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Scaling-continuous variation: supporting students' algebraic reasoning

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Abstract
This paper introduces a new mode of variational and covariational reasoning, which we call scaling-continuous reasoning. Scaling-continuous reasoning entails (a) imagining a variable taking on all values on the continuum at any scale, (b) understanding that there is no scale at which the continuum becomes discrete, and (c) re-scaling to any arbitrarily small increment for x and coordinating that scaling with associated values for y. Based on the analysis of a 15-h teaching experiment with two 12-year-old pre-algebra students, we present evidence of scaling-continuous reasoning and identify two implications for students' understanding of rates of change: seeing constant rate as an equivalence class of ratios, and viewing instantaneous rate of change as a potential rate. We argue that scaling-continuous reasoning can support a robust understanding of function and rates of change.

Keywords Student reasoning · Algebra · Middle school · Teaching experiments

1 Introduction: the importance of variation for function understanding

Functions and relations comprise a critical aspect of algebra, with recommendations for supporting students' algebraic reasoning advocating the introduction of functional relationships in the middle grades (e.g., National Governor's Association Center for Best Practices, 2010; U.K. Department for Education, 2009). Despite the importance of functional reasoning, however, research indicates that students exit secondary school viewing functions in terms of symbolic manipulations rather than as a model of dynamic situations (Stephens, Ellis, Blanton, & Brizuela, 2017). These findings highlight the need to better support students' emerging function concepts, particularly in terms of understanding functions as representations of

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variation. Currently, typical curricular and pedagogical approaches rely on the correspondence view, which treats a function as a fixed relationship between the members of two sets. This static treatment is widespread in secondary instruction; for instance, Thompson and Carlson (2017) reviewed 17 US secondary textbooks, ranging from Algebra I through Precalculus, and found that all relied on a correspondence definition of function.

A contrasting approach to supporting students' functional thinking is an emphasis on variational and covariational reasoning (Thompson & Carlson, 2017). Researchers argue that attending to and coordinating changes in quantities that continuously covary is critical for students' development of a robust understanding of function, constant and varying rates of change, and the foundational ideas of calculus (e.g., Carlson, Smith, & Persson, 2003). Further, situating functional exploration within contexts leveraging covarying quantities that enable visualization, manipulation, and prediction has been found to foster students' abilities to reason flexibly about dynamically changing events (Castillo-Garsow, Johnson, & Moore, 2013). Covariation is an examination of coordinated changes between x- and y-values (Confrey & Smith, 1995; Saldanha & Thompson, 1998). Confrey and Smith (1995), for instance, defined a covariation approach as moving operationally from y_m to y_{m+1} in coordination with movement from x_m to x_{m+1}. Saldanha and Thompson (1998) then extended this idea to characterize covariation as imagining two quantities changing together, which, in turn, is dependent on the ability to envision each quantity varying. From this perspective, covariation is the coupling of two quantities, which enables tracking either quantity's value with an explicit understanding that at every instance, the associated quantity has a corresponding value (Saldanha & Thompson, 1998).

In this paper, we introduce a new mode of variational and covariational reasoning, which we call *scaling-continuous covariation*. Scaling-continuous covariation entails imagining the continuum as infinitely zoomable, coupled with the understanding that one can re-scale to any arbitrarily small increment for x and coordinate that scaling with associated values for y. This paper reports on a study addressing the following research questions: (a) How does scaling-continuous reasoning differ from prior examples of covariational reasoning? (b) What are the implications of scaling-continuous reasoning for students' understanding of functions and rates of change? In the findings below, we argue that scaling-continuous reasoning can support productive ways of thinking about key function ideas, including constant and instantaneous rates of change.

2 Theoretical framework and relevant literature: rate reasoning and covariational reasoning

Stroup (2002) introduced the term qualitative calculus, which entails an informal introduction of calculus concepts, such as rates of change, to students in upper elementary and middle school. Given that calculus is a dynamic discipline (Tall, 2009), an approach emphasizing rate reasoning can be fruitful for supporting students' emerging function understanding (Carlson et al., 2003). This can be especially important given the prevalence in many European countries of introducing calculus in secondary school (Artigue, 2002; Maschietto, 2004). Rate of change, however, remains a difficult concept for both secondary students (e.g., Herbert & Pierce, 2012) and university students (e.g., Ubuz, 2007). There is some evidence that situating students' early exposure to function and rate reasoning in varying (rather than constant) rate contexts can be beneficial (e.g., Herbert & Pierce, 2012; Stroup, 2002). Further, emphasizing covarying quantities in rate situations can support students' understanding of rate as a relationship, rather than as the outcome of a calculation (Herbert & Pierce, 2012). Consequently, we chose to introduce dynamic contexts in which students could explore constant and varying rates of change in order not only to foster robust rate conceptions, but also to support an understanding of continuous function concepts.

Representing continuous relationships is challenging for students, and they often resort to discrete graphs to depict continuous phenomena (de Beer, Gravemeijer, & van Eijck, 2015). It is also not uncommon to approach continuous functions additively by comparing the changes in the output variable with respect to equal increments of the input variable (Kertil, Erbas, & Cetinkaya, 2019). However, there have been reports of successes in students moving from discrete to continuous representations (Yerushalmy & Swidan, 2012). For instance, de Beer et al. (2015) found that 5th-grade students could bootstrap their discrete reasoning about speed to make sense of continuous graphs and, ultimately, reason qualitatively about instantaneous speed. In order to make sense of students' rate reasoning for continuous functions, we leveraged ideas from Castillo-Garsow's (2012) and Thompson and Carlson's (2017) covariational reasoning frameworks, which we detail below.

