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Steady state and intermittency in the critical branching random walk with arbitrary total number of offspring

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ABSTRACT
For the critical branching random walk on the lattice $\mathbb{Z}^d$, in the case of an arbitrary total number of produced offspring spreading on the lattice from the parental particle, the existence of a limit distribution (which corresponds to a steady state (or statistical equilibrium)) of the population is proved. If the second factorial moment of the total number of offspring is much larger than the square of the first factorial moment, then the limit particle field displays strong deviations from the uniformity: this is intermittency.

KEYWORDS
Branching random walk; critical case; steady state; Carleman conditions; intermittency

1. Introduction

Watson and Galton (1875) expressed the probabilities of long-term survival of family names. Branching processes are useful when each individual may die or have offspring independently of other members. Sevast'yanov (1971) presented branching processes.

Particles can move randomly. In this case, branching processes are also called branching random walks or diffusion. The walks take place usually on $\mathbb{Z}^d$ or $\mathbb{R}^d$. The Kolmogorov–Petrovskii–Piskunov (1937) (KPP) model originated from the spread of a biological species with a dominant gene.

A key problem is the existence of a limit particle field (a steady state (also called statistical equilibrium)). Dobrushin and Siegmund-Schultze (1982) used the theory of Gibbs fields to study branching particles, homogeneous in space. His program based on the forward integral equations for the correlation functions was implemented in Liemant (1969) and Debes et al. (1970), who proved the existence of a steady state in the three-dimensional KPP model, where the initial particle field is Poissonian, the evolution proceeds binary branching, and the dynamics are Brownian.

In the contact (or "forest") model of Kondratiev et al. (2008), particles in $\mathbb{R}^d$, $d \geq 3$, are assimilated to "trees". The initial field of "trees" is

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Poissonian and "trees" do not move. During the time interval $(t, t+dt)$, the "tree" either dies or produces the "seed," which spreads randomly from the parental "tree". Such a particle field is formalized by a Markov process in the space of the locally finite configurations of points in $\mathbb{R}^d$. In the critical case of equal death and birth rates, Kondratiev et al. (2008) proved the ergodicity of this Markov process, that is, the convergence to the steady state. The correlation function is no longer convenient for the lattice models because each site $x \in \mathbb{Z}^d$ may contain many particles and Kondratiev et al. (2008)'s method no longer applies.

Molchanov and Whitmeyer (2017) presented the total population of the particles as the sum of independent subpopulations, generated by different initial particles. For subpopulations, Kolmogorov backward equations apply. The existence of the steady state now depends on the asymptotic moments of the total population.

We assume that the particles walk randomly on $\mathbb{Z}^d$, $d \geq 1$, with law a. Particles produce an arbitrary total number of offspring, which spread independently of the parental particle with law b and behave as independent particles. The proof of the existence of a steady state extends Molchanov and Whitmeyer (2017) to multiple branching.

The transition to an arbitrary total number of offspring leads to a different structure of the limit population, where the limit field importantly deviates from uniformity. Zeldovich et al. (2014) describe the applications of the intermittency phenomenon in magnetic and temperature fields of the turbulent flow, in chemical kinetics, and in biological models. For example, the solar magnetic field has an intermittent structure because more than 99% of the magnetic energy concentrates on less than 1% of the surface area. Gärtner and Molchanov (1990) and Molchanov (1994) developed the mathematical theory of intermittency with applications to hydrodynamics.

In Section 2, we present the branching random process and the main result, which is about the convergence to the steady state (Theorem 1). The proof is in the Appendix. The random field contains a system of high and well-separated peaks, which reflect the presence of intermittency (Section 3).

2. Existence of a steady state

$\left\{N(t,y),\, y \in \mathbb{Z}^d\right\}$ designates a particle field of a population at time $t \geq 0$, starting with a single particle at each point $y \in \mathbb{Z}^d$ $N(0,y) = 1$, $n(t,x,y)$ a subpopulation at $y \in \mathbb{Z}^d$ at time $t \geq 0$, generated by a single particle at

$x \in \mathbb{Z}^d$ at time $t = 0$, hence $n(0, x, y) = 1$ if and only if $x = y$. The subpopulations are independent of one another and

$$N(t, y) = \sum_{x \in \mathbb{Z}^d} n(t, x, y). \tag{1}$$

Each subpopulation changes according to a random walk of each of its particle. The random walk generator is

$$\kappa(\mathcal{L}_a \psi)(x) = \kappa \sum_{v \in \mathbb{Z}^d \setminus \{0\}} a(v)(\psi(x + v) - \psi(x)), \tag{2}$$

where $a(v)$ is a coefficient such that

$$\begin{aligned} &a(v) \geq 0, \quad a(v) = a(-v), \quad v \in \mathbb{Z}^d \setminus \{0\}, \\ &\sum_{v \in \mathbb{Z}^d \setminus \{0\}} a(v) = 1. \end{aligned} \tag{3}$$

Eq. (2) formalizes the fact that the particle spends an exponentially distributed random time, with parameter κ, in each site x, then jumps from x to $x + v$, $v \in \mathbb{Z}^d \setminus \{0\}$, with probability $a(v)$. Annihilation or death occurs with rate μ and splitting into l particles with rates β_l, where $l \geq 2$. At splitting, one offspring (considered as the parental particle) remains at x and the other $l - 1$ particles jump independently from x to $x + v$, with probability $b(v)$, where

$$\begin{aligned} &b(v) \geq 0, \quad b(v) = b(-v), \quad v \in \mathbb{Z}^d \setminus \{0\}, \\ &\sum_{v \in \mathbb{Z}^d \setminus \{0\}} b(v) = 1. \end{aligned} \tag{4}$$

Without loss of generality, we study the population at $y = 0$. The generating function for an individual subpopulation

$$u_z(t, x) = \mathbb{E} z^{n(t,x,0)} \tag{5}$$

satisfies the Kolmogorov backward equation:

$$\begin{aligned} \frac{\partial u_z}{\partial t}(t, x) = \quad & \kappa(\mathcal{L}_a u_z)(t, x) - \left(\mu + \sum_{\infty}^{l=2} \beta_l\right) u_z(t, x) + \mu + \\ & u_z(t, x) \sum_{l=2}^{\infty} \beta_l (u_z * b)^{l-1}(t, x) \end{aligned} \tag{6}$$

with initial condition $u_z(0, x) = z$ if $x = 0$ and $u_z(0, x) = 1$ otherwise. The star denotes the convolution of two functions:

$$(u_z * b)(t, x) = \sum_{v \in \mathbb{Z}^d} u_z(t, x - v) b(v). \tag{7}$$

In Eq. (6), a particle located at x contributes to $u_z(t+dt,x)$ during the time interval $(0,\, dt)$ in either:

