

Alexander Gordon

Alexander (Sasha) Gordon, died in Chicago on May 13, 2019, after a long illness. Sasha was a brilliant mathematician, author of a number of beautiful and original results in diverse fields of spectral theory and other areas of analysis. His insights were behind some of the foundations of almost periodic operators as well as the theme of genericity of singular continuous spectrum developed by B. Simon and collaborators. Born in Kharkov, Ukraine, on May 10, 1947, Sasha has had an unusual path in mathematics. At the time of his death, he was an associate professor of mathematics at UNCC. His life story can be viewed as a triumph of human and mathematical spirit over the circumstances.

Sasha is best known for what is called Gordon's lemma, a statement about absence of decaying solutions of 1D equations close to periodic. It has led to the first example of an almost periodic operator with singular-continuous spectrum, to the understanding of importance of arithmetics in this area, to ultimately sharp arithmetic spectral transitions (e.g. [1, 17, 18]) as well as powerful consequences in the field of 1d quasicrystals (e.g. [4]).

Before Sasha's work it was not even known whether a condition on the frequency, imposed in various KAM arguments to prove point spectrum, is necessary (it was sometime before the singular continuous revolution of the 90s which allowed for an easy soft argument). His proof is based on a beautiful in its simplicity Lemma on $SL(2, \mathbb{R})$ matrices, formulated and proved in his two-page "Uspehi"¹ paper [5]. Later E.A. Gorin (1935-2018)[3] found a sleeker proof that easily generalized to a $GL(n, \mathbb{R})$ version, using the Cayley-Hamilton theorem, a result that remained unpublished. Gordon's paper appeared, in Russian, in 1975, and was not known in the west until B. Simon's Moscow visit in 1981, when S. Molchanov communicated to him, on the blackboard, Sasha's lemma with Gorin's proof. Avron and Simon at the time were trying to implement Sarnak's suggestion that spectral properties of quasiperiodic operators might depend on arithmetic properties of the frequencies, and this lemma then led to their first example of an operator with singular-continuous spectrum: the super-critical almost Mathieu operator with Liouville frequency [2]. This lemma was prominently featured in many of Simon's articles and textbooks under the name "Gordon's lemma",

¹The top Russian journal at the time, publishing mainly announcements with the two-page limit.

which Sasha made several futile attempts to change to Gordon-Gorin's, see, for instance, Footnote 2 in [12]. With the 1D result so natural and so simple, Sasha went on a life-long quest to find a multi-dimensional version. There have been attempts by various authors, not leading to anything significant. Sasha finally succeeded recently, in a highly original joint work with his friend from the undergraduate days A. Nemirovsky [15]. They provided the first quantitative condition for absence of point spectrum of multidimensional quasiperiodic operators, a result he was still not fully satisfied with and thriving to make stronger till his last days.

Another fundamental work of comparable importance is Gordon's theorem that 1D Schrodinger operators with interval spectrum have no point spectrum for generic boundary values [10, 11]. This indirectly influenced B. Simon's discovery of the Wonderland theorem [21] and other results on generic nature of singular continuous spectrum, a big theme in the 90s.

A certain prelude to that was a remarkable paper [9] where Gordon constructed an explicit potential with Green's function decay outside a set of energies of measure zero (leading to localization for a.e. boundary value by spectral averaging). It motivated powerful results by Kirsch, Molchanov, Pastur and others, as well as the work described above because the example, being so explicit, led to a natural question whether the "a.e." in the boundary values is necessary - something that Sasha answered in a surprising and very general way in [10, 11].

Another fundamental contribution was the proof of measurability of eigenvalues of continuous operator families, first in a joint work with Jitomirskaya, Last, and Simon [13] and then in the work with Kechris [14]. The issue is that for ergodic families, say, the entire collection of eigenvalues is not a measurable object, because it is invariant, yet not a.e. constant. Gordon's work showed, for example, that, in a very general setting, there exists a measurable enumeration. This was fundamental for several further important advances, particularly recent results on localization as a corollary of dual reducibility for quasiperiodic operators, and first results on arithmetic localization in the multi-dimensional setting.

Finally, we mention Sasha's proof [6] of the Hartman-Putnam conjecture that the lengths of spectral gaps of 1-D Schrodinger operators with bounded potentials tend to zero at infinity, a long open problem. Hartman and Putnam in their 1950 paper [16] proved this under an additional condition of uniform continuity and asked whether it could be removed.

