Dispersion of waves in two and three-dimensional periodic media

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ABSTRACT
We consider the propagation of acoustic time-harmonic waves in homogeneous media containing periodic lattices of spherical or cylindrical inclusions. It is assumed that the wavelength has the order of the periods of the lattice while the radius a of inclusions is small. A new approach is suggested to derive the complete asymptotic expansions of the dispersion relations in two and three-dimensional cases as $a \to 0$ and first several terms of the expansions are evaluated explicitly. Our method is based on the reduction of the original singularly perturbed (by inclusions) problem to the regular one. The Dirichlet, Neumann, and transmission boundary conditions are considered. In the former case, we estimate the cutoff wavelength $\lambda_{\max}$ supported by the periodic medium in two and three dimensions. The effective wave speed is obtained as a function of the wave frequency, the filling fraction of the inclusions, and the physical properties of the constituents of the mixture. Dependence of the asymptotic formulas obtained in the paper on geometric and material parameters is illustrated by graphs.

KEYWORDS
Periodic medium; phononic crystal; acoustic waves; group velocity; dispersion relation; Dirichlet-to-Neumann operator

1. Introduction

Periodic media offer plenty of possibilities for manipulating wave propagation. These include electromagnetic waves in photonic crystals, where one can create bands and gaps in the wave spectrum, positive or negative group velocity [1], slowing down considerably the speed of light [2–4], nonreciprocal media [5], the self-collimation effect [6] and more that lead to the development of new devices [7].

Similar phenomena can be observed in phononic crystals [8] for elastic or acoustic waves [9] whose bandgap structure is employed in sound filters, transducer design and acoustic mirrors. Research on breaking time-reversal symmetry in wave phenomena is a growing area of interest in the field of phononic crystals and metamaterials aimed at realizing one-way propagation devices which have many potential technological applications [10].

Deriving an explicit dispersion relation for the Floquet-Bloch waves in two and three-dimensional periodic media is an arduous problem and is usually performed numerically [11]. However, assuming that the wavelength is long compared to the period of the lattice or a characteristic size of the scatterers one can obtain an asymptotic

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approximation. Typically, it employs the method of matched asymptotic expansions as in [12–15] for small scatterers with the Dirichlet or Neumann boundary conditions. A semi-analytical approach using the multipole expansion method is described in [16]. A rigorous analysis of a sub-wavelength plasmonic crystal was presented in [17], where a solution of a nonlinear eigenvalue problem is given in terms of convergent high-contrast power series for the electromagnetic fields and the first branch of the dispersion relation. Explicit formulas for the effective dielectric tensor and the dispersion relation are obtained in [18] assuming that the cell size is small compared to the wavelength, but large compared to the size of the inclusions. Some other approaches are presented in [19–24].

We consider the propagation of waves, governed by the Helmholtz equation in a medium containing periodic lattices of spherical or cylindrical inclusions of radius a. Our study can be applied to phononic, sonic crystals or metamaterials with positive wave speed. We assume transmission boundary conditions on the inclusions' interface and that a is small relative to the period of the lattice while the wavelength is comparable to (or larger than) the lattice size. The Dirichlet and Neumann boundary conditions are also discussed. We suggest a new method for determining the dispersion relations of the Floquet-Bloch waves. The method reduces the original singularly perturbed problem to the regular one and provides explicit formulas for the dispersion relations in two and three-dimensional settings with rigorous estimates of the remainders. Our approach can be extended to small inclusions of arbitrary shape. However, it will require serious modifications. The coefficients of the asymptotic expansion in this case will be expressed through the solutions of regular boundary value problems.

2. Formulation of the problem

We consider the propagation of acoustic waves through an infinite medium containing a periodic array $\mathcal{B}$ of spherical obstacles. The periods of the lattice $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and $\boldsymbol{\tau}_3$ are normalized in such a way that $\ell = \min\{|\boldsymbol{\tau}_1|, |\boldsymbol{\tau}_2|, |\boldsymbol{\tau}_3|\} = 1$, while the radius of the balls $r = a \ll 1$ (see Figure 1).

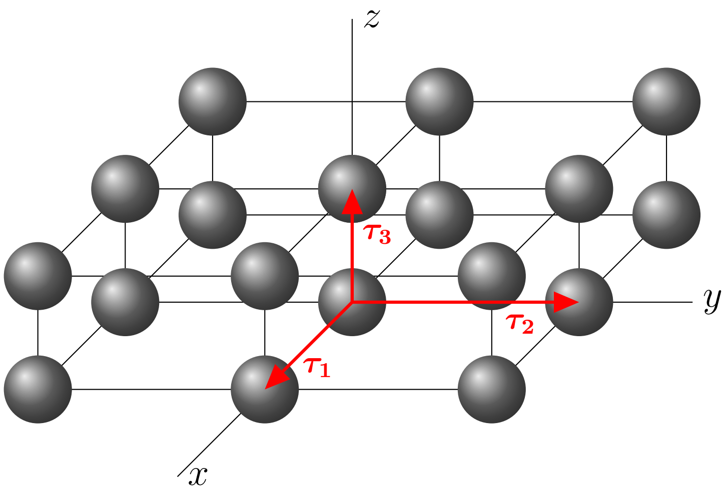

Figure 1. A fragment of periodic lattice of spherical inclusions of radius a generated by vectors $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and $\boldsymbol{\tau}_3$.

Assuming the excess pressure $p(\boldsymbol{r},t)$ to be time-harmonic, $p(\boldsymbol{r},t)=u(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t}$, the amplitude $u(\boldsymbol{r})$ should satisfy the equation

$$\nabla\cdot\frac{1}{\varrho(\boldsymbol{r})}\,\nabla u+\gamma(\boldsymbol{r})\,\omega^2 u=0,\quad \boldsymbol{r}\notin\partial\mathcal{B}, \tag{2.1}$$

where $\varrho(\boldsymbol{r})$ is the mass density and $\gamma(\boldsymbol{r})$ is the adiabatic bulk compressibility modulus of the media. Both $\varrho(\boldsymbol{r})$ and $\gamma(\boldsymbol{r})$ are periodic piecewise-constant functions with the periods of the lattice. In what follows, we denote the functions in the inclusions and the external medium by the subscripts $-$ and $+$, respectively. Thus,

$$\Delta u+k_{\pm}^2 u=0,\quad \boldsymbol{r}\notin\partial\mathcal{B}, \tag{2.2}$$

where

$$k_-=\sqrt{\varrho_-\gamma_-}\,\omega,\quad \boldsymbol{r}\in\mathcal{B}, \tag{2.3}$$

$$k_+=\sqrt{\varrho_+\gamma_+}\,\omega,\quad \boldsymbol{r}\notin\mathcal{B}. \tag{2.4}$$

We assume that the inclusions are penetrable and therefore impose the transmission conditions on their boundaries

$$[\![u(\boldsymbol{r})]\!]=0, \tag{2.5}$$

$$\left[\!\!\left[\frac{1}{\varrho(\boldsymbol{r})}\frac{\partial u(\boldsymbol{r})}{\partial n}\right]\!\!\right]=0, \tag{2.6}$$

where the brackets $[\![\cdot]\!]$ denote the jump of the enclosed quantity across the interface $\partial\mathcal{B}$ of the inclusions. The solution of (2.2) is sought in the form of Floquet-Bloch waves

$$u(\boldsymbol{r})=\Phi(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{2.7}$$

where $\boldsymbol{k}=(k_1,k_2,k_3)$ is the wave vector and $\Phi(\boldsymbol{r})$ is a periodic function with the periods of the lattice. The latter condition implies that function $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}u(\boldsymbol{r})$ is periodic. We write this as

$$]\!]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}u(\boldsymbol{r})[\![=0, \tag{2.8}$$

where the inverted brackets $]\!]\cdot[\![$ denote the jump of the enclosed expression and their first derivatives across the opposite sides of the cells of periodicity.

We reduce the above problem to the fundamental cell Π centered at the origin:

$$\begin{aligned}&\Delta u+k_-^2 u=0,\quad r<a,\\&\Delta u+k_+^2 u=0,\quad \boldsymbol{r}\in\Pi\cap\{r>a\},\end{aligned} \tag{2.9}$$

$$[\![u(\boldsymbol{r})]\!]=0,\quad \left[\!\!\left[\frac{1}{\varrho(\boldsymbol{r})}\frac{\partial u(\boldsymbol{r})}{\partial n}\right]\!\!\right]=0,\quad]\!]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}u(\boldsymbol{r})[\![=0. \tag{2.10}$$

In the inclusionless case, there is a simple dispersion relation between the time frequency ω and the spatial frequency $|\boldsymbol{k}|$. Namely, there are waves propagating in any direction and $\omega = c|\boldsymbol{k}|$, where $c = 1/\sqrt{\gamma_+ \varrho_+}$ is the speed of waves in the host medium. The dispersion relation $\omega = H(\boldsymbol{k}, a)$ in the presence of inclusions is more complicated and our goal is to find it when a is small. We will write the dispersion relation in the form $|\boldsymbol{k}|^2 = G(\hat{\boldsymbol{k}}, \omega, a)$, where $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$,

Interaction of the incident wave with inclusions creates multiple Floquet-Bloch waves in the same direction. We consider the waves for which $|\boldsymbol{k}|$ is close to ω/c when a is small. For other waves, $|\boldsymbol{k} + \boldsymbol{L}|$ is close to ω/c, where $\boldsymbol{L}$ is an arbitrary period of the dual lattice. These waves can be studied similarly.

We consider both three and two-dimensional cases ($d = 3, 2$). If $d = 2$, the inclusions have the shape of infinite cylinders shown in Figure 2.

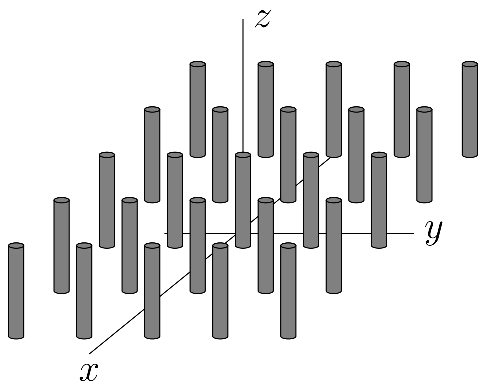

Figure 2. A fragment of periodic lattice of infinite cylinders of radius a.