2.1 Covariational reasoning frameworks

Thompson and Carlson (2017) provided an overview of research on student reasoning with continuous functions that synthesizes the work of a variety of researchers over the past several decades (e.g., Carlson et al., 2003; Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Our work here is situated in the context of this body of research and builds upon these frameworks.

Chunky-continuous variation and covariation Castillo-Garsow (2012) distinguished between two different images of change as students reason about variation: chunky and smooth reasoning. According to Castillo-Garsow et al. (2013), a *chunky* image of variation has two distinguishing features: "A unit chunk whose repetition makes up the variation, and a lack of image of variation within the unit chunk" (p. 33). One generates change by sequencing equal-sized chunks and measures change by counting the number of elapsed chunks. Castillo-Garsow and colleagues emphasized that a key aspect of chunky thinking is that one thinks *in* intervals, but not *about* intervals; intermediate values within a chunk may exist, but do not receive explicit attention.

Thompson and Carlson (2017) called this *chunky-continuous* variation: change occurs in completed chunks, but there is no clear image of how variation occurs within the chunk. For chunky-continuous covariation, the student imagines corresponding chunks in the other covarying quantity as well. When using chunky-continuous reasoning, a student may be able to reason with different chunk sizes; a given chunk is not necessarily indivisible. Rather, one chooses a chunk size and measures change in units of that chunk.

Smooth-continuous variation and covariation *Smooth* variation relies on an image of change experientially as it occurs (Castillo-Garsow et al., 2013); it is imagined in the present tense. In contrast, chunky thinking is in the past tense, imagining that the change has already happened and is now being analyzed. These authors add, "Smooth images of change are not the same as chunky images of change cut up really small. Smooth images of change involve an entirely different conceptualization of variation" (p. 34). The difference between smooth and chunky reasoning is like the difference between the experience of watching a movie and that of the movie-maker editing a sequence of already-filmed frames.

Thompson and Carlson (2017) described *smooth-continuous* variation in terms of projecting an image of one's own experiential time to a time period within the mathematical context. One imagines a value varying as its magnitude increases in bits while simultaneously anticipating smooth variation within each bit, passing through all of the intermediate values within any given bit. Smooth-continuous covariation then involves smooth variation in both quantities simultaneously. Furthermore, one can choose to consider change by intervals "while anticipating that within each interval the variable's value varies smoothly and continuously" (2017, p. 430). Thompson and Carlson therefore consider smooth-continuous reasoning to be more powerful than chunky-continuous reasoning, in that a student who can use smooth-continuous reasoning can also reason about change in chunks if needed, while a student who can use chunky-continuous reasoning may not be able to reason smoothly. Castillo-Garsow et al. (2013) make no such claim, and they treat chunky thinking and smooth thinking as different types of imagery between which a student might switch back and forth.

Both accounts of smooth reasoning agree that it requires reasoning in terms of motion, always entailing imagining "something moving" (Thompson & Carlson, 2017, p. 430). A varying quantity is imagined as being tacitly parameterized by conceptual time. Thompson and Carlson (2017) note that this is the same imagery Isaac Newton appealed to when he defined a variable quantity to be a "fluent" that depended on and changed with time.

2.2 Scaling-continuous variation and covariation

We now define a new image of change: *scaling-continuous variational and covariational reasoning*. The fundamental image involved in this type of reasoning is zooming or scaling. Scaling-continuous variation entails imagining that a variable takes on all values in any continuum of values, and assuming this will always be the case no matter how much you zoom in. The image is that the continuum is infinitely "zoomable," in that the process of zooming never reveals discrete atoms, holes, or values that the variable skips. The continuum is imagined as always being a continuum at any scale.

Scaling-continuous covariational reasoning imagines scaling to see any arbitrarily small continuum of values for one variable, and that this increment will always correspond to a continuum of associated values for the other variable. For instance, one can imagine a window of x-values growing, and the corresponding window of y-values simultaneously growing as a correspondence between increments of x and y.

Scaling-continuous reasoning does not require an image of motion or assume an underlying time parameter. In this way, it differs from smooth reasoning and is similar to chunky reasoning. Like chunky reasoning, it also treats change as having already occurred; reasoning about change is not grounded in the reasoner's experiential time. Yet scaling-continuous reasoning is unlike chunky reasoning in several ways. First, its fundamental image is of zooming, not of traversing an interval of change using a chosen chunk. Second, although a person using chunky reasoning does not attend to variation *within* a chunk, a person using scaling reasoning assumes there is always a correspondence between the values of the two variables at every scale within every chunk.

The third difference is the different form of generalizing each type of reasoning affords. Chunky reasoning entails imagining movement across a domain in chunks, so it can afford a generalization of some feature of the covariation across chunks. This could allow a student to imagine that feature being present in every chunk of the situation's domain. On the other hand, scaling-continuous covariational reasoning employs the image of repeated zooming or rescaling, which could enable one to generalize a feature of the covariation across *all* scales, including, potentially, the infinitesimal scale.

To illustrate this idea of generalizing across scale, we appeal to G. W. Leibniz' accounts of covariation in calculus. There are some notable parallels between scaling-continuous reasoning and the imagery G. W. Leibniz used when writing about infinitesimal calculus. We do not wish to speculate about what Leibniz thought, but rather to use Leibniz' imagery to motivate the idea of how scaling-continuous reasoning affords generalizing across scale.

