0|}} \right\|_{L^2(\Delta)} \lesssim \delta^{1-\alpha/2}$$

leads to

$$\left\| \frac{\widehat{E\varphi_1^{(2)}}}{\rho_0(\xi)} \right\|_{L^1(\Delta)} \lesssim \left(\delta^{-\alpha/2} + \delta^{-\beta/2}\right) \delta^{\alpha/4} \cdot \delta^{1-\alpha/2} = \delta^{1-\frac{3}{4}\alpha} + \delta^{1-\frac{\alpha}{4}-\frac{\beta}{2}},$$

$$\begin{aligned} \left\| \frac{\widehat{E\varphi_1^{(2)}}}{\sqrt{1+|\xi|}} \right\|_{L^2(\mathbb{R}\backslash\Delta)} &\lesssim \left(\delta^{3+\frac{\alpha}{2}} + \delta^{3+\frac{\alpha}{4}-\frac{\beta}{2}} + \delta^{1-\frac{\alpha}{4}-\frac{\beta}{2}}\sqrt{|\ln \delta|} + \delta^{1-\frac{\alpha}{2}}\sqrt{|\ln \delta|}\right) \delta^{1-\alpha/2} \\ &= \delta^4 + \delta^{4-\frac{\alpha}{4}-\frac{\beta}{2}} + \delta^{2-\frac{3\alpha}{4}-\frac{\beta}{2}}\sqrt{|\ln \delta|} + \delta^{2-\alpha}\sqrt{|\ln \delta|}. \end{aligned}$$

The proof is complete by combining the above estimates. ∎

The formulas (4.44) and (4.45) in the above theorem refer to the energy estimate for the solution of the operator equation (4.1) in the frequency band Δ and $\mathbb{R}\backslash\Delta$, respectively. The former corresponds to the energy of the excited surface plasmon wave, and the latter corresponds to the energy of the nonplasmoinc wave. It is observed that the surface plasmon resonance is not strong unless α is small and β is positive, which corresponds to small negative ε_m' and extremely small ε_m''.

Finally, as a consequence of the above result, we obtain the following decomposition for the wave field of the scattering problem (1.1).

Corollary 4.15. *Under Assumption* 1, *further assume that* $\alpha - \beta > -6$ *and* $\beta < 2$. *Then the solution* u *to the scattering problem* (1.1) *admits the decomposition* $u = u_0 + u_1 + u_2$ *for the even incident wave, where* u_0 *is the wave field in the absence of nano-gap,* u_1 *and* u_2 *correspond the plasmonic and the nonplasmonic part of the wave field, with the Fourier component localized in the frequency band* Δ *and* $\mathbb{R}\backslash\Delta$, *respectively. Moreover, there holds*

$$\begin{aligned} \|u_1(\cdot, 0+)\|_{L^\infty(\Gamma)} &\lesssim \delta^{3+\frac{\alpha-\beta}{2}} + \delta^3|\ln \delta| + \delta^{1-\frac{3}{4}\alpha} + \delta^{1-\frac{\alpha}{4}-\frac{\beta}{2}}, \\ \|u_2(\cdot, 0+)\|_{H^{1/2}(\Gamma)} &\lesssim \delta^3 + \delta^{4-\frac{\alpha}{4}-\frac{\beta}{2}} + \delta^{1-\alpha}\sqrt{|\ln \delta|} + \delta^{2-\frac{3\alpha}{4}-\frac{\beta}{2}}\sqrt{|\ln \delta|}. \end{aligned}$$

5. Solution of the integral equation for the odd case. Let us define $D := K_1^o + K_2$ and write the integral equation for the odd case as

$$D\varphi = -2u_o^i. \tag{5.1}$$

We would like to apply an analogous perturbation argument as in section 4 to obtain the solution φ. To this end, let $\tilde{D} : H^{-1/2}(\mathbb{R}^+) \to H^{1/2}(\mathbb{R}^+)$ be given by

$$\tilde{D}\varphi := -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\rho_0(\xi)} + \frac{\varepsilon_m}{\rho_m(\xi)}\right)\widehat{O\varphi}(\xi)e^{i\xi x_1}\,d\xi, \quad x_1 > 0.$$

It can be calculated that the solution of $\tilde{D}\tilde{\varphi}_0 = -2u_o^i$ takes the form of

$$\varphi_{00} = R\cdot\sin(k_1x_1)\cdot\chi_{(0,\infty)}, \tag{5.2}$$

where the coefficient $R = -\frac{2\rho_0(k_1)\rho_m(k_1)}{\rho_m(k_1)+\varepsilon_m\rho_0(k_1)}$. Now if one decomposes the solution of (5.1) as $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 = \varphi_{00}\cdot\chi_{(\delta,\infty)}$, then it is clear that φ_1 satisfies the equation

$$D\varphi_1 = \tilde{D}\varphi_{00} - D\varphi_0. \tag{5.3}$$

In order to distinguish the frequency component near and away from the surface plasmon frequency when solving the operator equation (5.3), we introduce the operator $D_0 : \tilde{H}^{-1/2}(\Gamma_\delta^+) \to H^{1/2}(\Gamma_\delta^+)$ that excludes the surface plasmonic resonances by letting

$$D_0\varphi := (K_{1,0}^o + K_2)\varphi = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\widehat{O\varphi}(\xi) + \frac{\varepsilon_m}{\rho_m(\xi)}e^{-i\delta\xi}\widehat{E\varphi}(\xi)\right)e^{i\xi x_1}\,d\xi.$$

Following the argument in Proposition 4.1, it holds that D_0 is invertible. As such (5.3) can be rewritten as

$$\varphi_1 + D_0^{-1}K_{1,1}^o\varphi_1 = D_0^{-1}f, \quad \text{where } f = \tilde{D}\varphi_{00} - D\varphi_0. \tag{5.4}$$

We discuss the solution of the above operator equation in the rest of this section. The derivation shares similarities with the one for the even case, and we skip some of technical calculations for conciseness.

