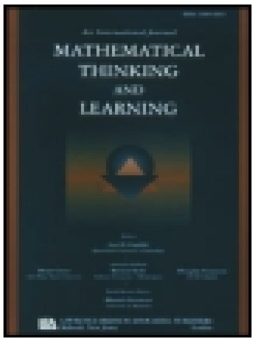

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Figurative and operative partitioning activity: students' meanings for amounts of change in covarying quantities

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ABSTRACT
Researchers have emphasized the importance of characterizing students' abilities to coordinate changes in covarying quantities. In this paper, we characterize three undergraduate students' coordination of covarying quantities' amounts of change during a teaching experiment. We adopt Piagetian notions of figurative and operative thought to describe the extent their meanings for covariational relationships are constrained to or supported by their *partitioning activity* – the mental and physical actions associated with constructing accruals in quantities' magnitudes. Our analysis suggests that students' construction of amounts of change is constrained by figurative partitioning activity that requires carrying out or emulating particular actions on perceptually available material. In contrast, operative partitioning activity supports the students' transformation and (anticipated) regeneration of partitioning activity in order to conceive equivalent covariational relationships among various situations and representational systems. We conclude by discussing how documenting these distinctive meanings contributes to extant literature on covariational reasoning and, more broadly, the theorization of mathematical concept construction.

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Partitioning activity; amount of change; covariational reasoning; piaget; cognition

Introduction

Researchers have argued that students' quantitative and covariational reasoning – the mental actions involved in conceiving an object's measurable attributes changing simultaneously (Thompson & Carlson, 2017; Thompson, 2010) – are critical for their learning of many mathematical topics such as function (Ellis, 2011; Oehrtman et al., 2008; Thompson & Carlson, 2017) and rate of change (Johnson, 2012, 2015a, 2015b). An important set of mental processes associated with quantitative and covariational reasoning is that in which students simultaneously coordinate and compare *amounts of changes* in two covarying quantities (Carlson et al., 2002; Johnson, 2015b). *Amount of change* refers to how much a quantity's value (or magnitude) varies from one state to another, and is foundational to numerous K-16 concepts, such as slope, average and instantaneous rate of change, derivative, and integration. Conceptual understandings of linear, quadratic, trigonometric, and exponential relationships from a covariational perspective also require students to compare and coordinate amounts of change in one variable with those in another (e.g., Ellis et al., 2015; Castillo-Garsow, 2010; Ellis, 2011; Lobato & Siebert, 2002). Stemming from the importance and complexity of amounts of change reasoning, researchers (e.g., Carlson et al., 2002; Johnson, 2015b) have called for studies that characterize and explain differences in students' amounts of change meanings.

We respond to this call by identifying a marked distinction in students' actions in constructing amounts of change in quantities' magnitudes: figurative and operative partitioning activity. We explore three undergraduate students' partitioning activities in what we perceive to be various

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situations and representational systems during a teaching experiment. Our analysis suggests that some students' activities foreground particular perceptual features of their actions and results, while other students' activities are grounded in coordinated mental operations regarding quantities' covariation. We organize this paper into five sections: (a) an overview of the literature on covariational reasoning and partitioning activity, (b) a synthesis of Piagetian constructs that informed our operationalization of figurative and operative partitioning activity, (c) our design of the teaching experiment, (d) three themes of partitioning activity, and (e) highlights of our theoretical and empirical contributions.

Literature review

Our approach to partitioning activity combines work in the area of covariational reasoning and partitioning in the context of fractional reasoning. We describe each in this section.

Covariational reasoning and magnitudes

Researchers have taken different approaches to investigate students' reasoning of amounts of change in covarying quantities. Confrey and her colleagues focused on students' coordination of changes regarding two sequences of values in tabular forms (Confrey & Smith, 1994, 1995). Ellis and her colleagues reported on students' meanings for exponential relationships in terms of their coordinating $f(x_2)/f(x_1)$ with $x_2 - x_1$ (Ellis et al., 2015). Johnson (2015a, 2015b) discussed the role of amounts of change reasoning in students' construction of rate of change and slope in graphical contexts. Commonly, these researchers provided students with opportunities to reason with quantities' differences when numerical values were available. They have illustrated such opportunities are productive for students to construct patterns in quantities' amounts of change and associated meanings for various function classes.

Taking a complementary focus to these researchers, Thompson, Moore and colleagues have drawn attention to students' covariational reasoning independent of specified values, attempting to understand students' images of covariation (Moore, accepted; Thompson & Carlson, 2017; Thompson et al., 2014). Specifically, they focus on characterizing students' mental activity in terms of perceptual material associated with a quantity's magnitude (or amount-ness). For example, a student can conceive the length of a segment as displaying a distance magnitude from a dynamic point to a fixed point and reason about the manner in which the segment varies, including by how much it changes from one state to another (Figure 1a); it is unnecessary for a student to use numerical values to reason about how a quantity varies (Moore, accepted; Moore, Stevens, Paoletti, Hobson, & Liang, 2019; Thompson, 2010).

