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On the critical behavior of a homopolymer model

In Memory of Professor Kai Lai Chung on the 100th Anniversary of His Birth

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Abstract We begin with the reference measure P^0 induced by simple, symmetric nearest neighbor continuous time random walk on $\mathbf{Z}^d$ starting at 0 with jump rate $2d$ and then define, for $\beta \geqslant 0$, $t > 0$, the Gibbs probability measure $P_{\beta,t}$ by specifying its density with respect to P^0 as

$$\frac{dP_{\beta,t}}{dP^0} = Z_{\beta,t}(0)^{-1}\mathrm{e}^{\beta\int_0^t \delta_0(x_s)ds}, \tag{0.1}$$

where $Z_{\beta,t}(0) \equiv E^0[\mathrm{e}^{\beta\int_0^t \delta_0(x_s)ds}]$. This Gibbs probability measure provides a simple model for a homopolymer with an attractive potential at the origin. In a previous paper (Cranston and Molchanov, 2007), we showed that for dimensions $d \geqslant 3$ there is a phase transition in the behavior of these paths from the diffusive behavior for β below a critical parameter to the positive recurrent behavior for β above this critical value. The critical value was determined by means of the spectral properties of the operator $\Delta + \beta\delta_0$, where Δ is the discrete Laplacian on $\mathbf{Z}^d$. This corresponds to a transition from a diffusive or stretched-out phase to a globular phase for the polymer. In this paper we give a description of the polymer at the critical value where the phase transition takes place. The behavior at the critical parameter is dimension-dependent.

Keywords Gibbs measure, homopolymer, phase transition, globular phase, diffusive phase

MSC(2010) 60K37, 60K35, 82B26, 82B27, 82D60, 35K10

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1 Introduction

In this paper we complete a picture of a homopolymer model, some of whose properties were discussed in the previous work of Cranston and Molchanov [2]. We can now give a fairly complete description of the polymer behavior at the critical parameter in all dimensions. Interest in polymer models is well developed. An early work on the subject [4] has been followed by myriad contributions and we refer the reader to an interesting paper [5] and its extensive bibliography. In contrast to other work on the homogeneous pinning model, our approach uses spectral theory and resolvent analysis in place of renewal theory. With our approach we are able to establish quite simply some interesting results about the behavior of the

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pinned homopolymer at the critical parameter which gives the point where a phase transition occurs. In order to describe the model, denote by Σ the space right continuous, left limit paths on $[0,\infty)$ into $\mathbf{Z}^d$. A typical element of Σ will be denoted by x and its position at time s by x_s. Sometimes we shall consider the restriction of elements of Σ to the interval $[0,t]$ and use Σ_t to denote these paths endowed with the natural σ-field $\mathcal{F}_{0,t}=\sigma(x_u : 0\leqslant u\leqslant t)$. We will later use the σ-fields $\mathcal{F}_{s,t}=\sigma(x_u : s\leqslant u\leqslant t)$ for $0\leqslant s<t\leqslant\infty$ generated by information from the paths on the interval $[s,t]$. The mapping $\theta_t:\Sigma\to\Sigma$ will be the usual shift operator. Our reference measure on Σ shall be P^0, where P^x denotes the measure induced by simple, symmetric nearest neighbor continuous time random walk on $\mathbf{Z}^d$ satisfying $P^x(x_0=x)=1$. This is the Markov process with generator the discrete Laplacian, i.e.,

$$\Delta\psi(x)=\sum_{y\in\mathbf{Z}^d:|x-y|=1}(\psi(y)-\psi(x)).$$

Define for $\beta\geqslant 0$ and $t>0$ the Gibbs (probability) measure $P_{\beta,t}$ on $(\Sigma_t,\mathcal{F}_{0,t})$ by specifying its density with respect to P^0 as

$$\frac{dP_{\beta,t}}{dP^0}=Z_{\beta,t}(0)^{-1}\mathrm{e}^{\beta\int_0^t\delta_0(x_s)ds}, \tag{1.1}$$

where

$$Z_{\beta,t}(0)\equiv E^0[\mathrm{e}^{\beta\int_0^t\delta_0(x_s)ds}]$$

is the usual normalizing factor called the partition function. Setting

$$Z_{\beta,t}(x)=E^x[\mathrm{e}^{\beta\int_0^t\delta_0(x_s)ds}]$$

enables us to define the Gibbs measure $P^x_{\beta,t}$ on paths started at x by

$$\frac{dP^x_{\beta,t}}{dP^x}=Z_{\beta,t}(x)^{-1}\mathrm{e}^{\beta\int_0^t\delta_0(x_s)ds}. \tag{1.2}$$

When $x=0$ we will write $Z_{\beta,t}$ in place of $Z_{\beta,t}(0)$. In the previous work [2], we demonstrated the existence of a phase transition in the parameter β at a particular parameter value which we shall denote by β_d. In all dimensions and all $\beta>\beta_d$, we proved that there is in a certain sense a limiting measure $P^x_{\beta,\infty}$. To be precise, for any $A\in\mathcal{F}_{0,t}$, the limit

$$\lim_{T\to\infty}P^x_{\beta,T}(A)=P^x_{\beta,\infty}(A)$$

determines a measure $P^x_{\beta,\infty}$ on $\mathcal{F}_{0,\infty}$. For $d=1$ or $d=2$, the process under the polymer measure is null recurrent for $\beta=0$ and positive recurrent for $\beta>0$. In dimensions $d\geqslant 3$, there is a dimension-dependent constant $\beta_d>0$ such that the process under the polymer measure $P^x_{\beta,\infty}$ is positive recurrent, for $\beta>\beta_d$, the globular phase. For $\beta<\beta_d$, the so-called diffusive phase, $x_t/\sqrt{t}$ is asymptotically normal with respect to $P^x_{\beta,t}$. We also proved there is a limit distribution for $x_t/\sqrt{t}$ at $\beta=\beta_d$ in dimensions 3 and 4. In this paper, we will recall our results about dimensions 3 and 4 and give a new proof which makes the result intuitively clear. We will establish the existence of $P^x_{\beta,\infty}$ at $\beta=\beta_d$ when $d\geqslant 5$ and give the value of β_d.

The phase transition described above corresponds to a transition for the operator

$$H_\beta=\Delta+\beta\delta_0.$$

In dimensions $d=1$ or $d=2$, this operator has a positive eigenvalue $\lambda_0(\beta)>0$ for all $\beta>0$. Thus we define $\beta_1=\beta_2=0$. In dimensions $d\geqslant 3$, for $\beta>\beta_d$, H_β has a positive eigenvalue $\lambda_0(\beta)$. Curiously, for $d\geqslant 5$, $\lambda_0(\beta_d)=0$ is an eigenvalue at the edge of the absolutely continuous part of the spectrum of H_β which is $[-4d,0]$. However, the situation in $d\leqslant 4$ is that there is no eigenvalue in the spectrum of H_β at $\beta=\beta_d$. In dimensions $d=3$ or $d=4$, the polymer paths at $\beta=\beta_d$ exhibit unusual behavior midway between the cases of $d\leqslant 2$ and $d\geqslant 5$.