5.1. Solution of $\mathbf{D_0\phi = K_{1,1}^o\psi}$. For a function ψ that satisfies $\|\frac{\widehat{O\psi}}{\sqrt{|\rho_0|}}\|_{L^2(\Delta)} < \infty$, consider solving the operator equation

$$D_0\phi = K_{1,1}^o\psi. \tag{5.5}$$

We decompose the operator D_0 as $D_0 = D_{0,0} + D_{0,1}$, where

$$D_{0,0} := (K_{1,0}^o + K_{2,0}) \quad \text{and} \quad D_{0,1} := K_{2,1}. \tag{5.6}$$

Define the operator $\tilde{D}_{0,0} : H^{-1/2}(\mathbb{R}^+) \to H^{1/2}(\mathbb{R}^+)$ as

$$\tilde{D}_{0,0}\psi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)} + \frac{\varepsilon_m}{\rho_m(\xi)}\right)\widehat{O\psi}(\xi)\,e^{i\xi x_1}\,d\xi, \quad x_1 > 0. \tag{5.7}$$

Let $\phi_{00,\psi}(x_1)$ be a function over the whole real line such that its Fourier transform is given by

$$\widehat{\phi_{00,\psi}}(\xi) = \frac{\chi_\Delta(\xi)\cdot\widehat{O\psi}(\xi)}{\rho_0(\xi)\left(\frac{\varepsilon_m}{\rho_m(\xi)}+\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\right)} = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m}\cdot\widehat{O\psi}(\xi)\cdot\chi_\Delta(\xi). \tag{5.8}$$

Then $\phi_{00,\psi}$ is a smooth and odd function, and it holds that

(5.9)
$$\|\widehat{\phi_{00,\psi}}\|_{L^1(\Delta)} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}\left\|\frac{\widehat{O\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)} \quad\text{and}\quad \|\phi_{00,\psi}\|_{C^2(\mathbb{R})} \lesssim \frac{1}{\sqrt{|\varepsilon_m'|}}\left\|\frac{\widehat{O\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}.$$

Let $\phi_{00,+} = \phi_{00,\psi}|_{(0,\infty)}$; then it follows that

$$\tilde{D}_{0,0}\,\phi_{00,+} = K_{1,1}^o\psi, \tag{5.10}$$

where we have extended $K_{1,1}^o$ naturally to $\mathbb{R}^+$. If one decomposes the solution ϕ of the operator equation (5.5) as $\phi_0+\phi_1$, where $\phi_0 = \phi_{00,\psi}\cdot\chi_{(\delta,\infty)}$, then ϕ_1 satisfies the equation

$$D_0\phi_1 = q, \quad \text{where} \quad q := \tilde{D}_{0,0}\,\phi_{00,+} - D_0\phi_0. \tag{5.11}$$

Lemma 5.1. *Let q be defined in (5.11); then $q(x_1) = q_1(x_1)+q_2(x_1)$, where*

$$\begin{aligned} q_1(x_1) = &-2\int_0^\infty g(x_1+y_1-\delta)\left(\phi_{00,\psi}(y_1+\delta)-\phi_{00,\psi}(y_1-\delta)\right)dy_1 \\ &-4\int_0^\infty g(x_1+y_1)\,\phi_{00,\psi}(y_1)\,dy_1 \\ &+4\int_0^\delta g(x_1-y_1)\,\phi_{00,\psi}(y_1)\,dy_1, \quad x_1>\delta, \end{aligned} \tag{5.12}$$

$$q_2(x_1) = -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\left(\phi_{00,\psi}\cdot\chi_{(-\delta,\delta)}\right)^\wedge(\xi)\,e^{i\xi x_1}\,d\xi, \quad x_1>\delta. \tag{5.13}$$

In addition, the following asymptotic expansions hold for $x_1>\delta$:

$$q_1(x_1) = \phi_{00,\psi}'(0)\cdot q_{1,0}(x_1)\cdot\varepsilon_m\delta^2 + O(\varepsilon_m\delta^3)\cdot\|\phi_{00,\psi}\|_{C^2(\mathbb{R})}, \tag{5.14}$$

$$q_2(x_1) = \phi_{00,\psi}'(0)\cdot q_{2,0}(x_1) + O\left(\delta^3\sqrt{\ln(1/\delta)}\right)\cdot\|\phi_{00,\psi}\|_{C^2(\mathbb{R})}, \tag{5.15}$$

where

$$\|q_{1,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim 1 \quad\text{and}\quad \|q_{2,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta^2\sqrt{|\ln\delta|}.$$

Proof. Define the operator $\bar{D}_{0,0}: H^{-1/2}(\mathbb{R}^+)\to H^{1/2}(\mathbb{R}^+)$ as

(5.16)
$$\bar{D}_{0,0}\psi(x_1) = -\frac{1}{2\pi}\int_{-\infty}^\infty\left(\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\widehat{O\psi}(\xi)+\frac{\varepsilon_m}{\rho_m(\xi)}\widehat{E\psi}(\xi)\right)e^{i\xi x_1}\,d\xi, \quad x_1>0.$$

We write q as

$$q = \left(\tilde{D}_{0,0}\,\phi_{00,+} - \bar{D}_{0,0}\,\phi_{00,+}\right) + \left(\bar{D}_{0,0}\,\phi_{00,+} - D_0\phi_0\right) =: J_1+J_2.$$