(1) dying with probability $\mu\, dt$;
(2) jumping with probability $\kappa a(v)\, dt$ from x to $x+v$, $v \in \mathbb{Z}^d \backslash \{0\}$. After the jump, during the remaining time interval $(dt, t+dt)$, the particle changes independently of its location, as if it had its parental particle at $x+v$. This case corresponds to the term $\kappa\, dt \sum\limits_{v \in \mathbb{Z}^d \backslash \{0\}} a(v) u_z(t, x+v)$;
(3) splitting into l-particles $(l \geq 2)$ with probability $\beta_l\, dt$. The parental particle located at the site x changes further during the remaining time interval $(dt, t+dt)$ independently of other particles. The other $(l-1)$ offspring jump independently of one another to the states $x+v_k$, $(v_k \in \mathbb{Z}^d \backslash \{0\})$ with corresponding probabilities $b(v_k)$, for all $k = 2, \ldots, l$, then they behave as independent particles. This case corresponds to the term

$$dt \sum_{l=2}^{\infty} \beta_l u_z(t,x) \sum_{k=2}^{l} \left(\sum_{v_k \in \mathbb{Z}^d \backslash \{0\}} b(v_k) u_z(t, x+v_k) \right) = \\ dt \sum_{l=2}^{\infty} \beta_l u_z(t,x) \left(\sum_{v \in \mathbb{Z}^d \backslash \{0\}} b(v) u_z(t, x+v) \right)^{l-1}; \tag{8}$$

4. or experiencing no event, with probability $1 - \mu\, dt - \kappa\, dt - dt \sum_{l=2}^{\infty} \beta_l$. The particle located at the site x changes further during $(dt, t+dt)$ as if it had a parent located at x. Then, its contribution is

$$\left(1 - \mu\, dt - \kappa\, dt - dt \sum_{l=2}^{\infty} \beta_l \right) u_z(t,x). \tag{9}$$

From Eq. (7) and the assumption of symmetry for b (posited in Eq. (4)):

$$\sum_{v \in \mathbb{Z}^d \backslash \{0\}} b(v) u_z(t, x+v) = \sum_{v \in \mathbb{Z}^d \backslash \{0\}} b(-v) u_z(t, x+v) = \\ \sum_{v \in \mathbb{Z}^d \backslash \{0\}} b(v) u_z(t, x-v) = (u_z \star b)(t,x). \tag{10}$$

Summing up and grouping:

$$u_z(t+dt, x) - u_z(t,x) = \kappa\, dt \left(\sum_{v \in \mathbb{Z}^d \backslash \{0\}} a(z) u_z(t, x+v) - u_z(t,x) \right) - \\ dt \left(\mu + \sum_{l=2}^{\infty} \beta_l \right) u_z(t,x) + \mu\, dt + dt u_z(t,x) \sum_{l=2}^{\infty} \beta_l (u_z \star b)^{l-1}. \tag{11}$$

From Eq. (2) and the assumption of normalization for a (posited in Eq. (3)):

$$\begin{aligned}&\kappa\left(\sum_{v\in\mathbb{Z}^d\setminus\{0\}} a(z)u_z(t,x+v)-u_z(t,x)\right)=\\&\kappa\sum_{v\in\mathbb{Z}^d\setminus\{0\}} a(z)(u_z(t,x+v)-u_z(t,x))=\kappa(\mathcal{L}_a u_z)(t,x).\end{aligned}\tag{12}$$

Using Eq. (12), dividing all terms of Eq. (11) by dt, and taking the limit when $dt\to 0$ yields Eq. (6).

The factorial moments

$$m_k(t,x)=\mathbb{E}(n(n-1)\cdots(n-k+1))=\left.\frac{\partial^k u_z}{\partial z^k}\right|_{z=1}(t,x),\tag{13}$$

where $n=n(t,x,0)$, $k=1,2,\ldots$, result from Eq. (6) after differentiation. The first moment satisfies

$$\begin{cases}\frac{\partial m_1}{\partial t}(t,x)= & \left(\left(\kappa\mathcal{L}_a+\sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b\right)m_1\right)(t,x)+\\ & \left(\sum_{l=2}^{\infty}(l-1)\beta_l-\mu\right)m_1(t,x),\\ m_1(0,x)= & \delta(x).\end{cases}\tag{14}$$

As for Eq. (2):

$$(\mathcal{L}_b\psi)(x)=\sum_{v\in\mathbb{Z}^d\setminus\{0\}} b(v)(\psi(x+v)-\psi(x)).\tag{15}$$

To derive Eq. (14), we use:

$$u_z(t,x)|_{z=1}=1 \text{ for all } t\geq 0, x\in\mathbb{Z}^{\mathrm{d}};\tag{16}$$

$$(u_z\star b)(t,x)|_{z=1}=\sum_{v\in\mathbb{Z}^d\setminus\{0\}} b(v)=1 \text{ for all } t\geq 0, x\in\mathbb{Z}^{\mathrm{d}};\tag{17}$$

$$\frac{\partial}{\partial z}(u_z\star b)(t,x)|_{z=1}=\sum_{v\in\mathbb{Z}^d\setminus\{0\}} b(v)m_1(t,x-v)=(m_1\star b)(t,x);\tag{18}$$

as a consequence of Eq. (4):

$$
\begin{aligned}
&\sum_{l=2}^{\infty}(l-1)\beta_l((m_1\star b)(t,x)-m_1(t,x))=\\
&\sum_{l=2}^{\infty}(l-1)\beta_l\sum_{v\in\mathbb{Z}^d\setminus\{0\}}b(v)(m_1(t,x-v)-m_1(t,x))=\\
&\sum_{l=2}^{\infty}(l-1)\beta_l\sum_{v\in\mathbb{Z}^d\setminus\{0\}}b(-v)(m_1(t,x-v)-m_1(t,x))=\\
&\sum_{l=2}^{\infty}(l-1)\beta_l\sum_{v\in\mathbb{Z}^d\setminus\{0\}}b(v)(m_1(t,x+v)-m_1(t,x))=\\
&\left(\sum_{l=2}^{\infty}(l-1)\beta_l\right)(\mathcal{L}_b m_1)(t,x).
\end{aligned}
\tag{19}
$$