Just as Newton's imagery of fluents has parallels with smooth reasoning, Leibniz' imagery of covariation has parallels with scaling-continuous reasoning. Rather than treating two covarying quantities as flowing simultaneously, Leibniz typically attended to the correspondence between increments of the two quantities, particularly infinitesimal increments (Bos, 1974). He distinguished among types of these increments based on their relative *scales*. For instance, for differential and integral calculus, Leibniz used infinitesimal increments. Starting with an algebraic relationship between the values of two variable quantities x and y, his differential calculus was then a way to derive a new equation describing the relationship between infinitesimal increments of the two quantities, dx and dy.

This idea can be illustrated by Leibniz' summary of the product rule:

> $d(xy)$ is the same as the difference between two adjacent xy, of which let one be xy, the other $(x + dx)(y + dy)$. Then, $d(xy) = (x + dx)(y + dy) - xy$, or $xdy + ydx + dxdy$, and this will be equal to $xdy + ydx$ if the quantity $dxdy$ is omitted, which is infinitely small with respect to the remaining quantities, because dx and dy are supposedly infinitely small (namely if the term of the sequence represents lines, increasing or decreasing continually by minima). (From Leibniz' *Elementa*, quoted in Bos, 1974, p. 16.)

First, we note the image of correspondence between the bits of two covarying quantities. Leibniz described a new variable quantity, xy, and he then sought to derive an equation describing the correspondence between an infinitesimal bit of this quantity, $d(xy)$, and infinitesimal bits (dx and dy) of the other two quantities x and y. Nothing about these increments is moving or changing, although their values depended upon where on the curve they were taken. Although Leibniz did not appeal to motion, the idea of covariation here entails the notion that every increment of one quantity, no matter how small, corresponds to an increment of another covarying quantity.

The second image we note is that of scaling. In the above example, Leibniz dismissed the quantity $dxdy$ because it is infinitely small even in comparison with other infinitely small quantities such as dx and xdy. Leibniz developed a scheme of orders of the infinitesimal and the infinite in order to systematize this idea of scaling. At the finite scale, infinitesimals such as dy are negligible, but at the first-order infinitesimal scale, they become significant, with second-order differences still negligible. Imagining correspondence at these different scales was crucial to a coherent system of calculus for Leibniz. At each such scale, the continuum was still continuous (Bos, 1974). The image of scaling can afford infinitesimal increments by generalizing features from finite increments in order to envision infinitesimal increments. One could imagine a process of infinite zooming, with the reification of such a process supporting an image of an increment at the infinitesimal scale that inherits properties from the finite cases (Ely, 2011).

One of the properties that can be projected onto infinitesimal increments is that of local straightness. For instance, Leibniz spoke of a curve as a polygon with infinitesimal sides, so that a difference triangle can be imagined at the infinitesimal scale with a straight hypotenuse of slope dy/dx. Scaling-continuous reasoning could afford, but by no means compels, this generalization of straightness to the infinitesimal scale. Differentiable curves appear straighter and straighter as you zoom in on them more and more. By attending to this property as one generalizes across scale, one might project onto the infinitesimal scale the image of local straightness. This is precisely the type of generalization Tall (1997) and Maschietto (2004) have sought to promote in their approaches to calculus through local linearity and the global/local game, respectively. Tall (2009), for instance, relied on the notion that a differentiable graph under infinite magnification is a straight line. If one zooms in dynamically on a graph with very small values of dx and dy, then the magnified graph looks like a straight line so that the graph and its tangent become indistinguishable. Both Tall (2009) and Maschietto (2004) leveraged the notions of embodiment and zooming to foster a shift from a global to a local perspective. These approaches, therefore, employ scaling-continuous imagery, which could afford the image of infinitesimal increments.

As illustrated in the "Results" section, this image of local straightness is not the only image that can be generalized across scale. If a student attends to different elements of the situation of covariation, then scaling-continuous reasoning might instead afford the projection of the image that a curve is always curved, even at infinitesimal scales. Scaling-continuous reasoning affords generalizing across scale, but does not by itself dictate what properties are generalized.

3 Methods

The study reported in this paper was part of a larger 3-year project aimed at understanding students' generalization processes in algebra, advanced algebra, and combinatorics. As part of this project, we implemented an exploratory teaching experiment in order to investigate students' generalizations of linear, quadratic, and higher-order polynomial functions from a rate-of-change perspective. The results of this study then supported the design of an instructional sequence for a series of larger-scale design experiments on function. The findings in this paper are from the first teaching experiment.

3.1 Participants and the teaching experiment

We conducted a 10-day, 15-h videoed teaching experiment (Steffe & Thompson, 2000) with two 12-year-old students in general mathematics (neither had yet taken algebra). We assigned gender-preserving pseudonyms to each student, Wesley and Olivia. The first author was the teacher-researcher, and a project member observed each session, which lasted between 1 and 2 h. The project team met daily to debrief.

We developed tasks to support a conception of linear growth as a representation of a constant rate of change and quadratic growth as a representation of a constantly changing rate of change. The tasks emphasized these ideas within the contexts of speed and area. The area tasks presented "growing rectangles," "growing stair steps," and "growing triangles" via dynamic geometry software, in which the students could manipulate a figure by extending the length and observing the associated growth in area (Fig. 1). Table 1 provides the mathematical topics and contexts addressed each day during the teaching experiment.