For $x_1 > \delta$,

$$\begin{aligned} J_1(x_1) &= \tilde{D}_{0,0}\,\phi_{00,+} - \bar{D}_{0,0}\,\phi_{00,+} \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} \left(\widehat{O\phi_{00,+}}(\xi) - \widehat{E\phi_{00,+}}(\xi)\right) e^{i\xi x_1}\,d\xi \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon_m}{\rho_m(\xi)} (\phi_{00,\psi}\cdot\chi_{(-\infty,0)})^{\wedge}(\xi)\,e^{i\xi x_1}\,d\xi \\ &= 4\int_{-\infty}^{0} g(x_1-y_1)\,\phi_{00,\psi}(y_1)\,dy_1 \\ &= -4\int_{0}^{\infty} g(x_1+y_1)\,\phi_{00,\psi}(y_1)\,dy_1. \end{aligned} \tag{5.17}$$

Note that $D_0 = D_{0,0} + D_{0,1}$, and we have

$$J_2 = \bar{D}_{0,0}\,\phi_{00,+} - D_{0,0}\phi_0 - D_{0,1}\phi_0. \tag{5.18}$$

For $x_1 > \delta$,

$$\begin{aligned} &\bar{D}_{0,0}\,\phi_{00,+} - D_{0,0}\phi_0 \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}\left(O(\phi_{00,\psi}\chi_{(0,\delta)})\right)^{\wedge}(\xi) + \frac{\varepsilon_m}{\rho_m(\xi)}\left(E(\phi_{00,\psi}\chi_{(0,\delta)})\right)^{\wedge}(\xi)\right] e^{i\xi x_1}\,d\xi \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)} (\phi_{00,\psi}\cdot\chi_{(-\delta,\delta)})^{\wedge}(\xi)\,e^{i\xi x_1}\,d\xi \\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon_m}{\rho_m(\xi)}\left(E(\phi_{00,\psi}\chi_{(0,\delta)})\right)^{\wedge}(\xi) e^{i\xi x_1}\,d\xi. \end{aligned}$$

Further calculation yields

$$\begin{aligned} &-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon_m}{\rho_m(\xi)}\left(E(\phi_{00,\psi}\chi_{(0,\delta)})\right)^{\wedge}(\xi)\,e^{i\xi x_1}\,d\xi \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon_m}{\rho_m(\xi)}(\phi_{00,\psi}\cdot\chi_{(-\delta,\delta)})^{\wedge}(\xi)\,e^{i\xi x_1}\,d\xi \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varepsilon_m}{\rho_m(\xi)}(\phi_{00,\psi}\cdot\chi_{(-\delta,0)})^{\wedge}(\xi)\,e^{i\xi x_1}\,d\xi \\ &= 2\int_{-\delta}^{\delta} g(x_1-y_1)\,\phi_{00,\psi}(y_1)\,dy_1 - 4\int_{-\delta}^{0} g(x_1-y_1)\,\phi_{00,\psi}(y_1)\,dy_1. \end{aligned}$$

Thus

$$\begin{aligned} \bar{D}_{0,0}\,\phi_{00,+} - D_{0,0}\phi_0 &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\chi_\Delta(\xi)}{\rho_0(\xi)}(\phi_{00,\psi}\cdot\chi_{(-\delta,\delta)})^{\wedge}(\xi)\,e^{i\xi x_1}\,d\xi \\ &\quad + 2\int_{-\delta}^{\delta} g(x_1-y_1)\,\phi_{00,\psi}(y_1)\,dy_1 \\ &\quad - 4\int_{-\delta}^{0} g(x_1-y_1)\,\phi_{00,\psi}(y_1)\,dy_1. \end{aligned} \tag{5.19}$$

On the other hand, from Lemma 4.4, it is known that

$$
\begin{aligned}
D_{0,1}\phi_0 &= 2\int_0^{\infty} g(x_1+y_1-\delta)\left(\phi_{00,\psi}(y_1+\delta)-\phi_{00,\psi}(y_1-\delta)\right)dy_1 \\
&\quad + 2\int_{-\delta}^{\delta} g(x_1+y_1)\,\phi_{00,\psi}(y_1)\,dy_1 \\
&= 2\int_0^{\infty} g(x_1+y_1-\delta)\left(\phi_{00,\psi}(y_1+\delta)-\phi_{00,\psi}(y_1-\delta)\right)dy_1 \\
&\quad - 2\int_{-\delta}^{\delta} g(x_1-y_1)\,\phi_{00,\psi}(y_1)\,dy_1.
\end{aligned} \tag{5.20}
$$

Then (5.12) and (5.13) follow by combining (5.17)–(5.20). The asymptotics of q_1 and q_2 can be obtained in an analogous way as in Lemma 4.4, by noting that $\phi_{00,\psi}$ is now an odd function. ∎

THEOREM 5.2. *Let ϕ be the solution of the equation $D_0\phi = K^o_{1,1}\psi$. Let $\phi_{00,\psi}$ be defined by*

$$\hat{\phi}_{00,\psi}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m}\cdot\widehat{O\psi}(\xi)\cdot\chi_{\Delta}(\xi).$$

Then $\phi = \phi_0 + \phi_1$, where

$$
\begin{aligned}
\phi_0 &= \phi_{00,\psi}\cdot\chi_{(\delta,\infty)}, \\
\phi_1 &= \phi'_{00,\psi}(0)\cdot\left(\delta^{2+\alpha}\cdot D_0^{-1}q_{1,0} + \delta^2\sqrt{|\ln\delta|}\cdot D_0^{-1}q_{2,0}\right) + D_0^{-1}q_h \\
&=: \phi'_{00,\psi}(0)\cdot\delta^{2+\alpha}\cdot D_0^{-1}q_0 + D_0^{-1}q_h.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&\|\phi_{00,\psi}\|_{C^2(\mathbb{R})} \lesssim \delta^{-\frac{\alpha}{2}}\cdot\left\|\frac{\widehat{O\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}, \\
&\|q_{1,0}\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim 1, \quad \|q_{2,0}\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim 1, \quad \|q_0\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim 1, \\
&\|q_h\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim \delta^{3+\alpha}\|\phi_{00,\psi}\|_{C^2(\mathbb{R})} \lesssim \delta^{3+\frac{\alpha}{2}}\left\|\frac{\widehat{O\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}, \\
&\|\phi_1\|_{H^{1/2}(\Gamma^+_\delta)} \lesssim \delta^2\left\|\frac{\widehat{O\psi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)}.
\end{aligned}
$$