Theorem 1. *Let $d = 3$. Then function G in the dispersion equation*

$$|\boldsymbol{k}|^2 = G(\hat{\boldsymbol{k}}, \omega, a), \quad \omega \neq 0, \tag{2.11}$$

is infinitely smooth. The first few terms of its Taylor expansion are given by

$$|\boldsymbol{k}|^2 = \frac{\omega^2}{c^2}\left(1 + c_1 a^3 + c_2 a^5\right) + g(\hat{\boldsymbol{k}}, \omega, a), \quad |g| < C(\omega) a^6, \quad a \to 0. \tag{2.12}$$

The same results are valid for the Neumann condition on inclusion boundaries, and (2.12) in this case has the form

$$|\boldsymbol{k}|^2 = \frac{\omega^2}{c^2}\left(1 + \frac{1}{2} f + \frac{3}{20}\left(\frac{\omega a}{c}\right)^2 f\right) + O\left(a^6\right), \quad f = \frac{4\pi a^3}{3|\Pi|}. \tag{2.13}$$

Here $c = 1/\sqrt{\gamma_+ \varrho_+}$ is the speed of waves in the host medium, $|\Pi|$ is the volume of the cell Π and f is the filling fraction of the inclusions.

For the transmission problem we have

$$c_1a^3 = \left(\frac{\varrho_- - 4\varrho_+}{\varrho_+ + 2\varrho_-} + \frac{\gamma_-}{\gamma_+}\right) f, \tag{2.14}$$

$$\begin{aligned} c_2a^5 = &\frac{1}{15}\left(\frac{\omega a}{c}\right)^2 f \left[\left(1 - \frac{\gamma_-}{\gamma_+}\right)\left(9 - \frac{5\gamma_-}{\gamma_+}\right) - \frac{\gamma_-}{\gamma_+}\left(1 - \frac{\gamma_-}{\gamma_+}\frac{\varrho_-}{\varrho_+}\right)\right] \\ &+ \frac{9}{5}\left(\frac{\omega a}{c}\right)^2 \frac{f}{(\varrho_+ + 2\varrho_-)^2}\left[\varrho_+^2 - \varrho_-^2 - \varrho_-\varrho_+\left(1 - \frac{\gamma_-}{\gamma_+}\frac{\varrho_-}{\varrho_+}\right)\right]. \end{aligned} \tag{2.15}$$

Theorem 2. *Let $d = 2$. Then the dispersion relation has the form*

$$|\boldsymbol{k}|^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \sigma_{ij}(\hat{\boldsymbol{k}}, \omega) a^{2i}\left(a^2 \ln a\right)^j, \quad a \to 0. \tag{2.16}$$

The first few terms of the asymptotic expansion of (2.16) are given by

$$|\boldsymbol{k}|^2 = \frac{\omega^2}{c^2}\left(1 + c_1a^2 - c_2a^4 \ln\frac{\omega a}{c}\right) + g(\hat{\boldsymbol{k}}, \omega, a), \quad |g| < C(\omega)a^4. \tag{2.17}$$

The same results are valid for the Neumann condition on inclusion boundaries, and (2.17) in this case has the form

$$|\boldsymbol{k}|^2 = \frac{\omega^2}{c^2}\left(1 + f - \frac{3}{2}\left(\frac{\omega a}{c}\right)^2 f \ln\frac{\omega a}{c}\right) + O\left(f^2\right), \quad f = \frac{\pi a^2}{|\Pi|}. \tag{2.18}$$

The constants c_i in (2.17) for the transmission problem are

$$c_1a^2 = \left(2\alpha - 1 + \frac{\gamma_-}{\gamma_+}\right) f, \qquad c_2a^4 = \left[\frac{1}{2}\left(1 - \frac{\gamma_-}{\gamma_+}\right)^2 + \alpha^2\right]\left(\frac{\omega a}{c}\right)^2 f, \tag{2.19}$$

where $\alpha = \dfrac{\varrho_- - \varrho_+}{\varrho_+ + \varrho_-}$.

Remark 1. The Neumann boundary condition can be obtained from the transmission conditions by passing to the limit $\lim \varrho_- \to \infty, \lim \gamma_-\varrho_- \to 0$. While formulas (2.13), (2.18) coincide with (2.14), (2.19), respectively, when $\varrho_-^{-1} = \gamma_- = 0$, we do not justify these limiting transitions in the formulas. Instead, one can obtain the results for the Neumann boundary conditions independently using the same approach applied below for the transmission boundary conditions.

Remark 2. Our approach can also be applied to the problem with the Dirichlet boundary condition (see the Appendix). The answer in this case differs considerably from the transmission problem. Namely,

- $d = 2$:

$$|\boldsymbol{k}|^2 = \frac{\omega^2}{c^2} + \frac{2\pi}{|\Pi| \ln\dfrac{\omega a}{c}} + O\left(\ln^{-2}\left(\frac{\omega a}{c}\right)\right), \quad a \to 0. \tag{2.20}$$

- $d = 3:$

$$|\boldsymbol{k}|^2 = \frac{\omega^2}{c^2} - \frac{4\pi a}{|\Pi|} + O\left(a^2\right), \quad a \to 0. \tag{2.21}$$

Remark 3. Unlike the transmission case, the propagating frequency ω for the Dirichlet boundary conditions is separated from zero by a cutoff frequency $\omega_{\min} > 0$. It follows from (2.21) that the maximum wavelength $\lambda_{\max}$ (the cutoff wavelength) supported by the periodic medium with the Dirichlet scatterers is given by

- $d = 2:$

$$\lambda_{\max} = \sqrt{2\pi|\Pi| \ln \frac{1}{a}} \left(1 + o\left(1\right)\right), \quad a \to 0, \tag{2.22}$$

- $d = 3:$

$$\lambda_{\max} = \sqrt{\frac{\pi|\Pi|}{a}} \left(1 + O\left(a\right)\right), \quad a \to 0. \tag{2.23}$$

It should be noted that the dispersion relation (2.20) is somewhat differs from that in [13] as the latter was derived under the assumption that the lattice length scale is much smaller than the wavelength. The correction term to the wavenumber in [13] does not depend on ω and as a result, the cutoff frequency in [13] is slightly higher.

3. Outline of the proof

We begin with the observation that the unperturbed eigenvalue problem

$$\begin{cases} (\Delta + k_+^2)u(\boldsymbol{r}) & = \lambda u(\boldsymbol{r}), \\ \quad \rrbracket \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} u(\boldsymbol{r}) \llbracket & = 0 \end{cases} \tag{3.1}$$

in the cell of periodicity Π and a fixed $\boldsymbol{k}$ with $|\boldsymbol{k}| = k_+$ has a simple eigenvalue $\lambda = 0$ with the eigenfunction $u = \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$. This statement is independent of whether we solve (3.1) in the Sobolev space $H^2(\Pi)$ or in the space of infinitely smooth functions, since the ellipticity of the problem implies that the solutions of (3.1) are infinitely smooth [25]. The simplicity of a zero eigenvalue can be easily established using the substitution $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} u = v$ and expanding the periodic function v into Fourier series.

The problem

$$\begin{cases} (\Delta + k_\pm^2)u(\boldsymbol{r}) & = \lambda u(\boldsymbol{r}), \quad \boldsymbol{r} \in \Pi, \quad r \neq a, \\ \qquad \llbracket u(\boldsymbol{r}) \rrbracket & = \left\llbracket \dfrac{1}{\varrho(\boldsymbol{r})} \dfrac{\partial u(\boldsymbol{r})}{\partial n} \right\rrbracket = 0, \quad \left\rrbracket \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} u(\boldsymbol{r}) \right\llbracket = 0 \end{cases} \tag{3.2}$$

can be considered as a perturbation of the problem (3.1). For arbitrary vector $\boldsymbol{k}$, we denote by $\boldsymbol{k}_+$ the vector with the same directions as $\boldsymbol{k}$ and with the magnitude k_+, i.e. $\boldsymbol{k}_+ = k_+\hat{\boldsymbol{k}}$. If a and $\varepsilon = k_+ - |\boldsymbol{k}|$ are small, then the perturbation is small from the point of view of physics. Since $\lambda = 0$ is a simple eigenvalue of the unperturbed problem, it follows that the eigenvalue $\lambda = \lambda(a, \varepsilon, \hat{\boldsymbol{k}})$ of the problem (3.2) is smooth when a and

$|\varepsilon|$ are small. Hence one could find the dispersion relation by finding $\lambda(a,\varepsilon,\hat{\boldsymbol{k}})$ and solving the equation $\lambda(a,\varepsilon,\hat{\boldsymbol{k}})=0$.

It is not easy to fulfill rigorously the approach discussed above, since the problem (3.2) with $a\neq 0$ is a singular perturbation of the problem (3.1), and the standard perturbation technique can not be applied. Thus we will consider the problem (3.2) only with $\lambda=0$:

$$\begin{cases} (\Delta+k_\pm^2)u(\boldsymbol{r}) &= 0, \quad \boldsymbol{r}\in\Pi, \quad r\neq a, \\ [\![u(\boldsymbol{r})]\!] &= \left[\!\!\left[\dfrac{1}{\varrho(\boldsymbol{r})}\dfrac{\partial u(\boldsymbol{r})}{\partial n}\right]\!\!\right]=0, \quad \left]\!\!\right] \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}u(\boldsymbol{r}) \left[\!\!\right[=0 \end{cases} \tag{3.3}$$

and will reduce it to an equivalent operator equation

$$(N_{\boldsymbol{k}}^+-N_a^-)\psi=0 \tag{3.4}$$

for which the standard perturbation theory is valid. Here, ψ is a function on a sphere $r=R$ of a fixed radius $R<1$ (a circle if $d=2$), and operator $N_{\boldsymbol{k}}^+-N_a^-$ (which will be defined below) is symmetric and Fredholm. Moreover, $N_{\boldsymbol{k}}^+-N_a^-$ is infinitely smooth in $\boldsymbol{k},a$, and has a simple zero eigenvalue when $\boldsymbol{k}=\boldsymbol{k}_+$, $a=0$. Thus, the dispersion relation can be found from (3.4) by the standard perturbation theory.

