Empirical Re-conceptualization as a Bridge to Insight

Amy Ellis, University of Georgia, amyellis@uga.edu
Elise Lockwood, Oregon State University, elise.lockwood@oregonstate.edu
Alison Lynch, CSU Monterey Bay, alynch@csumb.edu

Abstract: Identifying patterns is an important part of mathematical investigation, but many students struggle to justify their pattern-based generalizations. These findings have led some to argue for a de-emphasis on patterning, but others argue that it can support insight into a problem's structure. We introduce a phenomenon, empirical re-conceptualization, in which learners generalize based on an empirical pattern, and then re-interpret it from a structural perspective. We elaborate this construct by providing a representative example of empirical re-conceptualization from two secondary students. Our findings indicate that developing empirical results can foster subsequent insights, which can in turn lead to justification and proof.

Introduction: The affordances and constraints of empirical investigation

Developing patterns is a key aspect of mathematical activity, but students are often discouraged from relying on empirical evidence to defend mathematical claims. Researchers posit that a primary source underlying students' struggles to justify their conjectures concerns their treatment of empirical evidence. Students can be overly reliant on examples and often infer that a mathematical statement is true on the basis of checking a small number of cases (Knuth, Choppin, & Bieda, 2009). Students can be adept at leveraging patterns in order to develop generalizations, but then struggle to understand, explain, and justify their results (Čadež & Kolar, 2014). One potential solution is to help students understand the limitations of empirical evidence as a means of mathematical justification and thus recognize the need for proof (e.g., Stylianides & Stylianides, 2009). These approaches have shown some success in helping students learn the limitations of examples, but they also frame empirical reasoning strategies as stumbling blocks to overcome.

In contrast, we have identified a phenomenon that we call *empirical re-conceptualization*, in which students identify empirical patterns, form associated generalizations, and then re-interpret their findings from a structural perspective. Rather than positioning example-based reasoning as an unsophisticated approach to de-emphasize, we identify the ways in which students can bootstrap empirical reasoning into mathematically meaningful insights. In this paper, we address the following questions: (a) What characterizes students' abilities to leverage empirical patterns in order to develop mathematical insights? (b) What are the conceptual affordances of engaging in empirical patterning activity? We describe and elaborate this construct with an example with secondary students, and discuss the finding that developing results from empirical patterns can serve as a launch point for subsequent insight, including verification, justification, and proof.

The interplay between empirical reasoning and deductive reasoning

It is generally recognized that students' arguments are expected to progress from empirically-based justifications to deductive proofs. Indeed, various reasoning hierarchies have been proposed that reflect this expected progression (e.g., Balacheff, 1987). Although these hierarchies delineate levels of increasing sophistication in students' arguments, they do not sufficiently account for how students' empirical reasoning will make the transition to deductive reasoning. Indeed, many students find that transition challenging to navigate, and this is a challenge that persists even at the undergraduate level (Stylianides & Stylianides, 2009).

Despite these potential drawbacks, many researchers also point to the affordances of empirical investigation. Developing empirically-based generalizations can support the discovery of insight into a problem's underlying structure, which can, in turn, foster proof construction (de Villiers, 2010). Students can and do engage in a dynamic interplay between empirical patterning and deductive argumentation (Küchemann, 2010). de Villiers (2010) noted that mathematicians regularly engage in experimentation and deduction as complementary activities, and Tall et al. (2008) argued that students can bootstrap their empirical investigations into more sophisticated knowledge structures. It may be that students become stuck in a focus on empirical relationships because they lack sufficient experience with developing structural meaning from patterns. Curricular materials emphasize patterning activities that begin and end with the development of a generalization, typically presented as an algebraic rule. Forming a connection between the generalization and a structural justification for its reasonableness is seldom emphasized in standard classroom tasks. However, given the above evidence that meaningful connections can be developed from empirical investigation, we advocate for positioning empirical patterning as a bridge to insight and deduction.

Structural Reasoning and Figurative versus Operative Activity

Considering structure to be something made up of a number of parts that are held together in a particular manner, Harel and Soto (2017) introduced five major categories of structural reasoning: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category further distinguishes between two types of generalizing: Result pattern generalization (RPG) and process pattern generalization (PPG) (Harel, 2001). RPG is a way of thinking in which one attends solely to regularities in the result. The example Harel gave is observing that 2 is an upper bound for the sequence $\sqrt{2}$,$\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, … because the value checks for the first several terms. When we refer to empirical re-conceptualization and the identification of a pattern based on empirical evidence alone, we are referring to RPG. In contrast, PPG entails attending to regularity in the process. To extend the above example, Harel discussed how one might engage in PPG to determine that there is an invariant relationship between any two consecutive terms of the sequence, $a_{n+1} = \sqrt{a_n + 2}$, and therefore reason that all of the terms of the sequence are bounded by 2 because $\sqrt{2} < 2$.