An emphasis on quantities' magnitudes is critical for determining the extent students' reasoning about quantities and their covariation entails quantitative operations (Moore et al., 2019; Thompson & Carlson, 2017; Thompson, 2010). Segments such as those in graphical representations permit quantitative operations including partitioning, iterating, disembedding, and unit coordination (Steffe &

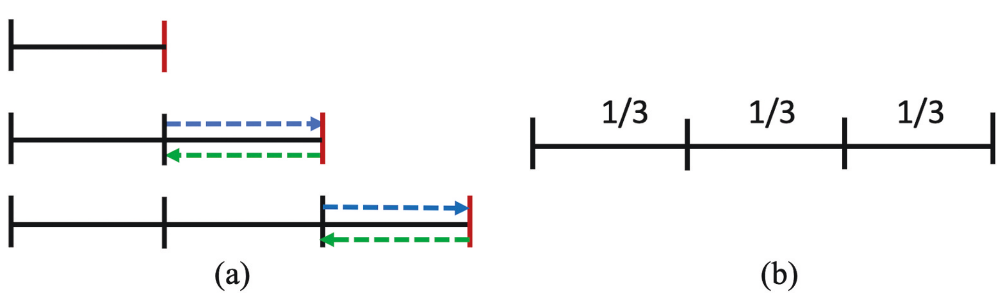


Figure 1. Partitioning activity for visualizing (a) directed changes and (b) equal parts.

Olive, 2010). Written numerical values are not conventionally designed to permit such activity, although they might symbolize it (Moore et al., 2019). Reflecting our interest in the present study, we draw on Thompson, Moore, and colleagues' approach to covariation and characterize threestudents' meanings for partitioning activity as it relates to quantities' magnitudes and associated perceptual material that permits quantitative operations.

Partitioning activity

Researchers have used the notion of *partitioning activity* in the areas of fractional and multiplicative reasoning including unit coordination (e.g., Hackenberg & Tillema, 2009; Izsák et al., 2008; Steffe & Olive, 2010). Hackenberg and Tillema (2009) defined *partitioning* as, "the process of dividing a unit into equal-sized parts (Kieren, 1980), either solely mentally or also materially" (p. 2). In their use, one partition serves as a landmark that separates two parts, and each part shares identical mathematical properties (e.g., sizes) (Figure 1b).

In our work, we extend the notion of partitioning to characterize students' reasoning about *(co) varying* quantities. In reasoning about varying quantities, partitioning can be for the purpose of characterizing accumulated quantities in terms of their accruals (Thompson, 1994a). A student can imagine a *single* quantity's magnitude accumulating by increments or dissipating by decrements in successive states (i.e., increasing or decreasing) (Figure 1a). To coordinate amounts of change in *two* quantities, a student also considers corresponding changes in another quantity (i.e., Mental Action 3, see Carlson et al. (2002)). As an example, consider a counterclockwise rotating Ferris wheel (Figure 2a; see [https://youtu.be/IKe6ry9Uqpo]), an arc length (quantity B, denoted in pink, Figure 2b), and a distance above the horizontal diameter of the wheel (quantity K, denoted in blue, Figure 2b). A student reasoning about covarying quantities B and K can envision the magnitude ||B|| accumulating in equal accruals (Figure 2b), construct the magnitude ||K|| accumulating in terms of corresponding accruals (Figure 2b), and coordinate those accruals in ||K|| (denoted in light blue, Figure 2c) to conceive ||K|| increasing by decreasing amounts with respect to ||B||.

A student can also regenerate the variations in ||B|| and ||K|| on two orthogonally-oriented or parallel bars (Figure 3a) and anticipate the blue bar to increase by decreasing amounts as the pink bar increases by equal amounts so as to match the circle situation (Figure 2c). Further, to display this relationship in Cartesian coordinate systems (Figure 3(b,c)) or a polar coordinate system (Figure 3d), a student can envision variations of both magnitudes, pair the pink and blue magnitudes at any states, and trace a point that unites both magnitudes as they vary simultaneously (see Thompson et al. (2017) for *emergent graphical shape thinking*).