We remark that for $d\geqslant 3$, the value β_d marks a transition in the free energy which is defined as

$$F(\beta)=\lim_{t\to\infty}\frac{1}{t}\ln Z_{\beta,t}.$$

Namely, for $\beta < \beta_d$ the free energy is $F(\beta) = 0$ while for $\beta > \beta_d$ one has $F(\beta) = \lambda_0(\beta) > 0$. In addition, this corresponds to the fact that $\int_0^\infty \delta_0(x_s)ds$ is an exponentially distributed random variable with parameter $2d\,r_d$ where $r_d = P^0$ (x never returns to the origin). One simply notes that

$$\int_0^\infty \delta_0(x_s)ds = \sum_{j=1}^N \tau_j,$$

where N, the number of visits to the origin, is a geometric random variable with parameter $1 - r_d$ which is independent of the i.i.d. sequence τ_j, $j \geqslant 1$ of sojourn times at the origin which are i.i.d. exponentially distributed random variables with parameter $2d$. Thus

$$Z_{\beta,\infty} = E^0[\mathrm{e}^{\beta \int_0^\infty \delta_0(x_s)ds}] < \infty$$

for $\beta < 2d\,r_d$ while this is infinite for $\beta \geqslant 2d\,r_d$ which shows that $\beta_d = 2d\,r_d$.

2 Behavior of the polymer at $\beta = \beta_d$

We now consider the behavior of the polymer at $\beta = \beta_d$. For $d = 1$ or 2, $\beta_d = 0$ and so the polymer measure $P_{\beta_d,\infty}$ is just P^0 and there is nothing new to add, but perhaps it is worth pointing out that the continuous time, simple, symmetric random walk is null recurrent in these dimensions. That the polymer in this case is weakly diffusive means $x_t/\sqrt{t}$ has a nondegenerate limiting distribution, which is of course Gaussian, yet x_t is recurrent.

For $d = 3$ or $d = 4$, the polymer is in a "weakly" diffusive phase at $\beta = \beta_d$. The potential has a weak, yet non-negligible, long-term effect in these dimensions at the critical value of the parameter β, but not strong enough to give a stationary probability distribution as is the case under $P_{\beta_d,\infty}$ when $d \geqslant 5$ described below. Here, we will show that the effect of the potential shows up in the behavior of σ_t/t, where

$$\sigma_t = \sup\{s \leqslant t : x_s = 0\}.$$

In dimensions 3 and 4 the variables σ_t/t have a limiting distribution under $P_{\beta_d,t}$ as $t \to \infty$. This distribution is more concentrated near 0 in dimension 3 than in dimension 4. For example, the mean of this limiting distribution is $1/3$ when $d = 3$ and $1/2$ when $d = 4$. We can derive the limiting distribution explicitly as well as that of $x_t/\sqrt{t}$ with respect to $P_{\beta_d,t}$ as $t \to \infty$. This relies on specifying the limiting distribution of σ_t/t. In these dimensions, the limiting distribution of $x_t/\sqrt{t}$ in the critical case $\beta = \beta_d$ is a mixture of Gaussians. This was first established in [2]. Here, we give a new proof showing how the mixture of Gaussians arises from the limiting behavior of σ_t/t. The reason is that the polymer is "free" of the influence of the potential after time σ_t and as a result, conditional on σ_t, the position x_t is approximately Gaussian with variance $t - \sigma_t = t(1 - \sigma_t/t)$. This will be made precise with a path decomposition at the time σ_t. One can think of the potential as providing a "sticky" boundary point in the critical case, but not "sticky" enough to create a bound state as in the cases $d \geqslant 5$.

In dimensions $d \geqslant 5$, the process under the polymer measure $P^x_{\beta_d,\infty}$ is positive recurrent. This curious case is due to the existence of 0 as an eigenvalue for the operator H_{β_d}. The associated eigenfunction for dimensions $d \geqslant 5$, denoted by ψ_{β_d}, provides the stationary probability measure for the time-homogeneous Markov process under $P_{\beta_d,\infty}$ in the form

$$\pi_{\beta_d} = \sum_{x\in\mathbf{Z}^d} \frac{\psi^2_{\beta_d}(x)}{\|\psi_{\beta_d}\|^2_{L^2(\mathbf{Z}^d)}}\delta_x.$$

This measure has fairly heavy tails, in the sense that only moments of order up to $d-5$ exist. By contrast, in the case $\beta > \beta_d$, all moments exist for the analogously defined measure π_β. This is why we say the polymer is in the "weakly" globular phase at $\beta = \beta_d$ for $d \geqslant 5$.

For $d = 3$ and $d = 4$, the following theorem describes the behavior at criticality.

Theorem 2.1. *For $\sigma_3(du) \equiv \frac{1}{2\sqrt{u}}du$, $0 \leqslant u \leqslant 1$,*

$$\lim_{t\to\infty} P_{\beta_3,t}(\sigma_t/t \in du) = \sigma_3(du), \tag{2.1}$$

$$\lim_{t\to\infty} E_{\beta_3,t}\left[\exp\left\{\mathrm{i}\left\langle \phi, \frac{x_t}{\sqrt{t}}\right\rangle\right\}\right] = \int_0^1 \mathrm{e}^{-|\phi|^2(1-u)}\sigma_3(du), \quad \phi \in \mathbb{R}^3. \tag{2.2}$$

For $\sigma_4(du) \equiv du$, $0 \leqslant u \leqslant 1$,

$$\lim_{t\to\infty} P_{\beta_4,t}(\sigma_t/t \in du) = \sigma_4(du), \tag{2.3}$$

$$\lim_{t\to\infty} E_{\beta_4,t}\left[\exp\left\{\mathrm{i}\left\langle \phi, \frac{x_t}{\sqrt{t}}\right\rangle\right\}\right] = \int_0^1 \mathrm{e}^{-|\phi|^2(1-u)}\sigma_4(du), \quad \phi \in \mathbb{R}^4. \tag{2.4}$$

For $0 < s < 1$, $0 < t < \infty$, let $F \in \mathcal{F}_{0,st}$ and $G \in \mathcal{F}_{st,t}$ be bounded random variables. Then we have the path decomposition for $d = 3, 4$, with μ the uniform distribution on $\{\pm e_j : j = 1, \dots, d\}$ where e_j are the unit vectors in $\mathbf{Z}^d$,

$$E_{\beta_d,t}[FG \,|\, \sigma_t/t = s] = P_{\beta_d,t}(F \,|\, \sigma_t/t = s)E^{\mu}[G \circ \theta_{-st} \,|\, x_u \neq 0, 0 \leqslant u \leqslant (1-s)t]. \tag{2.5}$$

The situation at criticality for $d \geqslant 5$ is described in the following theorem.