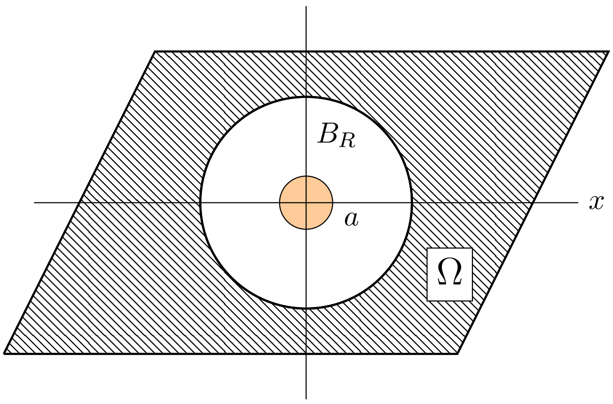

Figure 3. Decomposition of the cell of periodicity $\Pi=B_R\cup\Omega$.

Let us define operators in (3.4). We split Π into two domains $B_R=\{\boldsymbol{r}\mid r<R\}$, such that $B_R\subset\Pi$, and its compliment $\Omega=\Pi\setminus B_R$ (see Figure 3). Constant R will be fixed later, and $a\ll R$. Rather than (3.3), we consider two separate problems in Ω and B_R:

$$\left(\Delta+k_+^2\right)u^+(\boldsymbol{r})=0, \quad \boldsymbol{r}\in\Omega, \quad \left]\!\!\right]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}u^+(\boldsymbol{r})\left[\!\!\right[=0, \quad u^+|_{r=R}=\psi, \tag{3.5}$$

$$\left(\Delta+k_\pm^2\right)v(\boldsymbol{r})=0, \quad \boldsymbol{r}\in B_R\cap\{r\gtrless a\}, \quad [\![v(\boldsymbol{r})]\!]=\left[\!\!\left[\frac{1}{\varrho(\boldsymbol{r})}\frac{\partial v(\boldsymbol{r})}{\partial n}\right]\!\!\right]=0, \quad v|_{r=R}=\psi. \tag{3.6}$$

Both problems with $\psi\in C^\infty$ have unique C^∞-solutions for all values of R, except possibly a discrete set $\{R_i\}$. We fix an $R\notin\{R_i\}$ and define operators $\mathcal{N}_{\boldsymbol{k}}^+$ and $\mathcal{N}_a^-$ as the Dirichlet-to Neumann operators with the derivatives in the direction of r:

$$\mathcal{N}_{\boldsymbol{k}}^+, \mathcal{N}_a^- : H^1(\partial B_R) \to L^2(\partial B_R), \qquad \mathcal{N}_{\boldsymbol{k}}^+ \psi = \left.\frac{\partial u^+}{\partial r}\right|_{r=R}, \qquad \mathcal{N}_a^- \psi = \left.\frac{\partial v}{\partial r}\right|_{r=R}, \quad (3.7)$$

where $H^s(\partial B_R)$ is the Sobolev space of functions on ∂B_R. Constant a will always be assumed to be less than R.

4. Analysis of operators $\mathcal{N}_{\boldsymbol{k}}^+, \mathcal{N}_a^-$

Lemma 1.

(1) Operators $\mathcal{N}_{\boldsymbol{k}}^+$, $\mathcal{N}_a^-$ and

$$\mathcal{N}_{\boldsymbol{k}}^+ - \mathcal{N}_a^- : H^1(\partial B_R) \to L^2(\partial B_R) \qquad (4.1)$$

are Fredholm.

(2) Operators $\mathcal{N}_{\boldsymbol{k}}^+$ and $\mathcal{N}_a^-$ are symmetric in $L^2(\partial B_R)$. Thus, for example,

$$\int_{r=R} (\mathcal{N}_{\boldsymbol{k}}^+ \psi)\, \overline{\phi}\, \mathrm{d}S = \int_{r=R} \psi\, \overline{(\mathcal{N}_{\boldsymbol{k}}^+ \phi)}\, \mathrm{d}S, \qquad \psi, \phi \in H^1(\partial B_R). \qquad (4.2)$$

Proof. Dirichlet-to-Neumann operators (for elliptic equations of the second order) are elliptic pseudo-differential operators of the first order (see [26]). Thus, the first statement for operators $\mathcal{N}_{\boldsymbol{k}}^+, \mathcal{N}_a^-$ is immediately apparent. If one proves that the difference $\mathcal{N}_{\boldsymbol{k}}^+ - \mathcal{N}_a^-$ is still elliptic (i.e., the principal symbol of the difference is not vanishing), then the first statement for (4.1) becomes an immediate consequence of the ellipticity. This approach could be completed, since a calculation of the symbols for the Dirichlet-to-Neumann operators can be found in [26]. We prefer to prove the ellipticity of the difference by considering a simplified version $n^+ - n^-$ of operator $\mathcal{N}_{\boldsymbol{k}}^+ - \mathcal{N}_a^-$, where

$$n^{\pm}\psi = \left.\frac{\partial u^{\pm}}{\partial r}\right|_{r=R}$$

are the Dirichlet-to-Neumann operators related to the following problems (which are simplified versions of (3.5), (3.6)):

$$\Delta u^+ = 0, \quad R < r < 1, \quad u^+|_{r=R} = \psi, \quad u^+|_{r=1} = 0,$$

$$\Delta u^- = 0, \quad r < R, \quad u^-|_{r=R} = \psi.$$

From the local a priori estimates for elliptic operators, it follows that a perturbation of the boundary value problem outside a neighbourhood of the boundary on which the Dirichlet-to-Neumann operator is defined does not change the symbol of a Dirichlet-to-Neumann operator. The principal symbols of the Dirichlet-to-Neumann operators do not depend also on the lower order terms of the equations. Thus, the principal symbols of the operators $n^{\pm}$ are the same as the ones for the operators $\mathcal{N}_{\boldsymbol{k}}^+, \mathcal{N}_a^-$, respectively.

Hence, the first statement of the lemma will be proved if we prove that $n^+ - n^-$ is an elliptic operator of the first order. For this purpose, it is enough to show that operator

$$n^+ - n^- : H^1(\partial B_R) \to L^2(\partial B_R) \tag{4.3}$$

is an isomorphism. Consider the three-dimensional case (the two-dimensional case is similar). Denote by $\widetilde{\Delta}$ the Laplace-Beltrami operator on the sphere ∂B_R. Its spectrum consists of the eigenvalues $\lambda_n = n(n{+}1),\ n \geqslant 0$, of multiplicity $2n{+}1$. Let $\psi \in L^2(\partial B_R)$ and let ψ_n be the projection of ψ into the eigenspace of $\widetilde{\Delta}$ with the eigenvalue $\lambda_n = n(n+1)$. Thus $\psi = \sum_{n=0}^{\infty} \psi_n$, and the norms of ψ can be defined as follows:

$$\|\psi\|^2_{L^2(\partial B_R)} = \sum_{n=0}^{\infty} \|\psi_n\|^2_{L^2(\partial B_R)}, \tag{4.4}$$

$$\begin{aligned} \|\psi\|^2_{H^1(\partial B_R)} &= \|\psi_0\|^2_{L^2(\partial B_R)} + \sum_{n=1}^{\infty} \|\widetilde{\Delta}^{1/2}\psi_n\|^2_{L^2(\partial B_R)} \\ &= \|\psi_0\|^2_{L^2(\partial B_R)} + \sum_{n=1}^{\infty} n(n+1)\|\psi_n\|^2_{L^2(\partial B_R)}. \end{aligned} \tag{4.5}$$

We now estimate $(n^+ - n^-)\psi$. Obviously,

$$u^+ = \sum_{n=0}^{\infty} \frac{r^n - r^{-n-1}}{R^n - R^{-n-1}}\,\psi_n, \quad u^- = \sum_{n=0}^{\infty} \frac{r^n}{R^n}\,\psi_n,$$

and therefore

$$(n^+ - n^-)\psi = \left[\frac{\partial}{\partial r}\sum_{n=0}^{\infty} \frac{R^{-n-1}r^n - R^n r^{-n-1}}{R^n(R^n - R^{-n-1})}\,\psi_n\right]_{r=R} = \sum_{n=0}^{\infty} \frac{2n+1}{R + R^{2n+2}}\,\psi_n. \tag{4.6}$$

Since $R < 1$, the isomorphism of map (4.3) follows immediately from (4.4)-(4.6). This completes the proof of the first statement of the lemma.

Now we prove (4.2). From the symmetry of the problem (3.5) and Green's second identity, it follows that

$$\int_{r=R} (u_r\overline{v} - u\overline{v}_r)\,\mathrm{d}S = 0$$

for the solutions u, v of (3.5) with data ψ, ϕ at $r = R$, respectively. The latter relation is in agreement with (4.2). $\square$

Lemma 2.

(1) Relation $\psi = u|_{r=R}$ is a one-to-one correspondence between solutions $u \in C^\infty$ of (3.3) and solutions $\psi \in H^1(\partial B_R)$ of (3.4).