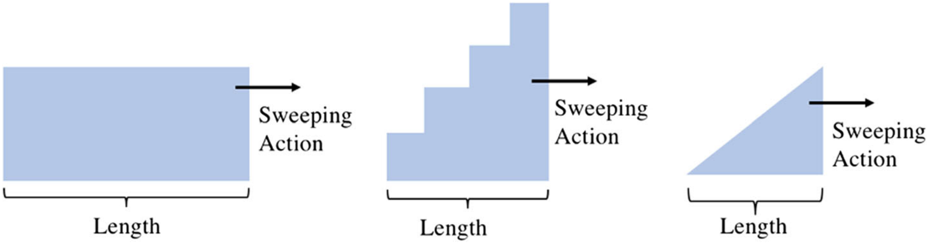


Fig. 1 Growing rectangle, stair step, and triangle tasks

Table 1 Overview of the teaching experiment unit

Day	Mathematical topics	Contexts
1	Linear growth, average rate of change	Speed
2	Linear growth, average rate of change	Speed, growing rectangles
3	Linear and piecewise linear growth	Growing rectangles, stair steps
4	Quadratic growth, average rate of change	Growing triangles
5	Quadratic growth, identifying constantly changing rates of change	Growing triangles
6	Quadratic growth, instantaneous rates of change	Growing triangles, trapezoids
7	Instantaneous rates of change	Growing rectangle, triangles, trapezoids
8	Cubic growth	Growing cubes and rectangular prisms
9	Cubic growth	Growing cubes and rectangular prisms
10	Higher-order polynomial functions	Growing 4D and n-dimensional figures

The researchers' goals informed the design and sequencing of the tasks. One goal was to create opportunities for students to reason about rates of change in progressively more sophisticated ways. The growing rectangle task provided an opportunity to establish that as the length and area grow together, the rate of change is fixed regardless of increment size. The stair step task had a fixed rate of change within each stair but increased by the same amount from stair to stair. The intent was to help the students begin to form the language and tools to model situations with non-constant rates of change.

The growing triangle task models quadratic growth. In this case, there are no periods of constant rate of change that can be calculated by dividing the displacement in area by the displacement in length. One can only calculate the average rate of change on that interval. We anticipated that students would partition the triangle into vertical columns of equal increments and reason about how the rates of change in area increased from one interval to the next. After working with different increments and varying their width, we hoped to encourage the students to begin reasoning informally about instantaneous rates of change by anticipating a rudimentary limiting process. Scaling-continuous reasoning emerged as a way of viewing functions that was adapted in part to these instructional materials and objectives.

3.2 Analysis

We employed retrospective analysis (Steffe & Thompson, 2000) in order to characterize the students' conceptions throughout the teaching experiment. We transcribed teaching session and then produced a set of enhanced transcripts that included all verbal utterances, images of students' work, descriptions of relevant gestures, and other non-verbal actions. Relying on the constant comparative method (Strauss & Corbin, 1990), we then analyzed the data in order to identify (a) students' forms of covariational reasoning and (b) students' conceptions of constant and changing rates of change. For the first round of analysis, we drew on Thompson and Carlson's (2017) framework of variational and covariational reasoning. We coded to infer categories of variation/covariation based on students' talk, figures, gestures, and task responses. We also developed emergent codes for students' understandings of constant, changing, and instantaneous rates of change. The first round then guided subsequent rounds of analysis in which the project team met to refine and adjust the codes in relation to one another. This iterative process continued until no new codes emerged. The final round of analysis was descriptive and supported the development of an emergent set of relationships between the students' covariational reasoning and their conceptions of constant, changing, and instantaneous rates of change.

4 Results

We focus on the distinction between two forms of reasoning in the teaching experiment, chunky-continuous and scaling-continuous covariation. Although these were not the only forms of reasoning we observed, they were the most prevalent forms that persisted throughout the teaching experiment. Further, we highlight chunky-continuous and scaling-continuous reasoning as a way to address the unique characteristics and affordances of scaling-continuous covariation as distinct from chunky-continuous covariation. In the sections below, we introduce evidence of the two forms of reasoning and then discuss the ways in which scaling-continuous covariation afforded generalizations about constant and instantaneous rates of change.

4.1 Chunky-continuous reasoning

On the fourth day of the teaching experiment, the students watched a video of a triangular region that grew in a smooth, continuous motion from left to right (Fig. 2a). The teacher-researcher (TR) asked the students to construct a graph to show the relationship between the total accumulated area and the length swept. Wesley explained his graph (Fig. 2b) by discussing segments of inches: "It's curved because as the length keeps going, for every inch it covers more area." Chunky-continuous covariation entails imagining each quantity's value changing by intervals of a fixed size. A student reasoning in this manner could re-chunk to different sizes, but would still lack an image of variation within a given chunk. In this case, Wesley considered amounts of length in 1-in. chunks, and he anticipated an amount of growth in area associated with each chunk. However, Wesley did not show evidence of considering how the length and area values accumulated together *within* each chunk.

In explaining the second graph Wesley produced on graph paper (Fig. 2c), he noted, "every inch it goes, it, like, it goes, it covers more area for that inch so it keeps getting steeper." The teacher-researcher asked Wesley whether the segments connecting the points were straight or curved:

W: I think they'd be a straight line too.
TR: Okay. When you say every inch it goes, it covers more area, is that true for only inches or is it true for any sort of increase?
W: Any sort of increase I think.