If $\mu=\sum_{l=2}^{\infty}(l-1)\beta_l$, Eq. (14) becomes:

$$
\begin{cases}
\frac{\partial m_1}{\partial t}(t,x)=\left(\left(\kappa\mathcal{L}_a+\sum_{l=2}^{\infty}(l-1\right)\beta_l\mathcal{L}_b)m_1\right)(t,x)\\
m_1(0,x)=\delta(x),
\end{cases}
\tag{20}
$$

whose solution is $m_1(t,x)=p(t,x,0)$, where $p(t,x,y)$ is the conditional probability of the event that a particle starting at $x\in\mathbb{Z}^d$ arrives at $y\in\mathbb{Z}^d$ at time $t>0$, when the random walk is defined by the symmetric isotropic generator $\kappa\mathcal{L}_a+\sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b$. That is why

$$
\begin{cases}
\sum_{y\in\mathbb{Z}^d}p(t,x,0)=1,\\
p(t,x,0)=\bar{p}(t,x-y),\quad \text{where } \bar{p}(t,z)=\bar{p}(t,-z)\quad \text{for all } z\in\mathbb{Z}^d.
\end{cases}
\tag{21}
$$

For each $x\in\mathbb{Z}^d$, $\nu_x(t)=\sum_{y\in\mathbb{Z}^d}n(t,x,y)$ is a Galton–Watson process and if $\mu=\sum_{l=2}^{\infty}(l-1)\beta_l$, from Eq. (21):

$$
\mathbb{E}\nu_x(t)=\sum_{y\in\mathbb{Z}^d}\mathbb{E}n(t,x,y)=\sum_{y\in\mathbb{Z}^d}p(t,x,y)=1,
\tag{22}
$$

which is the critical case ($\mathbb{E}\nu_x(t)=1$ for all $t\geq 0$) in Sevast'yanov (1971)'s classification of Galton–Watson processes. Sevast'yanov (1951) proved that if $\sum_{l=2}^{\infty}l^3\beta_l$ is finite then:

(1) $\nu_x(t)$ degenerates almost surely with rate $1-\frac{2}{t\sum_{l=2}^{\infty}l(l-1)\beta_l}$;
(2) for all $z>0$

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{2v_x(t)}{t\sum_{l=2}^{\infty} l(l-1)\beta_l} > z | v_x(t) > 0\right) = e^{-z},$$
$$\lim_{t\to\infty} \mathbb{E}(v_x(t)|v_x(t) > 0) = \frac{\sum_{l=2}^{\infty} l(l-1)\beta_l}{2} t, \tag{23}$$

which expresses the fact that if the population does not vanish, $v_x(t) > 0$, then the total number of particles is large, of order t.

Theorem 1 *For a random field $N(t,y)$, $y \in \mathbb{Z}^d$, as described in Eq. (1), (5), and (6), with additional conditions*:

(1) $\mu = \sum_{l=2}^{\infty}(l-1)\beta_l$;
(2) *the random walk on $\mathbb{Z}^d$ with generator $\kappa\mathcal{L}_a + \sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b$ is transient;*
(3) *for all $l \geq 2$, $\beta_l \leq \beta\delta^l$ for some $\beta > 0$, $\delta \in (0,1)$.*

Then, for all $y \in \mathbb{Z}^d$

$$N(t,y) \overset{\text{law}}{\to} N(\infty, y). \tag{24}$$

The transience of the random walk with generator $\kappa\mathcal{L}_a + \sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b$ implies that

$$G_0(0,0) = \int_0^{\infty} p(s,0,0)\,ds < \infty, \tag{25}$$

where $p(t,x,y)$ is the conditional probability corresponding to this random walk (that is, the probability that a particle starting at $x \in \mathbb{Z}^d$ arrives at $y \in \mathbb{Z}^d$ at time $t > 0$) (Eq. (21)). The Laplace transform of $p(t,x,y)$ $G_\lambda(x) = \int_0^{\infty} e^{-\lambda s} p(s,x,y)\,ds$ is the Green function (Spitzer, 1976).

The condition that β_l decreases geometrically implies that the generating function of the sequence $\sum_{l=2}^{\infty}\beta_l z^l$ is analytic on the disk $|z| < \epsilon$ for some suitable $\epsilon > 0$.

Because the particle field $N(t,y)$, $y \in \mathbb{Z}^d$, is homogeneous in space, it is sufficient to prove Theorem 1 for $y = 0$. We first estimate all factorial moments $m_k(t,x)$, $k \geq 1$, of the subpopulation in Eq. (13). We then estimate the moments for the total population $N(t,0)$ uniformly in t (the bounds do not depend on t). Because these moments $m_k(N(t,0))$ are monotonic in t and bounded, their limits at $t \to \infty$ exist. Then, we will use the Carleman conditions (Carleman, 1926; Shobat, 1943) to establish a unique limit distribution.

Proof of Theorem 1. If $\mu = \sum_{l=2}^{\infty}(l-1)\beta_l$, then the differential Eq. (6) for the generating function becomes:

$$\begin{aligned}\frac{\partial u_z}{\partial t}(t,x)= \quad & \kappa(\mathcal{L}_a u_z)(t,x)+\sum_{l=2}^{\infty}(l-1)\beta_l-\left(\sum_{l=2}^{\infty} l\beta_l\right)u_z(t,x)+ \\ & u_z(t,x)\sum_{l=2}^{\infty}\beta_l(u_z*b)^{l-1}(t,x).\end{aligned} \tag{26}$$

Differentiating Eq. (26) k times, $k\geq 2$, for the k-th factorial moment, leads to:

$$\begin{aligned}&\frac{\partial m_k}{\partial t}(t,x)=\left(\left(\kappa\mathcal{L}_a+\sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b\right)m_k\right)(t,x)+ \\ &\sum_{l=2}^{\infty}\beta_l\sum_{n=1}^{k-1}\frac{m_n(t,x)}{n!}\sum_{\sum_{i=1, j_i\geq 0}^{l-1} j_i=k-n,} k!\prod_{i=1}^{l-1}\frac{(m_{j_i}*b)(t,x)}{j_i!}+ \\ &\sum_{l=2}^{\infty}\beta_l\sum_{\sum_{i=1, 0\leq j_i\leq k-1}^{l-1} j_i=k,} k!\prod_{i=1}^{l-1}\frac{(m_{j_i}*b)(t,x)}{j_i!},\end{aligned} \tag{27}$$

where we assume $m_0(t,x)=u_z(t,x)|_{z=1}=1$ for all $t\geq 0$, $x\in\mathbb{Z}^d$, and the initial condition $m_k(0,x)=0$ for all $x\in\mathbb{Z}^d$, $k\geq 2$.