Theorem 2.2. *For $d \geqslant 5$ and $\beta = \beta_d$, there is a measure $P_{\beta_d,\infty}$ on Σ such that as $t \to \infty$ the process $(\Sigma_T, \mathcal{F}_{0,T}, P_{\beta_d,t})$ converges in law to $(\Sigma_T, \mathcal{F}_{0,T}, P_{\beta_d,\infty})$. The process $(\Sigma, \mathcal{F}_{0,\infty}, P_{\beta_d,\infty})$ is a stationary Markov process with Q-matrix*

$$q_d(x,y) = \begin{cases} 0, & \text{if } |x-y| > 1, \\ \dfrac{\psi_{\beta_d}(y)}{\psi_{\beta_d}(x)}, & \text{if } |x-y| = 1, \\ \beta_d\delta_0(x) - 2d, & \text{if } y = x, \end{cases} \tag{2.6}$$

and the generator

$$A_{\beta_d}f(x) = \sum_y q_d(x,y)(f(y) - f(x)),$$

where ψ_{β_d} denotes the eigenfunction of H_{β_d} corresponding to $\lambda_0(\beta_d) = 0$. The transition density for this ergodic, pure jump, Markov process on $\mathbf{Z}^d$ is given by

$$r_{\beta_d}(s,x,y) = \frac{p_{\beta_d}(s,x,y)\psi_{\beta_d}(y)}{\psi_{\beta_d}(x)}. \tag{2.7}$$

Its invariant probability distribution is

$$\pi_{\beta_d} = \sum_{x\in\mathbf{Z}^d} \frac{\psi^2_{\beta_d}(x)}{\|\psi^2_{\beta_d}\|^2}\delta_x. \tag{2.8}$$

The $2k$-th moment of π_{β_d} is finite if and only if $d \geqslant 2k+5$.

The distribution of the endpoint of the polymer, x_t, with respect to $P_{\beta_d,t}$ satisfies

$$\lim_{t\to\infty} P_{\beta_d,t}\,(x_t = x) = \frac{\psi_{\beta_d}(x)}{\langle\psi_{\beta_d}, \mathbf{1}\rangle}. \tag{2.9}$$

3 Resolvent analysis

Our analysis rests on properties of the resolvent, for $\lambda > 0$,

$$R_{0,\lambda}(x,y) = \int_0^\infty \mathrm{e}^{-\lambda s}p_0(s,x,y)ds,$$

where we have included a subscript of 0 in the notation since later the Gibbs measure with parameter β will be introduced which will have a parameter β and the quantities just defined will correspond to the parameter value $\beta = 0$. The resolvent satisfies the equation

$$(\Delta - \lambda)R_{0,\lambda}(x, y) = -\delta_y(x). \tag{3.1}$$

For $\phi \in \mathbf{T}^d$, the d-dimensional torus, with coordinates $\phi = (\phi_1, \dots, \phi_d)$, we use

$$\Phi(\phi) = 2\sum_{j=1}^{d}(1 - \cos\phi_j)$$

to denote the symbol (Fourier transform) of $-\Delta$. A useful fact is that $\Phi(\phi) \sim \|\phi\|^2$ as $\phi \to 0$. On applying the Fourier transform to (3.1), one obtains

$$\hat{R}_{0,\lambda}(\phi, y) = \frac{\mathrm{e}^{\mathrm{i}\langle\phi,y\rangle}}{\lambda + \Phi(\phi)}. \tag{3.2}$$

On inversion of (3.2), we have the representation

$$R_{0,\lambda}(0, y) = \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d}\frac{\mathrm{e}^{\mathrm{i}\langle\phi,y\rangle}}{\lambda + \Phi(\phi)}d\phi. \tag{3.3}$$

Since the function (of λ at $y = 0$) in (3.3) plays a central role in our development, we denote it for simplicity by

$$I(\lambda) \equiv \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d}\frac{1}{\lambda + \Phi(\phi)}d\phi = R_{0,\lambda}(0, 0). \tag{3.4}$$

The fundamental solution of the heat equation for H_β is the solution of

$$\frac{\partial p_\beta}{\partial t}(t, x, y) = H_\beta p_\beta(t, x, y), \quad p_\beta(0, x, y) = \delta_y(x). \tag{3.5}$$

By the Feynman-Kac formula,

$$\frac{\partial Z_{\beta,t}}{\partial t}(x) = H_\beta Z_{\beta,t}(x), \quad Z_{\beta,0}(x) \equiv 1.$$

This implies that the relation between $Z_{\beta,t}(x)$ and p_β is given by

$$Z_{\beta,t}(x) = \sum_{y\in\mathbf{Z}^d} p_\beta(t, x, y). \tag{3.6}$$

Analogously to the preceding development, define for $\beta > 0$, $\lambda > 0$,

$$R_{\beta,\lambda}(x, y) = \int_0^\infty \mathrm{e}^{-\lambda t} p_\beta(t, x, y)dt,$$
$$\hat{R}_{\beta,\lambda}(\phi, y) = \sum_{y\in\mathbf{Z}^d} R_{\beta,\lambda}(x, y)\mathrm{e}^{\mathrm{i}\langle\phi,x\rangle}.$$

Taking the Laplace transform in t of (3.5) yields the equation for the resolvent

$$H_\beta R_{\beta,\lambda} - \lambda R_{\beta,\lambda} = -\delta_y(x),$$

which implies that the Fourier transform

$$\hat{R}_{\beta,\lambda}(\phi, y) = \sum_{y\in\mathbf{Z}^d} R_{\beta,\lambda}(x, y)\mathrm{e}^{\mathrm{i}\langle\phi,x\rangle}$$

satisfies the equation

$$-\hat{R}_{\beta,\lambda}(\phi,y)\,(\Phi(\phi)+\lambda)+\beta R_{\beta,\lambda}(0,y)=-\mathrm{e}^{\mathrm{i}\langle\phi,y\rangle}.$$

Solving for $\hat{R}_{\beta,\lambda}(\phi,y)$ one arrives at

$$\hat{R}_{\beta,\lambda}(\phi,y)=\frac{\beta R_{\beta,\lambda}(0,y)+\mathrm{e}^{\mathrm{i}\langle\phi,y\rangle}}{\lambda+\Phi(\phi)}.$$