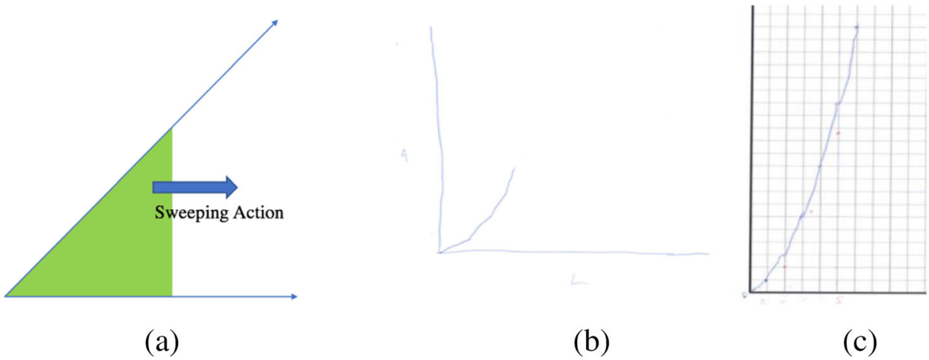


Fig. 2 A static image of the triangle's area swept from left to right (**a**) and Wesley's graphs (**b**, **c**)

Wesley stated that his image of growth would hold for "any sort of increase," suggesting that he may have been able to understand that the growth phenomenon was not dependent on the particular intervals he chose. The straight-line segments, however, suggest that any values occurring within chunks were tacit. A student reasoning with smooth-continuous covariation might instead create a curved graph that would reflect an image of continuous coordinated changes in x and y, understanding that for any arbitrary increment, both quantities will covary with smooth variation, moving through all values within the increment.

Olivia also showed evidence of chunky-continuous reasoning. When the teacher-researcher presented a growing triangle with a height-to-length ratio of 2:5, she invited the students to graph its area versus length. Olivia produced a piecewise linear graph with 1-in. increments for length, stating, "each line between each increment is just getting steeper":

> O: If you made the increments even smaller, like into 0.1 as your first point, then I think it'd be, all the little lines together I think they'd make a very subtle curve, but relatively straight. So when I did it with the increments as 1, I see them as straight, *but if they were smaller they might look as if they were curved to make one big curve.*

Olivia reasoned from one increment to the next, noting changes in steepness per line segment. She also affirmed that using smaller increments would change the appearance of the graph, an important hallmark of chunky-continuous covariation. When engaged in chunky reasoning, a student can "re-chunk" to different increments, but those increments still have a measurable length. Olivia reasoned that each interval was represented by a line segment; she could adjust the size of her increments, but without altering her view of the nature of change *within* each increment.

4.2 Scaling-continuous reasoning

On day 5 the teacher-researcher showed the students a video of another growing triangle, but this time asked them to draw a sketch of the area-length graph simultaneously as the video played. Wesley and Olivia worked together and both produced a smooth curve (Wesley's is shown in Fig. 3). The teacher-researcher asked Wesley why the graph was curved, in contrast to his prior piecewise graphs:

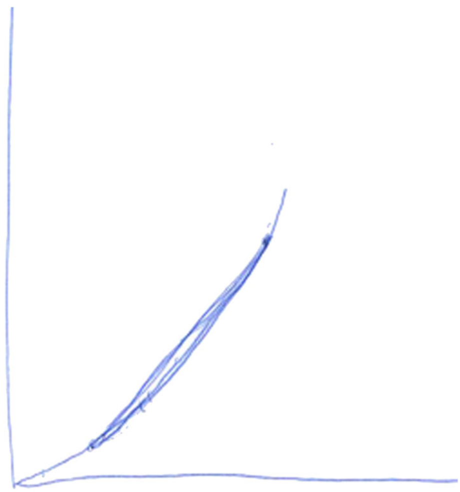


Fig. 3 Wesley's smooth area-length sketch reproduced from a dynamic sketch

W: Because, like, we were doing like big increments like here to like here (marks two points on the curve) and if you kind of draw a straight line (draws a line between the points) it's like not exactly on the curve. But if you add the tiny increments, like in-between, then it curves out.

The teacher-researcher then asked Wesley what the graph would look like between two points that were "super close together": would it be curved or straight? Wesley indicated that it would be curved, explaining that "there's tiny points *in between* those tiny points." The teacher-researcher further asked what would happen between two infinitesimally close points:

TR: What if I picked two points that were so close together that I couldn't, you couldn't even see the difference? They were just so close together there's like an infinitesimal difference in between them. Would the connection between them be a straight line, or a curve still?
O: Like the tiniest ones?
TR: Uh huh.
O: Then it would be a straight line.
TR: (Turns to Wesley). What do you think?
W: I think it'd be more of a curve because I think like it goes on infinitely, kind of, the points. So if you zoomed in really close on those it would like look like that and then *in between those there's still more points and it goes on forever*.
TR: (Turns to Olivia). What do you think?
O: I still think it'd be a straight line because to me it's just a whole bunch of little straight lines and so like to me *it would eventually stop* because you're graphing the triangle's, like, placing.

One way to distinguish Olivia's position from Wesley's is through the lens of potential versus actual infinity, a distinction that originates in Aristotle's work. Potential infinity is characterized by an ongoing process repeated over and over without end (Núñez, 2003). Núñez characterized this process by describing the action of imagining an unending sequence of regular polygons with more and more sides. The process, at any given stage, encompasses only a finite number of repetitions. As a whole, however, it does not end and therefore lacks a final resultant state. In contrast, actual infinity characterizes the infinite process as a realized thing. Even though the process lacks an end, it is conceived as being completed and as having a final resultant state. Following the same example, one can imagine an end *at infinity* "where the entire infinite sequence *does have* a final resultant state, namely a circle that is conceived as a regular polygon with an infinite number of sides" (p. 52, emphasis original). Similarly, one could imagine a curved graph as having an infinite number of straight sides (as Leibniz did), but, as we explain next, we do not believe that Olivia used actual infinity when imagining a curve made up of "a whole bunch of little straight lines."