5.2. Solution of $\mathbf{D}\boldsymbol{\varphi} = \boldsymbol{f}$ and excitation of surface plasmon. Let us introduce the function space

$$V_1 = \left\{\varphi\in\tilde{H}^{-1/2}(\Gamma^+_\delta) : \int\frac{1}{|\rho_0(\xi)|}\,|\widehat{O\varphi}(\xi)|^2\,d\xi < \infty\right\}.$$

First, by a parallel proof as in Theorem 4.2, it can be shown that D is invertible, and the following holds for the solution of the operator equation $D\varphi = f$:

$$\left\|\frac{\widehat{O\varphi}}{\sqrt{|\rho_0|}}\right\|_{L^2(\Delta)} \lesssim \|\varphi\|_{V_1} \lesssim \frac{|\varepsilon'_m|}{\varepsilon''_m}\|f\|_{V_2} = O(\delta^{\alpha-\beta})\|f\|_{V_2}. \tag{5.21}$$

Following (5.4), the operator equation $D\varphi = f$ is recast as

$$\varphi + D_0^{-1}K^o_{1,1}\varphi = D_0^{-1}f. \tag{5.22}$$

Using Theorem 5.2, it follows that $D_0^{-1}K_{1,1}^o\varphi = \phi_{00}\chi_{(\delta,\infty)} + \phi_1$, where ϕ_{00} is defined by

$$\widehat{\phi_{00}}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m} \cdot \widehat{O\varphi}(\xi) \cdot \chi_\Delta(\xi),$$

and

$$\phi_1 = \delta^{2+\alpha} \cdot \phi_{00}'(0) \cdot D_0^{-1}q_0 + D_0^{-1}q_h.$$

Moreover, in light of the estimates (5.9) and (5.21), we have

$$\|\widehat{\phi_{00}}\|_{L^1(\Delta)} \lesssim \frac{|\varepsilon_m'|^{\frac{1}{2}}}{\varepsilon_m''}\|f\|_{V_2}, \quad \|\phi_{00}\|_{C^2(\mathbb{R})} \lesssim \frac{|\varepsilon_m'|^{\frac{1}{2}}}{\varepsilon_m''}\|f\|_{V_2}. \tag{5.23}$$

Substituting into (5.22) and extending both sides of the above equation as odd functions over the whole real line, we obtain

$$O\varphi + \phi_{00}(1-\chi_{(-\delta,\delta)}) + \delta^{2+\alpha} \cdot \phi_{00}'(0) \cdot OD_0^{-1}q_0 + OD_0^{-1}q_h = O(D_0^{-1}f).$$

This is equivalent to the following equation in the Fourier domain:

$$\widehat{O\varphi}(\xi) + \widehat{\phi_{00}}(\xi) + \hat{Q}(\xi) = (OD_0^{-1}f)^\wedge(\xi), \tag{5.24}$$

where

$$\hat{Q}(\xi) := -(\phi_{00}\chi_{(-\delta,\delta)})^\wedge(\xi) + \delta^{2+\alpha} \cdot \phi_{00}'(0) \cdot (OD_0^{-1}q_0)^\wedge(\xi) + (OD_0^{-1}q_h)^\wedge(\xi).$$

Note that the Taylor expansion gives

$$(\phi_{00}\chi_{(-\delta,\delta)})^\wedge(\xi) = \phi_{00}'(0) \cdot \hat{q}_1(\xi) + \hat{q}_2(\xi),$$

where $\hat{q}_1(\xi) = \delta\sin(\delta\xi)/\xi$, and it holds that

$$\|\hat{q}_1(\xi)\|_{L^2(\Delta)} \lesssim \delta^2, \quad \left\|\frac{1}{\sqrt{1+|\xi|}}\hat{q}_1(\xi)\right\|_{L^2(\mathbb{R})} \lesssim \delta^2\sqrt{|\ln\delta|},$$

$$\|\hat{q}_2\|_{L^2(\Delta)} \lesssim \|\phi_{00}\|_{C^2(\mathbb{R})} \cdot \delta^3, \quad \left\|\frac{1}{\sqrt{1+|\xi|}}\hat{q}_2\right\|_{L^2(\mathbb{R})} \lesssim \|\phi_{00}\|_{C^2(\mathbb{R})} \cdot \delta^3\sqrt{|\ln\delta|}.$$

Hence $\hat{Q}(\xi)$ may be expressed as

$$\hat{Q}(\xi) = \phi_{00}'(0) \cdot \left[\delta^{2+\alpha}(OD_0^{-1}q_0)^\wedge(\xi) + \hat{q}_1(\xi)\right] + (OD_0^{-1}q_h + q_2)^\wedge(\xi). \tag{5.25}$$

Using the formula (5.8), we obtain the Fourier transform of $\tilde{\phi}_0$ as follows:

$$\widehat{\phi_{00}}(\xi) = \frac{\rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \cdot \left[(OD_0^{-1}f)^\wedge(\xi) - \hat{Q}(\xi)\right], \quad \xi \in \Delta. \tag{5.26}$$

This leads to the following equation for $\phi_{00}'(0)$:

$$\Pi \cdot \phi_{00}'(0) = b,$$

where

$$\Pi = \left[1 + \delta^{2+\alpha}A(OD_0^{-1}q_0)) + A(q_1)\right],$$
$$b = A(OD_0^{-1}f) - A(OD_0^{-1}q_h) - A(q_2),$$

and the functional A is defined by

$$A(\varphi) := \int_{\Delta} \frac{i\xi \cdot \rho_m(\xi)}{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)} \hat{\varphi}(\xi)\, d\xi.$$

Now a parallel proof of Lemma 4.9 gives the expansion of $\phi_{00}'(0)$ in the following lemma.