Lemma 1. *Under the conditions of Theorem 1, for all $k\geq 1$,*

$$m_k(t,x;0)\leq k!B^{k-1}D_k p(t,x,0), \tag{28}$$

where

$$B=\max\left\{1,\beta\int_0^{\infty}p(s,0,0)ds\right\}<\infty. \tag{29}$$

The sequence D_k is defined recursively as: $D_1=1$ and, for $k\geq 2$,

$$\begin{aligned}D_k= \quad & \sum_{l=2}^{\infty}\delta^l\sum_{n=1}^{k-1}D_n\sum_{r=1}^{l-1}\binom{l-1}{r}\sum_{\sum_{i=1, j_i\geq 1}^{r} j_i=k-n,} D_{j_1}\cdot\ldots\cdot D_{j_r}+ \\ & \sum_{l=2}^{\infty}\delta^l\sum_{r=2}^{l-1}\binom{l-1}{r}\sum_{\sum_{i=1, j_i\geq 1}^{r} j_i=k,} D_{j_1}\cdot\ldots\cdot D_{j_r}.\end{aligned} \tag{30}$$

Lemma 2. *The sequence D_k determined by $D_1=1$ and Eq. (30) does not increase faster than geometrically.*

Corollary 1. *There exists $c>0$ such that, for all $k\geq 1$,*

$$m_k(t,x)\leq c^k k!p(t,x,0), \tag{31}$$

$$\sum_{x\in\mathbb{Z}^d} m_k(t,x)\leq c^k k!. \tag{32}$$

The k-th factorial moment for the total population size $m_k(N(t,0))$ is not equal to $\sum_{x\in\mathbb{Z}^d} m_k(t,x)$ because factorial moments are not additive.

However, for independent random variables, cumulants have the property of additivity.

We recall the definition of the cumulants of a discrete random variable X. If, for all k, the factorial moments $m_k(X) = \mathbb{E}X(X-1)\cdot \ldots \cdot (X-k+1) < \infty$ exist, then the Taylor expansion

$$\Phi_X(z) = \mathbb{E}z^X = \sum_{k=0}^{\infty} \frac{m_k(X)}{k!}(z-1)^k \tag{33}$$

for the generating function is valid, and the derivatives $m_k(X) = \frac{d^k \Phi_X(z)}{dz^k}\big|_{z=1}$ exist and are continuous. $\Phi_X(1) = 1$ and Φ_X is a continuous function in a neighborhood of the point $z = 1$, which means that, in the neighborhood of $z = 1$, the continuous derivatives $\chi_k(X) = \frac{d^k \ln \Phi_X(z)}{dz^k}\big|_{z=1}$ exist and the log-generating function has the Taylor expansion

$$\ln \Phi_X(z) = \sum_{k=0}^{\infty} \frac{k(X)}{k!}(z-1)^k, \tag{34}$$

where the coefficients $\chi_k(X)$ are called "cumulants" or "semi-invariants" (Leonov and Shiryaev, 1959). Cumulants are additive: for independent random variables X and Y, $\varkappa_k(X+Y) = \varkappa_k(X) + \varkappa_k(Y)$.

From Eq. (33) and (34), cumulants and factorial moments are connected through each other:

$$\varkappa_n(X) = n! \sum_{\substack{\sum_{k=1}^{n} kj_k = n, \\ j_k \geq 0}} \frac{(-1)^{\sum_{k=1}^{n} j_k - 1}\left(\sum_{k=1}^{n} j_k - 1\right)!}{j_1! \cdots j_n!} \prod_{k=1}^{n} \left(\frac{m_k(x)}{k!}\right)^{j_k}. \tag{35}$$

From Eq. (35) and Corollary 1,

$$\varkappa_k(n(t,x,0)) = O(m_k(t,x)). \tag{36}$$

Using the additivity property, the cumulants of the random process $N(t,0)$ result from summing up the cumulants associated with the independent subpopulations:

$$\varkappa_k(N(t,0)) = \sum_{x \in \mathbb{Z}} \varkappa_k(n(t,x,0)) = O\left(c^k k!\right). \tag{37}$$

Lemma 3. *There exist $C_1 > 0$ and $C_2 > 0$ such that, for all $k \geq 1$, $\varkappa_k(N(t,y)) \leq C_1^k k!$ and $m_k(N(t,y)) \leq C_2^k k!$ uniformly in t.*

The monotonicity in t of the moments of the total population size $m_k(N(t,0))$ is proved in Lemma 2.4 of Molchanov and Whitmeyer (2017). Using this and Lemma 3, $\lim_{t\to\infty} m_k(N(t,0))$ exists and is bounded:

$$\lim_{t\to\infty} m_k(N(t,0)) \leq C_2^k k!. \tag{38}$$

The limit behavior of the moments of the total population size in Eq. (38) determines the limit distribution of $N(\infty, 0)$ uniquely. Indeed, the upper boundary in Eq. (38) implies that the generating function for the sequence

$$\lim_{t\to\infty} m_k(N(t,0)), \quad k \geq 1 \tag{39}$$

in Eq. (33) is analytical in a neighborhood of $z = 1$. This is why the sequence in Eq. (39) uniquely determines the probability distribution of $N(\infty, 0)$ (Feller, 1971: Chapter VII, § 6) and

$$m_k(N(\infty,0)) = \lim_{t\to\infty} m_k(N(t,0)), \quad k \geq 1. \tag{40}$$

These conditions called "Carleman conditions" dealing with the sequence of moments prove the existence of a uniquely determined distribution law. $N(\infty, \cdot)$ is the steady state of the population dynamics.

3. Intermittency

We definite intermittency: consider the random homogeneous-in-space (say, $\mathbb{Z}^d$) random field $N(\infty, y)$, which here is the steady state of the population. Assume that all statistical moments are defined and finite: $\mu_1 = \mathbb{E}N(\cdot)$, $\mu_2 = \mathbb{E}N^2(\cdot)$. The "intermittent" field if $\mu_j^{\frac{1}{j}} \ll \mu_{j+1}^{\frac{1}{j+1}}$ for all $j \geq 1$. One such inequality, say $\mu_2 \gg \mu_1^2$, implies all others because of Lyapunov's inequality: $\mu_1 \leq \mu_2^{1/2} \leq \mu_3^{1/3} \leq \dots$. For example, $\mu_2 \leq \mu_3^{2/3}$ and $\mu_2 \gg \mu_1^2$ imply $1 \ll \frac{\mu_2}{\mu_1^2} \leq \frac{\mu_3^{2/3}}{\mu_1^2}$, then $\frac{\mu_3}{\mu_1^3} \gg 1$ or $\mu_1 \ll \mu_3^{1/3}$.