To summarize, coordinating amounts of change requires a student to either mentally or physically partition the accumulated magnitude of one quantity and coordinate it with partitioning the accumulated magnitude of another quantity. We thus use *partitioning activity* to refer to students' mental and physical actions associated with their coordination of accruals in quantities' accumulation. We

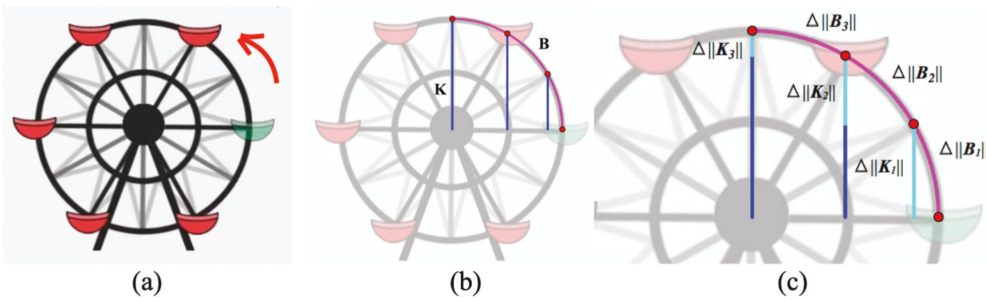


Figure 2. (a) Taking a Ride and (b, c) an illustration of amounts of change in ||B|| and ||K||.

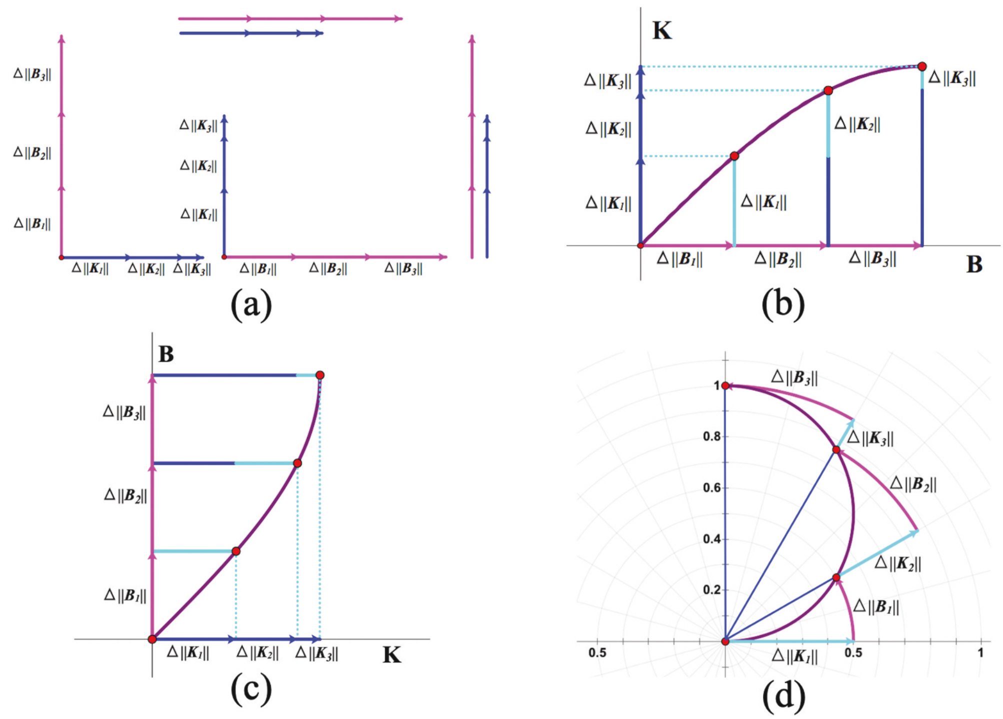


Figure 3. Displaying the relationship between arc length (B) and height (K) on (a) orthogonally-oriented and parallel bar pairs, (b) a Cartesian coordinate system with B on the horizontal axis, (c) a Cartesian coordinate system with B on the vertical axis, and (d) a polar coordinate system.

note that although it is important for students to conceive an accumulated magnitude as a varying quantity so that they anticipate its variation and accumulation occurring from on partition to another *continuously* (Ellis et al., 2020; Thompson & Carlson, 2017), we do not attend to this aspect here. We focus on students' meanings for the *resulting* partitions produced by their partitioning actions and the associated meanings for covariation with respect to those partitions.

Researchers have illustrated that students' meanings for fractions and number lines are not necessarily productive or connected to partitioning activity, although many students can carry out such activity (Izsák et al., 2008). Researchers have further noted the complex nature of the mental processes involved in partitioning displayed magnitudes, especially as it relates to coordinating the act of partitioning with the size of a partitioned magnitude and a whole (Hackenberg & Tillema, 2009; Izsák et al., 2008). We extend these findings by focusing on the extent students' meanings for covariational relationships are constrained to or supported by their partitioning activity. We note an important difference in partitioning activity in covariational contexts as compared to fraction and number line contexts: the latter foregrounds partitioning in equal-sized units. Partitioning in covariational contexts often does include incrementally partitioning a quantity's accumulation in an equal-sized unit, but the partitioning of a second quantity's accumulation can be unequal (i.e., a non-constant rate of change).