LEMMA 5.3. *Assume that $\alpha - \beta > -4$; then Π and b admit the expansions*

$$\Pi = 1 + \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right) \cdot O\left(\delta^{2+\alpha/2}\right),$$
$$b = A(OD_0^{-1}f) + \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right) \cdot O\left(\delta^{3+\alpha/2}\right) \|\widehat{\phi_{00}}\|_{L^1(\Delta)}.$$

Moreover, the following holds for $\phi_{00}'(0)$:

$$\phi_{00}'(0) = A(OD_0^{-1}f) \cdot (1 + o(1)) + \left(1 + \delta^{-\beta/2}\right) \cdot O\left(\delta^{3+\alpha/2}\right) \|\widehat{\phi_{00}}\|_{L^1(\Delta)}.$$

Remark 6. The assumption that $\alpha - \beta > -4$ is used throughout the subsequent analysis. This assumption is stricter than the one for the even case.

With the above preparation, we are ready to present the solution of the operator equation $D\varphi = f$. Again we distinguish two types of source function f when $f \in H^{1/2}(\Gamma_\delta^+)$ and

$$f = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi_\Delta(\xi)}{\rho_0(\xi)}\, \hat{\psi}(\xi)\, e^{i\xi x_1}\, d\xi.$$

The estimates for the energy of the solution in the frequency band Δ and $\mathbb{R}\backslash\Delta$ are given in Theorems 5.4 and 5.5, respectively. This can be established by estimating $Q(\xi)$ using the formula (5.25) and Lemma 5.3, which then leads to the estimation for $\widehat{\phi_{00}}$ and the solution φ. The proof is parallel to Theorems 4.10 and 4.12 and we omit it here for conciseness.

THEOREM 5.4. *Under Assumption 1, further assume that $\alpha - \beta > -4$. If $f \in H^{1/2}(\Gamma_\delta^+)$, the following holds for the solution of $D\varphi = f$:*

$$\left\|\frac{\widehat{O\varphi}}{\rho_0}\right\|_{L^1(\Delta)} \lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right) \|f\|_{H^{1/2}(\Gamma_\delta^+)}, \quad \left\|\frac{\widehat{O\varphi}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\backslash\Delta)} \lesssim \delta^{-\alpha/2} \|f\|_{H^{1/2}(\Gamma_\delta^+)}.$$

THEOREM 5.5. *Under Assumption 1, further assume that $\alpha - \beta > -4$. If*

$$f =-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\chi_\Delta(\xi)}{\rho_0(\xi)}\, \hat{\psi}(\xi)\, e^{i\xi x_1}\, d\xi,$$

where $\hat{\psi}(\xi)$ is odd and $\|\frac{\hat{\psi}}{\sqrt{\rho_0}}\|_{L^2(\Delta)} < \infty$, then the following holds for the solution of $D\varphi = f$:

$$\left\|\frac{\widehat{O\varphi}}{\rho_0}\right\|_{L^1(\Delta)} \lesssim \left(\delta^{-\alpha/2} + \delta^{-\beta/2}\right) \delta^{\alpha/4} \left\|\frac{\hat{\psi}}{\sqrt{\rho_0}}\right\|_{L^2(\Delta)}, \quad \left\|\frac{\widehat{O\varphi}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\backslash\Delta)} \lesssim \left(\delta^2 + \delta^{2+\frac{\alpha}{4}-\frac{\beta}{2}}\right) \left\|\frac{\hat{\psi}}{\sqrt{\rho_0}}\right\|_{L^2(\Delta)}.$$

5.3. Solution of the operator equation (5.1). We decompose the solution of the operator equation (5.1) as $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 = \varphi_{00} \cdot \chi_{(\delta,\infty)}$ and φ_{00} is given by (5.2). In addition, φ_1 satisfies the operator equation

$$D\varphi_1 = p, \quad \text{where} \quad p := \tilde{D}\varphi_0 - D\varphi_0. \tag{5.27}$$

From a parallel calcuation as in Lemma 5.1, we obtain the following lemma.

LEMMA 5.6. *Let p be defined in* (5.27)*; then $p = p_1 + p_2 + p_3$, and the following asymptotic expansions hold for $x_1 > \delta$:*

$$\begin{aligned} p_1(x_1) &= \varphi_{00}'(0) \cdot p_{1,0}(x_1) \cdot \varepsilon_m \delta^2 + O(\delta^{3+\alpha/2}) \quad \text{in } H^{1/2}(\Gamma_\delta^+), \\ p_2(x_1) &= \varphi_{00}'(0) \cdot p_{2,0}(x_1) + O\left(\delta^{3-\alpha/2}\sqrt{|\ln\delta|}\right) \quad \text{in } H^{1/2}(\Gamma_\delta^+), \\ p_3(x_1) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\chi_\Delta(\xi)}{\rho_0(\xi)} \left(\varphi_{00}\cdot\chi_{(-\delta,\delta)}\right)^\wedge(\xi)\, e^{i\xi x_1}\, d\xi, \end{aligned}$$

where

$$\|p_{1,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim 1 \quad \text{and} \quad \|p_{2,0}\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta^2\sqrt{|\ln\delta|}.$$

Moreover,

$$\|p_1\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta^{2+\alpha/2} \quad \text{and} \quad \|p_2\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta^{2-\alpha/2}\sqrt{|\ln\delta|}.$$