For an appropriate choice of the parameters μ and β_l, $l \geq 2$, the ratio $\frac{\mu_2}{\mu_1^2}$ can be arbitrary large, hence $\frac{\mu_2}{\mu_1^2} \gg 1$ or $\mu_1 \ll \mu_2^{1/2}$.

For example, for $N(y)$, $y \in \mathbb{Z}^d$, independent identically distributed random values and $\mathbb{P}\{N(y) = A\} = \frac{1}{A}$, $\mathbb{P}\{N(y) = 0\} = 1 - \frac{1}{A}$, $A \gg 1$, we have $\mu_1 = 1$, $\mu_2 = A$, $\mu_3 = A^2$, that is $\mu_2 \gg \mu_1^2$. This random field displays separated peaks of high magnitude. This characterizes an intermittency phenomenon.

The Fourier representation of the factorial moments of the subpopulation $n(t, x, 0)$:

$$\hat{m}_j(t,k) = \sum_{x\in\mathbb{Z}^d} e^{i(k,x)} m_j(t,x), \tag{41}$$

where $k \in T^d = [-\pi, \pi)^d$. Hence

$$\hat{m}_1(t,k) = \hat{p}(t,k,0) = e^{t\hat{\mathcal{L}}(k)} \tag{42}$$

is the solution of the Fourier representation of Eq. (20), which becomes:

$$\frac{\partial \hat{p}}{\partial t}(t,k,0) = \hat{\mathcal{L}}(k)\hat{p}(t,k,0) \tag{43}$$

with initial condition $\hat{p}(0,k,0) = 1$. Here

$$\begin{aligned} \hat{\mathcal{L}}(k) &= \kappa(\hat{a}(k) - 1) + \sum_{l=2}^{\infty}(l-1)\beta_l(\hat{b}(k) - 1), \\ \hat{a}(k) &= \sum_{z\in\mathbb{Z}^d} e^{i(k,z)}a(z) = \sum_{z\in\mathbb{Z}^d} \cos(k,z)a(z), \\ \hat{b}(k) &= \sum_{z\in\mathbb{Z}^d} e^{i(k,z)}b(z) = \sum_{z\in\mathbb{Z}^d} \cos(k,z)b(z), \end{aligned} \tag{44}$$

where $\hat{a}$, $\hat{b}$, and $\hat{\mathcal{L}}$ are symmetric functions in $T^d = [-\pi,\pi)^d$, $\hat{\mathcal{L}}$ is nonpositive function in T^d, $\hat{a}(0) = \hat{b}(0) = 1$, $\hat{\mathcal{L}}(0) = 0$, and

$$\mathbb{E}N(t,0) = \sum_{x\in\mathbb{Z}^d} m_1(t,x) = \hat{m}_1(t,0) = 1. \tag{45}$$

The Fourier analog of Eq. (27) for the second factorial moment is:

$$\frac{\partial \hat{m}_2}{\partial t}(t,k) = \hat{\mathcal{L}}(k)\hat{m}_2(t,k) + \hat{g}(t,k) \tag{46}$$

with

$$\begin{aligned} \hat{g}(t,k) = 2\frac{\sum_{l=2}^{\infty}(l-1)\beta_l}{(2\pi)^d}\int_{T^d}\hat{p}(t,k-\theta,0)\hat{b}(\theta)\hat{p}(t,\theta,0)\,d\theta + \\ \frac{\sum_{l=2}^{\infty}(l-1)(l-2)\beta_l}{(2\pi)^d}\int_{T^d}\hat{b}(k-\theta)\hat{p}(t,k-\theta,0)\hat{b}(\theta)\hat{p}(t,\theta,0)\,d\theta. \end{aligned} \tag{47}$$

The solution of Eq. (46) is $\hat{m}_2(t,k) = \int_0^t \hat{p}(t-s,k,0)\hat{g}(s,k)\,ds$. Hence

$$\begin{aligned} &\sum_{x\in\mathbb{Z}^d} m_2(t,x) = \hat{m}_2(t,0) = \int_0^t \hat{g}(s,0)\,ds = \\ &\frac{2}{(2\pi)^d}\sum_{l=2}^{\infty}(l-1)\beta_l\int_{T^d}\left(\int_0^t e^{2s\hat{\mathcal{L}}(\theta)}ds\right)\hat{b}(\theta)\,d\theta + \\ &\frac{1}{(2\pi)^d}\sum_{l=2}^{\infty}(l-1)(l-2)\beta_l\int_{T^d}\left(\int_0^t e^{2s\hat{\mathcal{L}}(\theta)}ds\right)\hat{b}^2(\theta)\,d\theta. \end{aligned} \tag{48}$$

The asymptotic behavior of the sum of the second factorial moments is

$$\begin{aligned} \sum_{x\in\mathbb{Z}^d} m_2(t,x) \underset{t\to\infty}{\longrightarrow} \quad & \frac{\sum_{l=2}^{\infty}(l-1)\beta_l}{(2\pi)^d}\int_{T^d}\frac{\hat{b}(\theta)}{-\hat{\mathcal{L}}(\theta)}\,d\theta + \\ & \frac{\sum_{l=2}^{\infty}(l-1)(l-2)\beta_l}{(2\pi)^d}\int_{T^d}\frac{\hat{b}^2(\theta)}{-2\hat{\mathcal{L}}(\theta)}\,d\theta. \end{aligned} \tag{49}$$

From Eq. (33), (34), and (35),

$$\chi_2(N(t,0)) = \sum_{x\in\mathbb{R}^d} \chi_2(n(t,x,0)) = \sum_{x\in\mathbb{Z}^d} m_2(t,x) - \sum_{x\in\mathbb{Z}^d} m_1^2(t,x) = \sum_{x\in\mathbb{Z}^d} m_2(t,x) - \sum_{x\in\mathbb{Z}^d} p^2(t,x,0) = \sum_{x\in\mathbb{Z}^d} m_2(t,x) - \sum_{x\in\mathbb{Z}^d} p(t,0,x)p(t,x,0) = \sum_{x\in\mathbb{Z}^d} m_2(t,x) - p(t,0,0) \tag{50}$$

where, from Eq. (25), the last term $p(t,0,0)$ tends to zero when $t\to\infty$.

