



Defect Modes for Dislocated Periodic Media

A. Drouot¹, C. L. Fefferman², M. I. Weinstein¹

¹ Mathematics Department, Columbia University, New York, NY 10027, USA.
E-mail: alexis.drouot@gmail.com; miw2103@columbia.edu

² Mathematics Department, Princeton University, Princeton, NJ 08544, USA.
E-mail: cf@math.princeton.edu

Received: 11 December 2018 / Accepted: 2 April 2020

Published online: 19 June 2020 – © Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract: We study defect modes in a one-dimensional periodic medium perturbed by an adiabatic dislocation of amplitude $\delta \ll 1$. If the periodic background admits a Dirac point—a linear crossing of dispersion curves—then the dislocated operator acquires a gap in its essential spectrum. For this model (and its honeycomb analog) Fefferman et al. (Proc Natl Acad Sci USA 111(24):8759–8763, 2014, Mem Am Math Soc 247(1173):118, 2017, Ann PDE 2(2):80, 2016, 2D Mater 3:1, 2016) constructed (at leading order in δ) defect modes with energies within the gap. These bifurcate from the eigenmodes of an effective Dirac operator. Here we address the following open problems:

- Do *all* defect modes arise as bifurcations from the Dirac operator eigenmodes?
- Do these modes admit expansions to all order in δ ?

We respond positively to both questions. Our approach relies on (a) resolvent estimates for the bulk operators; (b) scattering theory for highly oscillatory potentials [Dr18a, Dr18b, Dr18c]. It has led to an understanding of the topological stability of defect states in continuous dislocated systems—in connection with the bulk-edge correspondence [Dr18d].

1. Introduction

The study of systems which exhibit energy localization at material interfaces (line defects in 2D, facets in 3D) has a long history with many fundamental applications. Such models admit edge states, time-harmonic waves propagating *along* rather than *across* the interface. Recent research focuses on topologically robust edge states, i.e. states that are stable against large local perturbations. They arise e.g. in the quantum Hall effect [AMU75, KDP80, La81, Th83] and in topological insulators [KM11]. Haldane and Raghu [HR08, RH08] proposed a photonic analog of such states. This inspired investigations in electronic physics, photonics, acoustics and mechanics; see

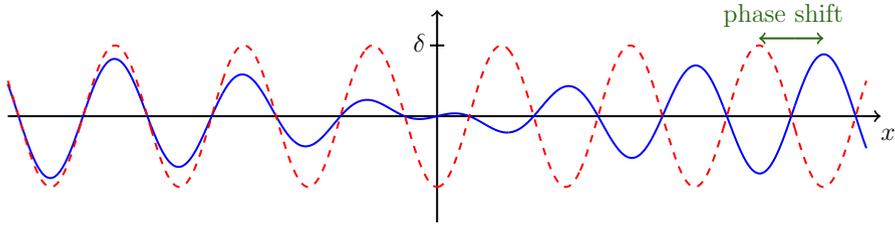


Fig. 1. The difference $\mathcal{P}_\delta - P_0$ (blue) is a potential of order δ that looks periodic near $\pm\infty$, but with different asymptotics on the left and right: it acquires a slow half-period phase-shift as x moves along \mathbb{R} . The operator \mathcal{P}_δ is not a local perturbation of a periodic operator

e.g. [KMT07, YVW08, WCJ08, SGK11, RZP13, NKR15, BPP17, OPA18]. In many examples, topological robustness relates to bifurcations from a *Dirac point*, a conical degeneracy in the band spectrum of the periodic background.

The theoretical analysis of such edge states has been mainly carried out in discrete systems, e.g. tight-binding operators. In this article we advance the mathematical theory behind a one-dimensional continuum model of Fefferman, Lee-Thorp and Weinstein [FLW14b, FLW17]. This model contributed to understanding topologically protected electronic transport in continuous honeycomb lattices with line defects [FLW16a, FLW16b, LWZ17, Dr19, DW19]. It also admits analogs in photonics. For instance, [LVX16, PWY16] theoretically and experimentally study the robustness properties of defect modes against imperfections in arrays of photonic waveguides, whose transverse structure is described by our model.

In our setup, the unperturbed Hamiltonian is a continuous periodic Schrödinger operator P_0 , with two dispersion curves intersecting transversely at a *one-dimensional Dirac point*, of energy E_* . We study an adiabatic dislocation \mathcal{P}_δ of P_0 : from $-\infty$ to $+\infty$, \mathcal{P}_δ acquires a slow phase-shift (of half-a-period). The *defect term* $\mathcal{P}_\delta - P_0$ is of order $\delta \ll 1$, but is supported on \mathbb{R} —see Fig. 1. This noncompact perturbation modifies the essential spectrum: while P_0 does not admit a spectral gap near energy E_* , \mathcal{P}_δ does—see Fig. 2. The present work studies eigenmodes (*defect states*) with energy within this gap, that bifurcate from the Dirac point. This contrasts with the more commonly studied situation where point spectrum bifurcates from a band edge; see for example, [HW11].

In [FLW17], the authors construct defect states of \mathcal{P}_δ , at leading order in δ . These admit a two-scale structure: they are slow linear combination of Dirac point Bloch modes, with eigenmodes of an emerging Dirac operator \mathcal{D} . Due to a chiral symmetry, \mathcal{D} always admits a zero mode; hence, [FLW17] is an existence result for defect modes of \mathcal{P}_δ , for $\delta \ll 1$.

The present work addresses the associated converse/uniqueness statement. We show that all defect states of \mathcal{P}_δ must arise as bifurcations from modes of \mathcal{D} —see Theorem 1. As a byproduct, we derive expansions of defect states of \mathcal{P}_δ at all order in δ (going beyond the leading order)—see Corollary 1. These results on dislocations now have analogs in the study of edge states in honeycomb lattices [Dr19, DW19]. This owes to the similarities between the two problems (Dirac point, gap via symmetry breaking, emerging Dirac operators).

Our results work in the adiabatic regime $\delta \ll 1$, where the slow and periodic scales decouple. However, certain qualitative features extend beyond this regime—for instance, the signed number of defect states, see [Dr18d]. The present work has thus improved our

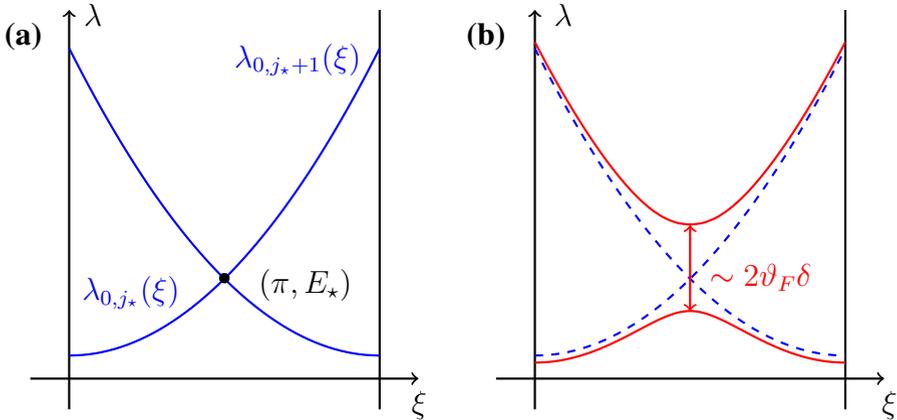


Fig. 2. Typical dispersion curves of **a** $P_0 = D_x^2 + V$ and **b** $P_\delta = P_0 + \delta W$. In **a**, they touch at a Dirac point (π, E_*) . In **b**, the perturbation δW breaks \mathcal{S} -invariance and a spectral gap of width $2\vartheta_F\delta + O(\delta^2)$ opens about E_*

understanding of topological protection (stability under local perturbation) in dislocation models.

1.1. Description of the model. Before stating our main results, we review the model of [FLW14b,FLW17]. We start with the periodic Schrödinger operator

$$P_0 \stackrel{\text{def}}{=} -\frac{\partial^2}{\partial x^2} + V = D_x^2 + V, \quad D_x \stackrel{\text{def}}{=} \frac{1}{i} \frac{\partial}{\partial x},$$

$$V \in C^\infty(\mathbb{R}, \mathbb{R}), \quad V(x + 1/2) = V(x).$$

Instead of seeing V as a $1/2$ -periodic potential, we regard V as a 1 -periodic potential, with an additional symmetry: $SV = V$, where

$$\mathcal{S}u(x) \stackrel{\text{def}}{=} u(x + 1/2).$$

The operator P_0 is 1 -periodic and its L^2 -spectrum is obtained from the family of operators $P_0(\xi)$, formally equal to $D_x^2 + V$ but acting on the space

$$L_\xi^2 \stackrel{\text{def}}{=} \left\{ f \in L_{\text{loc}}^2 : f(x + 1) = e^{i\xi} f(x) \right\}, \quad \xi \in [0, 2\pi].$$

The parameter ξ is the quasi-momentum. For each $\xi \in [0, 2\pi]$, the operator $P_0(\xi)$ has a compact resolvent; its spectrum consists of discrete real eigenvalues, listed below with multiplicity:

$$\lambda_{0,1}(\xi) \leq \lambda_{0,2}(\xi) \leq \dots \leq \lambda_{0,j}(\xi) \leq \dots$$

The eigenvalue curves $\xi \mapsto \lambda_{0,j}(\xi)$, $j = 1, 2, \dots$ are the dispersion curves of P_0 .

A *Dirac point* (ξ_*, E_*) is a quasi-momentum / energy pair corresponding to a transverse intersection of dispersion curves of P_0 ; see Sect. 2.3 for a detailed description. Due to \mathcal{S} -invariance, the Dirac points of P_0 with $\xi_* = \pi$ are precisely given by

$$(\pi, E_*) \quad \text{where } E_* \text{ is a } L_\pi^2\text{-eigenvalue of } P_0,$$

see e.g. [Dr18d, Lemma 2.3]. To each Dirac point (π, E_\star) of P_0 is associated a pair $(\phi_+^\star, \phi_-^\star) \subset L^2_\pi$ of eigenvectors of P_0 with energy E_\star , such that

$$\mathcal{S}\phi_+^\star = i\phi_+^\star, \quad \mathcal{S}\phi_-^\star = -i\phi_-^\star \quad \text{and} \quad \overline{\phi_+^\star} = \phi_-^\star, \tag{1.1}$$

see Sect. 2.3. We refer to a pair $(\phi_+^\star, \phi_-^\star)$ as a ‘‘Dirac eigenbasis’’.

We now introduce perturbations of P_0 that break \mathcal{S} -invariance. Pick $W \in C^\infty(\mathbb{R}, \mathbb{R})$, one periodic, such that $\mathcal{S}W = -W$; and introduce

$$P_{\pm\delta} \stackrel{\text{def}}{=} -\frac{\partial^2}{\partial x^2} + V \pm \delta W = D_x^2 + V \pm \delta W.$$

The operators P_δ and $P_{-\delta}$ are conjugated: $\mathcal{S}P_\delta = P_{-\delta}\mathcal{S}$. The addition of $\pm\delta W$ to $P_0 = D_x^2 + V$ breaks the \mathcal{S} -invariance. This perturbation opens a spectral gap \mathcal{G}_δ , which is an interval in \mathbb{R} , about the Dirac energy for $P_{\pm\delta}$ as long as

$$\vartheta_\star \stackrel{\text{def}}{=} \langle \phi_-^\star, W\phi_+^\star \rangle \neq 0 \quad \text{and} \quad \delta \ll 1. \tag{1.2}$$

See Fig. 2. Note that, given V (hence ϕ_+^\star and ϕ_-^\star), (1.2) holds unless W belongs to the hyperplane normal to $\phi_+^\star\phi_-^\star$. In particular, (1.2) is a generic assumption. The interval \mathcal{G}_δ is centered at $E_\star + O(\delta^2)$ and has width $2\vartheta_F\delta + O(\delta^2)$, where $\vartheta_F = |\vartheta_\star|$ —see e.g. [FLW17, Proposition 4.1].

The operator \mathcal{P}_δ studied in this article was introduced in [FLW14b,FLW17]. It interpolates between $P_{-\delta}$ at $x = -\infty$ and P_δ at $x = +\infty$. Specifically, we define

$$\mathcal{P}_\delta \stackrel{\text{def}}{=} -\frac{\partial^2}{\partial x^2} + V + \delta \cdot \kappa(\delta \cdot) \cdot W = D_x^2 + V + \delta \cdot \kappa(\delta \cdot) \cdot W.$$

Above $\kappa(\delta \cdot)(x) \stackrel{\text{def}}{=} \kappa(\delta x)$ and δ is a small parameter. The function κ is a smooth *domain wall*:

$$\text{for some } L > 0, \quad \kappa(y) = \begin{cases} 1 & \text{for } y \geq L, \\ -1 & \text{for } y \leq -L; \end{cases} \quad \kappa \in C^\infty(\mathbb{R}, \mathbb{R}). \tag{1.3}$$

The essential spectrum of \mathcal{P}_δ is determined by the bulk operators $P_{\pm\delta}$: $\Sigma_{\text{ess}}(\mathcal{P}_\delta) = \Sigma_{\text{ess}}(P_{\pm\delta})$ —see e.g. [FLW17, Proposition 4.1]. Hence \mathcal{P}_δ inherits the same essential spectral gap \mathcal{G}_δ as $P_{\pm\delta}$.

In [FLW14b,FLW17], Fefferman, Lee-Thorp and Weinstein developed a multiscale procedure to construct eigenmodes of \mathcal{P}_δ . It relies on the spectral properties of an emerging Dirac operator,

$$\begin{aligned} \mathcal{D} &\stackrel{\text{def}}{=} \nu_\star \sigma_3 D_y + \sigma_\star \kappa, \quad \text{where} \quad \nu_\star \stackrel{\text{def}}{=} 2 \langle \phi_+^\star, D_x \phi_+^\star \rangle \neq 0, \\ \sigma_3 &\stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_\star \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \overline{\vartheta_\star} \\ \vartheta_\star & 0 \end{bmatrix}. \end{aligned}$$

The operator \mathcal{D} was introduced in the context of quantum field theory [JR76] and plays an important role in the models of topological insulators; see, for example, [HR08,RH08, Ba17, Ba18]. We briefly summarize the spectral properties of \mathcal{D} :

- the essential spectrum of \mathcal{D} equals $\mathbb{R} \setminus (-\vartheta_F, \vartheta_F)$;
- the point spectrum $\{\vartheta_j\}_{j=-N}^N$ is simple and symmetric about the origin;

$$-\vartheta_F < \vartheta_{-N} < \dots < \vartheta_{-1} < \vartheta_0 = 0 < \vartheta_1 < \dots < \vartheta_N < \vartheta_F, \quad \vartheta_{-j} = -\vartheta_j.$$

See Sect. 6.2 for proofs. The paper [FLW17] shows that each eigenvalue ϑ_j of \mathcal{D} seeds an eigenvalue of \mathcal{P}_δ , with energy $E_j(\delta) = E_\star + \vartheta_j\delta + O(\delta^2)$. The corresponding eigenvector takes the form

$$u_{\delta,j}(x) = \alpha_+(\delta x)\phi_+^*(x) + \alpha_-(\delta x)\phi_-^*(x), \quad (\mathcal{D} - \vartheta_j) \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (1.4)$$

Since \mathcal{D} has at least one (zero) mode, this implies that \mathcal{P}_δ admits at least one defect state with energy in \mathcal{G}_δ , as long as δ is small enough.

The present paper addresses the questions:

- Are all eigenmodes of \mathcal{P}_δ in \mathcal{G}_δ of the form (1.4)?
- Do eigenpairs admit full expansions in powers of δ ?

1.2. *Main results.* We make the following hypotheses:

- (H1) $V, W \in C^\infty(\mathbb{R}, \mathbb{R})$ are 1-periodic and $\mathcal{S}V = V, \mathcal{S}W = -W$.
- (H2) (π, E_\star) is a Dirac point of P_0 with Dirac eigenbasis (ϕ_+^*, ϕ_-^*) ; see (1.1).
- (H3) $\vartheta_\star = \langle \phi_-^*, W\phi_+^* \rangle \neq 0$.

As mentioned above, when (H1)–(H3) are satisfied, the operator \mathcal{P}_δ has no essential spectrum in an interval centered at E_\star , of width $2\vartheta_F\delta + O(\delta^2)$.

Theorem 1. *Assume that (H1)–(H3) hold. For any $\vartheta_\sharp \in (\vartheta_N, \vartheta_F)$ and $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$,*

(A) *The spectrum of \mathcal{P}_δ in $[E_\star - \vartheta_\sharp\delta, E_\star + \vartheta_\sharp\delta]$ is contained in*

$$\bigcup_{j=-N}^N \mathcal{I}_{\delta,j}, \quad \text{where } \mathcal{I}_{\delta,j} \stackrel{\text{def}}{=} [E_\star + (\vartheta_j - \varepsilon)\delta, E_\star + (\vartheta_j + \varepsilon)\delta]. \quad (1.5)$$

(B) *The spectrum of \mathcal{P}_δ in each $\mathcal{I}_{\delta,j}$ consists of at most one eigenvalue.*

In addition to this theorem, the procedure of [FLW17] (reviewed in Sect. 3) constructs, for each j , an approximate defect states with energy within $\mathcal{I}_{\delta,j}$. Because of the variational description of eigenvalues for selfadjoint operators, Theorem 1 implies a much more powerful result. We introduce the space

$$X \stackrel{\text{def}}{=} \left\{ v \in C^\infty(\mathbb{R}^2, \mathbb{C}), \quad v(x+1, y) = -v(x, y), \quad \exists a > 0, \quad v(x, y) = O\left(e^{-a|y|}\right) \right\}.$$

Corollary 1. *Under the assumptions and notations of Theorem 1, the operator \mathcal{P}_δ has exactly one eigenvalue in each subinterval $\mathcal{I}_{\delta,j}$ of the spectral gap \mathcal{G}_δ .*

Moreover, for all $M \geq 1$ and $k \geq 0$, the associated eigenpair $(E_{\delta,j}, u_{\delta,j})$ expands as

$$\begin{aligned} E_{\delta,j} &= E_\star + \vartheta_j\delta + a_2\delta^2 + \dots + a_M\delta^M + O\left(\delta^{M+1}\right), \quad a_m \in \mathbb{R}; \\ u_{\delta,j}(x) &= v_0(x, \delta x) + \delta v_1(x, \delta x) + \dots + \delta^M v_M(x, \delta x) + o_{H^k}\left(\delta^M\right), \quad v_m \in X. \end{aligned} \quad (1.6)$$

The implicit constants involved in (1.6) depend on M and k and

- $a_m \in \mathbb{R}$ and $v_m \in X$ are constructed following a recursive procedure.

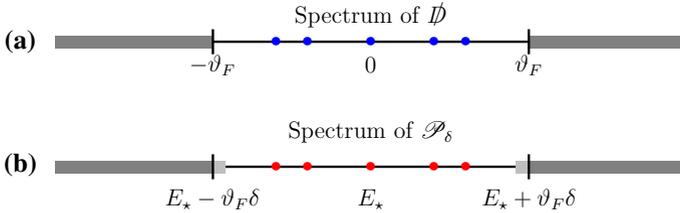


Fig. 3. Eigenvalues of **a** \mathcal{D} in $(-\vartheta_F, \vartheta_F)$ and **b** \mathcal{P}_δ in \mathcal{G}_δ . A rescaling equal to $\vartheta \mapsto E_* + \delta\vartheta$ modulo $O(\delta^2)$ maps **(a)** to **(b)**. Theorem 1 and Corollary 1 do not apply in the (arbitrarily small) light-gray region near the essential spectrum of \mathcal{P}_δ

- The function v_0 appearing in the leading order term satisfies

$$v_0(x, y) = \alpha_+(y)\phi_+^*(x) + \alpha_-(y)\phi_-^*(x), \quad (\mathcal{D} - \vartheta_j) \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (1.7)$$

Corollary 1 characterizes the spectrum of \mathcal{P}_δ in the gap \mathcal{G}_δ , arbitrarily close to its edges. It belongs to the general class of result that studies bound states in perturbed periodic backgrounds [DH86, G90, OI03, Bo07, BG08, PLA10, Bo11, HW11, Bo15, Ze16]. It shows that after suitable rescaling, the spectrum of \mathcal{D} and \mathcal{P}_δ are very similar; see Fig. 3. Figure 4 is a schematic of the “spray of eigenvalues” of \mathcal{P}_δ into the spectral gap \mathcal{G}_δ . The expression (1.7) relates the eigenfunctions of \mathcal{P}_δ to those of \mathcal{D} ; to leading order in δ , the eigenfunction $u_{\delta,j}$ is a modulation of the Dirac eigenbasis (ϕ_+^*, ϕ_-^*) , with slowly varying amplitudes that form eigenvectors of \mathcal{D} . The terms $v_m(x, \delta x)$ exhibit a two-scale structure: they oscillate on a unit length scale and decay on the length scale δ^{-1} .

The work [FLW17] had already constructed genuine eigenstates. The bifurcation argument used there is initiated with an approximate eigenstate, constructed via a multiscale procedure. Our approach is different: we work instead directly at the level of operators. This allows us to address existence *together with absence* of eigenvalues in \mathcal{G}_δ : we do not need an initiating point to start our procedure (i.e. an approximate eigenstate as in [FLW17]). This completes the results of [FLW17] with a full characterization of the spectrum.

1.3. Strategy. We now present a detailed outline of our approach:

- (A) To prove Theorem 1 and Corollary 1 we show:
 - (a) There is no point spectrum in $\mathcal{G}_\delta \setminus \bigcup_{j=N}^{-N} \mathcal{I}_{\delta,j}$ —the subintervals $\mathcal{I}_{\delta,j} \subset \mathcal{G}_\delta$ are defined in (1.5).
 - (b) Each interval $\mathcal{I}_{\delta,j} \subset \mathcal{G}_\delta$ of \mathcal{P}_δ contains at most one eigenvalue of \mathcal{P}_δ .
 - (c) Each interval $\mathcal{I}_{\delta,j} \subset \mathcal{G}_\delta$ contains at least an eigenvalue of \mathcal{P}_δ ; the associated eigenpair admits a full expansion in powers of δ .

The main part of this paper concerns (a) and (b). Part (c) follows from (b); a variational principle for self-adjoint operators; and the multiscale expansion of [FLW17]. We show that (b) implies (c) in Sect. 3.
- (B) In Sect. 5.2 we construct a parametrix $Q_\delta(\lambda)$ for $\mathcal{P}_\delta - \lambda$ when $\lambda \in \mathbb{D}(E_*, \vartheta_\sharp \delta)$, $\vartheta_\sharp \in (\vartheta_N, \vartheta_F)$; we recall that $\vartheta_N > 0$ denotes the largest eigenvalue of \mathcal{D} . Specifically,

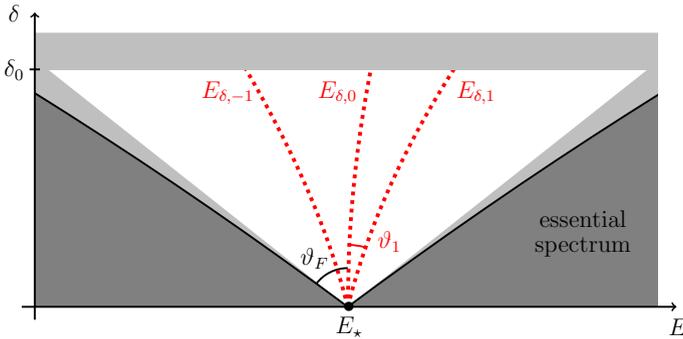


Fig. 4. Eigenvalues of \mathcal{D} seed eigenvalues of \mathcal{P}_δ (red dotted curves) from the Dirac energy E_* , into the gap \mathcal{G}_δ . The slopes of the red curves at $\delta = 0$ are the eigenvalues of \mathcal{D} . Our results do not apply in the (arbitrarily small) light-gray region $\vartheta_\pm \delta < |E - E_*| < \vartheta_\mp \delta$, nor for $\delta > \delta_0$

there exists $Q_\delta(\lambda) : L^2 \rightarrow H^2$ such that for $\lambda \in \mathbb{D}(E_*, \vartheta_\pm \delta)$,

$$(\mathcal{P}_\delta - \lambda)Q_\delta(\lambda) - \text{Id} \stackrel{\text{def}}{=} K_\delta(\lambda) \text{ is a compact operator on } L^2. \tag{1.8}$$

The family $\lambda \in \mathbb{D}(E_*, \vartheta_\pm \delta) \mapsto K_\delta(\lambda)$ depends holomorphically on λ . Analytic Fredholm theory shows that $(\text{Id} + K_\delta(\lambda))^{-1}$ is meromorphic. We can then write

$$(\mathcal{P}_\delta - \lambda)^{-1} = Q_\delta(\lambda) \cdot (\text{Id} + K_\delta(\lambda))^{-1}. \tag{1.9}$$

Thus each eigenvalue of \mathcal{P}_δ in $[E_* - \vartheta_\pm \delta, E_* + \vartheta_\pm \delta]$ is a pole of $(\text{Id} + K_\delta(\lambda))^{-1}$.