We also draw on a distinction between forms of mental activity. Figurative activity involves attending to similarity in perceptual or sensorimotor characteristics (Piaget, 2001). For instance, one could associate the sine curve with circular motion through conceiving both the graphical register and the physical register as representations of smooth continuous motion (Moore et al., in press). In contrast, operative activity entails attending to similarity in structure or function through the coordination and transformation of mental operations. To continue the above example, a student could associate the sine curve with circular motion through conceiving both as representing an invariant relationship of co-varying quantities. A shift from RPG to PPG is often accompanied by a shift from figurative to operative mental activity, and we consider operative activity to be a hallmark of the ability to reason structurally. We therefore define empirical re-conceptualization as the process of re-interpreting an empirically-determined generalization from a structural perspective, which can include engaging in any of the five major categories of structural reasoning, with the exception of the RPG sub-category of pattern generalization, or shifting from a figurative to an operative association.

Methods

We conducted a paired teaching experiment over 5 sessions, with each session lasting between 30 and 90 minutes. Homer was a 9th-grade student who had completed Algebra I, and Barney was a 7th-grade student who had completed pre-algebra. Our aim was to investigate the students' conjectures and generalizations about the areas and volumes of growing figures, and then to investigate their development of combinatorial reasoning by exploring the growing volumes of hypercubes and other objects in 4 dimensions and beyond.

All teaching sessions were videoed and transcribed. Using the constant-comparative method, we analyzed the data in order to identify the participants' generalizations and to characterize the mental activity that fostered them. For the first round of analysis we drew on Ellis et al.'s (2017) RFE Framework, and in subsequent rounds we used open coding to infer categories of generalizing based on the participants' talk, gestures, and task responses. The next round of analysis supported the development of an emergent set of relationships between the participants' patterning activities and their generalizations; this yielded the emergent category of empirical re-conceptualization. In a final round we re-visited the data corpus in order to identify all instances of empirical re-conceptualization and the initial generalizations that led to each instance. In this manner we were able to determine the characteristics of empirical re-conceptualization and track the changes in students' generalizing after engaging in re-conceptualizing, which led to the identification of the affordances detailed below.

Results

In order to characterize the participants' abilities to leverage empirical patterns to develop mathematical insights, we present an exemplar case. Barney and Homer initially worked with the following task: "Say you have an *n* by *n* by *n* cube, and you add 1 cm to the height, width, and length. What is the added volume of the new cube?" Both students worked with blocks to think about the component pieces of a larger cube and reason that the added volume would be $3n^2 + 3n + 1$. The teacher-researcher then gave the same task for a four-dimensional *n* by *n* by *n* by *n* hypercube. In order to introduce combinatorial reasoning as a way to ground sense-making, she asked the students to consider the dimensions height, width, length, and a fourth dimension introduced as "slide". Homer and Barney determined that when adding 1 cm to the *n* by *n* by *n* slice of a hypercube, they would have four outcomes: a) $1 \times n \times n \times n$; b) $n \times 1 \times n \times n$; c) $n \times n \times 1 \times n$; and d) $n \times n \times n \times 1$. Both students could then justify why the n term and the n^3 term should have a coefficient of 4. The students then re-wrote their expressions to be "$4n^3 + 4n^2 + 4n + 1^4$". This expression is incorrect – the coefficient for the n^2 term is not 4 – but it is an understandable generalization. It was a result of Barney and Homer generalizing from their operative activity of determining the coefficient of n^3 to be 4, as well as from a figurative extension of the numeric structure of the three-dimensional case to the four-dimensional case.

The teacher-researcher then asked the students to list out all of the options for the n^2 coefficient. They correctly listed 6 options for arranging two n's and two 1's into four slots. By this point, the students had now correctly determined expressions for the second, third, and fourth dimensions, and the teacher-researcher wrote down their expressions in Figure 1. The expressions caused Homer to have a realization:

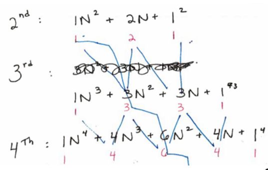

Figure 1. Expressions for added volume in the 2nd, 3rd, and 4th dimensions

Homer: I know what is happening Barney. It is simple, as 2 – sorry I'm writing on it. [Begins to draw the blue lines]. Two plus 1 is 3, and 2 plus 1 is 3, 3 plus 3 is 6, 3 plus 1 is 4, 1 plus 3 is 4. [Writes the red numbers into the figures.]