(2) Zero is a simple eigenvalue of the operator $(\mathcal{N}^+_{\boldsymbol{k}_+} - \mathcal{N}^-_0)$ *with the eigenfunction*

$$\hat{\psi} := \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}|_{r=R} = \mathrm{e}^{-\mathrm{i}k_+R\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}}. \tag{4.7}$$

Proof. Let $u \in C^\infty$ be a solution of (3.3). Then its restrictions to Ω and B_R satisfy (3.5), (3.6), respectively, with $\psi = u|_{r=R}$. Moreover, relation (3.4) holds, since $\mathcal{N}^+_{\boldsymbol{k}}\psi = \mathcal{N}^-_a\psi = u_r|_{r=R}$. Conversely, let $\psi \in H^1(\partial B_R)$ satisfy (3.4). The ellipticity of the equation (3.4) implies that $\psi \in C^\infty$. Let u coinside with solutions of (3.5), (3.6) in Ω and B_R, respectively. Then from (3.4) it follows that equation $(\Delta + k_+^2)u = 0$ holds also on ∂B_R, and therefore u is a solution of (3.3). The first statement of the lemma is proved. The second statement follows from the first one applied to the unperturbed problem $(\boldsymbol{k} = \boldsymbol{k}_+, a = 0)$ and the fact that zero is a simple eigenvalue of the problem (3.1) with the eigenfunction $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}|_{r=R}$. $\square$

We will write each element f in the domain $H^1(\partial B_R)$ and the range $L^2(\partial B_R)$ of operator (4.1) in the vector form $f = (f_1, f_L)$, where $f_1 = c\psi_0$ is the projection of f in $L^2(\partial B_R)$ on element (4.7) and f_L is orthogonal to $\hat{\psi}$ in $L^2(\partial B_R)$. Denote by $L_1,\ L_0$ the subspaces of $H^1(\partial B_R),\ L^2(\partial B_R)$, respectively, that consist of the functions orthogonal in $L^2(\partial B_R)$ to ψ_0. Then, due to Lemmas 1,2, operator (4.1) has the following matrix form

$$\mathcal{N}^+_{\boldsymbol{k}_+} - \mathcal{N}^-_0 = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \tag{4.8}$$

where $A : L_1 \to L_0$ is an isomorphism.

Obviously, operator $\mathcal{N}^+_{\boldsymbol{k}}$ is an infinitely smooth function of $\boldsymbol{k}$. Thus the operator $\mathcal{N}^+_{\boldsymbol{k}} - \mathcal{N}^-_0$ in the same basis chosen for $\boldsymbol{k} = \boldsymbol{k}_+$ has a matrix representation

$$\mathcal{N}^+_{\boldsymbol{k}} - \mathcal{N}^-_0 = \begin{bmatrix} C\varepsilon + O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & A + O(\varepsilon) \end{bmatrix} = \begin{bmatrix} C\varepsilon + \varepsilon^2 D_{11} & \varepsilon D_{12} \\ \varepsilon D_{21} & A + \varepsilon D_{22} \end{bmatrix}, \quad \varepsilon = k_+ - |\boldsymbol{k}|, \tag{4.9}$$

where C is a constant, $|\varepsilon| \ll 1$ and $D_{ij} = D_{ij}(\varepsilon, k_+, \hat{\boldsymbol{k}}),\ k_+ > 0$, are infinitely smooth functions of all the arguments. We will evaluate constant C in the next Lemma.

Lemma 3. *Constant C in the matrix expansion (4.9) of the operator $\mathcal{N}^+_{\boldsymbol{k}} - \mathcal{N}^-_0$ is equal to*

$$C = 2k_+|\Pi| \tag{4.10}$$

in both dimensions $d = 2$ and $d = 3$.

Proof. Recall that $\varepsilon = k_+ - |\boldsymbol{k}|$ and $\boldsymbol{k}_+ = k_+\hat{\boldsymbol{k}}$. Hence $\boldsymbol{k} = \boldsymbol{k}_+ - \varepsilon\hat{\boldsymbol{k}}$, and the boundary condition on $\partial\Pi$ in (3.5) can be written in the form $\left]\!\!\right]\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_+ - \varepsilon\hat{\boldsymbol{k}})\cdot\boldsymbol{r}}u(\boldsymbol{r})\left[\!\!\right[= 0$. Then problem (3.5) in $\Omega = \Pi \setminus B_R$ becomes a regular perturbation of the same problem with $\boldsymbol{k} = \boldsymbol{k}_+$. Thus, constant C in (4.9) is the coefficient in the linear term of the

Taylor expansion of functions $q^+ - q^-$ as $\varepsilon \to 0$ (i.e., $\boldsymbol{k} \to \boldsymbol{k}_+$), where

$$q^+ = \left(\mathcal{N}^+_{\boldsymbol{k}}\hat{\psi}, \hat{\psi}\right) = \int_{r=R} \frac{\partial u^+}{\partial r}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot \boldsymbol{r}}\, \mathrm{d}S, \quad q^- = \left(\mathcal{N}^-_0\hat{\psi}, \hat{\psi}\right) = \int_{r=R} \frac{\partial v}{\partial r}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot \boldsymbol{r}}\, \mathrm{d}S. \tag{4.11}$$

Here $\hat{\psi}$ and u^+, v are solutions of (3.5), (3.6), respectively, with $\psi = \hat{\psi}$ and without the inclusion in problem (3.6).

We now evaluate q^+. By the choice of R, problem (3.5) with $\boldsymbol{k} = \boldsymbol{k}_+$ is uniquely solvable. If $\psi = \hat{\psi}$, then its solution is $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}$. Thus, solution u^+ of (3.5) can be expanded into the Taylor series in ε with the zero order term being $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}$, i.e.,

$$u^+(\boldsymbol{r}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}} + \varepsilon u_1(\boldsymbol{r}) + O(\varepsilon^2), \quad \boldsymbol{r} \in \Omega, \tag{4.12}$$

where the remainder is uniformly small together with all its derivatives in $\boldsymbol{r}$, and u_1 is the solution of the problem

$$\begin{aligned} &(\Delta + k_+^2) u_1(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \Omega, \\ &\rrbracket \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}} u_1(\boldsymbol{r}) \llbracket = \rrbracket \mathrm{i}\hat{\boldsymbol{k}}\cdot\boldsymbol{r} \llbracket, \quad u_1(\boldsymbol{r})|_{r=R} = 0. \end{aligned} \tag{4.13}$$

We note that

$$\int_{r=R} \frac{\partial \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}}{\partial r}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\, \mathrm{d}S = -\mathrm{i}\int_{r=R} \boldsymbol{k}_+ \cdot \hat{\boldsymbol{r}}\, \mathrm{d}S = 0. \tag{4.14}$$

Thus, from (4.11) and (4.13) it follows that

$$q^+ = \varepsilon \int_{r=R} \frac{\partial u_1(\boldsymbol{r})}{\partial r}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\, \mathrm{d}S + O(\varepsilon^2), \quad \varepsilon \to 0. \tag{4.15}$$

In order to evaluate the integral above, we make the substitution $u_1(\boldsymbol{r}) = w(\boldsymbol{r}) + \mathrm{i}(\hat{\boldsymbol{k}}\cdot\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}$, which reduces (4.13) to the following problem for w with the homogeneous boundary condition on $\partial\Pi$:

$$\begin{aligned} &(\Delta + k_+^2) w(\boldsymbol{r}) = -2k_+ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}, \quad \boldsymbol{r} \in \Omega, \\ &\rrbracket \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}} w(\boldsymbol{r}) \llbracket = 0, \quad w(\boldsymbol{r})|_{r=R} = -\mathrm{i}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}) R\, \mathrm{e}^{-\mathrm{i}k_+ R\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}}. \end{aligned} \tag{4.16}$$

Denote the integral in (4.15) by q_1^+. Then

$$q_1^+ = \mathrm{i}\int_{r=R} \frac{\partial((\hat{\boldsymbol{k}}\cdot\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}})}{\partial r}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\, \mathrm{d}S + \int_{r=R} \frac{\partial w(\boldsymbol{r})}{\partial r}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\, \mathrm{d}S := q_{11}^+ + q_{12}^+.$$

We have

$$q_{11}^+ = \mathrm{i}\int_{r=R} (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}})\, \mathrm{d}S + k_+ R \int_{r=R} (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}})^2\, \mathrm{d}S.$$

The first integrand above is odd, and the corresponding integral is zero. Depending on the dimension $d = 2, 3$, the second integral above is $1/d$ times the Lebesgue measure of ∂B_R. Thus $q_{11}^+ = \dfrac{k_+R}{d}|\partial B_R|$, where $|\partial B_R| = 2\pi R$ in the dimension $d = 2$ and $|\partial B_R| = 4\pi R^2$ if $d = 3$.

Using Green's formula and the symmetry of the boundary problem (4.16), we obtain

$$q_{12}^+ = \int_{r=R} w(\boldsymbol{r})\frac{\partial \mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}}{\partial r}\,\mathrm{d}S - \int_\Omega [(\Delta + k_+^2)w(\boldsymbol{r})]\mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,d\boldsymbol{r} = \frac{k_+R}{d}|\partial B_R| + 2k_+|\Omega|.$$

Hence,

$$q_1^+ = 2\frac{k_+R}{d}|\partial B_R| + 2k_+|\Omega| = 2k_+|\Pi|. \tag{4.17}$$

Now we evaluate

$$\left(\mathcal{N}_0^-\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\right) = \int_{r=R}\frac{\partial v}{\partial r}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}S.$$

Function v solves the problem

$$\Delta v + k_+^2 v = 0, \quad r < R,$$
$$v|_{r=R} = \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}.$$

Then

$$\begin{aligned} q^- &= \int_R \frac{\partial v}{\partial r}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}S = \int_R v\frac{\partial}{\partial r}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}S \\ &= \int_R \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\frac{\partial}{\partial r}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}S = \mathrm{i}k_+\int_R \boldsymbol{k}_+\cdot\hat{\boldsymbol{r}}\,\mathrm{d}S. \end{aligned}$$

The latter integrand is odd, and therefore $q^- = 0$. This fact, together with (4.17), implies (4.10).

□

The next two lemmas are crucial in what follows, since they allow one to circumvent the singularity of the problem (3.3) as $a \to 0$.

Lemma 4. *Let $d = 3$. Then the operator function $\mathcal{N}_a^- - \mathcal{N}_0^- : H^1(B_R) \to L^2(B_R)$ is infinitely smooth in $a \geqslant 0$, and there are bounded operators P and Q such that*

$$\mathcal{N}_a^- - \mathcal{N}_0^- = Pa^3 + Qa^5 + O(a^6), \quad a \to 0, \tag{4.18}$$

Furthermore,

$$\left((\mathcal{N}_a^- - \mathcal{N}_0^-)\hat{\psi}, \hat{\psi}\right) = \frac{\omega^2}{c^2}\,|\Pi|\left(c_1a^3 + c_2a^5 + O\left(a^6\right)\right), \quad a \to 0, \tag{4.19}$$

where c_1 and c_2 are given by (2.14), (2.15) in Theorem 1.