Olivia could re-imagine the individual chunks to be smaller and smaller, but that process did not have a realized end state whereby the smooth curve and the piecewise linear approximation were one and the same. Olivia's "bunch of little straight lines" were not necessarily *infinitely* many straight lines; hence, Olivia noted that "it would eventually stop." In contrast, Wesley's claim that the segment would be curved is consistent with a notion of actual infinity. He spoke of the ability to zoom in such a way that it "goes on forever." Wesley could imagine points in between points, at any scale, even zooming in indefinitely. Wesley consequently treated the quantities' values as varying continuously, taking on all possible values within the interval, even if the interval was infinitesimal. An always-curved curve is the generalization Wesley made by employing actual infinity; he projected the property of "curving over every interval" onto infinitesimal intervals. In contrast, one could instead generalize, as did Leibniz, that the curve is made up of infinitely many straight lines.

The difference between Olivia and Wesley's reasoning illustrates an important distinction between scaling-continuous and chunky-continuous covariation. We also note that Wesley did not use smooth-continuous covariation, because his language contained no references to motion. His imagery was of scaling, not of a variable moving and tracing out values as it moved.

4.3 Affordances of scaling-continuous reasoning

In the following sections, we outline a critical way in which scaling-continuous covariation supported Wesley's thinking about rates of change: Namely, he was able to construct a multiplicative rate object by appealing to a figure's height, for both constant and changing rate of change figures. Below, we illustrate how this occurred in relation to Wesley's scaling-continuous covariational reasoning, and contrast his thinking with what is afforded by chunky-continuous covariation.

Constant rates of change Both students could reason about length and area growing together, but their conceptions of the ratio of area to length differed. Olivia conceived of this relationship as a ratio, by considering an amount of elapsed area in comparison with an amount of elapsed length. Wesley, in contrast, developed an understanding of the ratio as a rate of change. For instance, on day 6, Olivia and Wesley examined a dynamic figure in which the area was swept out under the curve in Fig. 4. The teacher-researcher positioned the mouse near the left end, where the figure's height is 3 cm. She asked the students, "At what rate is the area increasing in this portion?"

Olivia answered, "The area, if you break it into parts, the area would just keep growing." She concluded that the rate would be "consistent" because the amount of area gained for each

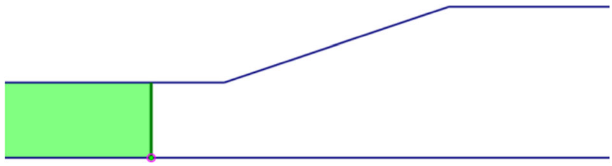


Fig. 4 Dynamically growing figure depicting constant and constantly changing rates of change

"part" was the same: Olivia compared the accumulated area across same-length increments. In contrast, Wesley answered 3 cm^2 per cm swept. He explicitly referenced both quantities, and when justifying his answer, Wesley said, "Because the height is 3 centimeters." As we detail below, we believe that Wesley's appeal to the height offers evidence that he conceived of the ratio of 3 cm^2 of area for 1 cm of length to be a rate.

Thompson and Thompson (1992) described a rate as a reflectively abstracted constant ratio. A ratio is a multiplicative comparison of two quantities; a rate requires viewing two such quantities as changing together, and treating the collection of equal ratios they generate as a single quantity of its own. It symbolizes the ratio structure as a whole while giving prominence to the constancy of the result of the multiplicative comparison. Wesley's appeal to the height, a single quantity, to justify the rate suggests that he saw it as a representation of the collection of equal ratios. In order to understand the height as a rate, Wesley needed to have an image of change such that 3 cm^2:1 cm represented an equivalence class of ratios.

Wesley appealed to the height as a representation of the area's rate on a number of other tasks. For instance, later on the same day, the teacher-researcher adjusted Fig. 4, making the rightmost region's height 10 cm. Wesley said the area's rate of change there was 10 cm^2 per cm, because its height was 10 cm. On yet another similar task where the rectangular region's height was 4 cm, Wesley justified the area's growth rate to be 4 cm^2 per cm because its height was always 4 cm regardless of how much length had been swept out. Wesley generalized that all of the ratios were instantiated in the same rectangle height, which did not depend on a specified amount of length. This idea later supported his reasoning with instantaneous rate of change, as we describe in the next section.

Wesley's work on the 4-cm rectangle task also provided additional evidence that he had constructed a rate. The teacher-researcher asked the students to create multiple ratios representing a 4-cm^2 per cm rate of change. Both students produced a number of equivalent ratios. Olivia, however, could not produce ratios for length increments less than 1 cm, and she did not perceive 8 cm^2:2 cm to be the same as 4 cm^2:1 cm. Olivia struggled to disentangle the rate of growth of the area with the amount of length lapsed. Wesley generated ratios that included length increments greater than and less than 1 cm, such as 12 cm^2:3 cm, 8 cm^2:2 cm, 2 cm^2:0.5 cm, and 0.4 cm^2:0.1 cm. More importantly, Wesley indicated that he would be able to create an equivalent ratio given *any* elapsed length, even an unspecified length of x, simply by multiplying it by 4. In contrast, in order to determine the area for an unspecified length smaller than 1 cm, Olivia needed to first find an elapsed length as some fraction of 1 cm, and then take the height of 4 cm and multiply it by that same fraction to get the proportional amount of area. This method depended on identifying an extant length first. Wesley's ability to appeal to only the height of the rectangle, rather than an increment representing an elapsed amount of length, suggests that he was able to see the area-to-length ratio as a rate.