We use the connection between the moments and the cumulants, given in Eq. (50), and the asymptotic behavior given in Eq. (49):

$$\mathbb{E}N^2(t,y) = \mathbb{E}N(t,y)(N(t,y)-1) + \mathbb{E}N(t,y) = \chi_2(N(t,y)) + (\mathbb{E}N(t,y))^2 + \mathbb{E}N(t,y) = \chi_2(N(t,y)) + 2 \underset{t\to\infty}{\longrightarrow} 2 + \frac{1}{(2\pi)^d}\sum_{l=2}^{\infty}(l-1)\beta_l\int_{T^d}\frac{\hat{b}(\theta)}{-\hat{\mathcal{L}}(\theta)}\,d\theta + \frac{1}{(2\pi)^d}\sum_{l=2}^{\infty}(l-1)(l-2)\beta_l\int_{T^d}\frac{\hat{b}^2(\theta)}{-2\hat{\mathcal{L}}(\theta)}\,d\theta. \tag{51}$$

The first term on the right-hand side of Eq. (51) is the constant 2, the second is proportional to the average total number $\sum_{l=2}^{\infty}(l-1)\beta_l$ (the first moment) of new particles per time unit, and the third is proportional to the second factorial moment of the total number $\sum_{l=2}^{\infty}(l-1)(l-2)\beta_l$ of new particles per time unit. By assumption of critical states, the first moment of new particles per time unit coincides with the intensity of mortality: $\sum_{l=2}^{\infty}(l-1)\beta_l=\mu$, and the second term on the right-hand side of Eq. (51) is constant, too. Hence, the intermittency phenomenon takes place if

$$\sum_{l=2}^{\infty}(l-1)(l-2)\beta_l \gg \left(\sum_{l=2}^{\infty}(l-1)\beta_l\right)^2 = \mu^2. \tag{52}$$

Conclusion

In the particle field defined on the lattice $\mathbb{Z}^d$, $d\geq 1$, a single particle starts from each point. Every particle is endowed with a homogeneous random migration process with zero drift. It produces an arbitrary total number of offspring, which spread symmetrically away from the parental particle and behave as independent particles. In this frame, we have proved the existence of a limit (stationary) distribution of the population under the conditions:

(a) the sum of the generator for the migration process of each particle and the generator for the spreading of offspring is a generator of a transient random walk;

(b) the tail of the distribution of the total number of offspring decreases at least geometrically;
(c) the mortality rate coincides with the average total number of new particles per time unit (this is a critical case of branching processes).

When the second factorial moment of the total number of offspring is much larger than the square of the first factorial moment, the limit particle field deviates importantly from uniformity: the field then displays intermittency.

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References

Bakry, D., Gill, R., and Molchanov, S. 1994. *Lecture Notes in Mathematics*. Berlin Heidelberg: Springer-Verlag.

Carleman, T. 1926. *Les fonctions quasi analytiques*. Paris, France: Gauthier-Villars.

Debes, H., Kerstan, J., Liemant, A., and Matthes, K. 1970. Verallgemeinerungen eines Satzes von Dobruschin I. In Robert Denk, WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim eds., *Mathematische nachrichten*. (Vol. 47, pp. 183–244). Retrieved from https://onlinelibrary.wiley.com/doi/pdf/10.1002/mana.19700470120

Dobrushin, R. L. and Siegmund-Schultze, R. 1982. The hydrodynamic limit for systems of particles with independent evolution. *Mathematische Nachrichten*, *105*(1), 199–224.

Feller, W. 1971. *An Introduction to Probability Theory and Its Applications*. (2nd ed.), New York, NY: John Wiley.

Flajolet, P. and Sedgewick, R. 2009. *Analytic Combinatorics*. Cambridge: Cambridge University Press.

Friedman, A. 1964. *Partial Differential Equations of Parabolic Type*. Englewood Cliffs, NJ: Prentice-Hall.

Gärtner, J. and Molchanov, S. A. 1990. Parabolic problems for the Anderson model. *Intermittency and Related Topics. Communications in Mathematical Physics*, *132*(3), 613–655.

Kolmogorov, A. N., Petrovskii, I. G., and Piskunov, N. S. 1937. A study of the diffusion equation with increase in the quantity of matter, and its application to a biological problem. *Moscow University Mathematics Bulletin*, *Series A*(1), 1–25.

Kondratiev, Y., Kutovyi, O., and Pirogov, S. 2008. Correlation functions and invariant measures in continuous contact model. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, *11*(2), 231–258.

Leonov, V. P. and Shiryaev, A. N. 1959. On a method of calculation of semi-invariants. *Theory of Probability and Its Applications*, *4*(3), 319–329.

Liemant, A. 1969. lnvariante zufällige Punktfolgen. *Wissenschaftliche Zeitschrift Der Friedrich-Schiller-Universität Jena*, *19*, 361–372.

Molchanov, S. and Whitmeyer, J. 2017. Stationary distributions in Kolmogorov–Petrovski–Piskunov-type models with an infinite number of particles. *Mathematical Population Studies*, *24*(3), 147–160.

Sevast'yanov, B. A. 1951. The theory of branching random processes. *Ycnexu Mamemamureckux Hayk*, 6(6), 47–99.
Sevast'yanov, B. A. 1971. *Branching Processes*. Moscow, Russia: Nauka.
Shobat, J. T. and Tamarkin, J. 1943. *The Problem of Moment*. New York, NY: American Mathematical Society.
Spitzer, F. 1976. *Principle of Random Walk*. New York: Springer-Verlag.
Watson, H. W. and Galton, F. 1875. On the probability of the extinction of families. *The Journal of the Anthropological Institute of Great Britain and Ireland*, *4*, 138–144.
Zeldovich, Y. B., Molchanov, S., and Ruzmaikin, A., and Sokoloff, D. D. 2014. Intermittency, diffusion, and generation in a nonstationary random medium (2nd Ed.). *Reviews in Mathematics and Mathematical Physics*. *15*(1), 1–112.