Proof. We prove this lemma using explicit construction of the operator $\mathcal{N}_a^- - \mathcal{N}_0^-$. We denote the Laplace-Beltrami operator on the sphere ∂B_R by $\widetilde{\Delta}$. Its spectrum consists of the eigenvalues $\lambda_n = n(n+1)$, $n \geqslant 0$, of multiplicity $2n+1$ while the eigenfunctions are spherical harmonics. Let ψ_n be the projection in $L^2(\partial B_R)$ of ψ on the eigenspace of $\widetilde{\Delta}$ with the eigenvalue $\lambda_n = n(n+1)$. Thus $\psi = \sum_{n=0}^{\infty} \psi_n$. Then solution of problem (3.6) has the form

$$u = \begin{cases} \displaystyle\sum_{n=0}^{\infty} a_n\, j_n(k_- r)\, \psi_n, & 0 \leqslant r < a, \\ \displaystyle\sum_{n=0}^{\infty} [b_n\, j_n(k_+ r) + c_n\, y_n(k_+ r)]\, \psi_n, & a < r < R, \end{cases} \tag{4.20}$$

where j_n, y_n are the spherical Bessel functions and constants a_n, b_n, c_n are determined from the the boundary conditions in (3.6). Solving (4.20) we obtain that

$$a_n = \frac{k_+}{\varrho_+} \left[j_n(k_+ a) y_n'(k_+ a) - j_n'(k_+ a) y_n(k_+ a) \right] d_n^{-1} = \left(\varrho_+ k_+ a^2 d_n \right)^{-1}, \tag{4.21}$$

$$b_n = \left[\frac{k_+}{\varrho_+}\, j_n(k_- a) y_n'(k_+ a) - \frac{k_-}{\varrho_-}\, j_n'(k_- a) y_n(k_+ a) \right] d_n^{-1}, \tag{4.22}$$

$$c_n = \left[\frac{k_-}{\varrho_-}\, j_n'(k_- a) j_n(k_+ a) - \frac{k_+}{\varrho_+}\, j_n(k_- a) j_n'(k_+ a) \right] d_n^{-1}, \tag{4.23}$$

where

$$\begin{aligned} d_n = {} & \frac{k_+}{\varrho_+}\, j_n(k_- a) \left[j_n(k_+ R) y_n'(k_+ a) - j_n'(k_+ a) y_n(k_+ R) \right] \\ & + \frac{k_-}{\varrho_-}\, j_n'(k_- a) \left[j_n(k_+ a) y_n(k_+ R) - j_n(k_+ R) y_n(k_+ a) \right]. \end{aligned} \tag{4.24}$$

Solution of the problem (3.6) without the inclusion has the form

$$u = \sum_{n=0}^{\infty} \frac{j_n(k_+ r)}{j_n(k_+ R)}\, \psi_n.$$

Hence, $\mathcal{N}_0^- \psi = k_+ \displaystyle\sum_{n=0}^{\infty} \frac{j_n'(k_+ R)}{j_n(k_+ R)}\, \psi_n$, and

$$\begin{aligned} \mathcal{N}_a^- \psi - \mathcal{N}_0^- \psi &= k_+ \sum_{n=0}^{\infty} \left[b_n\, j_n'(k_+ R) + c_n\, y_n'(k_+ R) - \frac{j_n'(k_+ R)}{j_n(k_+ R)} \right] \psi_n \\ &= \sum_{n=0}^{\infty} \frac{f_n}{d_n\, j_n(k_+ R)}\, \psi_n, \end{aligned} \tag{4.25}$$

where

$$f_n = \frac{1}{k_+ R^2} \left[\frac{k_-}{\varrho_-}\, j_n(k_+ a) j_n'(k_- a) - \frac{k_+}{\varrho_+}\, j_n(k_- a) j_n'(k_+ a) \right].$$

From the asymptotic behavior of Bessel functions $J_n(x), Y_n(x)$ [27] we have for $0 < x \ll \sqrt{n}$ and $n \to \infty$:

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) \sim \sqrt{\frac{\pi}{2x}}\, \frac{1}{\Gamma\left(n+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{n+\frac{1}{2}} \sim \frac{1}{2}\sqrt{\frac{\pi}{n}}\, \frac{1}{n!} \left(\frac{x}{2}\right)^{n}, \tag{4.26}$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}}\, Y_{n+\frac{1}{2}}(x) \sim -\sqrt{\frac{\pi}{2x}}\, \frac{\Gamma(n+\frac{1}{2})}{\pi} \left(\frac{2}{x}\right)^{n+\frac{1}{2}} \sim -\frac{n!}{2\sqrt{\pi n}} \left(\frac{2}{x}\right)^{n+1}. \tag{4.27}$$

Then

$$|f_n| \leqslant \frac{C_1^n a^{2n-1}}{(n!)^2}, \qquad |d_n| \geqslant \frac{C_2^n}{a^2 n!}, \qquad n \geqslant 0, \tag{4.28}$$

where constants C_i do not depend on n, and therefore $\left|\dfrac{f_n}{d_n\, j_n(k_+R)}\right| \sim a^{2n+1}$ uniformly in $n \geqslant 1$ as $a \to 0$ and $\left|\dfrac{f_0}{d_0\, j_0(k_+R)}\right| \sim a^3$. The latter relations and (4.25) justify the first statement of the lemma.

In order to obtain the exact values of coefficients c_1, c_2 in (4.19) and complete the proof of Lemma 4, we need asymptotic expansions of f_n, d_n for $n = 0, 1$ when $a \to 0$. They look as follows:

$$d_0 = \frac{j_0(k_+R)}{\varrho_+ k_+ a^2} \left[1 + \frac{1}{6}\left(3k_+^2 - 2\,\frac{\varrho_+}{\varrho_-}\, k_-^2 - k_-^2\right) a^2 + O\left(a^3\right)\right], \tag{4.29}$$

$$f_0 = \frac{a}{3k_+R^2} \left[\frac{k_+^2}{\varrho_+} - \frac{k_-^2}{\varrho_-} - \frac{a^2}{10}\left(\frac{{k_+}^4}{\varrho_+} - \frac{{k_-}^4}{\varrho_-}\right) + \frac{a^2}{6}\, k_+^2 k_-^2 \left(\frac{1}{\varrho_-} - \frac{1}{\varrho_+}\right)\right] + O\left(a^5\right), \tag{4.30}$$

$$\begin{aligned} d_1 &= \frac{1}{3\varrho_-\varrho_+}\, \frac{k_-}{(k_+a)^2}\, j_1(k_+R) \\ &\times \left[\varrho_+ + 2\varrho_- - \frac{1}{10}\left(2\varrho_- + 3\varrho_+\right)(k_-a)^2 + \frac{1}{6}\, \varrho_+(k_+a)^2 + O\left(a^3\right)\right], \end{aligned} \tag{4.31}$$

$$f_1 = \frac{k_-a}{9R^2} \left[\frac{1}{\varrho_-} - \frac{1}{\varrho_+} - \frac{3a^2}{10}\left(\frac{k_-^2}{\varrho_-} - \frac{k_+^2}{\varrho_+}\right) + \frac{a^2}{10}\left(\frac{k_-^2}{\varrho_+} - \frac{k_+^2}{\varrho_-}\right)\right] + O\left(a^5\right). \tag{4.32}$$

Then

$$\mathcal{N}_a^-\psi - \mathcal{N}_0^-\psi = \frac{\varrho_+ a^3}{3R^2}\left[\frac{k_+^2}{\varrho_+} - \frac{k_-^2}{\varrho_-} - \frac{a^2}{6}\left(\frac{k_+^2}{\varrho_+} - \frac{k_-^2}{\varrho_-}\right)\left(3k_+^2 - 2\,\frac{\varrho_+}{\varrho_-}\,k_-^2 - k_-^2\right)\right.$$
$$\left. -\frac{a^2}{10}\left(\frac{{k_+}^4}{\varrho_+} - \frac{{k_-}^4}{\varrho_-}\right) + \frac{a^2}{6}\,k_+^2 k_-^2\left(\frac{1}{\varrho_-} - \frac{1}{\varrho_+}\right)\right]\frac{\psi_0}{j_0^2(k_+R)}$$
$$+ \frac{k_+^2 a^3}{3R^2}\,\frac{\varrho_-\varrho_+}{\varrho_+ + 2\varrho_-}\left[\frac{1}{\varrho_-} - \frac{1}{\varrho_+} + \frac{1}{2\varrho_-\varrho_+}\,\frac{\varrho_+ - \varrho_-}{\varrho_+ + 2\varrho_-}\left(\frac{1}{5}\,(2\varrho_- + 3\varrho_+)(k_-a)^2\right.\right.$$
$$\left.\left. +\ \varrho_+(k_+a)^2\right) - \frac{3a^2}{10}\left(\frac{k_-^2}{\varrho_-} - \frac{k_+^2}{\varrho_+}\right) + \frac{a^2}{10}\left(\frac{k_-^2}{\varrho_+} - \frac{k_+^2}{\varrho_-}\right)\right]\frac{\psi_1}{j_1^2(k_+R)} + O\left(a^6\right). \tag{4.33}$$

We now substitute $\hat{\psi}$ for ψ in (4.33) and evaluate quadratic form (4.19). We have

$$\psi_0 = \frac{1}{4\pi R^2}\int_{r=R} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}S, \tag{4.34}$$

$$\psi_1 = (\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})\left(\int_{r=R}(\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})^2\,\mathrm{d}S\right)^{-1}\int_{r=R}(\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}S. \tag{4.35}$$

Note that

$$\int_{r=R} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}S = 4\pi R^2 j_0(k_+R), \qquad \int_{r=R}(\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})^2\,\mathrm{d}S = \frac{4\pi R^4}{3}.$$

Differentiation of the first equality above with respect to $|\boldsymbol{k}_+|$ leads to

$$\int_{r=R}(\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}S = -4\pi\mathrm{i}R^3 j_1(k_+R).$$