It is plausible that Wesley's reasoning about constant rate was a consequence of a generalization enabled by his scaling-continuous covariational reasoning. He coordinated increments of length with corresponding increments of area across smaller and smaller scales of length. The concept of actual infinity could also assist with the conception of a height as a rate. If one can imagine smaller and smaller length scales and then characterize that process as having a resultant state, the resultant state would be not a column with a tiny amount of length, but a line (i.e., a height) with no associated length. Wesley could then generalize across all scales the property that the ratio was always constant.

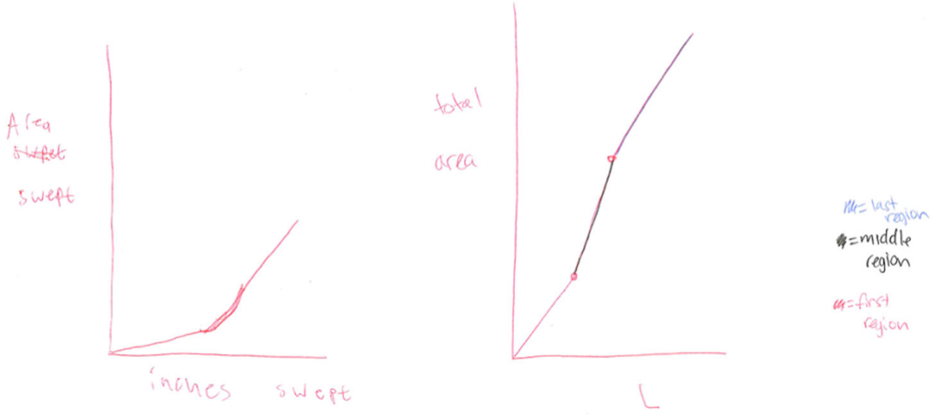
Instantaneous rates of change Wesley began to identify an instantaneous rate as also determined by a figure's height at the relevant location. In an initial activity to investigate the instantaneous rate of change, the teacher-researcher directed the students' attention to the trapezoidal middle region in Fig. 4. She asked the students to describe the rate of change of the area at the halfway mark of this region (where the height was 4.5 cm). Olivia explained that she saw the rate as "constantly getting larger than the previous increment," indicating a need to compare an amount of area for an elapsed increment with that for a previous elapsed increment. The teacher-researcher pushed Olivia to be more specific by asking, "Could we come up with a rate?" Olivia struggled to make sense of this question. She asked, "From, throughout the time?" and the teacher-researcher replied, "No, at that moment." Olivia answered, "It's not increasing, because you're not really going."

Wesley disagreed with Olivia's answer, stating, "I think it might be zero point – or, one-third plus 3 centimeters. *Because the slope of this is one-third*, but then you also have to take into account the rectangle." His answer reflected the fact that he viewed the trapezoidal region as a triangle atop a rectangle. The triangle's slope was 1/3, which means its area grew at a changing rate, according to 1/3 of its length. Thus, the area's rate of change halfway through the region was the rectangle area's growth rate, 3 cm^2 per cm, plus 1/3 of 4.5 (cm^2 per cm).

The teacher-researcher then asked for the instantaneous rate of change one-third of the way through the trapezoidal region. Wesley immediately said 4, which was the height of the trapezoid at that location: "Because it'd be 3 plus 1, since if it's a third [the slope] you can divide it 3 into 3 parts, which is 1." Likewise he said the area's instantaneous growth rate two-thirds of the way through the trapezoidal region the rate was 5 cm^2 per cm, again because the height of the figure was 5 cm there. In each case, Wesley used the slope of the region to determine the figure's height for a given swept length, and then said the area's rate of change was the region's height.

Wesley had initially generalized that the area's rate was the figure's height for rectangles, which have constant rates of change. One way to develop the more general understanding for a non-rectangular region is to consider how much area would be produced if the length were to begin sweeping out a bit. One could think of a bit as 1 cm, as Olivia did, but it is not necessary to sweep an entire centimeter in order to calculate the area's rate of change. Imagining this rate occurring at any scale, even an infinitesimal one, could allow one to see the rate as a "height," not as an extensive quantity, but as a ratio. Once the length sweeps out any amount, it turns the potential rate into an amount of area depending on how much length has been swept. All that matters is the region's height just at the moment when the area begins to grow.

A comparison of the students' graphs suggests that Wesley's image was of the area's growth rate continuously increasing throughout the sloped region, as evidenced by his attempt to draw a curved middle portion (Fig. 5a). In contrast, Olivia's graph represented the area's growth rate as constant (Fig. 5b). Olivia's explanation of the trapezoidal region on her graph suggests that she relied on chunky-continuous covariation: "In between each very small increment it's still growing, but you have to connect the two increments together *because it's the same shape*. It keeps growing at the same rate relatively." This suggests that Olivia did



(a) (b)

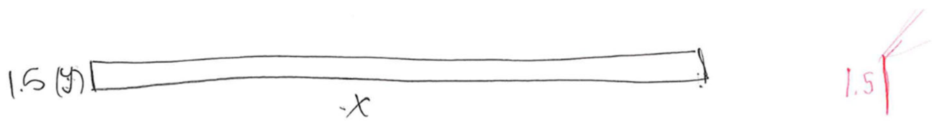
Fig. 5 Wesley (**a**) and Olivia's (**b**) graphs depicting accumulated area versus accumulated length

not imagine the nature of change within a given chunk, but she could imagine very tiny chunks and link them together.

In subsequent tasks, the teacher-researcher continued to probe the students' ideas about instantaneous rate. When discussing a growing rectangle the next day, Olivia stated that she believed the instantaneous rate of change of the area at any given amount of length swept would be the same, regardless of how much length had been swept. Wesley agreed, but provided a caveat: "I think so too, but with a triangle, it would be different, because the height is always increasing." This again suggests that he saw the triangle's height at a given length swept as indicative of the rate at that location.