THEOREM 5.7. *Under Assumption* 1*, further assume that $\alpha - \beta > -4$. Let φ_{00} be defined in* (5.2)*. Then the solution of* (5.1) *admits the decomposition $\varphi = \varphi_0 + \varphi_1$, where*

$$\varphi_0 = \varphi_{00}\cdot\chi_{(\delta,\infty)} \quad \text{and} \quad \varphi_1 = D^{-1}p_1 + D^{-1}p_2 + D^{-1}p_3.$$

In addition,

$$\left\|\frac{\widehat{E\varphi_1}}{\rho_0}\right\|_{L^1(\Delta)} \lesssim \delta^{2+\frac{\alpha-\beta}{2}} + \delta^{1-\frac{\alpha}{4}-\frac{\beta}{2}} \quad \text{and} \quad \left\|\frac{\widehat{E\varphi_1}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\backslash\Delta)} \lesssim \delta^2 + \delta^{3-\frac{\alpha}{4}-\frac{\beta}{2}}.$$

Proof. Based on the decomposition of the source function p in the above lemma, we decompose the solution of (5.27) as $\varphi_1 = \varphi_1^{(1)} + \varphi_1^{(2)}$, where

$$D\varphi_1^{(1)} = p_1 + p_2 \quad \text{and} \quad D\varphi_1^{(2)} = p_3.$$

Now if one applies Theorem 5.4 for the equation $D\varphi_1^{(1)} = p_1 + p_2$, it follows that from the estimate in Lemma 5.6 that

$$\begin{aligned} \left\|\frac{\widehat{O\varphi_1^{(1)}}}{\rho_0(\xi)}\right\|_{L^1(\Delta)} &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\|p_1+p_2\|_{H^{1/2}(\Gamma_\delta^+)} \\ &\lesssim \left(\delta^{-\alpha/2}|\ln\delta| + \delta^{-\beta/2}\right)\delta^{2+\alpha/2}, \\ \left\|\frac{\widehat{O\varphi_1^{(1)}}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\backslash\Delta)} &\lesssim \delta^{-\alpha/2}\,\|p_1+p_2\|_{H^{1/2}(\Gamma_\delta^+)} \lesssim \delta^2. \end{aligned}$$

On the other hand, applying Theorem 5.5 to the equation $D\varphi_1^{(2)} = p_3$, we obtain

$$\left\|\frac{\widehat{O\varphi_1^{(2)}}}{\rho_0(\xi)}\right\|_{L^1(\Delta))} \lesssim \left(\delta^{-\alpha/2} + \delta^{-\beta/2}\right)\delta^{\alpha/4}\left\|\frac{\widehat{\varphi_{00}\cdot\chi_{(-\delta,\delta)})}}{\rho_0}\right\|_{L^2(\Delta)}$$
$$\lesssim \left(\delta^{-\alpha/2} + \delta^{-\beta/2}\right)\delta^{1-\alpha/4},$$
$$\left\|\frac{\widehat{O\varphi_1^{(2)}}}{\sqrt{1+|\xi|}}\right\|_{L^2(\mathbb{R}\backslash\Delta)} \lesssim \left(\delta^2 + \delta^{2+\frac{\alpha}{4}-\frac{\beta}{2}}\right)\left\|\frac{\widehat{\varphi_{00}\cdot\chi_{(-\delta,\delta)})}}{\rho_0}\right\|_{L^2(\Delta)} \lesssim \left(\delta^2 + \delta^{2+\frac{\alpha}{4}-\frac{\beta}{2}}\right)\delta^{1-\alpha/2}.$$

The proof is complete by combining the above estimation. ◻

COROLLARY 5.8. *Under Assumption* 1, *further assume that* $\alpha - \beta > -4$. *Then the solution* u *to the scattering problem* (1.1) *admits the decomposition* $u = u_0 + u_1 + u_2$ *for the odd incident wave, where* u_0 *is the wave field in the absence of nano-gap, and* u_1 *and* u_2 *correspond the plasmonic and the nonplasmonic part of the wave field, with the Fourier component localized in the frequency band* Δ *and* $\mathbb{R}\backslash\Delta$, *respectively. In addition, there holds*

$$\|u_1(\cdot,0+)\|_{L^\infty(\Gamma)} \lesssim \delta^{2+\frac{\alpha-\beta}{2}} + \delta^{1-\frac{\alpha}{4}-\frac{\beta}{2}}, \quad \|u_2(\cdot,0+)\|_{H^{1/2}(\Gamma)} \lesssim \delta^2 + \delta^{3-\frac{\alpha}{4}-\frac{\beta}{2}}.$$

Appendix A. Proof of Lemma 3.1.

We first prove some auxiliary results.

LEMMA A.1. *For any* L^2 *even function* $f : \mathbb{R} \to \mathbb{R}$, *we have*

$$\|\langle x\rangle^{-\frac{1}{2}}\mathcal{H}(\langle y\rangle^{\frac{1}{2}} f)\|_2 \lesssim \|f\|_2,$$

where $\mathcal{H}$ *is the usual Hilbert transform on* $\mathbb{R}$ *and* $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$.