Therefore

$$\psi_0 = j_0(k_+R), \quad \psi_1 = -3\mathrm{i}(\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})R^{-1}j_1(k_+R), \tag{4.36}$$

and

$$\|\psi_0\|^2 = 4\pi R^2 j_0^2(k_+R), \quad \|\psi_1\|^2 = 12\pi R^2 j_1^2(k_+R). \tag{4.37}$$

Hence

$$\left((\mathcal{N}_a^- - \mathcal{N}_0^-)\psi_0, \psi_0\right) = \frac{4\pi\varrho_+ a^3}{3}\left(\frac{k_+^2}{\varrho_+} - \frac{k_-^2}{\varrho_-}\right)$$
$$- \frac{4\pi a^5}{45\varrho_-^2}\left(9k_+^4\varrho_-^2 + k_-^2\varrho_-\varrho_+(k_-^2 - 15k_+^2) + 5k_-^4\varrho_+^2\right) + O\left(a^6\right),$$

$$\left((\mathcal{N}_a^- - \mathcal{N}_0^-)\psi_1, \psi_1\right) = \frac{4\pi a^3 k_+^2(\varrho_+ - \varrho_-)}{\varrho_+ + 2\varrho_-} - \frac{12\pi a^5 k_+^2}{5(\varrho_+ + 2\varrho_-)^2}\left(k_+^2(\varrho_+^2 - \varrho_-^2) - \varrho_-\varrho_+(k_+^2 - k_-^2)\right) + O\left(a^6\right).$$

Now formula (4.19) follows from the last two relations and (4.33). □

Lemma 5. *Let $d = 2$. Then the operator function $\mathcal{N}_a^- - \mathcal{N}_0^- : H^1(B_R) \to L^2(B_R)$ has the following asymptotic expansion as $a \to 0$:*

$$\mathcal{N}_a^- - \mathcal{N}_0^- \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} N_{ij} a^{2i}\left(a^2 \ln a\right)^j, \quad N_{00} = 0. \tag{4.38}$$

Furthermore,

$$\left((\mathcal{N}_a^- - \mathcal{N}_0^-)\hat{\psi}, \hat{\psi}\right) = \frac{\omega^2}{c^2}|\Pi|\left(c_1 a^2 + c_2 a^4 \ln a + O\left(a^4\right)\right), \quad a \to 0, \tag{4.39}$$

where c_1 and c_2 are given in Theorem 2.

Proof. The proof is similar to that in Lemma 4. Namely, solution v of the problem (3.6) is represented in the form

$$v = \begin{cases} \sum\limits_{n=0}^{\infty} A_n\, J_n(k_- r)\, \psi_n, & 0 \leqslant r < a, \\ \sum\limits_{n=0}^{\infty} \left[B_n\, J_n(k_+ r) + C_n\, Y_n(k_+ r)\right]\psi_n, & a < r < R, \end{cases} \tag{4.40}$$

where $J_n(x), Y_n(x)$ are Bessel functions. Constants A_n, B_n, C_n are determined from the boundary conditions in (3.6):

$$A_n = \frac{k_+}{\varrho_+}\left[J_n(k_+ a)Y_n'(k_+ a) - J_n'(k_+ a)Y_n(k_+ a)\right] D_n^{-1} = \frac{2}{\pi \varrho_+ a D_n}, \tag{4.41}$$

$$B_n = \left[\frac{k_+}{\varrho_+} J_n(k_- a)Y_n'(k_+ a) - \frac{k_-}{\varrho_-} J_n'(k_- a)Y_n(k_+ a)\right] D_n^{-1}, \tag{4.42}$$

$$C_n = \left[\frac{k_-}{\varrho_-} J_n'(k_- a)J_n(k_+ a) - \frac{k_+}{\varrho_+} J_n(k_- a)J_n'(k_+ a)\right] D_n^{-1}, \tag{4.43}$$

where

$$D_n = Y_n(k_+ R)\left[\frac{k_-}{\varrho_-} J_n'(k_- a)J_n(k_+ a) - \frac{k_+}{\varrho_+} J_n'(k_+ a)J_n(k_- a)\right] - J_n(k_+ R)\left[\frac{k_-}{\varrho_-} J_n'(k_- a)Y_n(k_+ a) - \frac{k_+}{\varrho_+} J_n(k_- a)Y_n'(k_+ a)\right]. \tag{4.44}$$

Then

$$\mathcal{N}_a^- \psi - \mathcal{N}_0^- \psi = k_+ \sum_{n=0}^{\infty} \left[B_n\, J_n'(k_+R) + C_n\, Y_n'(k_+R) - \frac{J_n'(k_+R)}{J_n(k_+R)} \right] \psi_n$$
$$= \sum_{n=0}^{\infty} \frac{F_n}{D_n\, J_n(k_+R)}\, \psi_n, \tag{4.45}$$

where

$$F_n = \frac{2}{\pi R} \left[\frac{k_-}{\varrho_-}\, J_n(k_+a) J_n'(k_-a) - \frac{k_+}{\varrho_+}\, J_n(k_-a) J_n'(k_+a) \right].$$

Using the asymptotic behavior of the Bessel function as $x \to 0$

$$J_0(x) \sim 1, \qquad Y_0(x) \sim \frac{2}{\pi}\, \ln \frac{x}{2},$$
$$J_n(x) \sim \frac{1}{n!} \left(\frac{x}{2}\right)^n, \quad n \geqslant 1, \qquad Y_n(x) \sim -\frac{(n-1)!}{\pi} \left(\frac{2}{x}\right)^n, \quad n \geqslant 1,$$
$$J_0'(x) = -J_1(x) \sim -\frac{x}{2},$$

one can derive the asymptotic expansions of D_n and F_n as $a \to 0$:

$$D_n = \sum_{j=0}^{\infty} \alpha_{j,n}\, a^{2j-1} + \ln(k_+a) \sum_{j=\max(n,1)} \beta_{j,n}\, a^{2j-1}, \quad F_n = \sum_{j=n}^{\infty} \gamma_{j,n}\, a^{2j-1}. \tag{4.46}$$

In particular, for $n = 0, 1$ we have

$$D_0 = \frac{2}{\pi \varrho_+ a}\, J_0(k_+R) + \frac{a}{\pi} \left(\frac{k_-^2}{\varrho_-} - \frac{k_+^2}{\varrho_+} \right) J_0(k_+R) \ln(k_+a) + O\left(a\right), \tag{4.47}$$

$$F_0 = \frac{a}{\pi R} \left[\frac{k_+^2}{\varrho_+} - \frac{k_-^2}{\varrho_-} + \frac{a^2}{8} \left(\frac{k_-^4}{\varrho_-} - \frac{k_+^4}{\varrho_+} - 2k_-^2 k_+^2 \left(\frac{1}{\varrho_+} - \frac{1}{\varrho_-} \right) \right) \right] + O\left(a^5\right), \tag{4.48}$$

$$D_1 = \frac{k_-}{\pi k_+ a} \left[\frac{1}{\varrho_+} + \frac{1}{\varrho_-} \right] J_1(k_+R) + \frac{k_+ a}{2\pi} \left(\frac{1}{\varrho_+} - \frac{1}{\varrho_-} \right) J_1(k_+R) \ln(k_+a) + O\left(a\right), \tag{4.49}$$

$$F_1 = \frac{k_+ k_- a}{2\pi R} \left[\frac{1}{\varrho_-} - \frac{1}{\varrho_+} + \frac{a^2}{8 \varrho_- \varrho_+} \left(k_-^2 (\varrho_- - 3\varrho_+) - k_+^2 (\varrho_+ - 3\varrho_-) \right) \right] + O\left(a^5\right). \tag{4.50}$$

As a result,

$$\mathcal{N}_a^-\psi - \mathcal{N}_0^-\psi = \left[\frac{\varrho_+ a^2}{2R}\left(\frac{k_+^2}{\varrho_+} - \frac{k_-^2}{\varrho_-}\right) + \frac{(k_-^2\varrho_+ - k_+^2\varrho_-)^2}{4R\varrho_-^2}\, a^4 \ln(k_+a)\right]\frac{\psi_0}{J_0^2(k_+R)}$$
$$+ \left[\frac{k_+^2 a^2}{2R}\,\frac{\varrho_+ - \varrho_-}{\varrho_+ + \varrho_-} + \frac{k_+^4}{4R}\left(\frac{\varrho_+ - \varrho_-}{\varrho_+ + \varrho_-}\right)^2 a^4 \ln(k_+a)\right]\frac{\psi_1}{J_1^2(k_+R)} + O\left(a^4\right). \tag{4.51}$$

Substitution of (4.46) into (4.45) provides first statement (4.38) of the Lemma.
Next, we use the values of the integrals

$$\int_{r=R} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}s = 2\pi R J_0(k_+R),$$

$$\int_{r=R} (\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}s = 2\pi\mathrm{i} R J_1(k_+R).$$

Hence we calculate projections ψ_0, ψ_1 of the function $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}$:

$$\psi_0 = \frac{1}{2\pi R}\int_{r=R} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}s = J_0(k_+R), \tag{4.52}$$

$$\psi_1 = (\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})\left(\int_{r=R}(\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})^2\,\mathrm{d}s\right)^{-1}\int_{r=R}(\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_+\cdot\boldsymbol{r}}\,\mathrm{d}s$$
$$= \frac{\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r}}{\pi R}\,2\pi\mathrm{i}RJ_1(k_+R) = 2\mathrm{i}(\hat{\boldsymbol{k}}_+\cdot\boldsymbol{r})J_1(k_+R). \tag{4.53}$$

Therefore

$$\|\psi_0\|^2 = 2\pi R J_0^2(k_+R), \quad \|\psi_1\|^2 = 4\pi R J_1^2(k_+R), \tag{4.54}$$

which implies

$$\left((\mathcal{N}_a^- - \mathcal{N}_0^-)\,\psi_0, \psi_0\right) = \varrho_+\pi a^2\left[\frac{k_+^2}{\varrho_+} - \frac{k_-^2}{\varrho_-}\right] + \frac{\pi\ln(k_+a)}{2\varrho_-^2}\left(k_-^2\varrho_+ - k_+^2\varrho_-\right)^2 a^4 + O\left(a^4\right), \tag{4.55}$$

$$\left((\mathcal{N}_a^- - \mathcal{N}_0^-)\,\psi_1, \psi_1\right) = 2k_+^2\pi a^2\,\frac{\varrho_+ - \varrho_-}{\varrho_+ + \varrho_-} + \pi k_+^4\left(\frac{\varrho_+ - \varrho_-}{\varrho_+ + \varrho_-}\right)^2 \ln(k_+a)\,a^4 + O\left(a^4\right). \tag{4.56}$$