(C) We then seek “zeros” of $\text{Id} + K_\delta(\lambda)$ using holomorphic tools. This requires estimates of $K_\delta(\lambda)$ for δ near 0. This happens to be related to an expansion of the resolvent difference

$$(P_\delta - \lambda)^{-1} - (P_{-\delta} - \lambda)^{-1}.$$

(D) A key ingredient is Theorem 2. It shows that modulo rescaling and matrix-valued multiplications related to ϕ_+^* and ϕ_-^* , the resolvent $(P_\delta - \lambda)^{-1}$ behaves like a 2×2 Fourier multiplier:

$$\begin{aligned} &\vartheta_\pm \in (0, \vartheta_F), \delta \in (0, \delta_0), z \in \mathbb{D}(0, \vartheta_\pm), \lambda = E_* + z\delta \\ \Rightarrow (P_\delta - \lambda)^{-1} &= \frac{1}{\delta} \begin{bmatrix} \phi_+^* \\ \phi_-^* \end{bmatrix}^\top \mathcal{U}_\delta \cdot (\mathcal{D}_+ - z)^{-1} \cdot \mathcal{U}_\delta^{-1} \begin{bmatrix} \overline{\phi_+^*} \\ \phi_-^* \end{bmatrix} + \mathcal{O}_{L^2}(\delta^{1/3}), \end{aligned} \tag{1.10}$$

where $\mathcal{U}_\delta f(x) = \delta^{1/2} f(\delta x)$ and $\mathcal{D}_+ = v_* \sigma_3 D_x + \sigma_*$.

(E) The expansion (1.10) allows us to identify the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$ with the eigenvalues of a matrix Schrödinger operator with highly oscillatory 2×2 potential $\mathcal{M}(x/\delta, x)$ where $\mathcal{M}(x, y)$ is one-periodic in x , compactly supported in y and smooth in both variables. The first author’s work [Dr18b] shows that the eigenvalues of such operators are approached by those of their average—see also [GW05, BG06, DW11, DVW14, Dr18a, Dr18c, DR18]. The homogenized operator happens to be \mathcal{D}^2 . This identifies the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$ in the spectral gap \mathcal{G}_δ with the eigenvalues of \mathcal{D}^2 modulo negligible errors.

(F) But there's a glitch. The operator \mathcal{D}^2 has twice as many eigenvalues as predicted in Theorem 1 and Corollary 1. This is due to the factorization (1.9) of $\mathcal{P}_\delta - \lambda$: the parametrix $Q_\delta(\lambda)$ in (1.9) happens to cancel some of the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$. This phenomenon is however unstable and the parametrix is not unique: we may for example perturb $Q_\delta(\lambda)$ by a rank-one operator and still respect (1.8). We use these extra degrees of freedom to produce a *twisted parametrix*, which *deforms away* the spurious poles. With this final piece in place, we obtain Theorem 1.

1.4. Related work and perspectives. The most important advance of this work is the precise counting of eigenvalues of \mathcal{P}_δ in the gap \mathcal{G}_δ . Corollary 1 has since served to demonstrate the topological stability of defect states in dislocated systems [Dr18d].

In rough terms, [Dr18d] shows that for a (non-empty) open set of values of $t \in \mathbb{R}$, systems with t -phase-shifts (instead of half-period) admit defect states, even in the presence of large local perturbations. This relates to previous results [Ko00, Ko05, HK11a, DPR09] (when $V = 0$ and $\kappa(x) = \text{sgn}(x)$); see [HK11b, HKS15] for two-dimensional analogs.

In details, the Hamiltonian \mathcal{P}_δ is embedded in a one-parameter family of operators $t \in \mathbb{R}/\mathbb{Z} \mapsto \mathcal{P}_\delta(t)$. Following the mathematical physics literature, we set:

- An edge index: the spectral flow of $t \in \mathbb{R}/\mathbb{Z} \mapsto \mathcal{P}_\delta(t)$ at E_* , i.e. the signed number of eigenvalues of $\mathcal{P}_\delta(t)$ crossing E_* as t runs through \mathbb{R}/\mathbb{Z} .
- A bulk index: the Chern number of an eigenbundle associated to the asymptotic operator $P_\delta(t)$ for $\mathcal{P}_\delta(t)$.

In [Dr18d], we proved the bulk-edge correspondence for $\mathcal{P}_\delta(t)$: the bulk and edge indexes are equal. This belongs to a general class of results—see e.g. [KRS00, EGS05, GP13, PS16, Ba17, Sh17, ASV18, Br18, GS18, GT18, ST18, BR18] for studies in tight-binding models and [KS04a, KS04b, Ta14, FSF12, Ba17, Ba18] for analysis in the continuum. Corollary 1 ended up being a key tool in computing the edge index for the dislocation systems $\mathcal{P}_\delta(t)$. When $V = 0$, its value was derived by Korotyaev [Ko00, Ko05, HK11a, DPR09]: it is n for the n -th gap. This is no longer true when $V \neq 0$, see [Dr18d, Sect. 3]; in this case, the edge index admits an explicit formula as a winding number.

The dislocation model studied here shares many features with those of honeycomb structures with line defects, e.g. Dirac points at high-symmetry quasimomenta, opening of a spectral gap via symmetry breaking, effective Dirac operators with spatially varying mass. It has been used successfully as a stepping stone towards the analysis of edge states in continuous honeycomb structures. For instance, [FLW16a, FLW16b, LWZ17] constructed edge states as bifurcations from the Dirac points and [Dr19, DW19] showed that all edge states arise in this manner.

We hope to adapt our methods to deal with failures of the *spectral no-fold condition* [FLW16a, Sect. 1.3]. This condition requires that small parity-breaking perturbations open a Dirac point into a spectral gap. While it must hold in one dimension, it can fail in higher dimensions—see [FLW16a, Sect. 8]. Fefferman, Lee-Thorp and Weinstein [FLW16a, Sect. 1.4] conjectured that there should nonetheless exist long-lived edge states. Their multiscale expansion produces arbitrarily accurate approximate edge states (edge quasimodes); this formally indicates the existence of resonances (long-lived but finite lifetime states) with complex energies near the Dirac energy. We refer to [SV96, TZ98, St99, St00, Ga15] for the relation between quasimodes and resonances in other settings; and to [GS92] for their time-dependent dynamics.

We speculate that some of the results presented here have analogs in random situations where averaging effect arise—such as the settings of [ZB14a,ZB14b,GB15a,GB15b,GB16,Dr18c]. For instance, the bulk-edge correspondence has been investigated for the quantum Hall effect with random potential [EGS05,Ta14]. In our situation the discussion has remained speculative: it is not clear what random model to consider.

Notations We will use the following notations:

- $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$ and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.
- If $z \in \mathbb{C}$, \bar{z} is its complex conjugate and $\mathbb{D}(z, r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$.
- If \mathcal{H} is a Hilbert space and $\psi \in \mathcal{H}$, we write $\|\psi\|_{\mathcal{H}}$ for the norm of \mathcal{H} ; if $A : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator of \mathcal{H} , the operator norm of A is

$$\|A\|_{\mathcal{H}} \stackrel{\text{def}}{=} \sup \{|A\psi|_{\mathcal{H}} : \|\psi\|_{\mathcal{H}} = 1\}.$$

- If A is an operator, $\Sigma(A)$ denotes the spectrum of A ; $\Sigma_{\text{ess}}(A)$ denotes the essential spectrum of A ; $\Sigma_{\text{pp}}(A)$ denotes the pure point spectrum of A .
- If $\psi_{\varepsilon} \in \mathcal{H}$ (resp. $A_{\varepsilon} : \mathcal{H} \rightarrow \mathcal{H}$ is a linear operator) and $f : \mathbb{R} \rightarrow \mathbb{R}$, we write $\psi_{\varepsilon} = \mathcal{O}_{\mathcal{H}}(f(\varepsilon))$ (resp. $A_{\varepsilon} = \mathcal{O}_{\mathcal{H}}(f(\varepsilon))$) when there exists $C > 0$ such that $\|\psi_{\varepsilon}\|_{\mathcal{H}} \leq C f(\varepsilon)$ (resp. $\|A_{\varepsilon}\|_{\mathcal{H}} \leq C f(\varepsilon)$).
- D_x is the operator $\frac{1}{i} \partial_x$.
- To a kernel distribution $K(x, y)$ taking values in \mathbb{R}^2 , we associate the operator

$$(Kf)(x) = \int_{\mathbb{R}} K(x, y) f(y) dy.$$

- L^2 denotes the space of square-summable functions on \mathbb{R} and H^k , for $k \in \mathbb{R}$, are the classical Sobolev spaces.
- The space L^2_{ξ} consists of ξ -quasiperiodic functions:

$$L^2_{\xi} \stackrel{\text{def}}{=} \left\{ u \in L^2_{\text{loc}} : u(x+1) = e^{i\xi} u(x) \right\}.$$

It splits orthogonally as $L^2_{\xi} = L^2_{\xi,e} \oplus L^2_{\xi,o}$ with

$$L^2_{\xi,e} \stackrel{\text{def}}{=} \left\{ u \in L^2_{\text{loc}} : u(x+1/2, \xi) = e^{i\xi/2} u(x) \right\},$$

$$L^2_{\xi,o} \stackrel{\text{def}}{=} \left\{ u \in L^2_{\text{loc}} : u(x+1/2, \xi) = -e^{i\xi/2} u(x) \right\}.$$

- V, W always denote potentials that are smooth, real-valued and such that $V(x+1/2) = V(x)$ and $W(x+1/2) = -W(x)$.
- P_0 is the operator $D_x^2 + V$.
- $P_{\pm\delta}$ is the operator $D_x^2 + V \pm \delta \cdot W$.
- \mathcal{P}_{δ} is the operator $D_x^2 + V + \delta \cdot \kappa(\delta \cdot) \cdot W$
- (π, E_{\star}) is a Dirac point of P_0 with a Dirac eigenbasis $(\phi_{\star}^+, \phi_{\star}^-) \in L^2_{\pi,e} \times L^2_{\pi,o}$.
- The Pauli matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.11}$$

These matrices satisfy $\sigma_j^2 = \text{Id}_2$ and $\sigma_i \sigma_j = -\sigma_j \sigma_i$ for $i \neq j$.

- $\nu_\star = 2 \langle D_x \phi_\star^+, \phi_\star^+ \rangle$; $\vartheta_\star = \langle \phi_\star^-, W \phi_\star^+ \rangle$; $\vartheta_F = |\vartheta_\star|$; and

$$\sigma_\star \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \overline{\vartheta_\star} \\ \vartheta_\star & 0 \end{bmatrix} = \text{Re}(\vartheta_\star)\sigma_1 + \text{Im}(\vartheta_\star)\sigma_2.$$

- \mathcal{D} is the operator $\nu_\star \sigma_3 D_x + \sigma_\star \kappa$; \mathcal{D}_\pm is the operator $\nu_\star \sigma_3 D_x \pm \sigma_\star$.
- If $E, F \subset \mathbb{C}$, $\text{dist}(E, F)$ denotes the Euclidean distance between E and F .

2. Preliminaries

2.1. *Floquet–Bloch theory.* We review here how to study periodic operators from their Floquet–Bloch representation. Basic references are [RS78, Chapter XIII] and [Ku16]. We equip the space L_ξ^2 of ξ -quasiperiodic functions

$$L_\xi^2 \stackrel{\text{def}}{=} \left\{ f \in L_{\text{loc}}^2(\mathbb{R}) : u(x+1) = e^{i\xi} u(x) \right\}$$

with the Hermitian inner product $\langle f, g \rangle_{L_\xi^2} = \int_0^1 \overline{f} g$ and the norm $\|f\|_{L_\xi^2}^2 = \int_0^1 |f(x)|^2 dx$.

In particular, L_0^2 is the space of periodic square-summable functions.

Assume that T is a 1-periodic operator, i.e. $f \in \text{dom}(T)$ implies $(Tf)(\cdot + 1) = T(f(\cdot + 1))$. Then T maps L_ξ^2 to itself and we denote this operator by $T(\xi)$. If $f \in C_0^\infty(\mathbb{R}, \mathbb{C})$, then f has a L_ξ^2 -decomposition, based on the Fourier inversion formula:

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(\xi, x) d\xi, \quad \tilde{f}(\xi, x) \stackrel{\text{def}}{=} e^{i\xi x} \sum_{k \in \mathbb{Z}} \hat{f}(\xi - 2k\pi) e^{-2\pi i k x} \in L_\xi^2.$$

The action of T on L^2 can be studied using the reconstruction formula

$$Tf(x) = \frac{1}{2\pi} \int_0^{2\pi} (T(\xi) \tilde{f}(\xi, \cdot))(x) d\xi.$$

This formula extends to all $f \in L^2(\mathbb{R}, \mathbb{C})$ by density. We will use the [RS78] notation:

$$T = \frac{1}{2\pi} \int_{[0, 2\pi]}^\oplus d\xi T(\xi), \tag{2.1}$$

emphasizing that each $T(\xi)$ acts fiberwise on L_ξ^2 .

If now $T = D_x^2 + \mathcal{V}$ is a Schrödinger operator where $\mathcal{V} \in C^\infty(\mathbb{R}, \mathbb{R})$ is 1-periodic then the resolvent of $T(\xi)$ is selfadjoint and compact. The spectrum of $T(\xi)$ consists of point eigenvalues $E_1(\xi) \leq \dots \leq E_j(\xi) \leq \dots$, listed with multiplicity and tending to $+\infty$. The curves $\xi \mapsto E_j(\xi)$ obtained by varying $\xi \in [0, 2\pi]$ are called *dispersion curves* of the periodic operator T . The collection of L_ξ^2 -spectra of $T(\xi)$ sweeps out the L^2 -spectrum of T :

$$\Sigma_{L^2}(T) = \bigcup_{\xi \in [0, 2\pi]} \Sigma_{L_\xi^2}(T) = \{E_j(\xi) : j \geq 1, \xi \in [0, 2\pi]\}.$$

We will use the following stability result for dispersion surfaces. It follows from [FW14, Appendix A], a general stability result for selfadjoint eigenvalue problems.

Lemma 2.1. *Let \mathcal{V} and \mathcal{W} be two periodic potentials. Denote by $E_{\mathcal{V},\ell}(\xi)$ (resp. $E_{\mathcal{V}+\mathcal{W},\ell}(\xi)$) the ℓ -th L^2_{ξ} -eigenvalue of $D_x^2 + \mathcal{V}$ (resp. $D_x^2 + \mathcal{V} + \mathcal{W}$). For any $\ell \geq 1$, there exists a constant $C(\ell)$ such that*

$$|E_{\mathcal{V}+\mathcal{W},\ell}(\xi) - E_{\mathcal{V},\ell}(\xi)| \leq C(\ell)|\mathcal{W}|_{\infty}.$$

2.2. *The spaces $L^2_{\xi,e}$ and $L^2_{\xi,o}$. Fix $\xi \in [0, 2\pi]$. The operator \mathcal{S} is unitary on L^2_{ξ} and $\mathcal{S}^2 = e^{i\xi} \cdot \text{Id}_{L^2_{\xi}}$. Hence, \mathcal{S} has eigenvalues $\{e^{i\xi/2}, -e^{i\xi/2}\}$. Since \mathcal{S} is a normal operator on L^2_{ξ} , there exists an orthogonal decomposition of L^2_{ξ} in terms of eigenspaces of \mathcal{S} :*

$$L^2_{\xi} = L^2_{\xi,e} \oplus L^2_{\xi,o}, \quad L^2_{\xi,e} \stackrel{\text{def}}{=} \ker(\mathcal{S} - e^{i\xi/2}), \quad L^2_{\xi,o} \stackrel{\text{def}}{=} \ker(\mathcal{S} + e^{i\xi/2}). \quad (2.2)$$

Functions in $L^2_{\xi,e}$ (resp. $L^2_{\xi,o}$) admit even-indexed Fourier series representations (resp. odd-indexed Fourier series representations)—see [FLW17, Definition 3.3]. This explains the subscripts e/o .

When $\xi \in (0, 2\pi)$, these spaces are also related via complex conjugation: $\overline{L^2_{\xi,e}} = L^2_{2\pi-\xi,o}$. Indeed, if $f \in L^2_{\xi,e}$ then

$$\overline{f(x + 1/2)} = e^{-i\xi/2} \overline{f(x)} = -e^{i(2\pi-\xi)/2} \overline{f(x)},$$

which means that $\overline{f} \in L^2_{2\pi-\xi,o}$.

We recall that $P_0 = D_x^2 + V$, where $V \in C_0^{\infty}(\mathbb{R}, \mathbb{R})$ is one-periodic and satisfies $SV = V$. Because P_0 is \mathcal{S} -invariant, we can regard P_0 as an operator acting on $L^2_{\xi,e}$ or $L^2_{\xi,o}$. When $\xi \in (0, 2\pi)$ the eigenvalues of $P_0(\xi)$ on $L^2_{\xi,e}$ and on $L^2_{\xi,o}$ are simple. Indeed, since V is real-valued,

$$(P_0 - E)\psi = 0, \quad \psi \in L^2_{\xi,e} \implies (P_0 - E)\overline{\psi} = 0, \quad \overline{\psi} \in L^2_{2\pi-\xi,o}.$$

If E were a $L^2_{\xi,e}$ -eigenvalue of $P_0(\xi)$ of multiplicity at least 2, then the solution space of the ODE $(P_0(\xi) - E)u = 0$ would have dimension at least 4. This is impossible because the dimension of this space is at most 2.

2.3. *P_0 and Dirac points.* We recall that $V \in C^{\infty}(\mathbb{R}, \mathbb{R})$ is 1-periodic and satisfies $SV = V$. The operator $P_0(\xi)$ denotes the operator $P_0 = D_x^2 + V$ acting in the space L^2_{ξ} . Let $\lambda_{0,j}(\xi)$ denote the j -th eigenvalue (listed with multiplicity) of $P_0(\xi)$. We are interested in the situation where two dispersion curves intersect at a quasi-momentum / energy pair (π, E_{\star}) .

Definition 2.2. We say that P_0 has a Dirac point at the quasi-momentum / energy pair (π, E_{\star}) if there exist an integer j_{\star} and a constant $\nu_F > 0$ such that

$$\begin{aligned} \lambda_{0,j_{\star}}(\xi) &= E_{\star} - \nu_F \cdot |\xi - \pi| + O(\xi - \pi)^2, \\ \lambda_{0,j_{\star}+1}(\xi) &= E_{\star} + \nu_F \cdot |\xi - \pi| + O(\xi - \pi)^2. \end{aligned}$$

Remark 2.1. The expression “Dirac point” typically refers to the conical intersection of dispersion surfaces for Schrödinger operators in two or more dimensions. They arise e.g. for honeycomb structures – see [Co91,BC18,Le18] and notably [FW12] which shows that they generically exist. In [FLW17, Appendix D] the authors show that 1D Dirac points also arise generically in one-dimensional lattices with an extra symmetry. Watson [Wa17, Appendix B.1] gives a proof that in one dimension every eigenvalue $\lambda_{0,j}(\pi)$ of $P_0(\pi)$ corresponds to a Dirac point $(\pi, \lambda_{0,j}(\pi))$; this was used in [WW18]. We provide a simpler proof in [Dr18d, Appendix A.2].

The notation v_F is used since this constant plays the role of the Fermi velocity arising in mathematical theory of graphene; see, for example, [CGP09,FW12,FLW18,FW18].

Proposition 2.1. *Let (π, E_\star) denote a Dirac point of P_0 in the sense of Definition 2.2. Then for each $\xi \in (0, 2\pi)$, there exist L^2_ξ -normalized eigenpairs $(\lambda_+(\xi), \phi_+(\xi))$ and $(\lambda_-(\xi), \phi_-(\xi))$ of $P_0(\xi)$ with the following properties:*

- (1) $\phi_+(\xi) \in L^2_{\xi,e}$ and $\phi_-(\xi) \in L^2_{\xi,o}$.
- (2) If $\phi_+^\star \stackrel{\text{def}}{=} \phi_+(\pi)$ and $\phi_-^\star \stackrel{\text{def}}{=} \phi_-(\pi)$ then $\overline{\phi_+^\star} = \phi_-^\star$.
- (3) The maps $\xi \mapsto (\lambda_+(\xi), \phi_+(\xi))$ and $\xi \mapsto (\lambda_-(\xi), \phi_-(\xi))$ are real-analytic.
- (4) For ξ in a neighborhood of π ,

$$\begin{aligned} \lambda_+(\xi) &= E_\star + v_\star \cdot (\xi - \pi) + O(\xi - \pi)^2, & v_\star &\stackrel{\text{def}}{=}} 2\langle \phi_+^\star, D_x \phi_+^\star \rangle. \\ \lambda_-(\xi) &= E_\star - v_\star \cdot (\xi - \pi) + O(\xi - \pi)^2, \end{aligned} \tag{2.3}$$

Furthermore, if $(\mu_+(\xi), \psi_+(\xi))$ and $(\mu_-(\xi), \psi_-(\xi))$ are other normalized eigenpairs of $P_0(\xi)$ satisfying (1)–(4) then

$$\text{for some } \omega \in \mathbb{S}^1, \text{ we have: } \omega \cdot \phi_+^\star = \psi_+(\pi), \quad \overline{\omega} \cdot \phi_-^\star = \psi_+(\pi). \tag{2.4}$$

Proof. 1. Let (π, E_\star) be a Dirac point of P_0 . Because of the decomposition (2.2), we can write

$$\ker(P_0(\pi) - E_\star) = \left(\ker(P_0(\pi) - E_\star) \cap L^2_{\xi,e} \right) \oplus \left(\ker(P_0(\pi) - E_\star) \cap L^2_{\xi,o} \right).$$

Since (π, E_\star) is a Dirac point of P_0 , the space on the LHS has dimension 2. The subspaces on the RHS have dimension at most 1—see the discussion at the end of Sect. 2.2. We deduce that

$$\dim \left(\ker(P_0(\pi) - E_\star) \cap L^2_{\xi,e} \right) = \dim \left(\ker(P_0(\pi) - E_\star) \cap L^2_{\xi,o} \right) = 1. \tag{2.5}$$

Let ϕ_+^\star be a normalized element of $\ker(P_0(\pi) - E_\star) \cap L^2_{\xi,e}$; define $\phi_-^\star = \overline{\phi_+^\star}$.

- 2. Because of Step 1, E_\star is a $L^2_{\xi,e}$ -eigenvalue of $P_0(\pi)$. For $\xi \in (0, 2\pi)$ the $L^2_{\xi,e}$ -eigenvalues of P_0 are simple; and $P_0(\xi)$ is real analytic in ξ in the sense that

$$e^{-i\xi x} P_0(\xi) e^{i\xi x} = (D_x + \xi)^2 + V : L^2_0 \rightarrow L^2_0$$

is real-analytic in ξ . Therefore standard results on perturbation of eigenvalues show that for $\xi \in (0, 2\pi)$, (E_\star, ϕ_+^\star) seeds a unique $L^2_{\xi,e}$ -eigenpair $(\lambda_+(\xi), \phi_+(\xi))$ of P_0 which is real-analytic in ξ . Similarly, (E_\star, ϕ_-^\star) seeds a unique $L^2_{\xi,o}$ -eigenpair $(\lambda_-(\xi), \phi_-(\xi))$ of P_0 which is real-analytic in ξ .