A. Appendix

Proof of Lemma 1: Denote $\tilde{m}_k(t,x) = \frac{m_k(t,x)}{k!}$. Then, Eq. (27) becomes

$$\begin{aligned} \frac{\partial \tilde{m}_k}{\partial t}(t,x) &= \left(\left(\kappa\mathcal{L}_a + \sum_{l=2}^{\infty}(l-1)\beta_l\mathcal{L}_b\right)\tilde{m}_k\right)(t,x) + \\ &\sum_{l=2}^{\infty}\beta_l\sum_{n=1}^{k-1}\tilde{m}_n(t,x)\sum_{\sum_{i=1}^{l-1} j_i=k-n,\ j_i\geq 0}\prod_{i=1}^{l-1}\left(\tilde{m}_{j_i} * b\right)(t,x) + \\ &\sum_{l=2}^{\infty}\beta_l\sum_{\sum_{i=1}^{l-1} j_i=k,\ 0\leq j_i\leq k-1}\prod_{i=1}^{l-1}\left(\tilde{m}_{j_i} * b\right)(t,x). \end{aligned} \tag{53}$$

Recall that Duhamel's principle (Friedman, 1964) states that if $f(t,x)$, $t \geq 0$, $x \in \mathbb{Z}^d$, is the fundamental solution of the operator $\mathcal{L}$, that is, f is the solution of the homogeneous equation:

$$\frac{\partial f}{\partial t}(t,x) = \mathcal{L}f(t,x) \tag{54}$$

with initial condition $f(0,x) = \delta(x)$, then the equation

$$\frac{\partial F}{\partial t}(t,x) = \mathcal{L}F(t,x) + g(t,x) \tag{55}$$

with initial condition $F(0,x) = 0$ has the solution

$$F(t,x) = \int_0^t ds \sum_{z\in\mathbb{Z}^d} f(t-s, x-z)g(s,z). \tag{56}$$

"Fundamental" means that the solution of the nonhomogeneous Eq. (55) is the convolution of the fundamental solution and the nonhomogeneous term g.

Using Duhamel's formula, the solution of Eq. (53) is:

$$\begin{aligned} &\tilde{m}_k(t,x) = \\ &\int_0^t ds \sum_{z\in\mathbb{Z}^d} p(t-s, x-z, 0)\sum_{l=2}^{\infty}\beta_l\sum_{n=1}^{k-1}\tilde{m}_n(s,z)\sum_{\sum_{i=1}^{l-1} j_i=k-n,\ j_i\geq 0}\prod_{i=1}^{l-1}\left(\tilde{m}_{j_i} * b\right)(s,z) + \\ &\int_0^t ds \sum_{z\in\mathbb{Z}^d} p(t-s, x-z, 0)\sum_{l=2}^{\infty}\beta_l\sum_{\sum_{i=1}^{l-1} j_i=k,\ 0\leq j_i\leq k-1}\prod_{i=1}^{l-1}\left(\tilde{m}_{j_i} * b\right)(s,z). \end{aligned} \tag{57}$$

Excluding $(\tilde{m}_0 * b)(s,z) = 1$ for all $s \geq 0$ and $z \in \mathbb{Z}^d$, the inner sum of the first term in Eq. (57) becomes:

$$\sum_{n=1}^{k-1} \tilde{m}_n(s,z) \sum_{\sum_{i=1}^{l-1} j_i = k-n,\ j_i \geq 0} \prod_{i=1}^{l-1} \big(\tilde{m}_{j_i} * b\big)(s,z) = \sum_{n=1}^{k-1} \tilde{m}_n(s,z) \sum_{r=1}^{l-1} \binom{l-1}{r} \sum_{\sum_{i=1}^{r} j_i = k-n,\ j_i \geq 1} \prod_{i=1}^{r} \big(\tilde{m}_{j_i} * b\big)(s,z). \tag{58}$$

The inner sum of the second term in Eq. (57) becomes:

$$\sum_{\sum_{i=1}^{l-1} j_i = k,\ 0 \leq j_i \leq k-1} \prod_{i=1}^{l-1} \big(\tilde{m}_{j_i} * b\big)(s,z) = \sum_{r=2}^{l-1} \binom{l-1}{r} \sum_{\sum_{i=1}^{r} j_i = k,\ j_i \geq 1} \prod_{i=1}^{r} \big(\tilde{m}_{j_i} * b\big)(s,z). \tag{59}$$

We verify the lemma using the induction principle.

$\tilde{m}_1(t,x) = p(t,x,0)$ (p is the fundamental solution of Eq. (20)) and Eq. (28) holds true for $k = 1$. If Eq. (28) holds for $k-1$, then the right-hand side of Eq. (58) does not exceed

$$B^{k-1} p(s,z,0) \sum_{n=1}^{k-1} D_n \sum_{r=1}^{l-1} \binom{l-1}{r} \left(\frac{(p*b)(s,z)}{B}\right)^r \sum_{\sum_{i=1}^{r} j_i = k-n,\ j_i \geq 1} D_{j_1} \cdot \ldots \cdot D_{j_r} \leq B^{k-1} p(s,z,0) \frac{p(s,0,0)}{B} \sum_{n=1}^{k-1} D_n \sum_{r=1}^{l-1} \binom{l-1}{r} \sum_{\sum_{i=1}^{r} j_i = k-n,\ j_i \geq 1} D_{j_1} \cdot \ldots \cdot D_{j_r}, \tag{60}$$

where we use the fact that:

$$p(s, z-v, 0) \leq p(s,0,0) \text{ for all z, v} \in \mathbb{Z}^{\mathrm{d}}, \mathrm{s} \geq 0. \tag{61}$$

Indeed, from Eq. (42) and (44) and the symmetry of $\hat{\mathcal{L}}$, the inverse Fourier transform for all $x \in \mathbb{Z}^d$, $t \geq 0$ gives:

$$p(t,x,0) = \int_{[-\pi,\pi)^d} e^{-i(k,x)} \hat{p}(t,k,0)\, dk = \int_{[-\pi,\pi)^d} \cos(k,x) \hat{p}(t,k,0)\, dk \leq \int_{[-\pi,\pi)^d} \hat{p}(t,k,0)\, dk = p(t,0,0) \tag{62}$$

From Eq. (61) and the definition of B in Eq. (29), for all $r \geq 1$,

$$\left(\frac{(p*b)(s,z)}{B}\right)^r = \left(\frac{\sum_{v \in \mathbb{Z}^d} b(v) p(s, z-v, 0)}{B}\right)^r \leq \frac{p(s,0,0)}{B}. \tag{63}$$