Addition of the above formulas gives (4.39) in terms of notation (2.3),(2.4). □

5. Proof of Theorems 1, 2

In order to prove Theorem 1, we use Lemma 2 and reduce problem (3.3) to the equivalent equation (3.4). Due to Lemma 1, the operator in the latter equation is Fredholm and symmetric. We rewrite (3.4) in the form

$$[(\mathcal{N}_{\boldsymbol{k}}^{+}-\mathcal{N}_0^{-})-(\mathcal{N}_a^{-}-\mathcal{N}_0^{-})]\psi=0. \tag{5.1}$$

The first term on the left is infinitely smooth in $\boldsymbol{k}$ and has the matrix representation (4.9) in the decompositions $(\hat{\psi}, L)$ of the domain and range of the operator. Recall that $\hat{\psi}$ is defined by (4.7) and L is the subspace of the domain or range, respectively, which consists of the functions orthogonal to $\hat{\psi}$ in $L^2(\partial B_R)$. Due to Lemma 4, the second operator in (5.1) is an infinitely smooth function of a. Hence equation (5.1) can be written in the form

$$\begin{bmatrix} C\varepsilon+\varepsilon^2 D_{11}+B_{11}(a) & \varepsilon D_{12}+B_{12}(a) \\ \varepsilon D_{21}+B_{21}(a) & A+\varepsilon D_{22}+B_{22}(a)\end{bmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=0, \quad \varepsilon=k_+-|\boldsymbol{k}|, \tag{5.2}$$

where $\phi_1=\sigma\hat{\psi}$ is the projection of ψ on $\hat{\psi}$, component ϕ_2 of ψ is orthogonal to $\hat{\psi}$, operator functions $D_{ij}=D_{ij}(\varepsilon,k_+,\hat{\boldsymbol{k}}),\ k_+>0$, and $B_{1j}(a)$ are infinitely smooth, and asymptotic expansions of operators B_{ij} as $a\to 0$ are given in Lemma 4. In particular, $\|B_{ij}\|=O(a^3),\ a\to 0$.

Since operator A is invertible, the second equation of the system (5.2) can be solved for ϕ_2, and the solution is proportional to the norm of ϕ_1, i.e.,

$$\phi_2=\sigma\hat{\phi}_2,$$

where $\hat{\phi}_2$ is a σ-independent element of L which is infinitely smooth in $\varepsilon, k_+, \hat{\boldsymbol{k}}$ and such that $\|\hat{\phi}_2\|=O(|\varepsilon|+a^3),\ |\varepsilon|+a\to 0$. Then the first equation of (5.2) implies

$$[C\varepsilon+B_{11}(a)+O((|\varepsilon|+a^3)^2)]\sigma=0.$$

Hence, a non-trivial solution of (5.2) for small ε, a exists if and only if

$$C\varepsilon+B_{11}(a)+O((|\varepsilon|+a^3)^2)=0, \tag{5.3}$$

where the left-hand side is an infinitely smooth function of $\varepsilon, k_+>0, \hat{\boldsymbol{k}}$, and operator $B_{11}(a)$ coincides with the left-hand side in (4.19). The dispersion relation is defined by solving (5.3) for ε. We obtain that $|\boldsymbol{k}|-k_+$ is an infinitely smooth function of $a, k_+, \hat{\boldsymbol{k}}$ and

$$|\boldsymbol{k}|-k_+=C^{-1}B_{11}+O(a^6). \tag{5.4}$$

This justifies the infinite smoothness of the dispersion relation (2.11). In order to justify expansion (2.12), we use (4.19) for $B_{11}(a)$, formula for C from Lemma 3, and arrive at

$$2k_+(|\boldsymbol{k}|-k_+)=\frac{\omega^2}{c^2}\left(c_1a^3+c_2a^5+O\left(a^6\right)\right), \quad a\to 0. \tag{5.5}$$

From here it follows that $|\boldsymbol{k}|-k_+=O(a^3)$ and therefore $2k_+=|\boldsymbol{k}|+k_++O(a^3)$. The latter relation together with (5.5) implies (2.12) and completes the proof of Theorem

1. We also note that the first correction term in (2.12) agrees with that provided in [28].

Theorem 2 can be proved similarly. One only needs to use Lemma 5 instead of Lemma 4. Hence, all the functions of a will have asymptotic expansions as in (4.38) instead of being infinitely smooth. Operators B_{ij} will have order $O(a^2)$, $a \to 0$, instead of $O(a^3)$, B_{11} will coincide with the left-hand side in (4.39), and (5.3) will have the form

$$C\varepsilon + B_{11}(a) + O\left((|\varepsilon| + a^2)^2\right) = 0. \tag{5.6}$$

Formula (5.6) implies the following analog of (5.4)

$$|\boldsymbol{k}| - k_+ = C^{-1} B_{11} + O(a^4). \tag{5.7}$$

Further arguments are the same as in the dimension $d = 3$.

The case of the Dirichlet boundary condition is considered in the Appendix.

6. Effective wave velocity

Asymptotic formulas (2.12), (2.17) allow us to calculate both the effective phase velocity $c_{\text{ph}} = \dfrac{\omega}{k}$ and the group velocity $c_* = \dfrac{\partial\omega}{\partial k}$. With the accuracy of the asymptotic formulas we have

- $d = 2$, transmission boundary conditions:

$$c_* = c\left(1 - \frac{1}{2}\,c_1 a^2 + \frac{3}{2}\,c_2 a^4 \ln\frac{\omega a}{c}\right), \tag{6.8}$$

$$c_{\text{ph}} = c\left(1 - \frac{1}{2}\,c_1 a^2 + \frac{1}{2}\,c_2 a^4 \ln\frac{\omega a}{c}\right). \tag{6.9}$$

- $d = 2$, Neumann boundary conditions:

$$c_* = c\left(1 - \frac{1}{2}\,f + \frac{9}{4}\left(\frac{\omega a}{c}\right)^2 f \ln\frac{\omega a}{c}\right), \tag{6.10}$$

$$c_{\text{ph}} = c\left(1 - \frac{1}{2}\,f + \frac{3}{4}\left(\frac{\omega a}{c}\right)^2 f \ln\frac{\omega a}{c}\right). \tag{6.11}$$

- $d = 3$, transmission boundary conditions:

$$c_* = c\left(1 - \frac{1}{2}\,c_1 a^3 - \frac{3}{2}\,c_2 a^5\right), \tag{6.12}$$

$$c_{\text{ph}} = c\left(1 - \frac{1}{2}\,c_1 a^3 - \frac{1}{2}\,c_2 a^5\right), \tag{6.13}$$

- $d = 3$, Neumann boundary conditions:

$$c_* = c\left(1 - \frac{1}{4}f - \frac{9}{40}\left(\frac{\omega a}{c}\right)^2 f\right), \tag{6.14}$$

$$c_{\text{ph}} = c\left(1 - \frac{1}{4}f - \frac{3}{40}\left(\frac{\omega a}{c}\right)^2 f\right), \tag{6.15}$$

Here c_1, c_2 are defined by (2.14),(2.15) in three dimensions and are given by (2.19) for $d = 2$. The formulas are consistent with the results reported in [13,29–31]. If $\gamma_-/\gamma_+ \gg 1$ as in the case of air bubbles in water where $\gamma_-/\gamma_+ \sim 1.5 \cdot 10^4$, then even a tiny concentration of the air bubbles causes a dramatic reduction of the wave speed. This effect was confirmed numerically in [32] and [30], however it is not attributed to the Minnaert resonance or wave localization. We rewrite first correction terms (2.14) and (2.19) in formulas (2.12) and (2.17), respectively, in terms of the wave speed $c_\pm$ in the constituent media

$$c_1 a^2 = \frac{(\varrho_- c_- - \varrho_+ c_+)^2 + \varrho_- \varrho_+ (c_+ - c_-)(c_+ + 3c_-)}{\varrho_- c_-^2 (\varrho_+ + \varrho_-)} f, \tag{6.16}$$

$$c_1 a^3 = \frac{(\varrho_- c_- - \varrho_+ c_+)^2 + 2\varrho_- \varrho_+ (c_+ - c_-)(c_+ + 2c_-)}{\varrho_- c_-^2 (\varrho_+ + 2\varrho_-)} f. \tag{6.17}$$

The first terms in the numerators of the formulas indicate the mismatch in the characteristic impedances ϱc of the media. These formulas show that if the scatterers are "slow" ($c_- < c_+$) then the effective wave speed will *always* be slower than the wave speed in the host medium. On the other hand, the presence of "fast" scatterers ($c_- > c_+$) does not guarantee that the effective wave velocity will be larger than that in the matrix. It will be so only when the characteristic impedances of the two media are close enough so that their mismatch does not cause strong scattering.

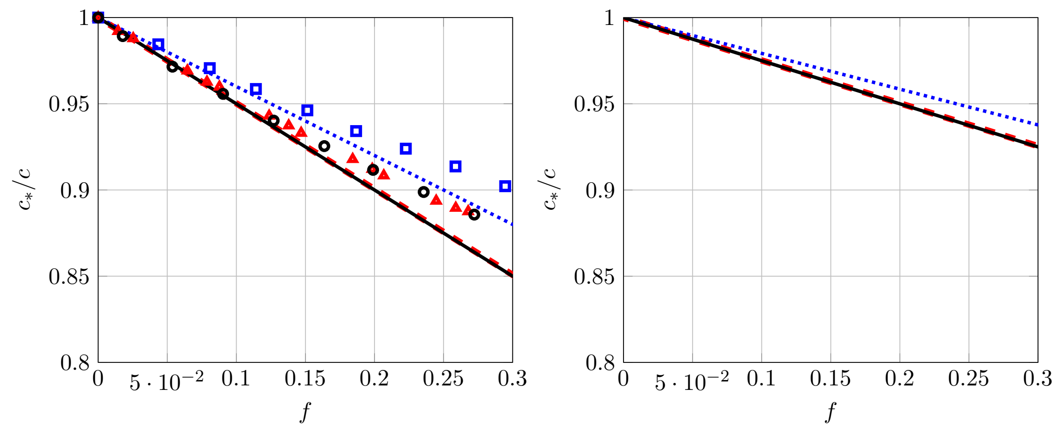

Figure 4. Long wavelength sound velocity for a hexagonal lattice of cylindrical (left) and an arbitrary three-dimensional lattice of spherical (right) inclusions as a function of the filling fraction, f. Lines correspond to the asymptotic formulas (6.8), (6.10), (6.12), (6.14) for water in air (red dashed), mercury in water (blue dotted) and rigid inclusions in air (solid black). Symbols depict results of numerical calculations for a cluster made of 151 cylinders [30] of water in air (red triangles), mercury in water (blue squares), and rigid cylinders (black circles).