Integrating over $\mathbf{T}^d$ and combining this with (3.3) and (3.4), we get

$$R_{\beta,\lambda}(0,y)=\beta I(\lambda)R_{\beta,\lambda}(0,y)+R_{0,\lambda}(0,y)$$

and so

$$R_{\beta,\lambda}(0,y)=\frac{R_{0,\lambda}(0,y)}{1-\beta I(\lambda)}. \tag{3.7}$$

From the resolvent formula (3.7), we can read off the value for β_d in terms of the function I. In dimensions $d=1$ or $d=2$, it is easy to see that $I(0)=\infty$ and so for every $\beta>0$, the value λ which satisfies $\beta=\frac{1}{I(\lambda)}$ provides a singularity for the resolvent and this λ is therefore the eigenvalue for H_β. Thus, $\beta_d=0$ for these dimensions. For $d\geqslant 3$, we have $I(0)<\infty$ and given $\beta>0$, the equation $\beta I(\lambda)=1$ can only be solved for λ when $\beta>\frac{1}{I(0)}$. It follows that β_d must satisfy $\beta_d I(0)=1$. Below we shall relate β_d to the parameter r_d. The following result enables one to derive large time asymptotics for $p_\beta(t,0,0)$ by means of a Tauberian theorem.

Theorem 3.1. *The following $\lambda\to 0$ asymptotics hold for $I(\lambda)$. For $d=3$ or $d=4$, $I(0)<\infty$ and*

$$I(\lambda)\sim\begin{cases} I(0)-\dfrac{\sqrt{\lambda}}{4\pi}, & d=3,\\ I(0)-\dfrac{\lambda}{8\pi}\ln\frac{1}{\lambda}, & d=4.\end{cases}$$

Proof. For $d=3$, this was established in [2, Theorem 3.2]. For $d=4$, it was proved in [2] without specifying the constant so we only prove this result for $d=4$. For $d=4$,

$$\begin{aligned} I(0)-I(\lambda)&=\frac{\lambda}{(2\pi)^3}\int_{\mathbf{T}^3}\frac{1}{(\lambda+\Phi(\phi))\Phi(\phi)}d\phi\\ &\sim\frac{2\pi^2\lambda}{(2\pi)^3}\int_0^\delta\frac{r^3}{(\lambda+r^2)r^2}dr\\ &\sim\frac{\lambda}{4\pi}\int_0^\delta\frac{r}{\lambda+r^2}dr\\ &\sim\frac{\lambda}{8\pi}\ln\frac{1}{\lambda}.\end{aligned}$$

Solving for $I(\lambda)$ gives

$$I(\lambda)=I(0)-\frac{\lambda}{8\pi}\ln\frac{1}{\lambda}+\cdots.$$

This completes the proof. □

According to Theorem 3.1, since $\beta_d I(0)=1$, for $d=3$, the Laplace transform of p_{β_3} satisfies

$$\begin{aligned} R_{\beta_3,\lambda}(0,0)&=\frac{I(\lambda)}{1-\beta_3 I(\lambda)}\\ &\sim\frac{4\pi I(0)}{\beta_3\sqrt{\lambda}}\\ &=\frac{4\pi}{\beta_3^2\sqrt{\lambda}},\quad \lambda\to 0,\end{aligned} \tag{3.8}$$

while for $d=4$,

$$R_{\beta_4,\lambda}(0,0) = \frac{I(\lambda)}{1-\beta_4 I(\lambda)} \sim \frac{8\pi I(0)}{\beta_4\lambda\ln\frac{1}{\lambda}} = \frac{8\pi}{\beta_4^2\lambda\ln\frac{1}{\lambda}}, \quad \lambda\to 0. \tag{3.9}$$

We obtain the following lemma by standard Tauberian arguments (see [3, Theorem 2, p. 443]). This was [2, Lemma 6.1] but without paying attention to constants. We need the exact constants in order to specify β_d so we reproduce the proof here with more attention to detail.

Lemma 3.1. *For $d=3$ and $c_3=\frac{8\sqrt{\pi}}{\beta_3^2}$,*

$$p_{\beta_3}(t,0,0) \sim \frac{c_3}{\sqrt{t}}, \tag{3.10}$$

$$Z_{\beta_3,t} \sim 2c_3\beta_3\sqrt{t}, \quad t\to\infty. \tag{3.11}$$

For $d=4$ and $c_4=\frac{8\sqrt{\pi}}{\beta_4^2}$,

$$p_{\beta_4}(t,0,0) \sim \frac{c_4}{\ln t}, \tag{3.12}$$

$$Z_{\beta_4,t} \sim c_4\beta_4\frac{t}{\ln t}, \quad t\to\infty. \tag{3.13}$$

Proof. The asymptotics for $p_{\beta_3}(t,0,0)$ and $p_{\beta_4}(t,0,0)$ are direct applications of a Tauberian theorem to Theorem 3.1.

For $d=3$ at $\beta=\beta_3$, by (3.6) and (3.10), we have

$$Z_{\beta_3,t} = \sum_{x\in\mathbf{Z}^d} p_{\beta_3}(t,0,x) = \hat{p}_{\beta_3}(t,0,0) = 1+\beta_3\int_0^t p_{\beta_3}(s,0,0)ds \sim 2c_3\beta_3\sqrt{t}, \quad t\to\infty. \tag{3.14}$$

For $d=4$ at $\beta=\beta_4$, by (3.6) and (3.12), we have

$$Z_{\beta_4,t} = \sum_{x\in\mathbf{Z}^d} p_{\beta_4}(t,0,x) = \hat{p}_{\beta_4}(t,0,0) = 1+\beta_4\int_0^t p_{\beta_4}(s,0,0)ds \sim c_4\beta_4\frac{t}{\ln t}, \quad t\to\infty. \tag{3.15}$$

This completes the proof. □

4 Spectrum of H_β

In this section we discuss the spectrum of the operator H_β which plays a major role in the behavior of the measure $P_{\beta,t}$. The operator $H_0=\Delta$ has purely absolutely continuous spectrum equal to $[-4d,0]$. The spectrum of H_β consists of an absolutely continuous part, $[-4d,0]$, and at most one simple eigenvalue

$\lambda_0(\beta)$ which, for $d \geqslant 5$, can be 0, i.e., on the edge of the absolutely continuous part of the spectrum there is an embedded eigenvalue. The first part of the next result, about the $\beta > \beta_d$ case, was established in [2] and we state it here for comparison purposes with the case $\beta = \beta_d$ in dimensions $d \geqslant 5$.