Proof. First observe that the regime $|y| \lesssim 1$ is easily handled. Therefore we may assume that f is supported in $|y| \gtrsim 1$. It then suffices for us to prove the inequality (for f even and supported in $|y| \gtrsim 1$):

$$\||x|^{-\frac{1}{2}}\mathcal{H}(|y|^{\frac{1}{2}} f)\|_2 \lesssim \|f\|_2.$$

By using parity, this is equivalent to showing

$$\left\|x^{-\frac{1}{2}} PV \int_{y\in(0,\infty)} \frac{x}{x^2-y^2} y^{\frac{1}{2}} f(y)dy\right\|_{L^2(0,\infty)} \lesssim \|f\|_{L^2(0,\infty)}.$$

Now introduce change of variable $x = e^t$, $y = e^s$, $y^{\frac{1}{2}} f(y) = \tilde{f}(s)$, $x^{\frac{1}{2}} f(x) = \tilde{f}(t)$. Note that $s, t \in \mathbb{R}$. Then we just need to show

$$\left\|PV \int K(t-s)\tilde{f}(s)ds\right\|_{L^2(\mathbb{R})} \lesssim \|\tilde{f}\|_2,$$

where the kernel K is given by

$$K(z) = \frac{1}{e^z - e^{-z}}.$$

It is easy to check that K is a standard Calderon–Zygmund operator (in particular K is an *odd* function and the Hörmander gradient condition $|K'(z)| \lesssim |z|^{-2}$ on $\mathbb{R} \setminus \{0\}$ is obviously true). The desired result then easily follows. ◻

LEMMA A.2. *Suppose* $f \in H^{-\frac{1}{2}}(\mathbb{R})$ *and* f *is supported on* $(0,\infty)$. *Then*

$$\|f\|_{H^{-\frac{1}{2}}(\mathbb{R})} \lesssim \|Ef\|_{H^{-\frac{1}{2}}(\mathbb{R})}.$$

Proof. Denote $g = Ef$; then obviously $f = g \cdot \chi_{(0,\infty)}$, i.e., f is simply the restriction of g to the half line. Denote $g = \langle \nabla \rangle^{\frac{1}{2}} h$ with $h \in L^2$; then the desired inequality is equivalent to

$$\|\langle \nabla \rangle^{-\frac{1}{2}} \chi_{(0,\infty)} \langle \nabla \rangle^{\frac{1}{2}} h\|_2 \lesssim \|h\|_2.$$

Observe that the Fourier transform of $\chi_{(0,\infty)}$ is simply the sum of a delta distribution and the Hilbert transform. The contribution of the delta part is harmless. Now denote $F = \hat{h}$. Then we only need to show for even function $F: \mathbb{R} \to \mathbb{R}$

$$\|\langle x \rangle^{-\frac{1}{2}} \mathcal{H}(\langle y \rangle^{\frac{1}{2}} F)\|_2 \lesssim \|F\|_2.$$

The result then follows from Lemma A.1. ∎

Proof of Lemma 3.1. It is straightforward to check that

$$\widehat{E\varphi}(\xi) = \widehat{\varphi}(\xi) + \widehat{\varphi}(-\xi), \quad \widehat{O\varphi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(-\xi)$$

and

$$||E\varphi||_{H^{-1/2}(\mathbb{R})} \lesssim ||\varphi||_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)}, \quad ||O\varphi||_{H^{-1/2}(\mathbb{R})} \lesssim ||\varphi||_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)}.$$

On the other hand, Lemma A.2 implies that

$$||\varphi||_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)} \lesssim ||E\varphi||_{H^{-1/2}(\mathbb{R})}.$$

A similar argument as in Lemmas A.1 and A.2 yields that

$$||\varphi||_{\tilde{H}^{-\frac{1}{2}}(\Gamma_\delta^+)} \lesssim ||O\varphi||_{H^{-1/2}(\mathbb{R})}.$$

This completes the proof of Lemma 3.1. ∎

Appendix B. Proof of Lemma 4.8.

Proof. We first prove (4.31). Let

$$r(\xi) := \frac{\rho_0(\xi)\varepsilon_m}{\rho_m(\xi)} \quad \text{and} \quad w(\xi) := \frac{\rho_0(\xi)\varepsilon_m + \rho_m(\xi)}{\rho_m(\xi)} = 1 + r(\xi).$$

Without loss of generality, we assume that $k = 1$ and write $\varepsilon_m = -a + bi$, where $a, b > 0$ and $a \gg 1, b > 0$, and $a \gg b$.

For $\xi \in (0,1)$, note that $\rho_0 = i\sqrt{1-\xi^2}$ and it can be shown that $|\mathrm{Re}\, r| \sim O(b/a)$. Thus

$$|w| = |1 + r| \geq 1 - O(b/a) > 1/2.$$

On the other hand,

$$\mathrm{Im}\, r(\xi) \geq \sqrt{1-\xi^2} \cdot \frac{a}{2\sqrt{1+a}}.$$

We obtain

(B.1) $$\int_0^1 \frac{1}{|w(\xi)|^2}\,d\xi \le \int_0^{1-1/a} \frac{1}{|\mathrm{Im}\, r(\xi)|^2} d\xi + \int_{1-1/a}^1 \frac{1}{|w(\xi)|^2} d\xi \lesssim \frac{1}{a}\ln a.$$

Next, we aim to show

(B.2) $$\int_1^2 \frac{1}{|w(\xi)|^2}\,d\xi \lesssim \frac{1}{|b|}.$$

To this end, note that

$$\frac{1}{\rho_m} = \frac{1}{\sqrt{\xi^2 + a - bi}} = \frac{1}{\sqrt{\xi^2+a}}\left(1 + \frac{b}{2(\xi^2+a)}i + \frac{3b^2}{4a^2} + O\left(\frac{b^3}{a^3}\right)\right).$$

Let $t = t(\xi) = \frac{\sqrt{\xi^2-1}}{\sqrt{\xi^2+a}}$. Then $1 \le \xi \le 2$ is equivalent to $0 \le t \le \sqrt{\frac{3}{a+4}}$. Moreover, we have

$$\mathrm{Re}\ r(\xi) = t\left(-a - \frac{b^2}{2(\xi^2+a)} + O\left(\frac{b^2}{a^2}\right)\right) = -t\left(a + O\left(\frac{b^2}{a}\right)\right),$$

$$\mathrm{Im}\ r(\xi) = t\left(b - \frac{ab}{2(\xi^2+a)} + O\left(\frac{b^3}{a^2}\right)\right).$$