In order to further support the conception that a height can determine the area's rate, the teacher-researcher asked the students to draw a line that was *about* to sweep out an area at a rate of 1.5 cm^2 per cm swept, but had not yet done so. Olivia drew a rectangle with an unspecified length x and a height y of 1.5 cm (Fig. 6a). Olivia still required two extant quantities in order to address the task. She explained, "I did a line with thickness so that we could write down the height and then the length." Wesley drew just a vertical line with a height of 1.5 cm (Fig. 6b). The teacher-researcher asked Wesley, "Does it makes sense to draw just a line as having a rate of change?" Wesley said yes, and then hesitated, amending his answer: "Well, I guess maybe for *instantaneous*, but not for average (rate of change)." He also indicated that his height line could represent a moment in a sweeping journey for any figure: "It could really be any (figure) because maybe as you keep sweeping it out it gets bigger and bigger, or, it just stays the same."

Scaling-continuous covariation and an associated concept of actual infinity could have supported Wesley's identification of a figure's height as a representation of the area's rate of



(a) (b)

Fig. 6 Olivia's (**a**) and Wesley's (**b**) drawings of a line about to sweep out an area at a rate of 1.5 cm^2 per cm change. Wesley saw the height line as a potential rate that could sweep out any amount. His comfort with relying on a *line*, rather than the column that Olivia required, suggests that his ability to imagine smaller and smaller length increments was a completed infinite process that had a resultant state, that of a height line.

5 Discussion

We found evidence that scaling-continuous reasoning afforded productive thinking about constant and instantaneous rates of change. Wesley was able to develop an understanding of a constant rate of change as a rate that represented an equivalence class of ratios. Further, Wesley constructed an understanding of the height of a figure as the area's rate of change. Chunky-continuous covariation can afford an image of re-chunking to very small increment sizes, which can in turn support generalizations about equivalent ratios across different increments. Scaling-continuous covariation, in contrast, enables one to extend such generalizations to any increment size, even the infinitesimal, which can support one's ability to develop and represent rates of change, a foundational idea for algebra and calculus. Thompson and Carlson (2017) noted that "The idea of a function having a nonconstant rate of change is actually constituted by thinking of the function having constant rates of change over small (infinitesimal) intervals of its argument, but different constant rates of change over different infinitesimal intervals of the argument" (p. 452). Olivia's reasoning approached this conception, as she did consider quadratic functions to have constant rates of change over small intervals, but for her, those intervals were not infinitesimal. Scaling-continuous covariation could support this understanding for infinitesimal intervals, as we saw with Wesley's belief that the constantly changing rate of change for triangles and trapezoids could be represented by the figure's height. We also observed in Wesley the less standard belief that a graph of such a function would have an always-changing rate of change even for an infinitesimal interval. As evidenced by Wesley's explanations, his images were a direct outcome of scaling-continuous covariation, in which he could imagine zooming to an infinitesimal scale at the level of actual infinity, with the infinitesimal interval being the final resultant state of an infinite zooming process.

We do not claim that scaling-continuous covariation necessarily preceded and therefore was the only support for Wesley's sensemaking about rates of change. Wesley could have been positioned to develop scaling-continuous covariation because he had already begun to construct equivalence classes of ratios. It could also be the case that Wesley developed a number of these forms of reasoning in tandem, each mutually supporting the other. In addition, we cannot ignore the influence of the task sequence and the teacher-researcher's pedagogical moves in supporting the students' development of chunky-continuous and scaling-continuous covariation. In particular, four salient themes emerged in the role played by the tasks and teacher questioning: (a) a repeated emphasis on continuous motion; (b) directing students' attention to what happens within intervals; (c) encouraging attention to increment size, particularly small and infinitesimal increments, and (d) pushing students to describe and identify rates.

We relied on tasks that used simulations of growing figures that emphasized continuous motion and sweeping actions. The teacher-researcher also emphasized such movement and encouraged the students to first represent rates of change in a non-quantified manner through descriptions and graphs before then transitioning to tasks with measurement. She encouraged the students to imagine change within an increment for a given figure or graph, asking them to describe what occurred within a given chunk. As part of this, the teacher-researcher used tasks that required students to consider rates for different increment sizes, including increments less than 1, and she regularly drew the students' attention to tiny increments and asked them to imagine infinitesimal increments. Finally, both the task sequence and the nature of the teacher-researcher's questioning required students to describe, identify, and ultimately quantify ratios and rates. The students experienced many opportunities to attend to changes in area for corresponding length changes, and the teacher-researcher pushed the students to first describe and then quantify these rates. By emphasizing constant and changing rates and the ways in which they remained invariant across increment sizes, even infinitesimal increments, the nature of instruction in the teaching experiment explicitly supported the development of an image of multiple increment scales, even (for Wesley) infinitely many small increments.

We do not suggest that smooth-continuous covariational reasoning is unimportant for the development of key ideas about function and rate. Indeed, smooth-continuous reasoning is a critical aspect of understanding the mathematics of change, including the ideas of calculus, and we support instructional efforts at all grade levels to develop conceptions of continuous covariation. Instead, we suggest that scaling-continuous covariation offers an additional form of reasoning that may plausibly foster productive understandings to support students' algebraic thinking. Given the potential for this form of reasoning to support key constructs in algebra and calculus, we advocate for additional research to better understand the nature of scaling-continuous variation and covariation and its affordances for productive mathematical thinking.

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