Theorem 4.1. *For any d and any $\beta > \beta_d$, the operator H_β has one simple eigenvalue $\lambda_0(\beta) > 0$. This eigenvalue is the root of the equation $\beta I(\lambda) = 1$. Corresponding to this eigenvalue there exists a unique eigenfunction, ψ_β, with the Fourier transform*

$$\hat{\psi}_\beta(\phi) = \frac{\beta}{\lambda_0(\beta) + \Phi(\phi)}. \tag{4.1}$$

This eigenfunction has the representation

$$\psi_\beta(x) = \frac{\beta}{(2\pi)^d} \int_{\mathbf{T}^d} \frac{e^{i\langle \phi, x\rangle}}{\lambda_0(\beta) + \Phi(\phi)} d\phi. \tag{4.2}$$

For $\beta > \beta_d$, the invariant measure

$$\pi_\beta \equiv \sum_{x \in \mathbf{Z}^d} \frac{\psi_\beta^2(x)}{\|\psi_\beta\|_2^2} \delta_x$$

for $r_{\beta_d}(s, x, y)$ defined at (2.7) has finite moments of all orders.

For $d \geqslant 5$ and $\beta = \beta_d$, $\lambda_0(\beta_d) = 0$ is an eigenvalue. Its eigenfunction ψ_{β_d} has the Fourier transform

$$\hat{\psi}_{\beta_d}(\phi) = \frac{\beta_d}{\Phi(\phi)}. \tag{4.3}$$

The function $\Phi(\phi)^{-1}$ is in $L^2(\mathbf{T}^d)$ and so $\hat{\psi}_{\beta_d} \in L^2(\mathbf{T}^d)$. The measure π_{β_d} has moments of order $2k$ only for $k \leqslant d - 5$.

Proof. The claims for $\beta > \beta_d$ were established in [2, Theorem 4.1], so we will not prove them here.

To see that $\frac{\beta_d}{\Phi(\phi)}$ is in $L^2(\mathbf{T}^d)$, just observe that $\Phi^2(\phi) \sim \|\phi\|^4$, $\|\phi\| \sim 0$ and integrating in polar coordinates introduces a factor of $\|\phi\|^{d-1}$ and thus $\Phi^{-2}(\phi)$ becomes integrable for $d \geqslant 5$. Thus

$$\psi_{\beta_d}(x) = \frac{\beta_{cr(d)}}{(2\pi)^d} \int_{\mathbf{T}^d} \frac{e^{-i\langle \phi, x\rangle}}{\Phi(\phi)} d\phi$$

is an eigenfunction corresponding to the eigenvalue 0.

The other claim is about the moments of π_β. Assuming that $d \geqslant 5$ and $\beta > \beta_{cr}$, we see that for any $(j_1, j_2, \ldots, j_d) \in \mathbf{N}^d$,

$$\left(\prod_{i=1}^d \frac{\partial^{j_i}}{\partial \phi_i^{j_i}}\right) \hat{\psi}_\beta(\phi) = \left(\prod_{i=1}^d \frac{\partial^{j_i}}{\partial \phi_i^{j_i}}\right) \frac{1}{\lambda_0(\beta) + \Phi(\phi)}$$

has moments of all orders, since the denominator is bounded from 0 and the integration is over the compact space $\mathbf{T}^d$. Since this is the Fourier transform of $(\prod_{i=1}^d x_{j_i})\psi_\beta(x)$, Plancherel's identity implies that the latter has finite second moments, which is the claim to be proved. However, at $\beta = \beta_d$, since $\lambda_0(\beta_d) = 0$,

$$\left(\prod_{i=1}^d \frac{\partial}{\partial \phi_i^{j_i}}\right) \hat{\psi}_\beta(\phi) = \left(\prod_{i=1}^d \frac{\partial}{\partial \phi_i^{j_i}}\right) \frac{1}{\Phi(\phi)}.$$

The rest of the proof is just verification; for example, when $k = 1$,

$$\begin{aligned} \frac{\partial}{\partial \phi_j} \frac{1}{\Phi(\phi)} &= \frac{2 \sin \phi_j}{\Phi(\phi)^2} \\ &\sim c \frac{\phi_j}{(\phi_j^2 + \sum_{i \neq j} \phi_i^2)^2} \end{aligned}$$

$$= c\frac{\psi_j}{(\psi_j^2+1)^2}\bigg(\sum_{i\neq j}\phi_i^2\bigg)^{-3/2},$$

where $\psi_j = \phi_j(\sum_{i\neq j}\phi_i^2)$. Thus,

$$\frac{\partial}{\partial\phi_j}\frac{1}{\Phi(\phi)} \in L^2(\mathbf{T}^d)$$

if and only if

$$\frac{\psi_j^2}{(\psi_j^2+1)^2}\bigg(\sum_{i\neq j}\phi_i^2\bigg)^{-2} \in L^2(\mathbf{T}^d),$$

which depends only on the integral near 0. Thus, we check in cylindrical coordinates,

$$\int_{\sum_{i\neq j}\phi_i^2<\delta^2}\int_0^\delta \frac{\phi_j^2}{(\phi_j^2+\sum_{i\neq j}\phi_i^2)^4} = \int_{\sum_{i\neq j}\phi_i^2<\delta^2}\int_0^{\delta/\sqrt{\sum_{i\neq j}\phi_i^2}} \frac{\psi_j^2}{(\psi_j^2+1)^4}\bigg(\sum_{i\neq j}\phi_i^2\bigg)^{-5/2} d\psi_j d\hat{\phi}_j$$

$$\leqslant c\int_0^\delta r^{d-2}r^{-5}dr$$

and the last integral is finite if and only if $d \geqslant 7$. The full claim that the $2k$-th moment is finite if and only if $d \geqslant 2k+5$ is a routine (though tedious) calculation which we shall omit. □

5 Proofs of Theorems 2.1 and 2.2

In this section we give the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. For the asymptotic distribution of σ_t/t described in (2.1) and (2.3) we observe, when $d=3$, for $s\in(0,1)$ and $\epsilon>0$ small enough so that $s+\epsilon\in(0,1)$,

$$\begin{aligned}P_{\beta_3,t}(\sigma_t\in(st,(s+\epsilon)t)) &= P_{\beta_3,t}(x_{st}=0, x_{(s+\epsilon)t}=e_1, x_u\neq 0, (s+\epsilon)t\leqslant u\leqslant t)+o(\epsilon)\\ &= \frac{E^0[\mathrm{e}^{\beta_3\int_0^{st}\delta_0(x_u)du}\delta_0(x_{st})]P^{e_1}(x_u\neq 0,\ 0\leqslant u\leqslant(1-(s+\epsilon))t)6\epsilon t}{Z_{\beta_3,t}}+o(\epsilon),\end{aligned}$$

where the term $6\epsilon t$ arises from the rate of jumping from 0 to the neighboring unit vectors, $2d=6$ neighbors and the time interval has length ϵt. Since the non-return probability is the same for all the neighbors of 0, we use $e_1\in\mathbf{Z}^3$ in this expression. As $t\to\infty$, the term

$$P^{e_1}(x_u\neq 0,\ 0\leqslant u\leqslant(1-(s+\epsilon))t)\to r_3\in(0,1)$$

due to the transience of the 3-dimensional random walk. Thus,

$$\begin{aligned}P_{\beta_3,t}\bigg(\frac{\sigma_t}{t}\in ds\bigg) &= Z_{\beta_3,t}^{-1}E^0[\mathrm{e}^{\beta_3\int_0^{st}\delta_0(x_u)du}\delta_0(x_{st})]6r_3tds\\ &= Z_{\beta_3,t}^{-1}\,p_{\beta_3}(st,0,0)6r_3tds.\end{aligned}$$