3. So far we have proved assertions (1)–(3) of Proposition 2.1. Because the eigenvalues of $P_0(\pi)$ have multiplicity at most 2, Step 2 implies that

$$\{\lambda_+(\xi), \lambda_-(\xi)\} = \{\lambda_{0,j_\star}(\xi), \lambda_{0,j_\star+1}(\xi)\}.$$

Since the maps $\xi \mapsto \lambda_\pm(\xi)$ are smooth, we deduce that

$$\lambda_+(\xi) = E_\star + \nu_\star(\xi - \pi) + O(\xi - \pi)^2, \quad \lambda_-(\xi) = E_\star - \nu_\star(\xi - \pi) + O(\xi - \pi)^2,$$

where ν_\star is equal to either ν_F or $-\nu_F$. We now prove that $\nu_\star = 2 \langle \phi_+^\star, D_x \phi_+^\star \rangle$. Let $p_+(\xi, x) = e^{-i\xi x} \phi_+(\xi, x)$. This forms a smooth family of functions in L^2_0 which satisfy the equation

$$\left((D_x + \xi)^2 + V - \lambda_+(\xi) \right) p_+(\xi) = 0.$$

Taking the derivative with respect to ξ yields

$$\left((D_x + \xi)^2 + V - \lambda_+(\xi) \right) \partial_\xi p_+(\xi) = \partial_\xi \lambda_+(\xi) \cdot p_+(\xi) - 2(D_x + \xi)p_+(\xi). \quad (2.6)$$

Setting $\xi = \pi$ and taking the scalar product of (2.6) with $p_+(\pi)$, we get

$$\left\langle \left((D_x + \pi)^2 + V - \lambda_+(\pi) \right) \partial_\xi p_+(\pi), p_+(\pi) \right\rangle = \partial_\xi \lambda_+(\pi) - 2 \langle (D_x + \pi)p_+(\pi), p_+(\pi) \rangle.$$

Observe that the LHS vanishes: $(D_x + \pi)^2 + V - \lambda_+(\pi)$ is selfadjoint and $((D_x + \pi)^2 + V - \lambda_+(\pi))p_+(\pi) = 0$. This yields (4):

$$\nu_\star = 2 \langle p_+(\pi), (D_x + \pi)p_+(\pi) \rangle = 2 \langle \phi_+^\star, D_x \phi_+^\star \rangle.$$

5. It remains to prove the characterization property (2.4). Assume that $(\mu_+(\xi), \psi_+(\xi))$ and $(\mu_-(\xi), \psi_-(\xi))$ are any normalized eigenpairs of $P_0(\xi)$ satisfying (1)–(4). Then $\psi_+(\pi) \in \ker(P_0(\pi) - E_\star)$. Because of (2.5), there exists $\omega \in \mathbb{S}^1$ such that $\psi_+(\pi) = \omega \cdot \phi_+^\star$. Because of (2), $\psi_-(\pi) = \bar{\omega} \phi_+^\star$. This concludes the proof. \square

We call the pair $(\phi_+^\star, \phi_-^\star)$ a “Dirac eigenbasis”. According to the characterization of Proposition 2.1, they are unique modulo a multiplicative factor in \mathbb{S}^1 . In particular, ν_\star is invariantly defined: if we change $(\phi_+^\star, \phi_-^\star)$ to $(\omega \cdot \phi_+^\star, \bar{\omega} \cdot \phi_-^\star)$, with $\omega \in \mathbb{S}^1$, then ν_\star in (2.3) is unchanged.

2.4. *The operators $P_{\pm\delta}$.* Recall that $P_0 = D_x^2 + V$, where $V \in L^2_{0,e}$: $V(x+1/2) = V(x)$; and $D_x = \frac{1}{i} \partial_x$. Let $0 \neq W \in L^2_{0,o}$: $W(x+1/2) = -W(x)$. We introduce the perturbed periodic Schrödinger operator

$$P_\delta \stackrel{\text{def}}{=} D_x^2 + V + \delta W.$$

For $\delta \neq 0$, $[S, P_\delta] = -2\delta \cdot W \neq 0$: the perturbation δW breaks the \mathcal{S} -invariance.

For the remainder of the paper, we assume that hypothesis (H3) holds: if $(\phi_+^\star, \phi_-^\star)$ is a Dirac eigenbasis for (π, E_\star) then

$$\vartheta_\star \stackrel{\text{def}}{=} \langle \phi_-^\star, W \phi_+^\star \rangle \neq 0. \quad (2.7)$$

It should be noted that ϑ_\star is *not* invariantly defined. Indeed, a change of Dirac eigenbasis $(\phi_+^\star, \phi_-^\star)$ to $(\omega \cdot \phi_+^\star, \bar{\omega} \cdot \phi_-^\star)$ — with $\omega \in \mathbb{S}^1$ — transforms ϑ_\star to $\omega^2 \vartheta_\star$. But the condition (2.7) is invariant under this change.

Proposition 2.2. *With the above notations,*

$$2 \begin{bmatrix} \langle \phi_+^*, D_x \phi_+^* \rangle \langle \phi_+^*, D_x \phi_-^* \rangle \\ \langle \phi_-^*, D_x \phi_+^* \rangle \langle \phi_-^*, D_x \phi_-^* \rangle \end{bmatrix} = \nu_\star \sigma_3,$$

$$\begin{bmatrix} \langle \phi_+^*, W \phi_+^* \rangle \langle \phi_+^*, W \phi_-^* \rangle \\ \langle \phi_-^*, W \phi_+^* \rangle \langle \phi_-^*, W \phi_-^* \rangle \end{bmatrix} = \begin{bmatrix} 0 & \vartheta_\star \\ \vartheta_\star & 0 \end{bmatrix} \stackrel{\text{def}}{=} \sigma_\star = \text{Re}(\vartheta_\star) \sigma_1 + \text{Im}(\vartheta_\star) \sigma_2$$

where the Pauli matrices σ_j are displayed in (1.11).

Remark 2.2. In [FLW17], the potentials V and W are assumed to be even (though the proof does not really require it). This implies $\vartheta_\star \in \mathbb{R}$. In general, ϑ_\star may be complex; for instance if $V \equiv 0$, $E_\star = \pi^2$ and $W(x) = 2 \sin(2\pi x)$ then $\phi_+^*(x) = e^{i\pi x}$, $\phi_-^*(x) = e^{-i\pi x}$ and $\vartheta_\star = i \int \mathbb{R}$.

Proof. In order to prove this proposition, it suffices to derive the following identities:

$$\begin{aligned} 2 \langle \phi_-^*, D_x \phi_-^* \rangle &= -\nu_\star, & \langle \phi_-^*, D_x \phi_+^* \rangle &= \langle \phi_+^*, D_x \phi_-^* \rangle = 0, \\ \langle \phi_+^*, W \phi_-^* \rangle &= \overline{\vartheta_\star}, & \langle \phi_+^*, W \phi_+^* \rangle &= \langle \phi_-^*, D_x \phi_-^* \rangle = 0. \end{aligned} \tag{2.8}$$

The first identity in (2.8) follows from $\phi_-^* = \overline{\phi_+^*}$ (Proposition 2.1) and $\overline{D_x} = -D_x$:

$$2 \langle \phi_-^*, D_x \phi_-^* \rangle = -2 \langle \overline{\phi_+^*}, \overline{D_x \phi_+^*} \rangle = -2 \langle D_x \phi_+^*, \phi_+^* \rangle = -2 \langle \phi_+^*, D_x \phi_+^* \rangle = -\nu_\star.$$

The second identity is a consequence of $L_{\pi,e}^2 \perp L_{\pi,0}^2$, and that D_x acts on both $L_{\pi,e}^2$ and $L_{\pi,0}^2$. Indeed, these facts imply that $D_x \phi_+^*$ is orthogonal to ϕ_-^* .

The third identity follows from $W = \overline{W}$:

$$\langle \phi_+^*, W \phi_-^* \rangle = \langle W \phi_+^*, \phi_-^* \rangle = \overline{\langle \phi_-^*, W \phi_+^* \rangle} = \overline{\vartheta_\star}.$$

Finally, the last identity is a consequence of $L_{\pi,e}^2 \perp L_{\pi,0}^2$ and $W \in L_{0,0}^2$. These facts imply $\phi_+^* \perp W \phi_+^*$ and $\phi_-^* \perp W \phi_-^*$. This completes the proof. \square

3. From Quasimodes to Eigenmodes

In this section we deduce Corollary 1 from Theorem 1 by using:

- The formal multiscale expansion of [FLW17, Sect. 4];
- A general principle for gapped self-adjoint problems; see Lemma 3.1.

3.1. The multiscale approach of [FLW14b, FLW17]. We review the formal multiscale analysis of [FLW14b, FLW17]. Recall that X is the function space of two-scale functions:

$$X = \left\{ v \in C^\infty(\mathbb{R}^2, \mathbb{C}), \quad v(x+1, y) = -v(x, y), \quad \exists a > 0, \quad |v(x, y)| = O\left(e^{-a|y|}\right) \right\}.$$

For each eigenvalue ϑ_j of \mathcal{D} , the procedure of [FLW14b, FLW17] constructs recursively:

- a sequence of numbers $a_m \in \mathbb{R}$ such that $a_0 = E_\star$ and $a_1 = \vartheta_j$;

- a sequence of two-scale functions $v_m(x, y) \in X$, starting with:

$$v_0(x, y) = \alpha_+(y)\phi_+^*(x) + \alpha_-(y)\phi_-^*(x), \quad (\mathcal{D} - \vartheta_j) \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

such that the following property holds: for any positive integer M , define

$$E_{\delta, M} \stackrel{\text{def}}{=} \sum_{m=0}^M a_m \delta^m = E_\star + \vartheta_j \delta + O(\delta^2),$$

$$v_{\delta, M}(x) \stackrel{\text{def}}{=} \delta^{1/2} \sum_{m=0}^M \delta^m v_m(x, \delta x) \quad \text{with} \quad |v_{\delta, M}|_{L^2} = 1. \tag{3.1}$$

Then, for any $k \in \mathbb{N}$, there exists a constant $C_{M, k}$ such that

$$\delta \in (0, 1) \Rightarrow |(\mathcal{P}_\delta - E_{\delta, M})v_{\delta, M}|_{H^k} \leq C_{M, k} \delta^M. \tag{3.2}$$

The energy $E_{\delta, M}$ is then an approximate eigenvalue of \mathcal{P}_δ with corresponding approximate eigenstate or *quasimode* $v_{\delta, M}$.

3.2. Abstract quasimode theory. We discuss how sufficiently accurate *approximate* eigenpairs imply existence of *genuine* eigenpairs.

Lemma 3.1. *Let T be a selfadjoint operator on a Hilbert space \mathcal{H} , with domain $\text{dom}(T)$. Assume:*

- *There exist E and $\epsilon_0 > 0$ such that $[E - \epsilon_0, E + \epsilon_0] \cap \Sigma_{\text{ess}}(T)$ is empty.*
- *There exist $v \in \text{dom}(T)$ and $\epsilon \in [0, \epsilon_0)$ such that*

$$|v|_{\mathcal{H}} = 1 \quad \text{and} \quad |(T - E)v|_{\mathcal{H}} \leq \epsilon. \tag{3.3}$$

Then T has a eigenvalue λ with $|\lambda - E| \leq \epsilon$.

Furthermore, if there is a constant $C > 0$ satisfying $C\epsilon \leq \epsilon_0$ and T has no other eigenvalue in the interval $[E - C\epsilon, E + C\epsilon]$ then T has an eigenvector u with

$$|v - u|_{\mathcal{H}} \leq C^{-1}.$$

The function v is called an (ϵ -precise) quasimode. It is generally not an eigenstate, unless $\epsilon = 0$; rather, it is an approximate eigenstate. Lemma 3.1 shows that a quasimode energy E lying in an essential spectral gap of a selfadjoint operator must be ϵ -close to an eigenvalue. We postpone the proof of this well-known lemma to Appendix A.

In the next section, we will apply this lemma with parameters related to the choice of quasimode: ϵ_0 of order δ and ϵ of order δ^M , where $M \geq 2$. In particular, C will be of order δ^{1-M} and C^{-1} is of order δ^{M-1} —hence small.

3.3. *The Proof of Corollary 1 assuming Theorem 1.* Let ε and δ_0 be as in Theorem 1. Fix $M \geq 2$, $j \in [-N, N]$ and $k = 0$. Let $E_{\delta,M}$, $v_{\delta,M}$ and $C_{M,0} = C_M$ be given by (3.1) and (3.2).

We begin by applying the first part of Lemma 3.1 with the choices:

$$T = \mathcal{P}_\delta, \quad E = E_{\delta,M}, \quad \epsilon_0 = \varepsilon\delta/2, \quad \epsilon = C_M\delta^M \quad \text{and} \quad v = v_{\delta,M}. \quad (3.4)$$

By (3.2), for $\delta \in (0, 1)$, we have:

$$|(T - E)v|_{L^2} \leq \epsilon.$$

This verifies the quasimode hypothesis (3.3) of Lemma 3.1

We next verify that the choices (3.4) ensure that the spectral gap assumption of Lemma 3.1 is satisfied. Since $E_{\delta,M} = E_\star + \delta\vartheta_j + O(\delta^2)$, for δ sufficiently small,

$$[E - \epsilon_0, E + \epsilon_0] = \left[E_{\delta,M} - \frac{\varepsilon\delta}{2}, E_{\delta,M} + \frac{\varepsilon\delta}{2} \right] \subset [E_\star + \delta(\vartheta_j - \varepsilon), E_\star + \delta(\vartheta_j + \varepsilon)].$$

In particular, Theorem 1 implies that

$$\Sigma_{\text{ess}}(\mathcal{P}_\delta) \cap [E - \epsilon_0, E + \epsilon_0] = \emptyset.$$

The first conclusion of Lemma 3.1 applies and shows that for any M ,

$$\mathcal{P}_\delta \text{ has an eigenvalue } E_{\delta,j} \in [E - \epsilon, E_{\delta,M} + \epsilon] = [E_{\delta,M} - C_M\delta^M, E_{\delta,M} + C_M\delta^M].$$

Now Theorem 1 asserts that for δ sufficiently small (depending on M),

$$\mathcal{P}_\delta \text{ has exactly one eigenvalue } E_{\delta,j} \text{ in } [E_{\delta,M} - C_M\delta^M, E_{\delta,M} + C_M\delta^M].$$

Since $M \geq 1$ is arbitrary, this proves that $E_{\delta,j}$ has an expansion to arbitrary finite order in the small parameter δ . This proves the first line of (1.6) in Corollary 1.

We now turn to arbitrary finite order expansions of the eigenvector corresponding to $E_{\delta,j}$. We keep the choices (3.4) made above. Let $C = \varepsilon/(2C_M\delta^{M-1}) = \varepsilon\delta/(2\epsilon)$. Since $E_{\delta,M} = E_\star + \delta\vartheta_j + O(\delta^2)$,

$$[E - C\epsilon, E + C\epsilon] \subset [E_\star + \delta(\vartheta_j - \varepsilon), E_\star + \delta(\vartheta_j + \varepsilon)],$$

as long as δ is sufficiently small. By Theorem 1, \mathcal{P}_δ has no other eigenvalues in the set $[E - C\epsilon, E + C\epsilon]$. The additional assumption of Lemma 3.1 holds. Thus this lemma produces $u_{\delta,j}$ that satisfies

$$|v_{\delta,M} - u_{\delta,j}|_{L^2} \leq C^{-1} = \frac{2C_M\delta^{M-1}}{\varepsilon}. \quad (3.5)$$

Since M was arbitrary, we can replace M by $M + 1$ in (3.5) and conclude the proof of Corollary 1 for the case $k = 0$.

In order to complete the proof of Corollary 1, we must show that the L^2 norm in (3.5) can be replaced to a H^k -norm. Specifically, we show by induction on k that for any $k \geq 0$, there exists $C_{M,k}$ with

$$|v_{\delta,M} - u_{\delta,j}|_{H^k} \leq C_{k,M}\delta^M. \quad (3.6)$$

Without loss of generality, we carry out the induction on k even. We have verified the case $k = 0$ above. Assume now that (3.6) holds for some value of k and let us show that it also holds for $k + 2$. The main point is the elliptic estimate; for any integer s if $f \in H^s$ and $\mathcal{P}_\delta f \in H^s$, then $f \in H^{s+2}$ and there is a constant A_s such that

$$|f|_{H^{s+2}} \leq A_s (|f|_{H^s} + |\mathcal{P}_\delta f|_{H^s}).$$

The constant A_s does not depend on $\delta \in (0, 1)$ because the derivatives of the coefficients of the potential $V + \delta\kappa(\delta \cdot)W$ arising in \mathcal{P}_δ are uniformly bounded in $\delta \in (0, 1)$. Applying this estimate to $f = v_{\delta,M} - u_{\delta,j}$ with $s = k$ yields

$$\begin{aligned} |v_{\delta,M} - u_{\delta,j}|_{H^{k+2}} &\leq A(|v_{\delta,M} - u_{\delta,j}|_{H^k} + |\mathcal{P}_\delta(v_{\delta,M} - u_{\delta,j})|_{H^k}) \\ &\leq A_k (|v_{\delta,M} - u_{\delta,j}|_{H^k} + |(\mathcal{P}_\delta - E_{\delta,M})v_{\delta,M}|_{H^k} + |E_{\delta,M} - E_{\delta,j}| \cdot |u_{\delta,j}|_{H^k}). \end{aligned} \tag{3.7}$$

We used $(\mathcal{P}_\delta - E_{\delta,j})u_{\delta,j} = 0$ above. Applying and iterating the elliptic estimate for $u_{\delta,j}$ shows that there exists B_k independent of δ such that $|u_{\delta,j}|_{H^k} \leq B_k$. Using this estimate together with the recursion hypothesis, (3.2), and $|E_{\delta,M} - E_{\delta,j}| \leq C_M \delta^M$, we obtain from (3.7) that

$$|v_{\delta,M} - u_{\delta,j}|_{H^{k+2}} \leq A_k \cdot (C_{M,k} \delta^M + C_M \delta^M + B_k C_M \delta^M).$$

This completes the induction and the proof of Corollary 1.

4. Resolvent of P_δ for Small δ

Recall that $V \in C^\infty(\mathbb{R}, \mathbb{R}) \cap L^2_{0,e}$ and $W \in C^\infty(\mathbb{R}, \mathbb{R}) \cap L^2_{0,o}$. Let $j_\star \geq 1$ and consider the corresponding Dirac point $(\pi, E_\star) = (\pi, \lambda_{0,j_\star}(\pi))$ of $P_0 = D_x^2 + V$. Proposition 2.1 yields a Dirac eigenbasis $(\phi_\star^+, \phi_\star^-)$; we set

$$\nu_\star = 2 \langle \phi_\star^+, D_x \phi_\star^+ \rangle, \quad \vartheta_\star = \langle \phi_\star^-, W \phi_\star^+ \rangle, \quad \sigma_\star = \begin{bmatrix} 0 & \overline{\vartheta_\star} \\ \vartheta_\star & 0 \end{bmatrix}. \tag{4.1}$$

As mentioned above, we assume that $\vartheta_\star \neq 0$. The operator $P_\delta = D_x^2 + V + \delta W$ has a L^2 -spectral gap near E_\star , of width $2\vartheta_F \delta + O(\delta^2)$. The main result of this section is an estimate on the resolvent of P_δ when the spectral parameter spans a slightly smaller gap. We introduce the operator \mathcal{D}_+ , which is the asymptotic limit of \mathcal{D} as $x \rightarrow +\infty$:

$$\mathcal{D}_+ \stackrel{\text{def}}{=} \nu_\star \sigma_3 D_x + \sigma_\star.$$

We will also need the dilation operator \mathcal{U}_δ defined by $\mathcal{U}_\delta f(x) = \delta^{1/2} f(\delta x)$.

Theorem 2. *Let $\vartheta_\sharp \in (0, \vartheta_F)$. There exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ and $\lambda \in \mathbb{D}(E_\star, \vartheta_\sharp \delta)$, the operator $P_\delta - \lambda$ is invertible $H^2 \rightarrow L^2$ and*

$$(P_\delta - \lambda)^{-1} = S_\delta \left(\frac{\lambda - E_\star}{\delta} \right) + \mathcal{O}_{L^2} \left(\delta^{-1/3} \right), \quad D_x (P_\delta - \lambda)^{-1} = S_\delta \left(\frac{\lambda - E_\star}{\delta} \right) + \mathcal{O}_{L^2} \left(\delta^{-1/3} \right),$$

where:

$$S_\delta(z) = \frac{1}{\delta} \begin{bmatrix} \phi_\star^+ \\ \phi_\star^- \end{bmatrix}^\top \cdot \mathcal{U}_\delta (\mathcal{D}_+ - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \begin{bmatrix} \overline{\phi_\star^+} \\ \phi_\star^- \end{bmatrix}, \tag{4.2}$$

$$S_\delta^D(z) = \frac{1}{\delta} \begin{bmatrix} D_x \phi_\star^+ \\ D_x \phi_\star^- \end{bmatrix}^\top \cdot \mathcal{U}_\delta (\mathcal{D}_+ - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \begin{bmatrix} \overline{\phi_\star^+} \\ \phi_\star^- \end{bmatrix}. \tag{4.3}$$

This theorem has a corollary for $P_{-\delta}$. We will need the operator \mathcal{D}_- , which is the asymptotic limit of \mathcal{D} as $x \rightarrow -\infty$:

$$\mathcal{D}_- \stackrel{\text{def}}{=} \nu_\star \sigma_3 D_x - \sigma_\star.$$

Corollary 2. *There exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ and $\lambda \in \mathbb{D}(E_\star, \vartheta_\# \delta)$,*

$$(P_{-\delta} - \lambda)^{-1} = S_{-\delta} \left(\frac{\lambda - E_\star}{\delta} \right) + \mathcal{O}_{L^2}(\delta^{-1/3}),$$

$$D_x(P_{-\delta} - \lambda)^{-1} = S_{-\delta}^D \left(\frac{\lambda - E_\star}{\delta} \right) + \mathcal{O}_{L^2}(\delta^{-1/3})$$

where

$$S_{-\delta}(z) = \frac{1}{\delta} \begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}^\top \cdot \mathcal{U}_\delta (\mathcal{D}_- - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \begin{bmatrix} \overline{\phi_+^\star} \\ \overline{\phi_-^\star} \end{bmatrix},$$

$$S_{-\delta}^D(z) = \frac{1}{\delta} \begin{bmatrix} D_x \phi_+^\star \\ D_x \phi_-^\star \end{bmatrix}^\top \cdot \mathcal{U}_\delta (\mathcal{D}_- - z)^{-1} \mathcal{U}_\delta^{-1} \cdot \begin{bmatrix} \overline{\phi_+^\star} \\ \overline{\phi_-^\star} \end{bmatrix}.$$

The proof of Corollary 2 follows from changing W to $-W$ and applying Theorem 2. The only parameter to change is ϑ_\star , which becomes $-\vartheta_\star$. This produces the operator \mathcal{D}_- instead of \mathcal{D}_+ . The proof of Theorem 2 occupies the remainder of the section. We prove it by analyzing the resolvent $(P_\delta - \lambda)^{-1} : L^2 \rightarrow H^2$ fiberwise:

$$(P_\delta - \lambda)^{-1} = \frac{1}{2\pi} \int_{[0, 2\pi]}^\oplus d\xi (P_\delta(\xi) - \lambda)^{-1}. \tag{4.4}$$

In light of (4.4), the analysis of $(P_\delta - \lambda)^{-1}$ reduces to

- Estimates on the fiber resolvents $(P_\delta(\xi) - \lambda)^{-1}$ —Sects. 4.1–4.2;
- A procedure to integrate these estimates over $[0, 2\pi]$ – Sect. 4.3.

In Sect. 4.1, we show that momenta away from π do not contribute to the leading terms in (4.4). In other words, the dominant contributions in $(P_\delta - \lambda)^{-1}$ must arise from the Dirac point (π, E_\star) . In Sect. 4.2, we study these contributions specifically.

4.1. *Estimates on $(P_\delta(\xi) - \lambda)^{-1}$ for quasi-momenta away from π .* Let $\Sigma_{L_\xi^2}(P_\delta)$ denote the spectrum of P_δ (acting on L_ξ^2). This spectrum consists of eigenvalues

$$\lambda_{\delta,1}(\xi) \leq \dots \leq \lambda_{\delta,j}(\xi) \leq \dots,$$

listed with multiplicity. Recall that (π, E_\star) is Dirac point of P_0 —see Definition 2.2.