Figure 4 depicts the comparison of the asymptotic formulas of the present paper with numerical calculations of the sound velocity for a hexagonal cluster of 151 fluid cylinders embedded in a fluid or gas (left panel) reported in [30] when $\omega \to 0$. The right panel shows the dependence of the long-wavelength sound group velocity for a lattice of spherical inclusions of the same constituents calculated by formulas (6.12), (6.14). Compared with the two-dimensional case, the sound velocity decays about twice as slowly. A qualitative explanation of this phenomenon is that the wave in 3D undergoes less scattering from the inclusions than in a similar 2D situation.

Increasing the wave frequency ω enhances the scattering and makes the decay of the sound velocity even faster as shown in Figure 5 for $\omega/c = 1$ and $|\Pi| = 1$. This effect is also illustrated in Figure 6 when the group velocity of sound in an orthorhombic lattice of rigid spherical inclusions decreases as ω/c increases from 0 to 3.

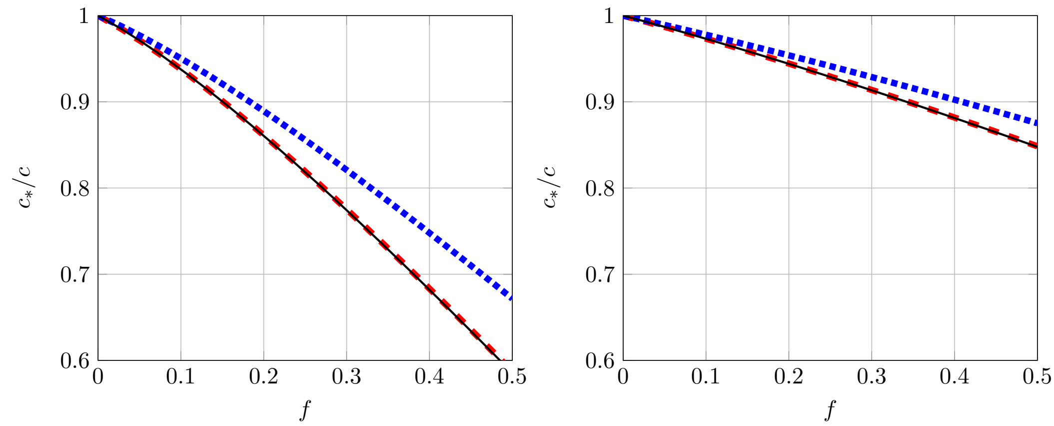

Figure 5. Same as Figure 4, but at the dimensionless wave frequency $\omega/c = 1$ and unit volume of the cell of periodicity $|\Pi| = 1$.

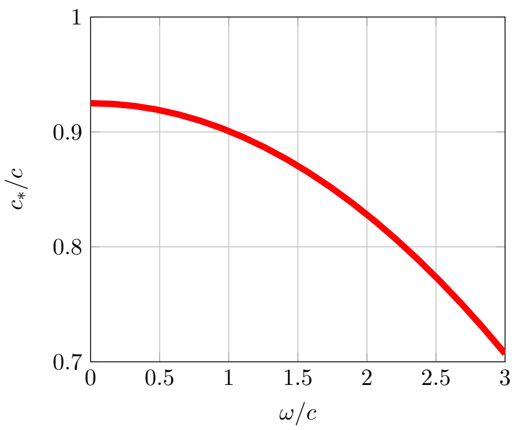

Figure 6. Effective group velocity for an infinite orthorhombic lattice of rigid spherical inclusions with the fundamental cell side ratios $1 : 1.5 : 2$ ($|\Pi| = 3$) as a function of dimensionless frequency ω/c when the filling fraction $f = 0.3$.

7. Conclusions

We have derived dispersion relations for the acoustic waves propagating in homogeneous medium containing a periodic lattice of spherical or cylindrical inclusions. The wavelength is assumed to be of the order of the periods of the lattice while the radius a of the inclusion is small compared to the periods. We suggest a new approach to derive and justify asymptotic expansions of the dispersion relations in two and three-dimensional cases as $a \to 0$ and evaluate explicitly first few terms. The approach is based on the reduction of the original singularly perturbed (by inclusions) problem to an equivalent regular one. To this end, we split the cell of periodicity into two parts by introducing an auxiliary spherical boundary of a fixed radius. We replace the original problem with the equality of the Dirichlet-to-Neumann maps on the auxiliary boundary of the new two subdomains. The Dirichlet, Neumann, and transmission boundary conditions on inclusion boundaries are considered. In the former case, we estimate the cutoff wavelength $\lambda_{\max}$ supported by the periodic medium in two and three dimensions. The effective wave speed is obtained as a function of the wave frequency, the filling fraction of the inclusions, and the physical properties of the constituents of the mixture. The method can also be extended to small inclusions of arbitrary shape.

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Appendix A. Dispersion relation for the Dirichlet boundary condition in 2D

We represent solution of the problem

$$\left(\Delta + k_+^2\right) v(\boldsymbol{r}) = 0, \quad a < r < R, \quad v|_{r=a} = 0, \quad v|_{r=R} = \psi, \tag{A1}$$

in the form

$$v = \sum_{n=0}^{\infty} \frac{J_n(k_+r)Y_n(k_+a) - J_n(k_+a)Y_n(k_+r)}{J_n(k_+R)Y_n(k_+a) - J_n(k_+a)Y_n(k_+R)}\, \psi_n. \tag{A2}$$

Then

$$\begin{aligned}\mathcal{N}_a^- \psi - \mathcal{N}_0^- \psi &= k_+ \sum_{n=0}^{\infty} \left[\frac{J_n'(k_+R)Y_n(k_+a) - J_n(k_+a)Y_n'(k_+R)}{J_n(k_+R)Y_n(k_+a) - J_n(k_+a)Y_n(k_+R)} - \frac{J_n'(k_+R)}{J_n(k_+R)} \right] \psi_n \\ &= -\frac{2}{\pi R} \sum_{n=0}^{\infty} \frac{J_n(k_+a)}{D_n J_n(k_+R)}\, \psi_n, \end{aligned}\tag{A3}$$

where $D_n = J_n(k_+R)Y_n(k_+a) - J_n(k_+a)Y_n(k_+R)$ and we used the identity $\mathscr{W}\left(J_n(z), Y_n(z)\right) = \frac{2}{\pi z}$ for the Wronskian $\mathscr{W}$ of the Bessel functions [27]. Asymptotic expansion of (A3) for $a \ll 1$ yields

$$\mathcal{N}_a^- \hat{\psi} - \mathcal{N}_0^- \hat{\psi} = -\frac{2}{\pi R} \left[\frac{\pi}{2\ln(k_+a) J_0^2(k_+R)} + O\left(\frac{1}{\ln^2(k_+a)} \right) \right] \psi_0. \tag{A4}$$

One can take into account more terms in expansion (A4). However, they will not provide a better approximation for the dispersion relation unless we increase the accuracy of (4.9), (5.3).

Evaluation of the functionals (4.55) together with (4.54) gives

$$\left(\left(\mathcal{N}_a^- - \mathcal{N}_0^-\right) \hat{\psi}, \hat{\psi} \right) = -\frac{2\pi}{\ln(k_+a)} + O\left(\frac{1}{\ln^2(k_+a)} \right). \tag{A5}$$

Substituting (A5) into (5.7), we find the dispersion relation (2.20).

Appendix B. Dispersion relation for the Dirichlet boundary condition in 3D

Solution of the three-dimensional problem (A1) can be represented in the form

$$v = \sum_{n=0}^{\infty} \frac{j_n(k_+r)y_n(k_+a) - j_n(k_+a)y_n(k_+r)}{j_n(k_+R)y_n(k_+a) - j_n(k_+a)y_n(k_+R)}\, \psi_n. \tag{B1}$$

Then

$$\mathcal{N}_a^- \psi - \mathcal{N}_0^- \psi = k_+ \sum_{n=0}^{\infty} \left[\frac{j_n'(k_+R) y_n(k_+a) - j_n(k_+a) y_n'(k_+R)}{j_n(k_+R) y_n(k_+a) - j_n(k_+a) y_n(k_+R)} - \frac{j_n'(k_+R)}{j_n(k_+R)} \right] \psi_n$$

$$= -\frac{1}{k_+ R^2} \sum_{n=0}^{\infty} \frac{j_n(k_+a)}{d_n j_n(k_+R)} \psi_n, \tag{B2}$$

where $d_n = j_n(k_+R) y_n(k_+a) - j_n(k_+a) y_n(k_+R)$ and, as in (A3), we used the identity $\mathscr{W}\left(j_n(z), y_n(z)\right) = \frac{1}{z^2}$ for the Wronskian $\mathscr{W}$ of the spherical Bessel functions [27]. Asymptotic expansion of (B2) for $a \ll 1$ yields

$$\mathcal{N}_a^- \hat{\psi} - \mathcal{N}_0^- \hat{\psi} = \frac{a \psi_0}{R^2 j_0^2(k_+R)} + O\left(a^2\right). \tag{B3}$$

As in (A4), the accuracy of (B3) is determined by the accuracy of (4.9), (5.3).

The dispersion relation (2.21) follows from (B3), (5.4), and (4.37).