By (3.10) and (3.14) we have for $s\in(0,1)$,

$$\begin{aligned}\lim_{t\to\infty} Z_{\beta_3,t}^{-1}p_{\beta_3}(st,0,0)6r_3t &= \lim_{t\to\infty}\frac{c_3\sqrt{st}^{-1}6r_3t}{2c_3\beta_3\sqrt{t}}\\ &= \frac{6r_3}{2\beta_3\sqrt{s}}\end{aligned}$$

and since this is a probability density we derive that $r_3 = 6\beta_3$ and so

$$\lim_{t\to\infty}P_{\beta_3,t}\bigg(\frac{\sigma_t}{t}\in ds\bigg) = \frac{1}{2\sqrt{s}}ds, \quad s\in(0,1)$$

as desired. Thus, (2.1) is proved.

For the distribution of σ_t/t when $d=4$ we observe as before, for $s\in(0,1)$ and $\epsilon>0$ small enough so that $s+\epsilon\in(0,1)$,

$$\begin{aligned}P_{\beta_4,t}(\sigma_t\in(st,(s+\epsilon)t)) &= P_{\beta_4,t}(x_{st}=0,x_{(s+\epsilon)t}=e_1,x_u\neq 0,(s+\epsilon)t\leqslant u\leqslant t)+o(\epsilon)\\ &= \frac{E^0[\mathrm{e}^{\beta_4\int_0^{st}\delta_0(x_u)du}\delta_0(x_{st})]P^{e_1}(x_u\neq 0,\ 0\leqslant u\leqslant(1-(s+\epsilon))t)8\epsilon t}{Z_{\beta_4,t}}+o(\epsilon),\end{aligned}$$

where again the term $8\epsilon t$ arises from the rate of jumping from 0 to the neighboring unit vectors in $\mathbf{Z}^d$ in the time interval $(st,(s+\epsilon)t)$. The term

$$P^{e_1}(x_u\neq 0,\ 0\leqslant u\leqslant(1-(s+\epsilon))t)\to r_4\in(0,1)$$

due to the transience of the 4-dimensional random walk. Thus,

$$\begin{aligned}P_{\beta_4,t}\left(\frac{\sigma_t}{t}\in ds\right) &= Z^{-1}_{\beta_4,t}E^0[\mathrm{e}^{\beta_4\int_0^{st}\delta_0(x_u)du}\delta_0(x_{st})]8r_4t\,ds\\ &= Z^{-1}_{\beta_4,t}p_{\beta_4}(st,0,0)8r_4t\,ds.\end{aligned}$$

By (3.12) and (3.15) we have for $s\in(0,1)$,

$$\begin{aligned}\lim_{t\to\infty}Z^{-1}_{\beta_4,t}p_{\beta_4}(st,0,0)rt &= \lim_{t\to\infty}\frac{c_4\,8r_4t/\ln st}{c_4\beta_4t/\ln t}\\ &= \frac{8r_4}{\beta_4}\end{aligned}$$

from which we derive, since this is a density on $[0,1]$, that $8r_4=\beta_4$ and so

$$\lim_{t\to\infty}P_{\beta_4,t}\left(\frac{\sigma_t}{t}\in ds\right)=ds,\quad s\in(0,1)$$

as claimed at (2.3).

For the proof of the path decomposition (2.5), first take $\delta>\epsilon$ and approximate $G\in\mathcal{F}_{st,t}$ by a random variable $G_\delta\in\mathcal{F}_{(s+\delta)t,t}$. Then if

$$\mu=\frac{1}{2d}\sum_{\|v\|=1}\delta_v,$$

we have for either $d=3$ or $d=4$ that as $\epsilon\to0$,

$$\begin{aligned}&E_{\beta_d,t}[FG_\delta\,|\,\sigma_t/t\in[s,s+\epsilon]]\\ &\quad=\frac{E_{\beta_d,t}[FG_\delta;\sigma_t/t\in[s,s+\epsilon]]}{P_{\beta_d,t}(\sigma_t/t\in[s,s+\epsilon])}\\ &\quad=\frac{\frac{1}{2d}\sum_{\|v\|=1}E_{\beta_d,t}[FG_\delta;x_{st}=0,x_{(s+\epsilon)t}=v,x_u\neq0,\ (s+\epsilon)t\leqslant u\leqslant t]2d\epsilon t+o(\epsilon)}{\frac{1}{2d}\sum_{\|v\|=1}P_{\beta_d,t}(x_{st}=0,x_{(s+\epsilon)t}=v,x_u\neq0,\ (s+\epsilon)t\leqslant u\leqslant t)2d\epsilon t+o(\epsilon)}\\ &\quad=\frac{E^0[F\mathrm{e}^{\beta_d\int_0^{st}\delta_0(x_u)du}\delta_0(x_{st})]E^\mu[G_\delta\circ\theta_{-st},\ x_u\neq0,\ 0\leqslant u\leqslant(1-(s+\epsilon)t)]+o(1)}{E^0[\mathrm{e}^{\beta_d\int_0^{st}\delta_0(x_u)du}\delta_0(x_{st})]P^\mu(x_u\neq0,\ 0\leqslant u\leqslant(1-(s+\epsilon)t))+o(1)}\\ &\quad\to\frac{E^0[F\mathrm{e}^{\beta_d\int_0^{st}\delta_0(x_u)du}\delta_0(x_{st})]}{p_{\beta_d}(st,0,0)}E^\mu[G_\delta\circ\theta_{-st}\,|\,x_u\neq0,\ 0\leqslant u\leqslant(1-s)t]\\ &\quad=\frac{Z_{\beta_d,st}}{p_{\beta_d}(st,0,0)}E_{\beta_d,st}[F\delta_0(x_{st})]E^\mu[G_\delta\circ\theta_{-st}\,|\,x_u\neq0,\ 0\leqslant u\leqslant(1-s)t]\\ &\quad=E_{\beta_d,st}[F\,|\,x_{st}=0]E^\mu[G_\delta\circ\theta_{-st}\,|\,x_u\neq0,\ 0\leqslant u\leqslant(1-s)t].\end{aligned}$$

Letting $\delta\to0$ completes the proof.