Proposition 4.1. *Let $\vartheta_\# \in (0, \vartheta_F)$. There exist $C > 0$ and $\delta_0 > 0$ such that*

$$\delta \in (0, \delta_0), \lambda \in \mathbb{D}(E_\star, \vartheta_\# \delta), |\xi - \pi| \geq \delta^{1/3} \Rightarrow \left\| (P_\delta(\xi) - \lambda)^{-1} \right\|_{L_\xi^2 \rightarrow H_\xi^2} \leq C \delta^{-1/3}.$$

Proof. 1. Thanks to the spectral theorem,

$$\left\| (P_\delta(\xi) - \lambda)^{-1} \right\|_{L_\xi^2} = \frac{1}{\text{dist}(\Sigma_{L_\xi^2}(P_\delta), \lambda)}. \tag{4.5}$$

Therefore a $L_\xi^2 \rightarrow L_\xi^2$ bound on $(P_\delta(\xi) - \lambda)^{-1}$ follows from a lower bound on the distance between λ and the eigenvalues of $P_\delta(\xi)$.

Let $j_\star \geq 0$ the index for which $E_\star = \lambda_{0,j_\star}(\pi) = \lambda_{0,j_\star+1}(\pi)$. We recall that

$$\begin{aligned} \lambda_{0,j_\star}(\xi) &= E_\star - |\nu_\star| \cdot |\xi - \pi| + O(\xi - \pi)^2, \\ \lambda_{0,j_\star+1}(\xi) &= E_\star + |\nu_\star| \cdot |\xi - \pi| + O(\xi - \pi)^2. \end{aligned}$$

Because of the monotonicity properties of the dispersion curves – see e.g. [RS78, Theorem XII.89]—the function $\xi \in [0, \pi] \mapsto \lambda_{0,j_\star}(\xi)$ is increasing and the function $\xi \in [\pi, 2\pi] \mapsto \lambda_{0,j_\star}(\xi)$ is decreasing; see Fig. 2. Assuming that $|\xi - \pi| \geq \delta^{1/3}$,

$$\lambda_{0,j_\star}(\xi) \leq E_\star - |\nu_\star| \cdot \delta^{1/3} + O\left(\delta^{2/3}\right). \tag{4.6}$$

Similarly, using the monotonicity properties of $\xi \mapsto \lambda_{0,j_\star+1}(\xi)$, we have

$$\lambda_{0,j_\star+1}(\xi) \geq E_\star + |\nu_\star| \cdot \delta^{1/3} + O\left(\delta^{2/3}\right). \tag{4.7}$$

Applying the stability estimate Lemma 2.1 with $\mathcal{V} = V$ and $\mathcal{W} = \delta W$ we have that if δ is sufficiently small, then

$$|\lambda_{\delta,j}(\xi) - \lambda_{0,j}(\xi)| \leq C(j)|W|_\infty \cdot \delta, \quad j \in \{j_\star, j_\star + 1\}.$$

Fix $c \in (0, |\nu_\star|)$. We deduce from (4.6) and (4.7) that for δ_0 sufficiently small,

$$\delta \in (0, \delta_0), \quad |\xi - \pi| \geq \delta^{1/3} \implies \lambda_{\delta,j_\star}(\xi) \leq E_\star - c\delta^{1/3} \text{ and } \lambda_{\delta,j_\star+1}(\xi) \geq E_\star + c\delta^{1/3}.$$

Because the dispersion curves are indexed in increasing order, this extends to all other dispersion curves: for $\delta \in (0, \delta_0)$ and $|\xi - \pi| \geq \delta^{1/3}$,

$$j \leq j_\star \implies \lambda_{\delta,j}(\xi) \leq E_\star - c\delta^{1/3}; \quad j \geq j_\star + 1 \implies \lambda_{\delta,j}(\xi) \geq E_\star + c\delta^{1/3}.$$

This yields that for $|\xi - \pi| \geq \delta^{1/2}$, $\text{dist}(\Sigma_{L_\xi^2}(P_\delta), E_\star) \geq c\delta^{1/3}$. If $\lambda \in \mathbb{D}(E_\star, \vartheta_\# \delta)$, we deduce $\text{dist}(\Sigma_{L_\xi^2}(P_\delta), \lambda) \geq c\delta^{1/3} - \vartheta_\# \delta$. Shrinking δ_0 if necessary, we get thanks to (4.5):

$$\delta \in (0, \delta_0), \quad |\lambda - E_\star| < \vartheta_\# \delta, \quad |\xi - \pi| \geq \delta^{1/3} \implies \left\| (P_\delta(\xi) - \lambda)^{-1} \right\|_{L_\xi^2} \leq C\delta^{-1/3}. \tag{4.8}$$

2. It remains to improve (4.8) by replacing the L_ξ^2 -operator norm by the H_ξ^2 -operator norm. We use an elliptic estimate, based on the inequality

$$|f|_{H_\xi^2} \leq |f|_{L_\xi^2} + \left| D_x^2 f \right|_{L_\xi^2} \leq (1 + |V|_\infty + \delta|W|_\infty + |\lambda|) \cdot |f|_{L_\xi^2} + |(P_\delta - \lambda)f|_{L_\xi^2}. \tag{4.9}$$

Since λ and δ vary within compact sets, we have for $|\xi - \pi| \geq \delta^{1/3}$ and $|\lambda - E| < \vartheta_\# \delta$:

$$\left\| (P_\delta - \lambda)^{-1} \right\|_{L_\xi^2 \rightarrow H_\xi^2} \leq C \left\| (P_\delta - \lambda)^{-1} \right\|_{L_\xi^2} + 1 \leq 2C\delta^{-1/3}.$$

This completes the proof. \square

4.2. *Estimates on $(P_\delta(\xi) - \lambda)^{-1}$ for quasi-momenta near π .* We now analyze $(P_\delta(\xi) - \lambda)^{-1}$ when $|\xi - \pi| \leq \delta^{1/3}$ and $\lambda \in \mathbb{D}(E_\star, \vartheta_\# \delta)$, with $0 < \vartheta_\# < \vartheta_F$. Recalling that (π, E_\star) is a Dirac point of P_0 , we associate eigenpairs $(\lambda_+(\xi), \phi_+(\xi))$ and $(\lambda_-(\xi), \phi_-(\xi))$ guaranteed by Proposition 2.1.

Define $\Pi_0(\xi) : L_\xi^2 \rightarrow \mathbb{C}^2$ by

$$\Pi_0(\xi)v \stackrel{\text{def}}{=} \begin{bmatrix} \langle \phi_+(\xi), v \rangle \\ \langle \phi_-(\xi), v \rangle \end{bmatrix}.$$

We introduce the matrix

$$M_\delta(\xi) \stackrel{\text{def}}{=} \begin{bmatrix} E_\star + \nu_\star(\xi - \pi) & \delta \overline{\vartheta_\star} \\ \delta \vartheta_\star & E_\star - \nu_\star(\xi - \pi) \end{bmatrix} = E_\star + \nu_\star(\xi - \pi)\sigma_3 + \delta\sigma_\star, \tag{4.10}$$

where ν_\star and σ_\star are displayed in (4.1). The eigenvalues of the matrix $M_\delta(\xi)$ are

$$E_\star \pm \sqrt{\vartheta_F^2 \delta^2 + \nu_\star^2(\xi - \pi)^2}.$$

Therefore, $M_\delta(\xi) - \lambda$ is invertible for every $\lambda \in \mathbb{D}(E_\star, \vartheta_\# \delta)$. Indeed, since $|\lambda - E_\star| < \vartheta_\# \delta$, its eigenvalues are $E_\star \pm \sqrt{\vartheta_F^2 \delta^2 + \nu_\star^2(\xi - \pi)^2} - \lambda$ and they satisfy

$$\begin{aligned} \left| E_\star \pm \sqrt{\vartheta_F^2 \delta^2 + \nu_\star^2(\xi - \pi)^2} - \lambda \right| &\geq \sqrt{\vartheta_F^2 \delta^2 + \nu_\star^2(\xi - \pi)^2} - |E_\star - \lambda| \\ &> (\vartheta_F - \vartheta_\#) \delta. \end{aligned}$$

The next proposition shows that for δ sufficiently small and (ξ, λ) near (π, E_\star) , the operator $P_\delta(\xi) - \lambda$ behaves like a 2×2 matrix $M_\delta(\xi)$.

Proposition 4.2. *Let $\vartheta_\# \in (0, \vartheta_F)$. There exists $\delta_0 > 0$ such that*

$$\begin{aligned} \delta \in (0, \delta_0), \quad |\xi - \pi| \leq 2\delta^{1/3}, \quad \lambda \in \mathbb{D}(E_\star, \vartheta_\# \delta) &\Rightarrow P_\delta(\xi) - \lambda \text{ is invertible; and} \\ (P_\delta(\xi) - \lambda)^{-1} &= \Pi_0(\xi)^\ast \cdot (M_\delta(\xi) - \lambda)^{-1} \cdot \Pi_0(\xi) + \mathcal{O}_{L_\xi^2 \rightarrow H_\xi^2}(1). \end{aligned} \tag{4.11}$$

Proof. 1. Let $V(\xi) = \mathbb{C}\phi_+(\xi) \oplus \mathbb{C}\phi_-(\xi) \subset L_\xi^2$. We write $P_\delta(\xi) - \lambda$ as a block matrix corresponding to the decomposition $L_\xi^2 = V(\xi) \oplus V(\xi)^\perp$:

$$P_\delta(\xi) - \lambda = \begin{bmatrix} A_\delta(\xi, \lambda) & B_\delta(\xi) \\ C_\delta(\xi) & D_\delta(\xi, \lambda) \end{bmatrix}. \tag{4.12}$$

Above, $A_\delta(\xi, \lambda) : V(\xi) \rightarrow V(\xi)$, $B_\delta(\xi) : V(\xi)^\perp \rightarrow V(\xi)$, $C_\delta(\xi) : V(\xi) \rightarrow V(\xi)^\perp$ and $D_\delta(\xi, \lambda) : V(\xi)^\perp \rightarrow V(\xi)^\perp$.

We will use the Schur complement Lemma to prove that $P_\delta(\xi) - \lambda$ is invertible and to get a useful formula for the inverse.

Lemma 4.1. *Suppose given an operator in block form*

$$\mathcal{M} \stackrel{\text{def}}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

such that the operators A and $E = A - BD^{-1}C$ are invertible (in their appropriate spaces). Then \mathcal{M} is invertible and its inverse is given by

$$\mathcal{M}^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}CE^{-1} & D^{-1} + D^{-1}CE^{-1}BD^{-1} \end{bmatrix}.$$

From (4.12), we conclude that $P_\delta(\xi) - \lambda$ is invertible if under the conditions of (4.11):

$$A_\delta(\xi, \lambda) : V(\xi) \rightarrow V(\xi) \text{ is invertible;}$$

$$E_\delta(\xi, \lambda) \stackrel{\text{def}}{=} D_\delta(\xi, \lambda) - C_\delta(\xi)A_\delta(\xi, \lambda)^{-1}B_\delta(\xi) : V(\xi)^\perp \rightarrow V(\xi)^\perp \text{ is invertible.}$$

We next verify these conditions in Steps 2 and 3.

2. Let $J_0(\xi) : V(\xi) \rightarrow \mathbb{C}^2$ be the coordinate map in the basis $\{\phi_+(\xi), \phi_-(\xi)\}$. In other words, $J_0(\xi)$ is the restriction of $\Pi_0(\xi)$ to L_ξ^2 . Thus, $J_0(\xi)^*$ maps a pair $(\alpha, \beta) \in \mathbb{C}^2$ to $\alpha\phi_+(\xi) + \beta\phi_-(\xi) \in V(\xi)$ and we have

$$\begin{aligned} A_\delta(\xi, \lambda) &= J_0(\xi)^*(P_\delta(\xi) - \lambda)J_0(\xi) \\ &= J_0(\xi)^* \begin{bmatrix} \lambda_+(\xi) - \lambda & \delta \langle \phi_+(\xi), W\phi_-(\xi) \rangle \\ \delta \langle \phi_-(\xi), W\phi_+(\xi) \rangle & \lambda_-(\xi) - \lambda \end{bmatrix} J_0(\xi). \end{aligned}$$

We show that $A_\delta(\xi, \lambda)$ is invertible. We recall that $\lambda_\pm(\xi) = E_\star \pm \nu_\star(\xi - \pi) + O(\xi - \pi)^2$ and we observe that by Propositions 2.1,

$$\delta \langle \phi_-(\xi), W\phi_+(\xi) \rangle = \delta \langle \phi_-^*, W\phi_+^* \rangle + O(\delta(\xi - \pi)) = \delta \vartheta_\star + O(\delta^2 + (\xi - \pi)^2).$$

Similarly, $\delta \langle \phi_+(\xi), W\phi_-(\xi) \rangle = \delta \overline{\vartheta_\star} + O(\delta^2 + (\xi - \pi)^2)$. Hence

$$A_\delta(\xi, \lambda) = J_0(\xi)^*(M_\delta(\xi) - \lambda)J_0(\xi) + \mathcal{O}_{V(\xi)}((\xi - \pi)^2 + \delta^2)$$

where $M_\delta(\xi)$ is displayed in (4.10).

In order to prove that $A_\delta(\xi, \lambda)$ is invertible when the conditions on ξ and λ in (4.11) are satisfied, we first observe that

$$\|(M_\delta(\xi) - \lambda)^{-1}\|_{\mathbb{C}^2} = O((\delta + |\xi - \pi|)^{-1}). \tag{4.13}$$

The matrix $M_\delta(\xi)$ is hermitian. Hence, a bound on $(M_\delta(\xi) - \lambda)^{-1}$ follows from an estimate on the distance between λ and the spectrum of $M_\delta(\xi)$. The eigenvalues of $M_\delta(\xi) - \lambda$ are $E_\star \pm \sqrt{\nu_\star^2(\xi - \pi)^2 + \delta^2\vartheta_F^2} - \lambda$. Using that for all $t \in (0, 1)$, $\sqrt{a^2 + b^2} \geq t|a| + (1-t)|b|$, we have :

$$\begin{aligned} \left| E_\star \pm \sqrt{\nu_\star^2(\xi - \pi)^2 + \delta^2\vartheta_F^2} - \lambda \right| &\geq \sqrt{\nu_\star^2(\xi - \pi)^2 + \vartheta_F^2\delta^2} - |\lambda - E_\star| \\ &\geq (1-t)\nu_\star|\xi - \pi| + t\vartheta_F\delta - \vartheta_\pm\delta \geq (1-t)\nu_\star|\xi - \pi| + \vartheta_F \left(t - \frac{\vartheta_\pm}{\vartheta_F} \right) \delta. \end{aligned}$$

Choosing t to satisfy $\frac{\vartheta_\pm}{\vartheta_F} < t < 1$, we have that under the conditions of (4.11), $M_\delta(\xi) - \lambda$ is invertible with the bound (4.13).

Because of (4.13) and a perturbative argument (based on a Neumann series), under the conditions of (4.11) $A_\delta(\xi, \lambda)$ is invertible and

$$\begin{aligned} A_\delta(\xi, \lambda)^{-1} &= J_0(\xi)^*(M_\delta(\xi) - \lambda)^{-1}J_0(\xi) + \mathcal{O}_{V(\xi)}\left(\frac{(\xi - \pi)^2 + \delta^2}{(\delta + |\xi - \pi|)^2}\right) \\ &= J_0(\xi)^*(M_\delta(\xi) - \lambda)^{-1}J_0(\xi) + \mathcal{O}_{V(\xi)}(1). \end{aligned}$$

In particular $A_\delta(\xi, \lambda)^{-1} = \mathcal{O}_{V(\xi)}((\delta + |\xi - \pi|)^{-1})$.

3. The operator $B_\delta(\xi) : V(\xi)^\perp \rightarrow V(\xi)$ is the composition of (i) the embedding $V(\xi)^\perp \rightarrow L^2_\xi$; (ii) the operator $P_\delta = P_0 + \delta W$; and (iii) the projection $L^2_\xi \rightarrow V(\xi)$. Note that $B_0(\xi) = 0$. This proves that $B_\delta(\xi)$ is more simply the composition of (i); the multiplication operator by δW ; and (iii). In particular $B_\delta(\xi) = \mathcal{O}_{V(\xi)^\perp \rightarrow V(\xi)}(\delta)$. Since $C_\delta(\xi) = B_\delta(\xi)^*$, we deduce have $C_\delta(\xi) = \mathcal{O}_{V(\xi) \rightarrow V(\xi)^\perp}(\delta)$.

The bounds obtained in Step 2 imply that

$$\begin{aligned} E(\xi, \lambda) &= D_\delta(\xi, \lambda) - C_\delta(\xi)A_\delta(\xi, \lambda)^{-1}B_\delta(\xi) = D_\delta(\xi, \lambda) + \mathcal{O}_{V(\xi)}(\delta) \\ &= (P_0(\xi) - \lambda)|_{V(\xi)^\perp} + \mathcal{O}_{V(\xi)^\perp}(\delta), \end{aligned} \tag{4.14}$$

where $(P_0(\xi) - \lambda)|_{V(\xi)^\perp}$ acts $V(\xi)^\perp \rightarrow V(\xi)^\perp$. In particular, $(P_0(\xi) - \lambda)|_{V(\xi)^\perp}$ has spectrum equal to $\Sigma_{L^2_\xi}(P_0) \setminus \{\lambda_{0,j_\star}(\xi), \lambda_{0,j_\star+1}(\xi)\}$.

Because of the monotonicity properties of dispersion curves [RS78, Theorem XIII.89], for every $\xi \in [0, 2\pi]$,

$$\lambda_{0,j_\star+2}(\xi) \geq \lambda_{0,j_\star+2}(0) \geq \lambda_{0,j_\star+1}(0) > \lambda_{0,j_\star+1}(\pi) = E_\star.$$

Hence, $\lambda_{0,j_\star+2}(\xi) > E_\star$ and similarly we can show that $\lambda_{0,j_\star-1}(\xi) < E_\star$. It follows that for a constant $c > 0$ and for any $j \notin \{j_\star, j_\star + 1\}$, $|\lambda_{0,j}(\xi) - E_\star| \geq c$. Because $\lambda \in \mathbb{D}(E_\star, \vartheta_\pi \delta)$, we deduce that

$$j \notin \{j_\star, j_\star + 1\} \Rightarrow |\lambda_{0,j}(\xi) - \lambda| \geq c - O(\delta).$$

Therefore $\left((P_0(\xi) - \lambda)|_{V(\xi)^\perp}\right)^{-1}$ exists and has norm $\mathcal{O}_{V(\xi)^\perp}(1)$. From (4.14), a Neumann series argument shows that $E_\delta(\xi, \lambda)$ is invertible with

$$E_\delta(\xi, \lambda)^{-1} = \left((P_0(\xi) - \lambda)|_{V(\xi)^\perp}\right)^{-1} + \mathcal{O}_{V(\xi)^\perp}(\delta) = \mathcal{O}_{V(\xi)^\perp \rightarrow V(\xi)^\perp}(1),$$

as long as the conditions on ξ and λ of (4.11) are satisfied.

4. By Lemma 4.1 and the previous bounds, when the conditions of (4.11) are fulfilled, the inverse of $P_\delta(\xi) - \lambda$ exists and satisfies

$$\begin{aligned} (P_\delta(\xi) - \lambda)^{-1} &= \begin{bmatrix} A_\delta(\xi, \lambda)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{O}_{L^2_\xi}(1) \\ &= \Pi_0(\xi)^* \cdot (M_\delta(\xi) - \lambda)^{-1} \cdot \Pi_0(\xi) + \mathcal{O}_{L^2_\xi}(1). \end{aligned} \tag{4.15}$$

5. Thanks to the elliptic estimate (4.9), the expansion (4.15) can be improved to one with an error term which is $\mathcal{O}_{L^2_\xi \rightarrow H^2_\xi}(1)$. The proof of Proposition 4.2 is complete. \square

4.3. *Expansion of $(P_\delta - \lambda)^{-1}$, near $\lambda = E_\star$.* By the Floquet–Bloch reconstruction formula (2.1) we have:

$$(P_\delta - \lambda)^{-1} = \frac{1}{2\pi} \int_{[0,2\pi]}^\oplus d\xi (P_\delta(\xi) - \lambda)^{-1}.$$

Proposition 4.2 implies that the dominant contributions to $(P_\delta(\xi) - \lambda)^{-1}$ are encoded in the RHS of (4.11). Thus we separately consider contributions coming from quasi-momenta near and away from $\xi = \pi$. Consider a function $\chi \in C_0^\infty(\mathbb{R})$ such that

$$\text{supp}(\chi) \subset (-2, 2), \quad \chi(x) = 1 \text{ for } x \in [-1, 1].$$

Then,

$$\begin{aligned} (P_\delta - \lambda)^{-1} &= \frac{1}{2\pi} \int_{[0,2\pi]}^\oplus d\xi \chi\left(\frac{\xi - \pi}{\delta^{1/3}}\right) \cdot (P_\delta(\xi) - \lambda)^{-1} \\ &\quad + \frac{1}{2\pi} \int_{[0,2\pi]}^\oplus d\xi \left(1 - \chi\left(\frac{\xi - \pi}{\delta^{1/3}}\right)\right) \cdot (P_\delta(\xi) - \lambda)^{-1}. \end{aligned}$$

By Proposition 4.1, the second integral satisfies the bound:

$$\left\| \frac{1}{2\pi} \int_{[0,2\pi]}^\oplus d\xi \left(1 - \chi\left(\frac{\xi - \pi}{\delta^{1/3}}\right)\right) \cdot (P_\delta(\xi) - \lambda)^{-1} \right\|_{L_\xi^2 \rightarrow H_\xi^2} \leq C\delta^{-1/3}.$$

By Proposition 4.2, the first integral may be expanded as:

$$\frac{1}{2\pi} \int_{[0,2\pi]}^\oplus d\xi \chi\left(\frac{\xi - \pi}{\delta^{1/3}}\right) \cdot (P_\delta(\xi) - \lambda)^{-1} = R_\delta(\lambda) + \mathcal{O}_{L_\xi^2 \rightarrow H_\xi^2}(1),$$

where $R_\delta(\lambda)$ is the operator acting on L^2 given by

$$R_\delta(\lambda) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{[0,2\pi]}^\oplus d\xi \chi\left(\frac{\xi - \pi}{\delta^{1/3}}\right) \cdot \Pi_0(\xi)^* (M_\delta(\xi) - \lambda)^{-1} \Pi_0(\xi).$$

Therefore,

$$(P_\delta - \lambda)^{-1} = R_\delta(\lambda) + \mathcal{O}_{L_\xi^2 \rightarrow H_\xi^2}(\delta^{-1/3}). \tag{4.16}$$

Theorem 2 follows from the following result:

Proposition 4.3. *Let $0 < \vartheta_\# < \vartheta_F$. There exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$ and $z \in \mathbb{D}(0, \vartheta_\#)$, then the operator $R_\delta(\lambda)$ satisfies*

$$R_\delta(E_\star + z\delta) = S_\delta(z) + \mathcal{O}_{L^2}(\delta^{-1/3}) \tag{4.17}$$

$$D_x R_\delta(E_\star + z\delta) = S_\delta^D(z) + \mathcal{O}_{L^2}(\delta^{-1/3}), \tag{4.18}$$

where the operators $S_\delta(z)$ and $S_\delta^D(z)$ are displayed in (4.2) and (4.3).