TR: Whoa. Huh.

Barney: Wow. It's just that one triangle, Pascal's triangle, right?

Homer recognized a pattern, and he knew that each coefficient for each term could be determined by adding the sum of the coefficients of the consecutive terms from the prior dimension. This was an empirically-based generalization. Pascal's triangle then became a mechanism for determining an expression for the additional volume of a fifth-dimensional figure. Homer said, "Right here, I'm going to write down what it would be if it followed the sequence." He and Barney both wrote "$5n^4 + 10n^3 + 10n^2 + 5n^1 + 1^5$", and then they decided to check their answer by listing out the arrangements of three ns and two 1s (the $10n^3$) case, which served to verify that the coefficient was indeed 10. As the students reflected on their activity, Homer explained that he did not want to do the tedious listing that would be required to double check each coefficient: "I don't want to check all of these. I was just going to check one, to kind of, maybe I'll check two or something." Barney then realized that given that they had verified the $10n^3$ case, they did not need to check the $10n^2$ case: "We can basically just take this and switch all the ns to 1s and 1s to ns." Explaining further, Barney said, "It will be the same combinations here, just substituting I for n and n for I" (he inadvertently starting calling the 1s "I"s). This explanation of symmetry caused Homer to then extend that finding to new cases: "Oh, and you know what? You can do the same for these (pointing to the $5n^4$ and the $5n^1$ terms)…you can just replace these 1s for ns."

Homer and Barney initially developed a generalization based on the empirical recognition of Pascal's triangle. This empirically-based generalization then provided them with a conjecture for the expression of the added volume in the fifth dimension, which they could then check through listing that there were indeed ten combinations of three ns and two 1s. Once they had confirmed that coefficient, they looked back at their conjectured expression and realized that they would again need to list ten outcomes to check the other coefficient. This sparked a desire to avoid repeating the listing process, which motivated a justification for symmetry. Barney was able to explain why the coefficients for the n^3 term and the n^2 term must be the same, which Homer could then extend to the n^4 and n terms. Homer was therefore able to re-interpret combinatorially what he had first conjectured using only the patterns in Pascal's triangle. The numerical pattern, which was developed from RPG, allowed the students to engage in a verification process and subsequently reason about outcomes to justify the symmetry in the coefficients. Barney's reasoning in particular was grounded in operative activity: He was able to reflect on his coordination of operations in listing the ten outcomes and realize that there was nothing special about the characters n and 1, and that they could simply be reversed in the case of determining the combinations of two ns and three 1s.

Discussion and implications

We have introduced a new phenomenon, empirical re-conceptualization, in which learners develop an initial generalization based on empirical evidence and then are able to re-conceptualize it from a structural perspective. The students both carried out structural operations in thought by justifying why it would be legitimate to replace *n*s with 1s for different listing options. Our findings indicate that empirical re-conceptualization can serve as a vehicle to transform empirical patterns into meaningful sources of verification, justification, and proof. This confirms de Villiers' (2010) claim that "experimental investigation can also sometimes contribute to the discovery of a hidden clue or underlying structure of a problem, leading eventually to the construction or invention of a proof" (p. 215).

Certainly, students may also identify and generalize patterns that they do not understand or cannot justify; this remains a common phenomenon. A danger is that students will engage in empirical investigation but then not seek to re-conceive their findings structurally. We find it useful to explore the conditions that can best support students' transition to the productive next step, that of engaging in empirical re-conceptualization. Homer and Barney had mechanisms by which they could draw their attention back to a combinatorial context. Even though they developed empirically-based generalizations, those statements were never far from their understanding of the combinatorial situations. This suggests that directing students back towards the contextual genesis of the patterns they generalize may be an effective strategy for supporting empirical re-conceptualization. Rather than discouraging reliance on empirical patterns or requiring students to prematurely shift to abstracted representations, we suggest situating instruction within particular, concrete contexts that can provide a meaningful foundation for empirical re-conceptualization. For Homer and Barney, this context was combinatorial; in other cases, we have found that contexts that leverage students' engagement with real-world quantities, such as distance, time, speed, length, area, and volume, can similarly provide fruitful supports for understanding and justifying conjectures. With the support of concrete contexts for meaning making combined with instructional moves that encourage students to consider their empirical findings in light of those contexts, our findings indicate that the activity of generalizing empirical patterns can serve as a bridge to more generative and productive mathematical activity.

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Acknowledgments

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