Likewise, the right-hand side of Eq. (59) does not exceed

$$B^{k-1}p(s,z,0)\sum_{r=2}^{l-1} l-1_r\left(\frac{p*b}{B}\right)^{r-1}\sum_{\substack{\sum_{i=1}^r j_i=k,\\ j_i\geq 1}} D_{j_1}\cdot\ldots\cdot D_{j_r}\leq$$
$$B^{k-1}p(s,z,0)\frac{p(s,0,0)}{B}\sum_{r=2}^{l-1}\binom{l-1}{r}\sum_{\substack{\sum_{i=1}^r j_i=k,\\ j_i\geq 1}} D_{j_1}\cdot\ldots\cdot D_{j_r}. \tag{64}$$

Substituting this term into Eq. (57):

$$\begin{aligned}\tilde{m}_k(t,x;0)\leq\ & B^{k-1}\int_0^t ds\frac{p(s,0,0)}{B}\sum_{z\in\mathbb{Z}^d}p(t-s,x-z,0)p(s,z,0)\cdot\\ &\sum_{l=2}^{\infty}\beta_l\Big(\sum_{n=1}^{k-1}D_n\sum_{r=1}^{l-1}\binom{l-1}{r}\sum_{\substack{\sum_{i=1}^r j_i=k-n,\\ j_i\geq 1}}D_{j_1}\cdot\ldots\cdot D_{j_r}+\\ &\sum_{r=2}^{l-1}\binom{l-1}{r}\sum_{\substack{\sum_{i=1}^r j_i=k,\\ j_i\geq 1}}D_{j_1}\cdot\ldots\cdot D_{j_r}\Big).\end{aligned} \tag{65}$$

The lemma results from the recursive definition of the sequence D_k in Eq. (30) and from the facts that:

$$\sum_{z\in\mathbb{Z}^d}p(t-s,x-z,0)p(s,z,0)=\sum_{z\in\mathbb{Z}^d}p(t-s,x,z)p(s,z,0)=p(t,x,0) \tag{66}$$

(this is the Chapman–Kolmogorov equation);

$$\beta_l\leq\beta\delta^l\quad\text{(assumption of thelemma)}; \tag{67}$$

$$\frac{\beta\int_0^t p(s,0,0)ds}{B}\leq 1\quad\text{(from Eq.(29))}. \tag{68}$$

Proof of Lemma 2: The generating function for the sequence $\{D_k\}_{k=1}^{\infty}$ is $D(w)=\sum_{k=1}^{\infty}D_kw^k$. We denote

$$[w^k]F(w)=F_k, \tag{69}$$

the extraction of the coefficient of w^k in the formal power series $F(w)=\sum_{k=1}^{\infty}F_kw^k$. Then

$$\sum_{\substack{\sum_{i=1}^r j_i=k-n,\\ j_i\geq 1}}D_{j_1}\cdot\ldots\cdot D_{j_r}=[w^{k-n}]D^r(w) \tag{70}$$

and the inner sum of the first term in Eq. (30) becomes

$$\begin{aligned}&\sum_{r=1}^{l-1}\binom{l-1}{r}\sum_{\substack{\sum_{i=1}^r j_i=k-n,\\ j_i\geq 1}}D_{j_1}\cdot\ldots\cdot D_{j_r}=[w^{k-n}]\sum_{r=1}^{l-1}\binom{l-1}{r}D^r(w)=\\ &[w^{k-n}]\Big((1+D(w))^{l-1}-1\Big)\end{aligned} \tag{71}$$

where $k-n\geq 1$.

For all $k\geq 2$, the sum in the middle of the first term in Eq. (30) becomes the convolution of two generating functions:

$$\begin{aligned}&\sum_{n=1}^{k-1} D_n \sum_{r=1}^{l-1} \binom{l-1}{r} \sum_{\substack{\sum_{i=1}^{r} j_i=k-n,\\ j_i\geq 1}} D_{j_1}\cdot\ldots\cdot D_{j_r} =\\ &\sum_{n=1}^{k-1} [w^n]D(w)[w^{k-n}]\Big((1+D(w))^{l-1}-1\Big) =\\ &[w^k]D(w)\Big((1+D(w))^{l-1}-1\Big).\end{aligned} \tag{72}$$

Likewise for Eq. (71):

$$\sum_{r=2}^{l-1} \binom{l-1}{r} \sum_{\substack{\sum_{i=1}^{r} j_i=k,\\ j_i\geq 1}} D_{j_1}\cdot\ldots\cdot D_{j_r} = [w^k]\Big((1+D(w))^{l-1}-(l-1)D(w)-1\Big). \tag{73}$$

Using Eq. (72) and (73), Eq. (30) becomes:

$$\begin{aligned}&\sum_{l=2}^{\infty}\delta^l \sum_{n=1}^{k-1} D_n \sum_{r=1}^{l-1} \binom{l-1}{r} \sum_{\substack{\sum_{i=1}^{r} j_i=k-n,\\ j_i\geq 1}} D_{j_1}\cdot\ldots\cdot D_{j_r}+\\ &\sum_{l=2}^{\infty}\delta^l \sum_{r=2}^{l-1} \binom{l-1}{r} \sum_{\substack{\sum_{i=1}^{r} j_i=k,\\ j_i\geq 1}} D_{j_1}\cdot\ldots\cdot D_{j_r} =\\ &[w^k]\sum_{l=2}^{\infty}\delta^l\Big(D(w)\Big((1+D(w))^{l-1}-1\Big)+(1+D(w))^{l-1}-(l-1)D(w)-1\Big) =\\ &[w^k]\left(\sum_{l=2}^{\infty}\delta^l(1+D(w))^l - D(w)\sum_{l=2}^{\infty} l\delta^l - \sum_{l=2}^{\infty}\delta^l\right) =\\ &[w^k]\left(\tfrac{\delta^2(1+D(w))^2}{1-\delta(1+D(w))} - \tfrac{\delta^2(2-\delta)}{(1-\delta)^2}D(w) - \tfrac{\delta^2}{1-\delta}\right).\end{aligned} \tag{74}$$

For the generating function $D(w)$, Eq. (30) is then equivalent to:

$$D(w) - w = \frac{\delta^2(1+D(w))^2}{1-\delta(1+D(w))} - \frac{\delta^2(2-\delta)}{(1-\delta)^2}D(w) - \frac{\delta^2}{1-\delta}. \tag{75}$$

Both sides of Eq. (75) have monomials not smaller than of second order.

The solution is a rational generating function. The growth coefficient is defined by the finite radius of convergence R, which is also the distance from the origin to the closest singularity (Flajolet and Sedgewick, 2009). Hence $D_k < \left(\frac{1}{R}\right)^k$. □