Proof of Proposition 4.3. Step 1. We begin with the proof of (4.17). The proof of (4.18) is very similar; we explain it in Step 4. The operator $R_\delta(\lambda)$ has kernel given by

$$R_\delta(x, y; \lambda) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{\mathbb{R}} \begin{bmatrix} \phi_+(\xi, x) \\ \phi_-(\xi, x) \end{bmatrix}^\top \cdot \chi \left(\frac{\xi - \pi}{\delta^{1/3}} \right) (M_\delta(\xi) - \lambda)^{-1} \cdot \overline{\begin{bmatrix} \phi_+(\xi, y) \\ \phi_-(\xi, y) \end{bmatrix}} d\xi. \tag{4.19}$$

Making the change of variables $\xi \mapsto \pi + \delta\xi$ and hence $d\xi \mapsto \delta d\xi$ in (4.19) shows that the kernel $R_\delta(x, y; \lambda)$ equals

$$\frac{\delta}{2\pi} \int_{\mathbb{R}} \chi \left(\delta^{2/3} \xi \right) \begin{bmatrix} \phi_+(\pi + \delta\xi, x) \\ \phi_-(\pi + \delta\xi, x) \end{bmatrix}^\top \cdot (M_\delta(\pi + \delta\xi) - \lambda)^{-1} \cdot \overline{\begin{bmatrix} \phi_+(\pi + \delta\xi, y) \\ \phi_-(\pi + \delta\xi, y) \end{bmatrix}} d\xi. \tag{4.20}$$

Introduce the rescaled energy parameter $z = \frac{\lambda - E_\star}{\delta} \in \mathbb{D}(0, \vartheta_\pm)$. By the definition of $M_\delta(\xi)$ —see (4.10)—we have:

$$(M_\delta(\pi + \delta\xi) - \lambda)^{-1} = \frac{1}{\delta} \begin{bmatrix} \nu_\star \xi - z & \overline{\vartheta_\star} \\ \vartheta_\star & -\nu_\star \xi - z \end{bmatrix}^{-1} = \frac{1}{\delta} (\mathcal{D}_+(\xi) - z)^{-1},$$

$$\mathcal{D}_+(\xi) \stackrel{\text{def}}{=} \nu_\star \xi \sigma_3 + \sigma_\star \in M_2(\mathbb{C}).$$

Substituting in (4.20) shows that $R_\delta(x, y; E_\star + z\delta)$ equals

$$\frac{1}{2\pi} \int_{\mathbb{R}} \begin{bmatrix} \phi_+(\pi + \delta\xi, x) \\ \phi_-(\pi + \delta\xi, x) \end{bmatrix}^\top \cdot \chi \left(\delta^{2/3} \xi \right) (\mathcal{D}_+(\xi) - z)^{-1} \cdot \overline{\begin{bmatrix} \phi_+(\pi + \delta\xi, y) \\ \phi_-(\pi + \delta\xi, y) \end{bmatrix}} d\xi. \tag{4.21}$$

To deal with the integrand in (4.21), we define $p_\pm(\xi)$, p_\pm^\star and $q_\pm(\xi)$ via:

$$\phi_\pm(\xi, x) \stackrel{\text{def}}{=} e^{i\xi x} p_\pm(\xi, x); \quad p_\pm^\star(x) \stackrel{\text{def}}{=} p_\pm(\pi, x); \quad p_\pm(\pi + \xi, x) - p_\pm^\star(x) \stackrel{\text{def}}{=} e^{-i\pi x} q_\pm(\xi, x).$$

A direct manipulation implies $\phi_\pm(\pi + \delta\xi, x) = e^{i\delta\xi x} \cdot (\phi_\pm^\star(x) + q_\pm(\delta\xi, x))$. Hence, the integrand in (4.21) becomes

$$e^{i\delta\xi(x-y)} \begin{bmatrix} \phi_+^\star(x) + q_+(\delta\xi, x) \\ \phi_-^\star(x) + q_-(\delta\xi, x) \end{bmatrix}^\top \cdot \chi \left(\delta^{2/3} \xi \right) (\mathcal{D}_+(\xi) - z)^{-1} \cdot \overline{\begin{bmatrix} \phi_+^\star(y) + q_+(\delta\xi, y) \\ \phi_-^\star(y) + q_-(\delta\xi, y) \end{bmatrix}}. \tag{4.22}$$

Observe that $\chi(\delta\xi)\chi(\delta^{2/3}\xi) = \chi(\delta^{2/3}\xi)$ —as long as δ is sufficiently small. Therefore, inserting cutoffs $\chi(\delta\xi)$ leaves (4.22) unchanged. It now writes as

$$e^{i\delta\xi(x-y)} \sum_{j,k=0}^1 f_j(\delta\xi, x)^\top \cdot \chi \left(\delta^{2/3} \xi \right) (\mathcal{D}_+(\xi) - z)^{-1} \cdot \overline{f_k(\delta\xi, y)},$$

$$f_0(\xi, x) \stackrel{\text{def}}{=} \chi(\xi) \begin{bmatrix} \phi_+^\star(x) \\ \phi_-^\star(x) \end{bmatrix}; \quad f_1(\xi, y) \stackrel{\text{def}}{=} \chi(\xi) \begin{bmatrix} q_+(\xi, y) \\ q_-(\xi, y) \end{bmatrix}. \tag{4.23}$$

The leading order term in (4.23) corresponds to $(j, k) = (0, 0)$. Coming back to (4.21), we deduce that

$$R_\delta(x, y; E_\star + \delta z) = R_\delta^0(x, y; z) + \mathcal{E}_\delta(x, y; z)$$

where

$$R_\delta^0(x, y; z) \stackrel{\text{def}}{=} \begin{bmatrix} \phi_+^*(x) \\ \phi_-^*(x) \end{bmatrix}^\top \cdot \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \cdot \chi(\delta^{2/3}\xi) (\mathcal{D}_+(\xi) - z)^{-1} \cdot d\xi \cdot \begin{bmatrix} \phi_+^*(y) \\ \phi_-^*(y) \end{bmatrix}$$

$$\mathcal{E}_\delta(x, y; z) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \sum_{(j,k) \neq (0,0)} f_j(\delta\xi, x)^\top \cdot \chi(\delta^{2/3}\xi) (\mathcal{D}_+(\xi) - z)^{-1} \cdot \overline{f_k(\delta\xi, y)} \cdot d\xi.$$

In Step 2 we will prove that $\mathcal{E}_\delta(x, y; z)$ defines an operator $\mathcal{E}_\delta(z)$ with $\mathcal{E}_\delta(z) = \mathcal{O}_{L^2}(\delta^{-1/3})$ and in Step 3 we will show that since the function $\chi(\delta^{2/3} \cdot)$ has slow variations, we can to replace it by 1. These considerations will imply Proposition 4.3.

Step 2. Motivated by the form error term $\mathcal{E}_\delta(z)$ we introduce a class of kernels. Let $\Phi \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{C})$ independent of δ be such that:

- (i) Φ is bounded together with all its derivatives;
- (ii) $\Phi(0, x, y) = 0$ for every $x, y \in \mathbb{R}$.

Let $s_\delta : \mathbb{R} \rightarrow \mathbb{C}$ be a symbol such that

$$\text{supp}(s_\delta) \subset [-2\delta^{-2/3}, 2\delta^{-2/3}]; \quad \text{for all } \ell, \quad \sup_{\xi \in \mathbb{R}, 0 < \delta < 1} |\langle \xi \rangle^{\ell+1} D_\xi^\ell s_\delta(\xi)| < \infty. \quad (4.24)$$

An operator bound on $\mathcal{E}_\delta(z)$ is a consequence of:

Lemma 4.2. *Let A denote the operator with kernel*

$$A(x, y) = \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \cdot s_\delta(\xi) \cdot \Phi(\delta\xi, x, y) d\xi,$$

where s_δ and Φ are as above. Then $A = \mathcal{O}_{L^2}(\delta^{-1/3})$.

Proof. By Schur’s test—see e.g. [So93, Theorem 0.3.1]—it suffices to show that

$$\sup_{y \in \mathbb{R}} \int_{\mathbb{R}} |A(x, y)| dx + \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |A(x, y)| dy = \mathcal{O}(\delta^{-1/3}).$$

We bound the first term; the second term is treated similarly. Note that

$$\int_{\mathbb{R}} |A(x, y)| dx = \int_{\mathbb{R}} |A(x+y, y)| dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{i\delta\xi x} \cdot s_\delta(\xi) \cdot \Phi(\delta\xi, x+y, y) \cdot d\xi \right| dx.$$

We first look at the integral for $|x| \leq L$, where $L > 0$ is to be determined later. Note that since $\Phi(0, \cdot) \equiv 0$ and $\Phi(\xi, \cdot)$ has bounded derivatives, there exists a constant C such that

$$(\xi, x, y) \in \mathbb{R}^3 \Rightarrow |\Phi(\xi, x, y)| \leq C|\xi|. \quad (4.25)$$

Using that $s_\delta \in S^{-1}$, the support condition (4.24), and the estimate (4.25),

$$\int_{|x| \leq L} |A(x, y)| dx \leq \int_{|x| \leq L} \int_{|\xi| \leq 2\delta^{-2/3}} C|\xi|^{-1} \delta\xi \cdot d\xi = \mathcal{O}(L\delta^{1/3}).$$

To we deal with the integral over $|x| \geq L$ we take advantage of the oscillatory integrand. Integration by parts gives:

$$\begin{aligned} & \int_{|x| \geq L} \left| \int_{\mathbb{R}} e^{i\delta\xi x} s_{\delta}(\xi) \Phi(\delta\xi, x + y, y) d\xi \right| dx \\ &= \int_{|x| \geq L} \left| \int_{\mathbb{R}} \frac{(D_{\xi}^2 e^{i\delta\xi x})}{\delta^2 x^2} s_{\delta}(\xi) \Phi(\delta\xi, x + y, y) d\xi \right| dx \\ &= \int_{|x| \geq L} \frac{1}{\delta^2 x^2} \left| \int_{\mathbb{R}} e^{i\delta\xi x} D_{\xi}^2 (s_{\delta}(\xi) \Phi(\delta\xi, x + y, y)) d\xi \right| dx \\ &\leq \int_{|x| \geq L} \frac{1}{\delta^2 x^2} \int_{\mathbb{R}} \left| D_{\xi}^2 (s_{\delta}(\xi) \cdot \Phi(\delta\xi, x + y, y)) \right| d\xi dx \\ &\leq \int_{|x| \geq L} \frac{C}{\delta^2 x^2} \int_{[-2\delta^{-2/3}, 2\delta^{-2/3}]} \left(\left| \langle \xi \rangle^{-3} \delta \xi \right| + \left| \langle \xi \rangle^{-2} \delta \right| + \left| \langle \xi \rangle^{-1} \delta^2 \right| \right) d\xi dx. \end{aligned}$$

In the last inequality, we used (4.25), the support condition (4.24), the estimates $D_{\xi}(\Phi(\delta\xi, \cdot)) = O(\delta)$ and $D_{\xi}^2(\Phi(\delta\xi, \cdot)) = O(\delta^2)$. We deduce that

$$\int_{|x| \geq L} \left| \int_{\mathbb{R}} e^{i\delta\xi x} \cdot s_{\delta}(\xi) \cdot \Phi(\delta\xi, x + y, y) d\xi \right| dx \leq \frac{C}{\delta^2 L} \cdot (\delta - \delta^2 \ln(\delta)) = O\left((\delta L)^{-1}\right).$$

Optimize the two bounds by picking $L = \delta^{-2/3}$ and get the bound $O(\delta^{-1/3})$. This concludes the proof. \square

The operator $\mathcal{E}_{\delta}(z)$ —with kernel $\mathcal{E}_{\delta}(x, y; z)$ —is a finite sum of operators handled by Lemma 4.2. The functions s_{δ} are among coefficients of the matrix $\chi(\delta^{2/3}\xi)(\mathcal{D}_{+}(\xi) - z)^{-1}$: they satisfy (4.24); the $\Phi(\xi, x, y)$ are among the coefficients of the matrix $f_k(\xi, y) \cdot f_j(\xi, x)^{\top}$ for some $(j, k) \neq (0, 0)$ – in particular (ii) holds. We deduce that $\mathcal{E}_{\delta}(z) = \mathcal{O}_{L^2}(\delta^{-1/3})$.

Step 3. So far we have shown that $R_{\delta}(E_{\star} + \delta z) = R_{\delta}^0(z) + \mathcal{O}_{L^2}(\delta^{-1/3})$ where the kernel of $R_{\delta}^0(z)$ is given in Step 1. We may write

$$R_{\delta}^0(z) = \begin{bmatrix} \phi_{+}^{\star} \\ \phi_{-}^{\star} \end{bmatrix}^{\top} \cdot \mathcal{D}_{\delta}(z) \cdot \overline{\begin{bmatrix} \phi_{+}^{\star} \\ \phi_{-}^{\star} \end{bmatrix}} + \mathcal{O}_{L^2}(\delta^{-1/3}) \tag{4.26}$$

where the Fourier multiplier $\mathcal{D}_{\delta}(z)$ has kernel

$$\mathcal{D}_{\delta}(x, y; z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \cdot \chi\left(\delta^{2/3}\xi\right) (\mathcal{D}_{+}(\xi) - z)^{-1} \cdot d\xi.$$

In terms of \mathcal{U}_{δ} —given by $\mathcal{U}_{\delta} f(x) \stackrel{\text{def}}{=} \delta^{1/2} f(\delta x)$ —we obtain

$$\left(\mathcal{U}_{\delta}^{-1} \mathcal{D}_{\delta}(\cdot, \cdot; z) \mathcal{U}_{\delta}\right)(x, y) = \frac{1}{2\pi \delta} \int_{\mathbb{R}} e^{i\xi(x-y)} \chi\left(\delta^{2/3}\xi\right) \cdot (\mathcal{D}_{+}(\xi) - z)^{-1} \cdot d\xi.$$

In particular, $\mathcal{U}_{\delta}^{-1} \mathcal{D}_{\delta}(z) \mathcal{U}_{\delta}$ is a Fourier multiplier with symbol $\delta^{-1} \cdot \chi\left(\delta^{2/3}\xi\right) (\mathcal{D}_{+}(\xi) - z)^{-1}$. Split this symbol as

$$\delta^{-1} \cdot (\mathcal{D}_{+}(\xi) - z)^{-1} + \delta^{-1} \cdot \left(\chi\left(\delta^{2/3}\xi\right) - 1\right) (\mathcal{D}_{+}(\xi) - z)^{-1}.$$

The second term has L^∞ -norm controlled by $\delta^{-1/3}$, therefore it is associated with a Fourier multiplier that is $\mathcal{O}_{L^2}(\delta^{-1/3})$. Therefore,

$$\mathcal{D}_\delta(z) = \mathcal{U}_\delta \cdot (\mathcal{D} - z)^{-1} \cdot \mathcal{U}_\delta^{-1} + \mathcal{O}_{L^2}(\delta^{-1/3}).$$

Thanks to (4.26), we deduce (4.17).

Step 4. We indicate the steps to prove (4.18). We first write the kernel of $D_x R_\delta$ as in (4.19):

$$(D_x R_\delta)(x, y; z) = \frac{1}{2\pi} \int_{\mathbb{R}} \left[\begin{matrix} (D_x \phi_+)(\xi, x) \\ (D_x \phi_-)(\xi, x) \end{matrix} \right]^\top \cdot \chi \left(\frac{\xi - \pi}{\delta^{1/3}} \right) (M_\delta(\xi) - \lambda)^{-1} \cdot \begin{bmatrix} \overline{\phi_+(\xi, y)} \\ \phi_-(\xi, y) \end{bmatrix} \cdot d\xi.$$

As in Step 1, we deduce (after setting $\xi = \pi + \delta\xi$) that $D_x R_\delta$ has kernel

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \sum_{j,k=0}^1 [\delta\xi \cdot f_j(\delta\xi, x)]^\top \cdot \chi(\delta^{2/3}\xi) (\mathcal{D}_+(\xi) - z)^{-1} \cdot \overline{f_k(\delta\xi, y)} \cdot d\xi \tag{4.27}$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \sum_{j,k=0}^1 (D_x f_j)(\delta\xi, x)^\top \cdot \chi(\delta^{2/3}\xi) (\mathcal{D}_+(\xi) - z)^{-1} \cdot \overline{f_k(\delta\xi, y)} \cdot d\xi \tag{4.28}$$

We introduce $g_j = \xi \cdot f_j$ so that (4.27) reduces to

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \sum_{j,k=0}^1 g_j(\delta\xi, x)^\top \cdot \chi(\delta^{2/3}\xi) (\mathcal{D}_+(\xi) - z)^{-1} \cdot \overline{f_k(\delta\xi, y)} \cdot d\xi.$$

As in Step 2, this kernel is a finite sum of operators handled by Lemma 4.2. The functions s_δ are among coefficients of the matrix $\chi(\delta^{2/3}\xi) (\mathcal{D}_+(\xi) - z)^{-1}$: they satisfy (4.24); the $\Phi(\xi, x, y)$ are among the coefficients of the matrix $\overline{f_k(\xi, y)} \cdot g_j(\xi, x)^\top$. Note that (i) holds; since $g_j(0, x) = 0$, (ii) holds. We deduce that (4.27) is the kernel of an operator $\mathcal{O}_{L^2}(\delta^{-1/3})$.

We now look at (4.27). In order to control the terms associated with $(j, k) \neq (0, 0)$, we can apply the same analysis as in the end of Step 2. We deduce that the kernel

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \sum_{(j,k) \neq (0,0)} (D_x f_j)(\delta\xi, x)^\top \cdot \chi(\delta^{2/3}\xi) (\mathcal{D}_+(\xi) - z)^{-1} \cdot \overline{f_k(\delta\xi, y)} \cdot d\xi$$

represents an operator that is $\mathcal{O}_{L^2}(\delta^{-1/3})$. Hence, the leading term in (4.27) corresponds to $j = k = 0$: it has kernel

$$\left[\begin{matrix} D_x \phi_+^*(x) \\ D_x \phi_-^*(x) \end{matrix} \right]^\top \cdot \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\delta\xi(x-y)} \cdot \chi \left(\delta^{2/3}\xi \right) (\mathcal{D}_+(\xi) - z)^{-1} \cdot d\xi \cdot \begin{bmatrix} \overline{\phi_+^*(y)} \\ \phi_-^*(y) \end{bmatrix}.$$

The same procedure as in Step 3 allows to replace χ by 1; and identify the resulting kernel with the operator on the RHS of (4.18). \square

Proposition 4.3 together with the estimate (4.16) imply Theorem 2 for P_δ .

5. Construction of a Parametrix

We are interested in the eigenvalues of the operator

$$\mathcal{P}_\delta = D_x^2 + V + \delta \cdot \kappa(\delta \cdot) \cdot W$$

that lie in the spectral gap about the Dirac energy E_\star . The operator \mathcal{P}_δ has both point and essential spectrum. Its essential spectrum is determined by the asymptotic operators $P_{\pm\delta}$, expanded in the previous section. It is natural to compose \mathcal{P}_δ with a bounded operator $Q_\delta(\lambda)$ (a *parametrix*, constructed using $P_{\pm\delta}$) which effectively “divides out” the essential spectrum. The resulting operator is of the form $\text{Id} + K_\delta(\lambda)$, where $K_\delta(\lambda)$ is compact and depends analytically on λ . The eigenvalues of \mathcal{P}_δ are among the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$.

Our strategy in Sects. 5 and 6 is:

- Construct explicitly $Q_\delta(\lambda)$ and $K_\delta(\lambda)$ from the asymptotic operators $P_{\pm\delta}$;
- Apply Theorem 2 to derive an expansion of $K_\delta(\lambda)$ for small δ ;
- Use the expansion of $K_\delta(\lambda)$ to locate and count the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$.

5.1. Analytic Fredholm theory. We briefly review analytic Fredholm theory. We refer to [DZ18, Appendix C]. Let $U \subset \mathbb{C}$ be open and \mathcal{H} be a Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the space of bounded operators on \mathcal{H} . A family $\zeta \in U \mapsto T(\zeta) \in \mathcal{B}(\mathcal{H})$ is holomorphic (resp. meromorphic) on U if for every $u, v \in \mathcal{H}$, the function

$$\zeta \in U \mapsto \langle u, T(\zeta)v \rangle_{\mathcal{H}} \in \mathbb{C}$$

is holomorphic (resp. meromorphic).

We recall that a bounded operator $T \in \mathcal{B}(\mathcal{H})$ is Fredholm if both its kernel and cokernel are finite dimensional. We will use the following result:

Lemma 5.1. [DZ18, Theorem C.5] *Let $\zeta \in U \mapsto T(\zeta) \in \mathcal{B}(\mathcal{H})$ be a holomorphic family of Fredholm operators. If there exists $\zeta_0 \in U$ such that $T(\zeta_0)$ is invertible, then $\zeta \in U \mapsto T(\zeta)^{-1}$ (initially defined in a small neighborhood of ζ_0) extends to a meromorphic family of operators with poles of finite rank.*

5.2. Parametrix. We define resonances for the operator \mathcal{P}_δ . We look for a parametrix of $\mathcal{P}_\delta - \lambda$, i.e. an operator $Q_\delta(\lambda) : L^2 \rightarrow H^2$ such that for $\lambda \in \mathbb{D}(E_\star, \vartheta_\sharp \delta)$,

$$(\mathcal{P}_\delta - \lambda)Q_\delta(\lambda) - \text{Id} \stackrel{\text{def}}{=} K_\delta(\lambda) \text{ is a compact operator on } L^2. \tag{5.1}$$

This requires $Q_\delta(\lambda)$ to be equal to the asymptotic resolvents $(P_{\pm\delta} - \lambda)^{-1}$ near $x = \pm\infty$. Since $\kappa(x) \rightarrow \pm 1$ as $x \rightarrow \pm\infty$, a natural parametrix is

$$Q_\delta(\lambda) = \frac{1 + \kappa_\delta}{2}(P_\delta - \lambda)^{-1} + \frac{1 - \kappa_\delta}{2}(P_{-\delta} - \lambda)^{-1}, \quad \kappa_\delta(x) = \kappa(\delta x).$$

For $\vartheta_\sharp \in (0, \vartheta_F)$, and for sufficiently small positive δ , $Q_\delta(\lambda)$ is well-defined for $\lambda \in \mathbb{C}^+ \cup \mathbb{D}(E_\star, \vartheta_\sharp \delta)$ because $P_\delta - \lambda$ is invertible for λ in this set—see Theorem 2. We next verify that with this choice of $Q_\delta(\lambda)$, assertion (5.1) holds for $\lambda \in \mathbb{C}^+ \cup \mathbb{D}(E_\star, \vartheta_\sharp \delta)$:

$$\begin{aligned}
 (\mathcal{P}_\delta - \lambda)Q_\delta(\lambda) &= \frac{1}{2} \sum_{\pm} (\mathcal{P}_\delta - \lambda)(1 \pm \kappa_\delta)(P_{\pm\delta} - \lambda)^{-1} \\
 &= \frac{1}{2} \sum_{\pm} (P_{\pm\delta} - \lambda)(1 \pm \kappa_\delta)(P_{\pm\delta} - \lambda)^{-1} \\
 &\quad + \frac{1}{2} \sum_{\pm} (\mathcal{P}_\delta - P_{\pm\delta})(1 \pm \kappa_\delta)(P_{\pm\delta} - \lambda)^{-1} \\
 &= \text{Id} + \frac{1}{2} \sum_{\pm} [D_x^2, \pm\kappa_\delta](P_{\pm\delta} - \lambda)^{-1} \\
 &\quad + \frac{\delta}{2} \sum_{\pm} (\kappa_\delta \mp 1)W(1 \pm \kappa_\delta)(P_{\pm\delta} - \lambda)^{-1} \\
 &= \text{Id} + \frac{1}{2} \sum_{\pm} \pm [D_x^2, \kappa_\delta](P_{\pm\delta} - \lambda)^{-1} \\
 &\quad + \frac{\delta}{2} \sum_{\pm} \pm (\kappa_\delta^2 - 1) W(P_{\pm\delta} - \lambda)^{-1}.
 \end{aligned}$$

Thus we have $(\mathcal{P}_\delta - \lambda)Q_\delta(\lambda) = \text{Id} + K_\delta(\lambda)$ where

$$K_\delta(\lambda) \stackrel{\text{def}}{=} \frac{1}{2} \left([D_x^2, \kappa_\delta] + \delta (\kappa_\delta^2 - 1) W \right) \cdot \left((P_\delta - \lambda)^{-1} - (P_{-\delta} - \lambda)^{-1} \right). \tag{5.2}$$

Proposition 5.1. *There exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and all $\lambda \in \mathbb{C}^+ \cup \mathbb{D}(E_*, \vartheta_{\pm\delta})$, the operator $K_\delta(\lambda)$ is compact and $\text{Id} + K_\delta(\lambda)$ forms a holomorphic family of Fredholm operators. In addition, $\text{Id} + K_\delta(i)$ is invertible on L^2 .*

Proof of Proposition 5.1. We show that (a) $K_\delta(\lambda)$ forms a holomorphic family of compact operators on L^2 and (b) $\|K_\delta(i)\|_{L^2} \leq 1/2$ for δ sufficiently small. Assertion (a) implies that $\text{Id} + K_\delta(\lambda)$ forms a holomorphic family of Fredholm operators and assertion (b) implies that $\text{Id} + K_\delta(i)$ is invertible for δ sufficiently small (by a Neumann series).