Using the path decomposition at (2.5), we now give a different proof of (2.2) and (2.4) than the one given in [2]. This proof clarifies the relation between (2.1) and (2.2) and as well, the relation between (2.3) and (2.4). Again working in dimensions 3 or 4, from the central limit theorem, it follows that for each fixed $u > 0$,

$$E^{\mu}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t-u}}{\sqrt{t}}\rangle}] \to \mathrm{e}^{-(1-s)\|\phi\|^2}, \quad t \to \infty. \tag{5.1}$$

Also, if $\tau_0 = \inf\{t > 0 : x_t = 0\}$ then

$$\begin{aligned} E^{\mu}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t}}{\sqrt{t}}\rangle}] = {} & E^{\mu}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t}}{\sqrt{t}}\rangle}; \tau_0 \leqslant (1-s)t] \\ & + E^{\mu}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t}}{\sqrt{t}}\rangle}; \tau_0 > (1-s)t]. \end{aligned} \tag{5.2}$$

But, by the strong Markov property,

$$E^{\mu}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t}}{\sqrt{t}}\rangle}; \tau_0 \leqslant (1-s)t] = E^{\mu}[E^0[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t-\tau_0}}{\sqrt{t}}\rangle}]; \tau_0 \leqslant (1-s)t].$$

Now, by (5.1) and the fact that

$$P^{\mu}(\tau_0 < (1-s)t) \to 1 - r_d \quad \text{as } t \to \infty,$$

we have by dominated convergence that

$$E^{\mu}[E^0[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t-\tau_0}}{\sqrt{t}}\rangle}]; \tau_0 \leqslant (1-s)t] \to (1-r_d)\mathrm{e}^{-(1-s)\|\phi\|^2}, \quad t \to \infty. \tag{5.3}$$

From (5.1)–(5.3) and $\lim_{t\to\infty} P^{\mu}(\tau_0 > (1-s)t) = r_d$, we conclude

$$E^{\mu}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t}}{\sqrt{t}}\rangle} \mid \tau_0 > (1-s)t] \to \mathrm{e}^{-(1-s)\|\phi\|^2}, \quad t \to \infty. \tag{5.4}$$

Thus, taking $F \equiv 1$ and $G = \mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_t}{\sqrt{t}}\rangle}$ we can apply (2.5) and (5.4) to get

$$\begin{aligned} E_{\beta_d,t}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_t}{\sqrt{t}}\rangle}] &= \int_0^1 E_{\beta_d,t}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_t}{\sqrt{t}}\rangle} \mid \sigma_t/t = s] P_{\beta_d,t}(\sigma_t/t \in ds) \\ &= \int_0^1 E^{\mu}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_{(1-s)t}}{\sqrt{t}}\rangle} \mid \tau_0 > (1-s)t] P_{\beta_d,t}(\sigma_t/t \in ds) \\ &\to \int_0^1 \mathrm{e}^{-(1-s)\|\phi\|^2} \sigma_d(ds), \end{aligned} \tag{5.5}$$

which completes the proof of (2.2) and (2.4). □

Remark 5.1. The proof of (2.2) and (2.4) in [2] used the easily verified perturbation formula, which holds for any d,

$$p_{\beta_d}(t,0,x) = p_0(t,0,x) + \beta_d \int_0^t p_0(t-s,0,x) p_{\beta_d}(s,0,0) ds. \tag{5.6}$$

In order to derive the asymptotic distribution of $x_t/\sqrt{t}$ for $d = 3$, evaluate $\hat{p}_{\beta_3}(t,0,\xi)$ at $\xi = \phi/\sqrt{t}$ with $\phi \in \mathbb{R}^3$ to get, by means of (5.6), the central limit theorem for the simple symmetric random walk,

$$\hat{p}_{\beta_3}\left(t, 0, \frac{\phi}{\sqrt{t}}\right) = \mathrm{e}^{-|\phi|^2}(1 + o(1)) + c_3\beta_3 \int_0^t (1 + o(1)) \mathrm{e}^{-|\phi|^2(1-\frac{s}{t})} (1+s)^{-1/2} ds.$$

After normalization by $Z_{\beta_3,t} \sim 2c_3\beta_3\sqrt{t}$ it follows that

$$E_{\beta_3,t}[\mathrm{e}^{\mathrm{i}\langle \phi, \frac{x_t}{\sqrt{t}}\rangle}] = \frac{\hat{p}_{\beta_3}(t,0,\frac{\phi}{\sqrt{t}})}{Z_{\beta_3,t}} \to \frac{1}{2}\int_0^1 \mathrm{e}^{-|\phi|^2(1-u)} \frac{1}{\sqrt{u}} du.$$

The proof for (2.4) was similar.

Proof of Theorem 2.2. The semigroup generated by p_{β_d} is

$$Q_t f(x) = \sum_{y\in\mathbf{Z}^d} p_{\beta_d}(t,x,y)f(y) = \mathrm{e}^{tH_{\beta_d}}f(x)$$

and Q_t acts on the space of bounded functions. Since ψ_{β_d} is the eigenfunction corresponding to $\lambda_0(\beta_d)=0$,

$$Q_t\psi_{\beta_d}(x) = \psi_{\beta_d}(x).$$

The computation of the limiting transition densities follows from considering for $s+u<t$, on letting $t\to\infty$, we see

$$\begin{aligned}
P_{\beta_d,t}(x_s = x, x_{s+u} = y) &= Z^{-1}_{\beta_d,t}E^0[\mathrm{e}^{\beta_d\int_0^t \delta_0(x_r)dr}; x_s = x,\ x_{s+u} = y]\\
&= Z^{-1}_{\beta_d,t}E^0[\mathrm{e}^{\beta_d\int_0^s \delta_0(x_r)dr}; x_s = x]\\
&\quad\times E^x[\mathrm{e}^{\beta_d\int_0^u \delta_0(x_r)dr}; x_u = y]\\
&\quad\times E^y[\mathrm{e}^{\beta_d\int_0^{t-s-u} \delta_0(x_r)dr}]\\
&= \frac{p_{\beta_d}(s,0,x)Z_{\beta_d,t}(x)}{Z_{\beta_d,t}}\frac{p_{\beta_d}(u,x,y)Z_{\beta_d,t-s-u}(y)}{Z_{\beta_d,t}(x)}\\
&\to \frac{p_{\beta_d}(s,0,x)\psi_{\beta_d}(x)}{\psi_{\beta_d}(0)}\frac{p_{\beta_d}(s,x,y)\psi_{\beta_d}(y)}{\psi_{\beta_d}(x)},
\end{aligned}$$

which establishes (2.7).