Sasha produced several gems also outside the spectral theory. The highlights include

- A counterexample to the long-standing conjecture of A. Kolmogorov (1953) about the impossibility of mixed spectrum for an analytic flow with an integral invariant on a 2-D torus.[7]
- An effective sufficient condition for the cohomological equation to have no measurable solutions [8]

It should be noted that most of the beautiful results described above were obtained when Sasha was doing math in his spare time. His story is in a way characteristic of time and circumstances. Sasha got his undergraduate degree from Moscow State University in 1970, with high honors. As an undergraduate, he was an active participant of the Banach Algebras seminar led by E. Gorin and V. Lin, published two independent papers in top Russian journals and wrote an excellent MS thesis. Should he have finished in 1968 or earlier, such a performance would have more than guaranteed him a recommendation to continue as a PhD student, requiring only to pass several formal examinations. Things had changed however in 1969, when 39 Jewish students recommended for the PhD by their advisors, got Cs on the “History of the Communist Party” exam and were not allowed in. This marked the beginning of an era of significantly increased antisemitism at the School of Mathematics at MSU and in Soviet math in general, an era that lasted till the end of the 80s and of which Sasha was one of the victims. From 1970 on, Jews were largely not even recommended to be admitted to the graduate admission exams, although the problem soon almost disappeared because they mostly stopped being admitted as undergrads already in 1968.² Michael Brin, Svetlana Katok, Yakov Pesin are some other names of mathematicians who finished their undergraduate studies in 1969-70 and were treated similarly. For an excellent account of related issues see [19].

For the next almost 25 years Sasha’s work in math was done purely for fun. He took a day job but was able to stay in Moscow which made it possible for him to attend research seminars at MSU. The MSU math department was then at the beginning of the end of its “golden age”, and the quality of seminars was still outstanding (see again [19] for more detail). Sasha regularly attended the one on Mathematical Physics led by S. Molchanov, A. Ruzmaikin and D. Sokoloff. There, he was exposed to problems in spectral theory that led to his papers.

²See, for instance, [20] for an account of the techniques used to achieve this in the society that proclaimed its ideals of equality and internationalism, and for a fascinating story of an underground school, Jewish People’s University, created in 1978 so that “Jewish children can learn math”.

His advisor was S. Molchanov, whose questions proposed at the seminar indeed stimulated much of Sasha's research in spectral theory, but who views his own role rather as that of a friend and benefactor than an advisor. Sasha's thesis featuring, among other things, Hartman-Putnam and absence of point spectrum, was ready by about 1980, but to get a PhD, it was necessary for the advisor to find a university which would agree to schedule a defense for an outsider. It usually involved also finding a fake advisor, thus providing no benefit to the academic record of the true advisor. Most Jewish PhDs in the 70s-80s were obtained this way. Yet it also took time and financial resources to travel and be away from the day job, something that Sasha didn't have, making the task especially complicated in his case. Molchanov's multiple attempts to organize the defense at various universities failed. The university in Sasha's native Kharkov that was successfully used for this purpose in the 70s, say for the defense of M. Brin, by the 80s would no longer allow their own Jews, with, for example, M. Lyubich having to defend in Uzbekistan. Things changed around 1987, and in 1988 Molchanov finally succeeded in organizing Sasha's defense at the Moscow Institute of Electronic Machine Building where V. Maslov (from Maslov's index) was a department chair and thought it would be easy, but at the end, it still took Maslov threatening to quit his job to make it happen. Finally, in 1993 Sasha got a job at a research laboratory that would utilize his PhD and pay for him doing mathematics. At about the same time, the Russian government essentially stopped paying not just living wages, but any wages at all, to scientists.

As the Iron Curtain fell, Sasha got emboldened by the realization that his earlier work, popularized by Molchanov and Simon, has found interest and acclaim, and started trying to realize the dream of his youth to become a professional mathematician. He had several short visiting positions in Europe, and when invited by Molchanov to visit UNCC for a year in 1995-96, he decided to try to make it in the US. Yet, despite the Gordon Lemma fame, for a 48 year old with a heavy accent and a short list of short publications full of 3-5 year gaps, this seemed an almost impossible quest. There was not much he could do about the age or the accent, but remarkably, in the next ten years he managed to change the publication list around, producing a steady record of publications in computational bio-statistics, other aspects of applied math (true random number generators), and squeezing in a few in his beloved spectral theory. His work in bio-statistics was related to his job as a programmer at the Department of Computational Biology, University of Rochester, 2001-2006. It has been published in some of the leading journals in the field, including *Annals of Applied Statistics* and *SIAM J.Appl.Math*,

so we believe he has brought his remarkable originality there as well. In 2006, Sasha's UNCC friends managed to achieve what seemed to be impossible, and, aged 59, Sasha became a tenure-track assistant professor there. He was happy and grateful for an opportunity to live a normal academic life. He continued working in both spectral theory and computational bio-statistics.

Sasha's papers were usually short and always struck with their simplicity, clarity and mathematical beauty. While his list of early publications is much shorter than it could have been hadn't he had so many obstacles, it is fair to say that most of his works from that period, those that found lots of resonance as well as those that went almost unnoticed outside Russia, are true mathematical masterpieces.

Outside mathematics, Sasha was a connoisseur of Russian literature, especially poetry. He was active at UNCC's Russian literature club, where he often led the readings. He knew thousands of poems by heart and also wrote beautiful poems himself.

Sasha was battling a grave illness for the last six years, yet he continued working and creating until the very end. He finished grading his students' exams from his hospital bed two days before his death. His last papers, one each in computational bio-statistics and in the spectral theory are being published posthumously, the latter one in the present issue. He will be greatly missed.

S. Jitomirskaya, S. A. Molchanov, B. Simon, B. R. Vainberg

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