We start with (a). The holomorphic dependence on λ is clear. To show that $K_\delta(\lambda)$ is compact on L^2 , we observe that $D_x \kappa_\delta$, $D_x^2 \kappa_\delta$ and $\kappa_\delta^2 - 1$ are all smooth, compactly supported functions (with support contained in that of $\kappa_\delta^2 - 1$); and that the operators $(P_{\pm\delta} - \lambda)^{-1}$ map L^2 to H^2 when λ is outside the spectrum of P_δ . Therefore, the range of $K_\delta(\lambda)$ consists of H^2 -functions with support contained in $\text{supp}(\kappa_\delta^2 - 1)$. This is a compact subset of L^2 by the Rellich–Kondrachov theorem.

We now prove (b). We first show that $(P_\delta + i)^{-1}$ is bounded from L^2 to H^2 , with uniform bound in $\delta \in (0, 1)$. By the spectral theorem, $\|(P_\delta + i)^{-1}\|_{L^2} \leq 1$. We now use the elliptic estimate: if $f \in H^2$ then

$$|f|_{H^2} \leq \left| D_x^2 f \right|_{L^2} + |f|_{L^2} \leq |(P_\delta + i)f|_{L^2} + C|f|_{L^2},$$

where C does not depend on δ when $0 < \delta < 1$. This shows that

$$\left| (P_\delta + i)^{-1} f \right|_{H^2} \leq |f|_{L^2} + C|(P_\delta + i)^{-1} f|_{L^2} \leq C|f|_{L^2}.$$

Thus $\|(P_\delta + i)^{-1}\|_{L^2 \rightarrow H^2} = O(1)$ uniformly in $\delta \in (0, 1)$ as claimed. Now, we note that $D_x \kappa_\delta, D_x^2 \kappa_\delta$ and $\delta(\kappa_\delta^2 - 1)$ are all $O_{L^\infty}(\delta)$. This together with the uniform bound on $(P_\delta + i)^{-1} : L^2 \rightarrow H^2$ implies that $K_\delta(i) = \mathcal{O}_{L^2}(\delta)$. Taking δ sufficiently small concludes the proof. \square

Proposition 5.1 implies assertion (5.1). It relates the eigenvalues of \mathcal{P}_δ with the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$. Indeed, Lemma 5.1 shows that $(\text{Id} + K_\delta(\lambda))^{-1} : L^2 \rightarrow L^2$, initially defined for λ in a neighborhood of i , extends analytically to a family of bounded operators on L^2 for $\lambda \in \mathbb{C}^+ \cup \mathbb{D}(E_\star, \vartheta_\pm \delta)$. From (5.1), we deduce that $\mathcal{P}_\delta - \lambda$ is invertible away from the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$; and

$$(\mathcal{P}_\delta - \lambda)^{-1} = Q_\delta(\lambda) \cdot (\text{Id} + K_\delta(\lambda))^{-1}.$$

In particular, the eigenvalues of \mathcal{P}_δ in $[E_\star - \vartheta_\pm \delta, E_\star + \vartheta_\pm \delta]$ are among the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$.

Remark 5.1. The converse inclusion does not necessarily hold. This is because $Q_\delta(\lambda)$ may cancel poles of $(\text{Id} + K_\delta(\lambda))^{-1}$. This phenomena unfortunately happens here. We will fix this issue in Sect. 6.4.

5.3. *Expansion of $\text{Id} + K_\delta(\lambda)$.* To expand the expression for $K_\delta(\lambda)$ in (5.2), we first study the resolvent difference $(P_\delta - \lambda)^{-1} - (P_{-\delta} - \lambda)^{-1}$. We apply Theorem 2 and Corollary 2. Observe that

$$(\mathcal{D}_+ - z)^{-1} - (\mathcal{D}_- - z)^{-1} = \frac{2\sigma_\star}{v_\star^2 D_x^2 + \vartheta_F^2 - z^2}.$$

Use $\lambda = E_\star + z\delta, z \in \mathbb{D}(0, \vartheta_\pm)$. Theorem 2 and Corollary 2 imply

$$\begin{aligned} (P_\delta - \lambda)^{-1} - (P_{-\delta} - \lambda)^{-1} &= \frac{2}{\delta} \begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}^\top \mathcal{U}_\delta \cdot \frac{\sigma_\star}{v_\star^2 D_x^2 + \vartheta_F^2 - z^2} \cdot \mathcal{U}_\delta^{-1} \begin{bmatrix} \overline{\phi_+^\star} \\ \overline{\phi_-^\star} \end{bmatrix} \\ &\quad + \mathcal{O}_{L^2}(\delta^{-1/3}), \\ D_x \left((P_\delta - \lambda)^{-1} - (P_{-\delta} - \lambda)^{-1} \right) &= \frac{2}{\delta} \begin{bmatrix} D_x \phi_+^\star \\ D_x \phi_-^\star \end{bmatrix}^\top \mathcal{U}_\delta \cdot \frac{\sigma_\star}{v_\star^2 D_x^2 + \vartheta_F^2 - z^2} \cdot \mathcal{U}_\delta^{-1} \begin{bmatrix} \overline{\phi_+^\star} \\ \overline{\phi_-^\star} \end{bmatrix} \\ &\quad + \mathcal{O}_{L^2}(\delta^{-1/3}). \end{aligned} \tag{5.3}$$

From (5.2) and (5.3), for $z \in \mathbb{D}(0, \vartheta_\pm)$, the operator $\text{Id} + K_\delta(E_\star + z\delta)$ splits as

$$\text{Id} + K_\delta(E_\star + z\delta) = \text{Id} + \mathcal{E}_\delta(z) + \mathcal{K}_\delta(z), \tag{5.4}$$

where $\mathcal{E}_\delta(z) = \mathcal{O}_{L^2}(\delta^{1/3}), z \in \mathbb{D}(0, \vartheta_\pm)$ and $\mathcal{K}_\delta(z)$ equals

$$\left(2(D_x \kappa)_\delta \begin{bmatrix} D_x \phi_+^\star \\ D_x \phi_-^\star \end{bmatrix}^\top + (\kappa_\delta^2 - 1)W \begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}^\top \right) \mathcal{U}_\delta \cdot \frac{\sigma_\star}{v_\star^2 D_x^2 + \vartheta_F^2 - z^2} \cdot \mathcal{U}_\delta^{-1} \begin{bmatrix} \overline{\phi_+^\star} \\ \overline{\phi_-^\star} \end{bmatrix}. \tag{5.5}$$

The operator $\mathcal{K}_\delta(z)$ is a trace-class operator. The trace-class property holds because $\mathcal{K}_\delta(z)$ maps L^2 to H^2 functions with fixed compact support. This is sufficient in dimension 1; see for instance [DZ18, (B.3.9)].

6. Eigenvalues of \mathcal{P}_δ and the Effective Dirac Operator \mathcal{D}

In this section we show that the expansion of $\mathcal{K}_\delta(z)$ given in (5.5) helps locate the eigenvalues of \mathcal{P}_δ . We will see that $\mathcal{K}_\delta(z)$ is related to a matrix Schrödinger operator with localized, highly oscillatory potential. The weak – or homogenized—limit of this operator is $\mathcal{D}^2 - z^2$, which is not invertible precisely when z or $-z$ is an eigenvalue of \mathcal{D} .

This yields only a weak form of Theorem 1—see Proposition 6.1 below. Indeed, the non-zero eigenvalues of \mathcal{D}^2 are double—because the spectrum of \mathcal{D} is symmetric about 0—while Theorem 1 predicts that the point spectrum of \mathcal{P}_δ is simple.

This paradox is illusive; zeros of $Q_\delta(\lambda)$ cancel poles of $(\text{Id} + K_\delta(\lambda))^{-1}$. This phenomena is annoying, but it is very unstable. In the last step of the proof of Theorem 1, we add a specifically-designed rank-one operator to the natural parametrix $Q_\delta(\lambda)$. This twists the parametrix and deforms away spurious modes.

6.1. Fredholm determinants. We refer to [DZ18, Appendix B]. Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces; let $\mathcal{B}(\mathcal{H}, \mathcal{H}')$ be the space of bounded operators $\mathcal{H} \rightarrow \mathcal{H}'$. A compact operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is trace-class if the eigenvalues $\{\mu_j\}_{j=1}^\infty$ of $(TT^*)^{1/2} \in \mathcal{B}(\mathcal{H}')$ (called singular values of T) are summable. The space of trace-class operators on \mathcal{H} is denoted $\mathcal{L}(\mathcal{H}, \mathcal{H}')$; when $\mathcal{H} = \mathcal{H}'$ we more simply write $\mathcal{L}(\mathcal{H})$. The trace-class norm is

$$\|T\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}')} \stackrel{\text{def}}{=} \sum_{j=1}^\infty \mu_j.$$

The space $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ is a two-sided ideal: if $A \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ and $T \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ then $AT \in \mathcal{L}(\mathcal{H})$ and $TA \in \mathcal{L}(\mathcal{H}')$. See [DZ18, (B.4.6)].

If T is trace-class (hence compact), we can approximate T by a sequence T_n of finite-rank operators in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$. When $\mathcal{H} = \mathcal{H}'$, the trace-class property shows that $\text{Det}(\text{Id} + T_n)$ (well-defined because T_n has finite rank) converges; and the limit depends on T only. The limit is the *Fredholm determinant* $\text{Det}(\text{Id} + T)$. See [DZ18, Sect. B.5].

Fredholm determinants inherit most properties of their finite-dimensional analogs. Three of them are particularly relevant here:

- If $T \in \mathcal{L}(\mathcal{H})$ then $\text{Det}(\text{Id} + T) = 0$ if and only if $\text{Id} + T$ is not invertible—see [DZ18, Proposition B.25].
- The Fredholm determinant is cyclic—see [DZ18, (B.5.13)]:

$$A \in \mathcal{B}(\mathcal{H}', \mathcal{H}), \quad T \in \mathcal{L}(\mathcal{H}, \mathcal{H}') \quad \Rightarrow \quad \text{Det}(\text{Id} + AT) = \text{Det}(\text{Id} + TA). \quad (6.1)$$

- If $z \in U \mapsto T(z)$ is holomorphic family of trace-class operators, then $z \in U \mapsto \text{Det}(\text{Id} + T(z))$ is holomorphic.

To prove the last property, we remark that it holds for holomorphic families of finite-rank operators. Hence $z \in U \mapsto \text{Det}(\text{Id} + T(z))$ is a uniform limit of holomorphic functions, thus holomorphic.

We will use the following lemma, proved in Appendix B:

Lemma 6.1. *For any $s \in (0, 1/2)$, there exists C with the following. Let $A, A' : L^2 \rightarrow L^2$ and $B : L^2 \rightarrow H^2$ such that B is continuous from H^{-s} to H^{2-s} . Then for any $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$, $A\chi B$ and $A'\chi B$ are trace-class and*

$$\left| \text{Det}(\text{Id} + A\chi B) - \text{Det}(\text{Id} + A'\chi B) \right| \leq C \|A - A'\|_{H^s \rightarrow H^{-s}} \cdot \|\chi B\|_{H^{-s} \rightarrow H^{2-s}}.$$

6.2. *The Dirac operator.* We introduce the Dirac operator $\mathcal{D} \stackrel{\text{def}}{=} \nu_\star \sigma_3 D_x + \sigma_\star \kappa$, where we recall that

$$\nu_\star = 2 \langle \phi_+^\star, D_x \phi_+^\star \rangle, \quad \vartheta_\star = \langle \phi_-^\star, W \phi_+^\star \rangle, \quad \sigma_\star = \begin{bmatrix} 0 & \overline{\vartheta_\star} \\ \vartheta_\star & 0 \end{bmatrix}.$$

It is a selfadjoint operator on L^2 with domain H^1 . It will emerge naturally in the study of the spectrum of \mathcal{P}_δ .

Theorem 3. *The following holds:*

- (1) $\Sigma(\mathcal{D})$ is independent of the choice of $(\phi_+^\star, \phi_-^\star)$ in Proposition 2.1.
- (2) The essential spectrum of \mathcal{D} is $\mathbb{R} \setminus (-\vartheta_F, \vartheta_F)$.
- (3) The pure point spectrum of \mathcal{D} consists of finitely many simple eigenvalues

$$-\vartheta_F < \vartheta_{-N} < \dots < \vartheta_{-1} < \vartheta_0 = 0 < \vartheta_1 < \dots < \vartheta_N < \vartheta_F,$$

satisfying $\vartheta_{-j} = -\vartheta_j$.

Proof. We start with the first point. If we choose another Dirac eigenbasis satisfying Proposition 2.1 then ν_\star remains the same; and ϑ_\star transforms to $\omega^2 \vartheta_\star$, where $\omega \in \mathbb{S}^1$. The operator \mathcal{D} transforms to

$$\nu_\star \sigma_3 D_x + \begin{bmatrix} 0 & \overline{\omega^2 \vartheta_\star} \\ \omega^2 \vartheta_\star & 0 \end{bmatrix} \kappa. \tag{6.2}$$

We observe that

$$\begin{aligned} \begin{bmatrix} \overline{\omega} & 0 \\ 0 & \omega \end{bmatrix} \sigma_3 \begin{bmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{bmatrix} &= |\omega|^2 \sigma_3 = \sigma_3; \\ \begin{bmatrix} \overline{\omega} & 0 \\ 0 & \omega \end{bmatrix} \sigma_\star \begin{bmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{bmatrix} &= \begin{bmatrix} \overline{\omega} & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} 0 & \overline{\vartheta_\star} \\ \vartheta_\star & 0 \end{bmatrix} \begin{bmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{bmatrix} = \begin{bmatrix} 0 & \overline{\omega^2 \vartheta_\star} \\ \omega^2 \vartheta_\star & 0 \end{bmatrix}. \end{aligned}$$

This implies that \mathcal{D} and the operator (6.2) are conjugated. Therefore they have the same spectrum. This proves (1).

The assertions regarding the essential spectrum of \mathcal{D} and the zero mode were proved in [FLW17, Sect. 4.1] when $\vartheta_\star \in \mathbb{R}$. The same proofs apply to the more general case $\vartheta_\star \in \mathbb{C}$. We next show that eigenvalues of \mathcal{D} do not accumulate at the edges $\pm \vartheta_F$ of the essential spectrum. A computation using $\sigma_\star \sigma_3 = -\sigma_3 \sigma_\star$ and $\sigma_3^2 = \text{Id}_2$, $\sigma_\star^2 = \vartheta_F^2 \text{Id}_2$ yields

$$\mathcal{D}^2 - \vartheta_F^2 = \nu_\star^2 D_x^2 + \mathcal{M}_0, \quad \mathcal{M}_0 \stackrel{\text{def}}{=} \vartheta_F^2 (\kappa^2 - 1) + \nu_\star (D_x \kappa) \sigma_3 \sigma_\star. \tag{6.3}$$

In particular, $\mathcal{D}^2 - \vartheta_F^2$ is a Schrödinger operator potential with support in $[-L, L]$ —see (1.3). Therefore, its resonances (in particular, its eigenvalues) are isolated—see e.g. [DZ18, Theorem 2.2]. It follows that \mathcal{D} has only isolated eigenvalues.

To show the symmetry of the spectrum, we note the identity

$$\sigma_3 \sigma_\star \cdot \mathcal{D} \cdot (\sigma_3 \sigma_\star)^{-1} = -\mathcal{D}, \tag{6.4}$$

which follows from the relations $\sigma_\star \sigma_3 = -\sigma_3 \sigma_\star$, $\sigma_3^2 = \text{Id}_2$ and $\sigma_\star^2 = \vartheta_F^2 \text{Id}_2$. The identity (6.4) implies that \mathcal{D} and $-\mathcal{D}$ have same spectra; in particular, $\Sigma(\mathcal{D})$ must be symmetric about 0.

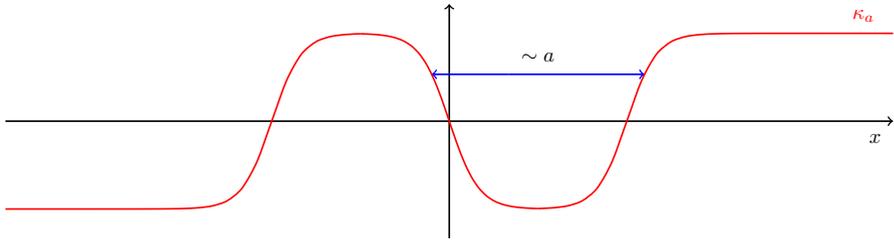


Fig. 5. The function κ_a is the concatenation of 3 “single-bump” domain walls separated by a

Finally we prove that any eigenvalue ϑ of \mathcal{D} is simple. Let α be an eigenvector. Because of (1.3),

$$(\nu_\star \sigma_3 D_y \pm \sigma_\star - \vartheta)\alpha \quad \text{for } \pm y \geq L.$$

It follows that α solves an ODE with constant coefficients for $\pm y \geq L$. In particular, in these regions, α is a linear combination of exponentials. Since $\alpha \in L^2(\mathbb{R}, \mathbb{C}^2)$, all these exponential terms must decay. This implies that eigenvectors of \mathcal{D} (and their derivatives) decay exponentially. Note now that if α and $\tilde{\alpha}$ are two eigenvectors for the same eigenvalue ϑ , then their Wronskian is a constant in y . Moreover the Wronskian decays exponentially and hence is identically zero. This implies that α and $\tilde{\alpha}$ are linearly dependent and we conclude that ϑ is a simple eigenvalue. \square

Remark 6.1. The Dirac operator \mathcal{D} may have arbitrarily many eigenvalues. Fix $N \in \mathbb{N}$. The paper [LWW18] studies the point spectrum of the Dirac operator $\mathcal{D}_a = \nu_\star \sigma_3 D_x + \sigma_\star \kappa_a$, where κ_a is constructed by gluing together $2N + 1$ reference domain walls κ separated by a distance a —see Fig. 5. If $\varepsilon > 0$ is given and $\nu_\star \sigma_3 D_x + \sigma_\star \kappa$ has a single eigenvalue then for a sufficiently large, [LWW18] shows that \mathcal{D}_a has precisely $2N + 1$ eigenvalues in $(-\vartheta_\sharp + \varepsilon, \vartheta_\sharp - \varepsilon)$.

6.3. Localization of eigenvalues.

Proposition 6.1. *Let $\vartheta_N < \vartheta_\sharp < \vartheta_F$ and define*

$$\mathcal{Z}_\delta \stackrel{\text{def}}{=} \{z \in \mathbb{D}(0, \vartheta_\sharp) : E_\star + z\delta \in \Sigma(\mathcal{P}_\delta)\}.$$

Then $\text{dist}(\mathcal{Z}_\delta, \Sigma_{\text{pp}}(\mathcal{D}))$ tends to 0 as δ tends to 0.

Proposition 6.1 shows that the eigenvalues of \mathcal{P}_δ in $[E_\star - \vartheta_\sharp\delta, E_\star + \vartheta_\sharp\delta]$ must necessarily be of the form $E_\star + \vartheta_j\delta + o(\delta)$, where the ϑ_j ’s are the eigenvalues of the effective Dirac operator \mathcal{D} . This proves the assertion (A) of Theorem 1.

Proof. 1. If λ is an eigenvalue of \mathcal{P}_δ , then $\text{Id} + K_\delta(\lambda)$ is not invertible. We recall that $\text{Id} + K_\delta(E_\star + z\delta)$ decomposes as

$$\text{Id} + \mathcal{E}_\delta(z) + \mathcal{K}_\delta(z),$$

where $\mathcal{E}_\delta(z) = \mathcal{O}_{L^2}(\delta^{1/3})$ and $\mathcal{K}_\delta(z)$ is trace-class, see (5.4). Hence, for δ sufficiently small, $\text{Id} + \mathcal{E}_\delta(z)$ is invertible and

$$\lambda = E_\star + z\delta \in \Sigma(\mathcal{P}_\delta) \cap \mathbb{D}(E_\star, \vartheta_\sharp\delta) \Rightarrow \text{Id} + (\text{Id} + \mathcal{E}_\delta(z))^{-1} \mathcal{K}_\delta(z) \text{ is not invertible.}$$

Since $\mathcal{K}_\delta(z)$ is a trace-class operator we deduce that

$$\begin{aligned} \lambda &= E_\star + z\delta \in \Sigma(\mathcal{P}_\delta) \cap \mathbb{D}(E_\star, \vartheta_\mp \delta) \\ \Rightarrow \mathcal{D}_\delta(z) &= 0, \quad \mathcal{D}_\delta(z) \stackrel{\text{def}}{=} \text{Det} \left(\text{Id} + (\text{Id} + \mathcal{E}_\delta(z))^{-1} \mathcal{K}_\delta(z) \right). \end{aligned} \tag{6.5}$$

Therefore we get $\mathcal{Z}_\delta \subset \mathcal{D}_\delta^{-1}(0) \cap \mathbb{D}(0, \vartheta_\mp)$.

2. We have reduced the proof of Proposition 6.1 to a problem in complex analysis: locate the zeros of \mathcal{D}_δ . Note first that since $\mathcal{E}_\delta(z) = \mathcal{O}_{L^2}(\delta^{1/3})$, uniformly in $z \in \overline{\mathbb{D}(0, \vartheta_\mp)}$, we have

$$\mathcal{D}_\delta(z) = D_\delta(z) + \mathcal{O} \left(\delta^{1/3} \right), \quad D_\delta(z) \stackrel{\text{def}}{=} \text{Det}(\text{Id} + \mathcal{K}_\delta(z)).$$

3. Using the cyclicity of the determinant (6.1), we rewrite the Fredholm determinant $D_\delta(z)$ as the determinant of an operator acting on $L^2(\mathbb{R}, \mathbb{C}^2)$. Specifically permuting the matrix term on the right implies that $D_\delta(z)$ equals

$$\begin{aligned} &\text{Det} \left(\text{Id} + \left(2(D_x \kappa)_\delta \begin{bmatrix} D_x \phi_\star^+ \\ D_x \phi_\star^- \end{bmatrix}^\top + (\kappa_\delta^2 - 1)W \begin{bmatrix} \phi_\star^+ \\ \phi_\star^- \end{bmatrix}^\top \right) \mathcal{U}_\delta \cdot \frac{\sigma_\star}{v_\star^2 D_x^2 + \vartheta_F^2 - z^2} \cdot \mathcal{U}_\delta^{-1} \overline{\begin{bmatrix} \phi_\star^+ \\ \phi_\star^- \end{bmatrix}} \right) \\ &= \text{Det} \left(\text{Id} + \mathcal{U}_\delta^{-1} \overline{\begin{bmatrix} \phi_\star^+ \\ \phi_\star^- \end{bmatrix}} \left(2(D_x \kappa)_\delta \begin{bmatrix} D_x \phi_\star^+ \\ D_x \phi_\star^- \end{bmatrix}^\top + (\kappa_\delta^2 - 1)W \begin{bmatrix} \phi_\star^+ \\ \phi_\star^- \end{bmatrix}^\top \right) \mathcal{U}_\delta \cdot \frac{\sigma_\star}{v_\star^2 D_x^2 + \vartheta_F^2 - z^2} \right) \\ &= \text{Det} \left(\text{Id} + \mathcal{M}_\delta \left(v_\star^2 D_x^2 + \vartheta_F^2 - z^2 \right)^{-1} \right). \end{aligned}$$

Here, $\mathcal{M}_\delta(x) = \mathcal{M}(x/\delta, x)$ is a matrix-valued multiscale highly oscillatory potential. Specifically, $\mathcal{M}(x, y)$ is 1-periodic in x , compactly supported in y , smooth in both variables equal to

$$\overline{\begin{bmatrix} \phi_\star^+(x) \\ \phi_\star^-(x) \end{bmatrix}} \left(2(D_x \kappa)(y) \begin{bmatrix} D_x \phi_\star^+(x) \\ D_x \phi_\star^-(x) \end{bmatrix}^\top + (\kappa(y)^2 - 1) W(x) \begin{bmatrix} \phi_\star^+(x) \\ \phi_\star^-(x) \end{bmatrix}^\top \right) \sigma_\star. \tag{6.6}$$