Now the kernel

$$r_{\beta_d}(t,x,y) = \frac{p_{\beta_d}(t,x,y)\psi_{\beta_d}(y)}{\psi_{\beta_d}(x)}$$

generates a semigroup, which we denote by R_t and $R_t\mathbf{1}=\mathbf{1}$. The generator A_{β_d} of R_t is calculated by the formula

$$A_{\beta_d}f(x) = \lim_{h\searrow 0}\frac{R_hf(x)-f(x)}{h}.$$

Making the computation, we find, using the facts that $p_{\beta_d}(t,x,y)$ solves (3.5) and that $\lambda_0(\beta_d)=0$, that

$$\begin{aligned}
\lim_{h\searrow 0}\frac{R_hf(x)-f(x)}{h} &= \frac{1}{\psi_{\beta_d}(x)}H_\beta(\psi_{\beta_d}f)(x)\\
&= \frac{1}{\psi_{\beta_d}(x)}\bigg(\sum_{\|y-x\|=1}(\psi_{\beta_d}(y)f(y)-\psi_{\beta_d}(x)f(x))\bigg) + \beta_d\delta_0(x)f(x)\\
&= \frac{1}{\psi_{\beta_d}(x)}\bigg(\sum_{\|y-x\|=1}(\psi_{\beta_d}(y)-\psi_{\beta_d}(x))f(x)\bigg)\\
&\quad + \frac{1}{\psi_{\beta_d}(x)}\bigg(\sum_{\|y-x\|=1}(f(y)-f(x))\psi_{\beta_d}(y)\bigg) + \beta_d\delta_0(x)f(x)\\
&= \sum_{\|y-x\|=1}\frac{\psi_{\beta_d}(y)}{\psi_{\beta_d}(x)}(f(y)-f(x)).
\end{aligned}$$

This results in the expression

$$A_{\beta_d}f(x) = \sum_{|y-x|=1} q_d(x,y)(f(y)-f(x)),$$

as claimed in Theorem 2.2. The last step regarding the Q-matrix involves the diagonal term which is determined by the condition $\sum_y q_d(x,y)=0$. Since

$$\begin{aligned}\sum_{y:y\neq x} q_d(x,y) &= \sum_{y:y\neq x}\frac{\psi_{\beta_d}(y)}{\psi_{\beta_d}(x)}\\ &= \sum_{y:y\neq x}\frac{\psi_{\beta_d}(y)-\psi_{\beta_d}(x)}{\psi_{\beta_d}(x)}+2d\\ &= \frac{1}{\psi_{\beta_d}(x)}\Delta\psi_{\beta_d}(x)+2d\\ &= -\beta_d\delta_0(x)+2d,\end{aligned}$$

it follows that $q_d(x,x)=\beta_d\delta_0(x)-2d$.

Using the asymptotic formula, which comes from the spectral theorem (recall the absolutely continuous part of the spectrum of H_{β_d} is $[-4d,0]$),

$$p_{\beta_d}(t,x,y)\sim\psi_{\beta_d}(x)\psi_{\beta_d}(y),\quad t\to\infty,$$

together with the definition of $r_{\beta_d}(t,x,y)$ we see that both

$$\lim_{t\to\infty} r_{\beta_d}(t,x,y)=\psi^2_{\beta_d}(y)$$

and

$$\sum_{x\in\mathbf{Z}^d}\psi^2_{\beta_d}(x)r_{\beta_d}(t,x,y)=\psi^2_{\beta_d}(y).$$

This proves that π_{β_d} defined in the theorem gives the stationary probability distribution. Also, the endpoint distribution $P_{\beta_d,t}(x_t=x)$ can be handled using the spectral theorem as follows:

$$\begin{aligned}P_{\beta_d,t}\,(x_t=x) &= \frac{p_{\beta_d}(t,0,x)}{Z_{\beta_d,t}}\\ &\to\frac{\psi_{\beta_d}(x)}{\sum_y\psi_{\beta_d}(y)}\\ &= \frac{\psi_{\beta_d}(x)}{\langle\psi_{\beta_d},\mathbf{1}\rangle}.\end{aligned}$$

In order to obtain weak converge of the measures $P_{\beta_d,t}$ on the space of trajectories up to time T we need to establish tightness. According to [1], this requires control of the oscillations. Set

$$\omega_T(x,[t_{i-1},t_i))\equiv\sup_{s,t\in[t_{i-1},t_i)}|x_t-x_s|$$

and

$$\omega'_T(x,\delta)\equiv\inf_{\{t_i\}}\max_{1\leqslant i\leqslant r}\omega_T(x,[t_{i-1},t_i)),$$

where the inf is taken over finite sets $\{t_i\}$ such that

$$0=t_0<t_1<\cdots<t_r=T,\quad t_i-t_{i-1}>\delta.$$

Then, since our paths all start at 0 under $P_{\beta_d,t}$, tightness will follow provided for each positive ϵ and η there exist $\delta\in(0,1)$ and T_0 such that

$$P_{\beta_d,t}(\omega'_T(x,\delta)\geqslant\epsilon)\leqslant\eta,\quad t\geqslant T_0.$$

However, we have

$$P_{\beta_d,t}(\omega'_T(x,\delta)\geqslant\epsilon)=Z^{-1}_{\beta_d,t}E^0[\mathrm{e}^{\beta_d\int_0^t\delta_0(x_s)ds}1_{\{\omega'_T(x,\delta)\geqslant\epsilon\}}]$$

$$
\begin{aligned}
&= Z_{\beta_d,t}^{-1} E^0[\mathrm{e}^{\beta_d \int_0^T \delta_0(x_s)ds} 1_{\{\omega'_T(x,\delta)\geqslant\epsilon\}} E^{x_T}[\mathrm{e}^{\beta_d \int_0^{t-T} \delta_0(x_s)ds}]] \\
&\leqslant E^0[\mathrm{e}^{\beta_d T} 1_{\{\omega'_T(x,\delta)\geqslant\epsilon\}}] \frac{Z_{\beta_d,t-T}}{Z_{\beta_d,t}} \\
&\leqslant \mathrm{e}^{\beta_d T} P^0(\omega'_T(x,\delta)\geqslant\epsilon) \\
&\to 0, \quad t\to\infty.
\end{aligned}
$$

This proves the convergence in distribution of the process on $[0,T]$ under $P_{\beta_d,t}$ as $t\to\infty$. □

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