Therefore, $|\mathrm{Im}\ r(\xi)| \ge \frac{2tb}{3}$. It follows that

$$|1+r(\xi)|^2 \ge |1 + \mathrm{Re}\ r(\xi)|^2 + |\mathrm{Im}\ r(\xi)|^2 \ge (1-ta)^2 + \frac{1}{4}t^2b^2.$$

On the other hand, since $t = t(\xi) = \sqrt{\frac{\xi^2-1}{\xi^2+a}}$, we have

$$\xi = \sqrt{\frac{at^2+1}{1-t^2}}.$$

A direct calculation shows that $|\xi'(t)| \lesssim at$. Therefore

$$\int_1^2 \frac{1}{|w(\xi)|^2}\,d\xi = \int_1^2 \frac{1}{(1+|r(\xi)|)^2}\,d\xi \le \int_0^{\sqrt{\frac{3}{a+4}}} \frac{at\,dt}{(1-at)^2 + \frac14 t^2b^2}.$$

We now consider three regions: $I_1 = \{0 \le t \le \frac{1}{a+\frac{b}{2}}\}$, where $|1-ta| \ge \frac{tb}{2}$; $I_2 = \{\frac{1}{a+\frac{b}{2}} \le t \le \frac{1}{a-\frac{b}{2}}\}$, where $|1-ta| \le \frac{tb}{2}$; and $I_3 = \{\frac{1}{a-\frac{b}{2}} \le t \le \sqrt{\frac{3}{a+4}}\}$, where $|1-ta| \ge \frac{tb}{2}$. We have

$$\int_{I_1} \frac{at\,dt}{(1-at)^2+\frac14 t^2b^2} \le \int_{I_1} \frac{at\,dt}{(1-at)^2} = \frac{1}{a}\int_0^{\frac{a}{a+b/2}} \frac{t\,dt}{(1-t)^2} \lesssim \frac{1}{b},$$

$$\int_{I_2} \frac{at\,dt}{(1-at)^2+\frac14 t^2b^2} \le \int_{I_2} \frac{at\,dt}{t^2b^2} = \int_{I_2} \frac{a\,dt}{tb^2} \lesssim \frac{1}{b},$$

$$\int_{I_3} \frac{at\,dt}{(1-at)^2+\frac14 t^2b^2} \le \int_{I_3} \frac{at\,dt}{(1-at)^2} = \frac{1}{a}\int_{\frac{a}{a-b/2}}^{\sqrt{\frac{3}{a+4}}} \frac{t\,dt}{(1-t)^2} \lesssim \frac{1}{b}.$$

It follows that

$$\int_0^2 \frac{1}{|w(\xi)|^2}\, d\xi = \int_0^1 \frac{1}{|w(\xi)|^2}\, d\xi + \int_1^2 \frac{1}{|w(\xi)|^2}\, d\xi \lesssim \frac{1}{a}\ln a + \frac{1}{b},$$

which proves the desired estimate (4.31).

We now prove (4.32). First,

$$\begin{aligned}\int_0^1 \frac{1}{|w(\xi)|^2\sqrt{1-\xi^2}}\, d\xi &\le \int_0^{1-1/a} \frac{1}{|\mathrm{Im}\, w(\xi)|^2\sqrt{1-\xi^2}} d\xi + \int_{1-1/a}^1 \frac{1}{|w(\xi)|^2\sqrt{1-\xi^2}} d\xi \\ &\lesssim \int_0^{1-1/a} \frac{1}{a(1-\xi^2)^{3/2}}\, d\xi + \int_{1-1/a}^1 \frac{1}{\sqrt{1-\xi^2}} d\xi \\ &\lesssim \frac{1}{\sqrt{a}}.\end{aligned}$$

On the other hand, note that for $1 \le \xi \le 2$,

$$\sqrt{\xi^2-1} = t\sqrt{\xi^2+a} \le t\sqrt{4+a} \lesssim 2t\sqrt{a}.$$

Therefore,

$$\begin{aligned}\int_1^2 \frac{1}{|w(\xi)|^2\sqrt{\xi^2-1}}\, d\xi &\le \int_0^{\sqrt{\frac{3}{a+4}}} \frac{at\, dt}{\left[(1-at)^2+\frac{1}{4}t^2b^2\right]\cdot 2t\sqrt{a}} \\ &= \int_0^{\sqrt{\frac{3}{a+4}}} \frac{\sqrt{a}\, dt}{(1-at)^2+\frac{1}{4}t^2b^2}.\end{aligned}$$

It is straightforward to show that

$$\begin{aligned}&\int_{I_1} \frac{\sqrt{a} dt}{(1-at)^2+\frac{1}{4}t^2b^2} \le \int_{I_1} \frac{\sqrt{a} dt}{(1-at)^2} \lesssim \frac{\sqrt{a}}{b},\\ &\int_{I_2} \frac{\sqrt{a} dt}{(1-at)^2+\frac{1}{4}t^2b^2} \le \int_{I_2} \frac{\sqrt{a} dt}{t^2b^2} = \int_{I_2} \frac{a\, dt}{tb^2} \lesssim \frac{\sqrt{a}}{b},\\ &\int_{I_3} \frac{\sqrt{a} dt}{(1-at)^2+\frac{1}{4}t^2b^2} \le \int_{I_3} \frac{\sqrt{a} dt}{(1-at)^2} \lesssim \frac{\sqrt{a}}{b}.\end{aligned}$$

Hence we have

$$\int_0^2 \frac{1}{|w(\xi)|^2\, |\sqrt{1-\xi^2}|}\, d\xi \lesssim \frac{1}{\sqrt{a}} + \frac{\sqrt{a}}{b},$$

which proves (4.32). Finally, the proof of (4.33) is similar. ∎

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