4. We would like to expand $D_\delta(z)$. Due to the oscillations of \mathcal{M}_δ , the operator $\text{Id} + \mathcal{M}_\delta \left(v_\star^2 D_x^2 + \vartheta_F^2 - z^2 \right)^{-1}$ does not converge in the operator norm. The L^2 -weak limit of \mathcal{M}_δ (corresponding to the average with respect to x) is computed using that as $\delta \rightarrow 0$:

$$2 \overline{\begin{bmatrix} \phi_\star^+(\cdot/\delta) \\ \phi_\star^-(\cdot/\delta) \end{bmatrix}} \begin{bmatrix} D_x \phi_\star^+(\cdot/\delta) \\ D_x \phi_\star^-(\cdot/\delta) \end{bmatrix}^\top \rightharpoonup v_\star \sigma_3 \quad \text{and} \quad \overline{\begin{bmatrix} \phi_\star^+(\cdot/\delta) \\ \phi_\star^-(\cdot/\delta) \end{bmatrix}} W \begin{bmatrix} \phi_\star^+(\cdot/\delta) \\ \phi_\star^-(\cdot/\delta) \end{bmatrix}^\top \rightharpoonup \sigma_\star. \tag{6.7}$$

We find $\mathcal{M}_\delta \rightharpoonup \mathcal{M}_0$, where \mathcal{M}_0 already appeared in (6.3):

$$\mathcal{M}_\delta \rightharpoonup \left((D_x \kappa) v_\star \sigma_3 + (\kappa^2 - 1) \sigma_\star \right) \sigma_\star = \vartheta_F^2 (\kappa^2 - 1) + v_\star (D_x \kappa) \sigma_3 \sigma_\star = \mathcal{M}_0. \tag{6.8}$$

We now show that $D_\delta(z)$ can be expanded about the formal weak limit

$$D_0(z) \stackrel{\text{def}}{=} \text{Det} \left(\text{Id} + \mathcal{M}_0 \left(v_\star^2 D_x^2 + \vartheta_F^2 - z^2 \right)^{-1} \right).$$

Specifically, using Lemma B.1, we have $|\mathcal{M}_\delta - \mathcal{M}_0|_{H^{-1/4}} = \mathcal{O}(\delta^{1/4})$. As a multiplication operator, $\mathcal{M}_\delta - \mathcal{M}_0$ is bounded from $H^{1/4}$ to $H^{-1/4}$ with bound

$$\|\mathcal{M}_\delta - \mathcal{M}_0\|_{H^{1/4} \rightarrow H^{-1/4}} = \mathcal{O} \left(\delta^{1/4} \right),$$

see [Dr18c, Lemma 2.1]. Because of Lemma 6.1 we deduce that

$$D_\delta(z) = D_0(z) + O\left(\delta^{1/4}\right). \tag{6.9}$$

5. A complex number $z \in \mathbb{D}(0, \vartheta_{\pm})$ is a zero of D_0 if and only if the operator $v_\star^2 D_x^2 + \vartheta_F^2 + \mathcal{M}_0 - z^2$ is not invertible. Because of (6.3), this operator is precisely $(\mathcal{D} - z)(\mathcal{D} + z)$. Hence, the zeros of $D_0(z)$ in $\mathbb{D}(0, \vartheta_{\pm})$ are exactly the eigenvalues of \mathcal{D} or $-\mathcal{D}$. Since \mathcal{D} and $-\mathcal{D}$ have the same spectrum—see Theorem 3—we deduce that the zeros of $D_0(z)$ are exactly the eigenvalues of \mathcal{D} , with twice their multiplicities.
6. From (6.9), $|\mathcal{D}_\delta(z) - D_0(z)| = O(\delta^{1/4})$ for $z \in \mathbb{D}(0, \vartheta_{\pm}\delta)$. Hurwitz’s theorem implies that the zeros of \mathcal{D}_δ converge to those of D_0 : $\text{dist}\left(\mathcal{D}_\delta^{-1}(0), D_0^{-1}(0)\right) \rightarrow 0$. Proposition 6.1 now follows from (6.5) and Step 5. \square

6.4. The twisted parametrix. In order to conclude the proof of Theorem 1, we must show that the eigenvalues of \mathcal{D} each seed at most one eigenvalue of \mathcal{P}_δ . However, a deeper look at the proof of Proposition 6.1 shows that the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$ have twice the multiplicity of the eigenvalues of \mathcal{D} .

If one believes in Theorem 1, there is only one explanation: $Q_\delta(\lambda)$ must cancel out half the multiplicity of the poles of $(\text{Id} + K_\delta(\lambda))^{-1}$. This surprising phenomena is however very unstable. We take advantage of this instability to produce a quick fix.

Given a family of rank-one operators $\Pi_\delta(\lambda) : L^2 \rightarrow H^2$ (depending on $\delta > 0$ and holomorphically on λ), we define a *twisted parametrix* $Q_\delta^\Pi(\lambda)$ by

$$Q_\delta^\Pi(\lambda) = Q_\delta(\lambda) + \Pi_\delta(\lambda).$$

To this twisted parametrix corresponds a family of operators $K_\delta^\Pi(\lambda)$ such that

$$(\mathcal{P}_\delta - \lambda)Q_\delta^\Pi(\lambda) = \text{Id} + K_\delta(\lambda) + (\mathcal{P}_\delta - \lambda)\Pi_\delta(\lambda) \stackrel{\text{def}}{=} \text{Id} + K_\delta^\Pi(\lambda).$$

Since $K_\delta^\Pi(\lambda)$ is a rank-one perturbation of $K_\delta(\lambda)$, it still forms a holomorphic family of compact operators on L^2 .

We show that $\text{Id} + K_\delta^\Pi(\lambda_\delta)$ is invertible for some $\lambda_\delta \in \mathbb{C}$; [DZ18, Theorem C.5] provides the meromorphic continuation of $(\text{Id} + K_\delta^\Pi(\lambda_\delta))^{-1}$. We deduce that the resolvent of \mathcal{P}_δ is expressed in terms of the twisted parametrix:

$$(\mathcal{P}_\delta - \lambda)^{-1} = Q_\delta^\Pi(\lambda) \cdot (\text{Id} + K_\delta^\Pi(\lambda))^{-1}.$$

In particular, the eigenvalues of \mathcal{P}_δ must be within the poles of $(\text{Id} + K_\delta^\Pi(\lambda))^{-1}$.

In the rest of the section, we consider operators $\Pi_\delta(\lambda)$ in the form

$$\Pi_\delta(\lambda) = \frac{1}{\delta} \begin{bmatrix} \phi_\star^+ \\ \phi_\star^- \end{bmatrix}^\top \cdot \mathcal{U}_\delta \cdot f \otimes g \cdot \mathcal{U}_\delta^{-1} \cdot \begin{bmatrix} \phi_\star^+ \\ \phi_\star^- \end{bmatrix}. \tag{6.10}$$

Above, $f, g \in H^2(\mathbb{R}, \mathbb{C}^2)$ may depend on $\delta \in (0, 1)$ and $\lambda \in \mathbb{D}(E_\star, \vartheta_{\pm}\delta)$. They will be explicitly determined.

Proposition 6.2. *Let $\tilde{\Pi}_\delta(z) \stackrel{\text{def}}{=} (\mathcal{P}_\delta - E_\star - z\delta) \cdot \Pi_\delta(E_\star + z\delta)$. There exists $C > 0$ such that for any $\delta > 0$,*

$$\|\tilde{\Pi}_\delta(z)\|_{L^2} \leq C(1 + |z|) \cdot |f|_{H^2} \cdot |g|_{L^2}.$$

Proof. Fix $f, g \in H^2(\mathbb{R}, \mathbb{C}^2)$. Then, using that $(D_x^2 + V - E_\star)\phi_\pm^\star = 0$ and that $D_x \mathcal{U}_\delta(f)(x) = \delta \mathcal{U}_\delta(D_x f)(x)$, we find

$$\begin{aligned} & (\mathcal{P}_\delta - E_\star - z\delta) \cdot \frac{1}{\delta} \begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}^\top \cdot \mathcal{U}_\delta(f \otimes g) \mathcal{U}_\delta^{-1} \cdot \overline{\begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}} \\ &= \frac{1}{\delta} \left(\begin{bmatrix} (\mathcal{P}_\delta - E_\star - z\delta)\phi_+^\star \\ (\mathcal{P}_\delta - E_\star - z\delta)\phi_-^\star \end{bmatrix}^\top \cdot \mathcal{U}_\delta f + 2 \begin{bmatrix} D_x \phi_+^\star \\ D_x \phi_-^\star \end{bmatrix}^\top \cdot D_x \mathcal{U}_\delta f + \begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}^\top \cdot D_x^2 \mathcal{U}_\delta f \right) \\ & \quad \otimes g \mathcal{U}_\delta^{-1} \cdot \overline{\begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}} \\ &= \left((\kappa_\delta W - z) \begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}^\top \cdot \mathcal{U}_\delta f + 2 \begin{bmatrix} D_x \phi_+^\star \\ D_x \phi_-^\star \end{bmatrix}^\top \cdot \mathcal{U}_\delta D_x f + \delta \mathcal{U}_\delta D_x^2 f \right) \otimes g \mathcal{U}_\delta^{-1} \cdot \overline{\begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}}. \end{aligned}$$

A direct estimate shows that the L^2 -operator norm of this projector is controlled by $C(1 + |z|) \cdot \|f\|_{H^2} \cdot \|g\|_{L^2}$, as claimed. \square

The family $\lambda \in \mathbb{D}(E_\star, \vartheta_\# \delta) \mapsto K_\delta^\Pi(\lambda)$ is analytic. To adapt the arguments of Sect. 6.3, we first show that $(\text{Id} + K_\delta^\Pi(\lambda))^{-1}$ defines a meromorphic family of operators. Because of [DZ18, Theorem C.5], it suffices to prove that for δ sufficiently small, there exists $\lambda_\delta \in \mathbb{C}^+ \cup \mathbb{D}(E_\star, \vartheta_\# \delta)$ with $\text{Id} + K_\delta^\Pi(\lambda_\delta)$ invertible. We set

$$\lambda_\delta \stackrel{\text{def}}{=} E_\star + i\delta\vartheta', \quad \vartheta' \stackrel{\text{def}}{=} \frac{\vartheta_\# + \vartheta_N}{2} \quad (\text{note } \vartheta_N < \vartheta' < \vartheta_F).$$

Proposition 6.3. *There exist $\delta_0 > 0, \theta_0 > 0$ such that if*

$$\|f\|_{H^2} \cdot \|g\|_{L^2} \leq \theta_0, \quad \delta \in (0, \delta_0) \tag{6.11}$$

then $\text{Id} + K_\delta^\Pi(\lambda_\delta)$ is invertible.

Proof of Proposition 6.3. 1. Recall that $\mathcal{K}_\delta(z)$ is given by (5.5). We first show that $\text{Id} + \mathcal{K}_\delta(i\vartheta')$ is invertible, with bounds on its inverse. In the proof of Proposition 6.1 (see (6.9)) we showed that

$$\text{Det}(\text{Id} + \mathcal{K}_\delta(i\vartheta')) = D_\delta(i\vartheta') = D_0(i\vartheta') + O(\delta^{1/4}).$$

In addition, we know that $D_0(i\vartheta') \neq 0$ since $i\vartheta'$ is not an eigenvalue of \mathcal{D} . Therefore for δ sufficiently small, $\text{Det}(\text{Id} + \mathcal{K}_\delta(i\vartheta'))$ is non-zero. This implies that $\text{Id} + \mathcal{K}_\delta(i\vartheta')$ is invertible. Furthermore, bounds on the inverse are obtained using Cramer’s rule, see e.g. [Dr18b, Lemma 5.1]:

$$\left\| (\text{Id} + \mathcal{K}_\delta(i\vartheta'))^{-1} \right\|_{L^2} \leq \frac{e^{2\|\mathcal{K}_\delta(i\vartheta')\|_{\mathcal{L}(L^2)}}}{|\text{Det}(\text{Id} + \mathcal{K}_\delta(i\vartheta'))|} \leq C e^{2\|\mathcal{K}_\delta(i\vartheta')\|_{\mathcal{L}(L^2)}}.$$

The trace-class norm of $\mathcal{K}_\delta(\lambda_\delta)$ is uniformly bounded because it is the product of a uniformly bounded multiplication operator with a δ -independent Fourier multiplier mapping L^2 to H^2 .

2. By (5.4) we have that

$$\text{Id} + K_\delta^\Pi(E_\star + z\delta) = \text{Id} + \mathcal{K}_\delta(z) + \mathcal{E}_\delta(z) + \tilde{\Pi}_\delta(z),$$

where $\tilde{\Pi}_\delta(z)$ was defined in Proposition 6.2. The operator $\text{Id} + \mathcal{K}_\delta(i\vartheta')$ is invertible, with uniform bounds on its inverse. Therefore for $\text{Id} + K_\delta^\Pi(\lambda_\delta)$ to be invertible it suffices that $\|\mathcal{E}_\delta(i\vartheta') + \tilde{\Pi}_\delta(i\vartheta')\|_{L^2}$ gets arbitrarily small as $\delta \rightarrow 0$. By (5.5) and Proposition 6.2,

$$\|\mathcal{E}_\delta(\lambda_\delta) + \tilde{\Pi}_\delta(i\vartheta')\|_{L^2} \leq C\delta^{1/3} + C(1 + \vartheta') \cdot |f|_{H^2} \cdot |g|_{L^2}.$$

Therefore, if δ and $|f|_{H^2} \cdot |g|_{L^2}$ are sufficiently small then $\text{Id} + K_\delta^\Pi(\lambda_\delta)$ is invertible. This completes the proof of Proposition 6.3. \square

By Proposition 6.3 and analytic Fredholm theory (Sect. 5.1), we know that whenever (6.11) is satisfied, the eigenvalues of \mathcal{P}_δ in the disk $\mathbb{D}(E_\star, \vartheta_\# \delta)$ must be among the poles of $(\text{Id} + K_\delta^\Pi(\lambda))^{-1}$. We now have all the ingredients to prove Theorem 1.

Proof of Theorem 1. The first point of Theorem 1 was proved in Propopsition 6.1. It remains to eliminate the spurious modes discussed above. We show that for every $j \in [-N, N]$, for some $\varepsilon > 0$ and $\delta_0 > 0$ and for $\delta \in (0, \delta_0)$, \mathcal{P}_δ has at most one eigenvalue in $[E_\star + \delta(\vartheta_j - \varepsilon), E_\star + \delta(\vartheta_j + \varepsilon)]$.

1. Assume that ϑ_j is an eigenvalue of \mathcal{D} . We now construct $\Pi_\delta(\lambda)$ of the form (6.10) by choosing $f \in \ker(\mathcal{D} - \vartheta_j)$ with $|f|_{L^2} = 1$. We next set

$$g \stackrel{\text{def}}{=} t \cdot \left(v_\star^2 D_x^2 + \vartheta_F^2 - \bar{z}^2 \right)^{-1} \chi f.$$

Above χ is a function with compact support which takes values in $[0, 1]$. We now fix

$$t \stackrel{\text{def}}{=} \frac{\theta_0 \cdot (\vartheta_F^2 - |\vartheta_\#|^2)}{2|f|_{H^2}}. \tag{6.12}$$

The L^2 -operator norm of $(v_\star^2 D_x^2 + \vartheta_F^2 - \bar{z}^2)^{-1}$ is bounded by $(\vartheta_F^2 - \bar{z}^2)^{-1}$. With the choice (6.12) of t , the functions f and g satisfy (6.11). Therefore $\lambda \in \mathbb{D}(E_\star, \vartheta_\# \delta) \mapsto (\text{Id} + K_\delta^\Pi(\lambda))^{-1}$ forms a meromorphic family.

The eigenvalues of \mathcal{P}_δ are among the poles of $(\text{Id} + K_\delta^\Pi(\lambda))^{-1}$. In other words, if λ is an eigenvalue of \mathcal{P}_δ then $\text{Id} + K_\delta^\Pi(\lambda)$ is not invertible. The decomposition (5.4) implies that

$$\text{Id} + K_\delta^\Pi(E_\star + \delta z) = \text{Id} + \mathcal{E}_\delta(z) + \mathcal{K}_\delta(z) + \tilde{\Pi}_\delta(z)$$

where $\mathcal{E}_\delta(\lambda) = \mathcal{O}_{L^2}(\delta^{1/3})$ and $\mathcal{K}_\delta(z) + \tilde{\Pi}_\delta(z)$ is a trace-class operator. Hence

$$\begin{aligned} \lambda = E_\star + z\delta \text{ is an eigenvalue of } \mathcal{P}_\delta &\Rightarrow \mathcal{D}_\delta^\Pi(z) = 0, \\ \mathcal{D}_\delta^\Pi(z) \stackrel{\text{def}}{=} \text{Det} \left(\text{Id} + (\text{Id} + \mathcal{E}_\delta(z))^{-1} \mathcal{K}_\delta^\Pi(z) \right), \quad \mathcal{K}_\delta^\Pi(z) \stackrel{\text{def}}{=} \mathcal{K}_\delta(z) + \tilde{\Pi}_\delta(z). \end{aligned} \tag{6.13}$$

2. The proof of Theorem 1 reduces now to a problem in complex analysis; we must locate the zeros of \mathcal{D}_δ^Π . Similarly to Step 2 in the proof of Proposition 6.1, we have

$$\mathcal{D}_\delta^\Pi(z) = D_\delta^\Pi(z) + \mathcal{O} \left(\delta^{1/3} \right), \quad D_\delta^\Pi(z) = \text{Det} \left(\text{Id} + \mathcal{K}_\delta^\Pi(z) \right).$$

3. Applying the same calculation as in Step 3 of Proposition 6.1, we get

$$D_\delta^\Pi(z) = \text{Det} \left(\text{Id} + (\mathcal{M}_\delta + X_\delta) \left(v_\star^2 D_x^2 + \vartheta_F^2 - z^2 \right)^{-1} \right).$$

Above, \mathcal{M}_δ is the multiscale matrix-valued highly oscillatory potential (6.6), and X_δ is a multiscale highly oscillatory rank-one operator, associated with $\Pi_\delta(\lambda)$:

$$X_\delta \stackrel{\text{def}}{=} t \cdot \mathcal{U}_\delta^{-1} \cdot \begin{bmatrix} \overline{\phi_+^\star} \\ \overline{\phi_-^\star} \end{bmatrix} \left((\kappa_\delta W - z) \begin{bmatrix} \phi_+^\star \\ \phi_-^\star \end{bmatrix}^\top \cdot \mathcal{U}_\delta f + 2 \begin{bmatrix} D_x \phi_+^\star \\ D_x \phi_-^\star \end{bmatrix}^\top \cdot \mathcal{U}_\delta D_x f \right) \otimes f \chi.$$

4. To anticipate the limiting behavior of $D_\delta^\Pi(z)$ we calculate the weak limit of the operator $\mathcal{M}_\delta + X_\delta$ as δ tends to zero. The weak limit \mathcal{M}_0 of \mathcal{M}_δ is displayed in (6.8). We now study the convergence of X_δ as δ tends to 0. We observe that

$$X_\delta = t \cdot (F_\delta + G_\delta) \otimes \chi f,$$

where $F_\delta(y) = F(y/\delta, y)$ and $G_\delta(y) = G(y/\delta, y)$. The functions F, G are both one-periodic in the first variable and exponentially decaying in the second:

$$F(x, y) \stackrel{\text{def}}{=} (\kappa(y)W(x) - z) \cdot f(y) \cdot \begin{bmatrix} \overline{\phi_+^\star(x)} \\ \overline{\phi_-^\star(x)} \end{bmatrix} \begin{bmatrix} \phi_+^\star(x) \\ \phi_-^\star(x) \end{bmatrix}^\top,$$

$$G(x, y) \stackrel{\text{def}}{=} 2(D_y f)(y) \cdot \begin{bmatrix} \overline{\phi_+^\star(x)} \\ \overline{\phi_-^\star(x)} \end{bmatrix} \begin{bmatrix} D_x \phi_+^\star(x) \\ D_x \phi_-^\star(x) \end{bmatrix}^\top.$$

Using the relations (6.7) we have

$$X_\delta \rightarrow t \cdot (\kappa \sigma_\star - z) f \otimes f \chi + t \cdot \sigma_3 v_\star D_y f \otimes f \chi$$

$$= t \cdot (\mathcal{D} - z) \cdot f \otimes f \chi \stackrel{\text{def}}{=} X_0.$$

Using Lemma B.1 we can obtain quantitative bounds in $H^{1/4}$ for the convergence of F_δ and G_δ to their weak limit as δ goes to zero—in the same spirit as Step 4 in the proof of Proposition 6.1. We deduce that $\|X_\delta - X_0\|_{H^{1/4} \rightarrow H^{-1/4}} = O(\delta^{1/4})$ because of [Dr18c, Lemma 2.1].

Thanks to Lemma 6.1, $D_\delta^\Pi(\lambda) = D_0^\Pi(\lambda) + O(\delta)$ with

$$D_0^\Pi(\lambda) \stackrel{\text{def}}{=} \text{Det} \left(\text{Id} + (\mathcal{M}_0 + X_0) \left(v_\star^2 D_x^2 + \vartheta_F^2 - z^2 \right)^{-1} \right). \tag{6.14}$$

5. Hence z is a zero of $D_0^\Pi(z)$ if and only the operator

$$\mathcal{D}^2 - z^2 + t (\mathcal{D} - z) \cdot f \otimes f \chi = (\mathcal{D} - z) (\mathcal{D} + z + t \cdot f \otimes f \chi)$$

is not invertible. Therefore, the zeros of $D_0^\Pi(z)$ are exactly the eigenvalues of \mathcal{D} and of $\mathcal{D} + t f \otimes f \chi$.

6. With t as in (6.12), we fix $\varepsilon \in (0, t)$ small enough, so that the only eigenvalue of \mathcal{D} in the interval $[\vartheta_j - \varepsilon, \vartheta_j + \varepsilon]$ is ϑ_j . The eigenvalues of $\mathcal{D} + t \cdot f \otimes f$ are given by

$$\mu_k \stackrel{\text{def}}{=} \vartheta_k \text{ if } k \neq j; \quad \mu_j = \vartheta_j + t \text{ (when } k = j\text{).}$$

In particular, they all lie outside $[\vartheta_j - \varepsilon, \vartheta_j + \varepsilon]$. By taking χ equal to 1 on a sufficiently large set we can arrange for all the eigenvalues of

$$\mathcal{D} + t \cdot f \otimes f \chi$$

to also lie outside $[\vartheta_j - \varepsilon, \vartheta_j + \varepsilon]$. Hence the only zero of D_0^Π in the disc $\overline{\mathbb{D}(\vartheta_j, \varepsilon)}$ is the eigenvalue ϑ_j of \mathcal{D} .

Using (6.14), Hurwitz’s theorem implies that for δ sufficiently small, D_δ^Π has precisely one zero in $\mathbb{D}(\vartheta_j, \varepsilon)$. From (6.13), this completes the proof of Theorem 1. \square

Acknowledgements. This research was supported in part by National Science Foundation Grants DMS-1800086 (AD), DMS-1265524 (CF), DMS-1412560, DMS-1620418 (MIW) and Simons Foundation Math + X Investigator Award #376319 (MIW and AD). The authors thank A. B. Watson for very stimulating discussions and S. Dyatlov for suggesting a more conceptual proof to Proposition 4.2.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A. From Quasimodes to Eigenvalues in Gapped Selfadjoint Problems

Proof of Lemma 3.1. If E is an eigenvalue of T , the lemma is proved. Otherwise $T - E : \mathcal{H} \rightarrow \mathcal{H}$ is invertible; from the assumption, E is neither in the pure point spectrum nor in the essential spectrum. From (3.3), $|(T - E)^{-1}|_{\mathcal{H}} \geq \varepsilon^{-1}$. But because of the spectral theorem,

$$\|(T - E)^{-1}\|_{\mathcal{H}} = \frac{1}{\text{dist}(E, \Sigma(T))}.$$

We deduce $\text{dist}(E, \Sigma(T)) \leq \varepsilon$, therefore E is ε -close to the spectrum of T . Since $0 < \varepsilon < \varepsilon_0$, and T has no essential spectrum in $[E - \varepsilon_0, E + \varepsilon_0]$, T must have an eigenvalue ε in $[E - \varepsilon, E + \varepsilon]$.

Assume now that T has only one eigenvalue—say λ —in $[E - C\varepsilon, E + C\varepsilon]$. Let u be a corresponding normalized eigenvector. Define T' formally as T , but acting on the space $\mathcal{H}' = \text{span}\{u\}^\perp$. The spectral theorem implies that

$$\|(T' - E)^{-1}\|_{\mathcal{H}'} = \frac{1}{\text{dist}(E, \Sigma(T'))} \leq \frac{1}{C\varepsilon}.$$

Therefore we have

$$w \in \mathcal{H}' \Rightarrow C\varepsilon \cdot |w|_{\mathcal{H}'} \leq |(T' - E)w|_{\mathcal{H}'}. \tag{A.1}$$

If v satisfies (3.3), write $v = \alpha u + w$ where $w \in \mathcal{H}'$ and $\alpha \in \mathbb{C}$. Then,

$$(T - E)v = \alpha(T - E)u + (T - E)w = \alpha(\lambda - E)u + (T' - E)w.$$

This is because $Tu = \lambda u$ and $w \in \mathcal{H}'$, hence $Tw = T'w$. The RHS is an orthogonal decomposition of $(T - E)v$. Thus $|(T - E)v|_{\mathcal{H}} \geq |(T' - E)w|_{\mathcal{H}} = |(T' - E)w|_{\mathcal{H}'}$. From (3.3) the lower bound (A.1) and the relation $w = v - \alpha u$, we deduce

$$\varepsilon \geq |(T - E)v|_{\mathcal{H}} \geq |(T' - E)w|_{\mathcal{H}'} \geq C\varepsilon|w|_{\mathcal{H}'} = C\varepsilon|v - \alpha u|_{\mathcal{H}'}$$

Thus, $|v - \alpha u|_{\mathcal{H}} \leq C^{-1}$; since αu is an eigenvector of T , the proof is complete. \square

Appendix B. Spectral Estimates for Highly Oscillatory Operators

Lemma B.1. *Let $F \in C^\infty(\mathbb{R} \times \mathbb{R})$ be 1-periodic in the first variable and such that there exists $\theta > 0$ with*

$$\text{for all } \alpha \geq 0, \quad \sup_{(x,y) \in \mathbb{R}^2} e^{\theta|y|} \cdot \left| \partial_y^\alpha F(x, y) \right| < \infty.$$

Set $f_\delta(y) = F(y/\delta, y)$. Then there exists $C > 0$ such that for $\delta \in (0, 1)$,

$$\|f_\delta - g\|_{H^{-1/4}} \leq C\delta^{1/4}, \quad g(y) = \int_0^1 F(x, y) dx.$$

Proof. This is mostly contained in the proof of [Dr18b, Theorem 1]. Write a Fourier decomposition of F :

$$F(x, y) = \sum_{m \in 2\pi\mathbb{Z}} a_m(y) e^{imx}, \quad a_m(y) = \int_0^1 F(x, y) e^{-imx} dx.$$

Then,

$$\begin{aligned} \|f_\delta - g\|_{H^{-1/4}} &= \|\langle D \rangle^{-1/4} (f_\delta - g)\|_{L^2} \leq \sum_{m \in 2\pi\mathbb{Z} \setminus \{0\}} \|\langle \cdot \rangle^{-1/4} \widehat{a}_m(\cdot - m/\delta)\|_{L^2} \\ &= \sum_{m \in 2\pi\mathbb{Z} \setminus \{0\}} \|\langle \cdot \rangle^{-1/4} \langle \cdot - m/\delta \rangle^{-1/4} \langle \cdot - m/\delta \rangle^{1/4} \widehat{a}_m(\cdot - m/\delta)\|_{L^2} \\ &\leq \sum_{m \in 2\pi\mathbb{Z} \setminus \{0\}} \left(\sup_{\zeta \in \mathbb{R}} \langle \zeta \rangle^{-1/4} \langle \zeta - m/\delta \rangle^{-1/4} \right) \cdot \|\langle \cdot - m/\delta \rangle^{1/4} \widehat{a}_m(\cdot - m/\delta)\|_{L^2}. \end{aligned}$$

We now apply Peetre’s inequality to the first factor and deduce that

$$\|f_\delta - g\|_{H^{-1/4}} \leq \sum_{m \in 2\pi\mathbb{Z} \setminus \{0\}} \langle m/\delta \rangle^{-1/4} \cdot \|\langle \cdot \rangle^{1/4} \widehat{a}_m\|_{L^2} \leq C\delta^{1/4} \sum_{m \in 2\pi\mathbb{Z} \setminus \{0\}} \frac{\|a_m\|_{H^{1/4}}}{m^{1/4}}.$$

Because $F \in C^\infty(\mathbb{R} \times \mathbb{R})$ decays sufficiently fast, the sum on the RHS is absolutely convergent. This completes the proof. \square

Proof of Lemma 6.1. 1. We first use the cyclicity of the determinant to write

$$\text{Det}(\text{Id} + A\chi B) = \text{Det}(\text{Id} + \langle D \rangle^{-s} A \langle D \rangle^{-s} \cdot \langle D \rangle^s \chi B \langle D \rangle^s).$$

The trace-class norm controls the determinant difference—see e.g. [DZ18, Proposition B.26]. Hence,

$$\begin{aligned} |\text{Det}(\text{Id} + A\chi B) - \text{Det}(\text{Id} + A'\chi B)| &\leq C \|\langle D \rangle^{-s} (A - A') \langle D \rangle^{-s} \cdot \langle D \rangle^s \chi B \langle D \rangle^s\|_{\mathcal{L}(L^2)} \\ &\leq C \|\langle D \rangle^{-s} (A - A') \langle D \rangle^{-s}\|_{L^2} \cdot \|\langle D \rangle^s \chi B \langle D \rangle^s\|_{\mathcal{L}(L^2)}. \end{aligned}$$

The space of bounded operators from L^2 to H^{2-2s} with range in function with fixed compact support embeds continuously in the space of trace-class operators on L^2 , because $2 - 2s > 1$ —see e.g. [DZ18, Equation (B.3.9)]. Therefore,

$$|\text{Det}(\text{Id} + A\chi B) - \text{Det}(\text{Id} + A'\chi B)| \leq C \|\langle D \rangle^{-s} (A - A') \langle D \rangle^{-s}\|_{L^2} \cdot \|\chi B\|_{H^{-s} \rightarrow H^{2-s}}.$$

This completes the proof of Lemma 6.1. \square

References

- [AMU75] Ando, T., Matsumoto, Y., Uemura, Y.: Theory of Hall effect in a two-dimensional electron system. *J. Phys. Soc. Jpn.* **39**(2), 279–288 (1975)
- [ASV18] Avila, J.C., Schulz-Baldes, H., Villegas-Blas, C.: Topological invariants of edge states for periodic two-dimensional models. *Math. Phys. Anal. Geom.* **16**(2), 137–170 (2013)
- [Ba17] Bal, G.: Topological Protection of Perturbed Edge States. *Math. Phys.* <https://doi.org/10.4310/cms.2019.v17.n1.a8>
- [Ba18] Bal, G.: Continuous Bulk and Interface Description of Topological Insulators. *J. Math. Phys.* **60**, 081506 (2019). <https://doi.org/10.1063/1.5086312>
- [BC18] Berkolaiko, G., Comech, A.: Symmetry and Dirac points in graphene spectrum. *J. Spectr. Theory* **8**(3), 1099–1147 (2018)
- [Bo07] Borisov, D.I.: Some singular perturbations of periodic operators. *Theoret. Math. Phys.* **151**, 614–624 (2007)
- [Bo11] Borisov, D.I.: On the spectrum of a two-dimensional periodic operator with a small localized perturbation. *Izv. Math.* **75**(3), 471–505 (2011)
- [Bo15] Borisov, D.I.: On the band spectrum of a Schrödinger operator in a periodic system of domains coupled by small windows. *Russ. J. Math. Phys.* **22**(2), 153–160 (2015)
- [BG06] Borisov, D.I., Gadyl'shin, R.R.: On the spectrum of the Schrödinger operator with a rapidly oscillating compactly supported potential. *Theoret. Math. Phys.* **147**(1), 496–500 (2006)
- [BG08] Borisov, D.I., Gadyl'shin, R.R.: On the spectrum of a periodic operator with small localized perturbation. *Izv. Math.* **72**(4), 659–688 (2008)
- [BR18] Bourne, C., Rennie, A.: Chern numbers, localization and the bulk-edge correspondence for continuous models of topological phases. *Math. Phys. Anal. Geom.* **21**(3), 62 (2018)
- [Br18] Braverman, M.: The Spectral Flow of a Family of Toeplitz Operators. Preprint, [arXiv:1803.11101](https://arxiv.org/abs/1803.11101)
- [BPP17] Brendel, C., Peano, V., Painter, O., Marquardt, F.: Snowflake Topological Insulator for Sound Waves. Preprint, [arXiv:1701.06330](https://arxiv.org/abs/1701.06330)
- [CGP09] Castro Neto, A.H., Guinea, F., Peres, N.M.R., Novoselov, K.S., Geim, A.K.: The electronic properties of graphene. *Rev. Mod. Phys.* **81**, 109–162 (2009)
- [Co91] Colin de Verdière, Y.: Sur les singularités de van Hove génériques. *Mém. Soc. Math. France (N.S.)* **46**, 99–110 (1991)
- [DH86] Deift, P., Hempel, R.: On the existence of eigenvalues of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$. *Commun. Math. Phys.* **103**(3), 461–490 (1986)
- [DPR09] Dohnal, T., Plum, M., Reichel, W.: Localized modes of the linear periodic Schrödinger operator with a nonlocal perturbation. *SIAM J. Math. Anal.* **41**(5), 1967–1993 (2009)
- [Dr18a] Drouot, A.: Bound states for highly oscillatory potentials in dimension 2. *SIAM J. Math. Anal.* **50**, 1471–1484 (2018)
- [Dr18b] Drouot, A.: Scattering resonances for highly oscillatory potentials. *Ann. Sci. Éc. Norm. Supér. (4)* **51**(4), 865–925 (2018)
- [Dr18c] Drouot, A.: Resonances for random highly oscillatory potentials. *J. Math. Phys.* **59**, 101506 (2018)
- [Dr18d] Drouot, A.: The Bulk-Edge Correspondence for Continuous Dislocated Systems. [arXiv:1810.10603](https://arxiv.org/abs/1810.10603)
- [Dr19] Drouot, A.: Characterization of edge states in perturbed honeycomb structures. *Pure Appl. Anal.* **1**(3), 385–445 (2019)
- [DW19] Drouot, A., Weinstein, M.I.: Edge states and the valley Hall effect. *Adv. Math.* **368**, 107142 (2020)
- [DR18] Duchêne, V., Raymond, N.: Spectral asymptotics for the Schrödinger operator on the line with spreading and oscillating potentials. *Doc. Math.* **23**, 599–636 (2018)
- [DVW14] Duchêne, V., Vukićević, I., Weinstein, M.I.: Scattering and localization properties of highly oscillatory potentials. *Commun. Pure Appl. Math.* **67**(1), 83–128 (2014)
- [DW11] Duchêne, V., Weinstein, M.I.: Scattering, homogenization, and interface effects for oscillatory potentials with strong singularities. *Multiscale Model. Simul.* **9**(3), 1017–1063 (2011)
- [DZ18] Dyatlov, S., Zworski, M.: Mathematical theory of scattering resonances. *Graduate Studies in Mathematics*, 200. American Mathematical Society, Providence, RI, pp. xi+634 (2019)
- [EGS05] Elgart, A., Graf, G.M., Schenker, J.H.: Equality of the bulk and edge Hall conductances in a mobility gap. *Commun. Math. Phys.* **259**(1), 185–221 (2005)
- [FLW14b] Fefferman, C.L., Lee-Thorp, J.P., Weinstein, M.I.: Topologically protected states in one-dimensional continuous systems and Dirac points. *Proc. Natl. Acad. Sci. USA* **111**(24), 8759–8763 (2014)
- [FLW16a] Fefferman, C.L., Lee-Thorp, J.P., Weinstein, M.I.: Edge states in honeycomb structures. *Ann. PDE* **2**(2), Art. 12, 80 (2016)
- [FLW16b] Fefferman, C.L., Lee-Thorp, J.P., Weinstein, M.I.: Bifurcations of edge states—topologically protected and non-protected in continuous 2D honeycomb structures. *2D Mater.* **3**, 1 (2016)

- [FLW17] Fefferman, C.L., Lee-Thorp, J.P., Weinstein, M.I.: Topologically protected states in one-dimensional systems. *Mem. Am. Math. Soc.* **247**(1173), 118 (2017)
- [FLW18] Fefferman, C.L., Lee-Thorp, J.P., Weinstein, M.I.: Honeycomb Schrödinger operators in the strong binding regime. *Commun. Pure Appl. Math.* **71**(6), 1178–1270 (2018)
- [FW12] Fefferman, C.L., Weinstein, M.I.: Honeycomb lattice potentials and Dirac points. *J. Am. Math. Soc.* **25**(4), 1169–1220 (2012)
- [FW14] Fefferman, C.L., Weinstein, M.I.: Wave packets in honeycomb structures and two-dimensional Dirac equations. *Commun. Math. Phys.* **326**(1), 251–286 (2014)
- [FW18] Fefferman, C.L., Weinstein, M.I.: Edge States of Continuum Schrödinger Operators for Sharply Terminated Honeycomb Structures. Preprint, [arXiv:1810.03497](https://arxiv.org/abs/1810.03497)
- [FK07] Fu, L., Kane, C.L.: Topological insulators with inversion symmetry. *Phys. Rev. B* **76**(4), 045302 (2007)
- [FSF12] Fukui, T., Shiozaki, K., Fujiwara, T., Fujimoto, S.: Bulk-edge correspondence for Chern topological phases: a viewpoint from a generalized index theorem. *J. Phys. Soc. Jpn.* **81**(11), 114602 (2012)
- [Ga15] Gannot, O.: From quasimodes to resonances: exponentially decaying perturbations. *Pac. J. Math.* **277**(1), 77–97 (2015)
- [Gé90] Gérard, C.: Resonance theory for periodic Schrödinger operators. *Bull. Soc. Math. France* **118**(1), 27–54 (1990)
- [GS92] Gérard, C., Sigal, I.M.: Space-time picture of semiclassical resonances. *Commun. Math. Phys.* **145**(2), 281–328 (1992)
- [GW05] Golowich, S., Weinstein, M.: Scattering resonances of microstructures and homogenization theory. *Multiscale Model. Simul.* **3**(3), 477–521 (2005)
- [GP13] Graf, G.M., Porta, M.: Bulk-edge correspondence for two-dimensional topological insulators. *Commun. Math. Phys.* **324**(3), 851–895 (2013)
- [GS18] Graf, G.M., Shapiro, J.: The Bulk-Edge Correspondence for Disordered Chiral Chains. *Math. Phys.* <https://doi.org/10.1007/s00220-018-3247-0>
- [GT18] Graf, G.M., Tauber, C.: Bulk-edge correspondence for two-dimensional Floquet topological insulators. *Ann. Henri Poincaré* **19**(3), 709–741 (2018)
- [GB15a] Gu, Y., Bal, G.: Homogenization of parabolic equations with large time-dependent random potential. *Stoch. Process. Appl.* **125**(1), 91–115 (2015)
- [GB15b] Gu, Y., Bal, G.: Fluctuations of parabolic equations with large random potentials. *Stoch. Partial Differ. Equ. Anal. Comput.* **3**(1), 1–51 (2015)
- [GB16] Gu, Y., Bal, G.: Weak convergence approach for parabolic equations with large, highly oscillatory, random potential. *Ann. Inst. Henri Poincaré Probab. Stat.* **52**(1), 261–285 (2016)
- [HR08] Haldane, F., Raghu, S.: Possible realization of directional optical waveguides in photonic crystals with broken time-reversal symmetry. *Phys. Rev. Lett.* **100**(1), 013904 (2008)
- [HK11a] Hempel, R., Kohlmann, M.: A variational approach to dislocation problems for periodic Schrödinger operators. *J. Math. Anal. Appl.* **381**(1), 166–178 (2011)
- [HK11b] Hempel, R., Kohlmann, M.: Spectral properties of grain boundaries at small angles of rotation. *J. Spectr. Theory* **1**(2), 197–219 (2011)
- [HKS15] Hempel, R., Kohlmann, M., Stautz, M., Voigt, J.: Bound states for nano-tubes with a dislocation. *J. Math. Anal. Appl.* **431**(1), 202–227 (2015)
- [HW11] Hoefler, M.A., Weinstein, M.I.: Defect modes and homogenization of periodic Schrödinger operators. *SIAM J. Math. Anal.* **43**(2), 971–996 (2011)
- [JR76] Jackiw, R., Rebbi, C.: Solitons with fermion number 1/2. *Phys. Rev. D* **13**(12), 3398–3409 (1976)
- [KM11] Kane, C.L., Moore, J.E.: Topological insulators. *Phys. World* **24**, 32–36 (2011)
- [KRS00] Kellendonk, J., Richter, T., Schulz-Baldes, H.: Edge current channels and Chern numbers in the integer quantum Hall effect. *Rev. Math. Phys.* **14**(1), 87–119 (2002)
- [KS04a] Kellendonk, J., Schulz-Baldes, H.: Quantization of edge currents for continuous magnetic operators. *J. Funct. Anal.* **209**, 388–413 (2004)
- [KS04b] Kellendonk, J., Schulz-Baldes, H.: Boundary maps for \mathbb{C}^* -crossed products with \mathbb{R} with an application to the quantum Hall effect. *Commun. Math. Phys.* **249**(3), 611–637 (2004)
- [KMT07] Khanikaev, A.B., Mousavi, S.H., Tse, W.-K., Kargarian, M., MacDonald, A.H., Shvets, G.: Topological insulators with inversion symmetry. *Phys. Rev. B* **76**(4), 045302 (2007)
- [KDP80] Klitzing, K.V., Dorda, G., Pepper, M.: New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance. *Phys. Rev. Lett.* **45**(6), 494–497 (1980)
- [Ko00] Korotyaev, E.: Lattice dislocations in a 1-dimensional model. *Commun. Math. Phys.* **213**(2), 471–489 (2000)
- [Ko05] Korotyaev, E.: Schrödinger operator with a junction of two 1-dimensional periodic potentials. *Asymptot. Anal.* **45**(1–2), 73–97 (2005)
- [Ku16] Kuchment, P.A.: An overview of periodic elliptic operators. *Bull. Am. Math. Soc.* **53**(3), 343–414 (2016)

- [La81] Laughlin, R.B.: Quantized Hall conductivity in two dimensions. *Phys. Rev. B.* **23**(10), 5632–5633 (1981)
- [Le18] Lee, M.: Dirac cones for point scatterers on a honeycomb lattice. *SIAM J. Math. Anal.* **48**(2), 1459–1488 (2016)
- [LWZ17] Lee-Thorp, J.P., Weinstein, M.I., Zhu, Y.: Elliptic operators with honeycomb symmetry: Dirac points, edge states and applications to photonic graphene. *Arch. Ration. Mech. Anal.* **232**, 1–63 (2019). <https://doi.org/10.1007/s00205-018-1315-4>
- [LVX16] Lee-Thorp, J.P., Vukicevic, I., Xu, X., Yang, J., Fefferman, C.L., Wong, C.W., Weinstein, M.I.: Photonic realization of topologically protected bound states in domain-wall modulated waveguide arrays. *Phys. Rev. A* **93**, 033822 (2016)
- [LWW18] Lu, J., Watson, A.B., Weinstein, M.I.: Dirac Operators and Domain Walls. *Siam J. Math. Anal.* **52**(2), 1115–45 (2020)
- [NKR15] Nash, L.M., Kleckner, D., Read, A., Vitelli, V., Turner, A.M., Irvine, W.T.M.: Topological mechanics of gyroscopic materials. *Proc. Natl. Acad. Sci. U.S.A.* **24**, 14495–14500 (2015)
- [PLA10] Parzygnat, A., Lee, K., Avniel, Y., Johnson, S.: Sufficient conditions for two-dimensional localization by arbitrarily weak defects in periodic potentials with band gaps. *Phys. Rev. B* **81**(15), 155324 (2010)
- [PWY16] Poo, Y., Lee-Thorp, J.P., Tan, Y., Wu, R., Weinstein, M.I., Yu, Z.: Observation of highly robust phase-defect induced photonic states. In: *Lasers and Electro-optics (CLEO)*, pp. 1–2 (2016)
- [OI03] Olaf, P.: Eigenvalues in spectral gaps of a perturbed periodic manifold. *Math. Nachr.* **261**(262), 141–162 (2003)
- [OPA18] Ozawa, T., Price, H.M., Amo, A., Goldman, N., Hafezi, M., Lu, L., Rechtsman, M., Schuster, D., Simon, J., Zilberberg, O., Carusotto, I.: Topological Photonics. [arXiv:1802.04173](https://arxiv.org/abs/1802.04173)
- [PS16] Prodan, E., Schulz-Baldes, H.: Bulk and Boundary Invariants for Complex Topological Insulators, From K-Theory to Physics. Springer, Berlin (2016)
- [RH08] Raghun, S., Haldane, F.D.M.: Analogs of quantum-hall-effect edge states in photonic crystals. *Phys. Rev. A* **78**(3), 033834 (2008)
- [RZP13] Rechtsman, M.C., Zeuner, J.M., Plotnik, Y., Lumer, Y., Podolsky, D., Dreisow, F., Nolte, S., Segev, M., Szameit, A.: Photonic Floquet topological insulators. *Nature* **496**, 196 (2013)
- [RS78] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics. IV. Analysis of Operators*. Academic Press, New York (1978)
- [Sh17] Shapiro, J.: The Bulk-Edge Correspondence in Three Simple Cases. <https://doi.org/10.1142/S0129055X20300034>
- [ST18] Shapiro, J., Tauber, C.: Strongly Disordered Floquet Topological Systems. *Ann. Henri Poincaré* **20**, 1837–1875 (2019). <https://doi.org/10.1007/s00023-019-00794-3>
- [SGK11] Singha, A., Gibertini, M., Karmakar, B., Yuan, S., Polini, M., Vignale, G., Kastnelson, M.I., Pinczuk, A., Pfeiffer, L.N., West, K.W., Pellegrini, V.: Two-dimensional Mott–Hubbard electrons in an artificial honeycomb lattice. *Science* **332**(6034), 1176–1179 (2011)
- [So93] Sogge, C.D.: *Fourier Integral Operators in Classical Analysis*. Cambridge University Press, Cambridge (1993)
- [St99] Stefanov, P.: Quasimodes and resonances: sharp lower bounds. *Duke Math. J.* **99**, 75–92 (1999)
- [St00] Stefanov, P.: Resonances near the real axis imply existence of quasimodes. *C. R. Acad. Sci. Paris Sér. I Math.* **330**(2), 105–108 (2000)
- [SV96] Stefanov, P., Vodev, G.: Neumann resonances in linear elasticity for an arbitrary body. *Commun. Math. Phys.* **176**, 645–659 (1996)
- [Ta14] Taarabt, A.: Equality of Bulk and Edge Hall Conductances for Continuous Magnetic Random Schrödinger Operators. Preprint, [arXiv:1403.7767](https://arxiv.org/abs/1403.7767)
- [TZ98] Tang, S.-H., Zworski, M.: From quasimodes to resonances. *Math. Res. Lett.* **5**(3), 261–272 (1998)
- [Th83] Thouless, D.J.: Quantization of particle transport. *Phys. Rev. B.* **27**(10), 6083–6087 (1983)
- [WCJ08] Wang, Z., Chong, Y.D., Joannopoulos, J.D., Soljacic, M.: Reflection-free one-way edge modes in a gyromagnetic photonic crystal. *Phys. Rev. Lett.* **100**, 013905 (2008)
- [Wa17] Watson, A.B.: Wave dynamics in locally periodic structures by multiscale analysis. Ph.D. Thesis, Columbia University (2017)
- [WW18] Watson, A., Weinstein, M.I.: Wavepackets in inhomogeneous periodic media: propagation through a one-dimensional band crossing. *Commun. Math. Phys.* **363**(2), 655–698 (2018)
- [YVW08] Yu, Z., Veronis, G., Wang, Z., Fan, S.: One-way electromagnetic waveguide formed at the interface between a plasmonic metal under a static magnetic field and a photonic crystal. *Phys. Rev. B* **100**, 023902 (2008)
- [Ze16] Zelenko, L.: Virtual bound levels in a gap of the essential spectrum of the weakly perturbed periodic Schrödinger operator. *Integral Equ. Oper. Theory* **85**(3), 307–345 (2016)
- [ZB14a] Zhang, N., Bal, G.: Convergence to SPDE of the Schrödinger equation with large, random potential. *Commun. Math. Sci.* **12**(5), 825–841 (2014)

[ZB14b] Zhang, N., Bal, G.: Homogenization of the Schrödinger equation with large, random potential. *Stoch. Dyn.* **14**(1), 29 (2014)

Communicated by W. Schlag