Theoretical framing

In this section, we first elaborate on Piagetian distinction between *figurative and operative thought* (Piaget, 2001; Piaget & Inhelder, 1971; Steffe, 1991; Thompson, 1985), and we introduce the notion of *re-presentation* to operationalize these two constructs. Lastly, we describe our definitions of *figurative* and *operative partitioning activity*.

Figurative and operative thought

Piaget (Chapman, 1988; Montangero & Maurice-Naville, 1997; Piaget, 2001; Piaget & Inhelder, 1971) differentiated between two primary forms of cognition: *operative* and *figurative*. Figurative thought is that which foregrounds states. Our use of states is not meant to imply a static picture. Rather, our use of states implies the absence of the coordination of mental actions with their results, and thus actions are indissociable from their results. Meanings foregrounding figurative thought or states are frequently constrained to carrying out activity including physical or mental actions, motion, gestural imitations, and the like so that such activity is subordinate to obtaining a particular resulting state (Chapman, 1988; Montangero & Maurice-Naville, 1997; Piaget, 2001; Piaget & Inhelder, 1971; Steffe & Olive, 2010; Von Glasersfeld, 1995). Piaget described operative thought as the coordination and transformation of mental operations so that this coordination includes but dominates figurative activity and material including that of the physical and perceptual kind. Operative thought rests on the coordination of actions including their "intrinsic necessity, as opposed to successful solutions by chance or successful solutions that have simply been observed" (Piaget, 2001, p. 272). Operative meanings are rooted in conceptual relations and mental operations so that they do not depend on specific perceptual material, activity, and configurations (Von Glasersfeld, 1995); results are subordinate to properties of the mental operations.

We underscore that the figurative and operative distinction does not imply that operative thought does not entail fragments of perceptual activity or material; "operations have to operate on something and that something is the figurative material contained in the operations, figurative material that has its origin in the construction of the operations" (L. P. Steffe, personal communication, July 24, 2019). Figurative meanings – those meanings constrained to particular states – are the foundation for the construction of operative meanings (Montangero & Maurice-Naville, 1997; Piaget, 2001; Von Glasersfeld, 1995). A researcher's sensitivity to these distinctions is thus an issue of "figure to ground" (Thompson, 1985, p. 195). When thinking foregrounds carrying out or imitating repeatable actions and their results, it is figurative. When thinking foregrounds the coordination and transformations of actions and their results so that they dominate perceptual activity or material, it is operative (Chapman, 1988; Montangero & Maurice-Naville, 1997; Piaget, 2001; Von Glasersfeld, 1995).

For example, Moore et al. (2019) has shown that characterizing students' graphing actions in terms of their foregrounding aspects of figurative or operative thought is useful to explain observed differences in students. Graphing necessarily entails perceptual activity and material (e.g., drawing a graph) *and* operative schemes (e.g., constructing a coordinate system and coordinating quantities' magnitudes within it). As the authors described, figurative or operative aspects of prospective secondary teachers' meanings had important implications for the generativity of their graphing activity. For instance, graphical meanings that necessitated drawing a graph in particular ways (e.g., from left-to-right) limited their abilities to construct a graph that was not compatible with this perceptual activity, while meanings that persistently foregrounded covariational relationships and the transformation of quantities supported them in graphing in various coordinate orientations. Like graphing, partitioning activity necessarily entails perceptual activity (e.g., drawing partitions) and operative schemes (e.g., coordinating mathematical properties of partitions or amounts of change), and thus our interest is to understand the extent students' activity is dominated by reasoning about particular states or by the coordination of mental actions and operations.

Re-presentation

A reader may ask: what evidence do we use to infer figurative or operative aspects of a student's partitioning activity, including which is foregrounded in thought? A student's ability to *re-present* their partitioning activity is a property of their meanings for that activity, and we draw attention to the extent they can re-present partitioning activity both within and among contexts that entail what we perceive as differences in their perceptual material. Drawing from Piaget, Von Glasersfeld (1995) defined *re-presentation* as re-playing or reconstructing something that was present in a subject's experiential world at some other time. Reflecting his primary study of language, this form of re-presentation is wholly self-generated and involves the individual mentally generating some substitute for the sensory material that was present in prior experience but is absent currently (e.g., speaking or writing an English word as opposed to recognizing it through reading or listening). In our contexts, when offered a blank sheet of a paper, this form of re-presentation involves a student self-generating or imagining *all* perceptual material associated with a prior partitioning activity (e.g., reproducing a Ferris wheel with spokes partitioning the circumference equally and drawing decreasing amounts of change in height; see Figure 2c).

Mathematics education researchers have further adopted and modified the notion of re-presentation with a primary focus on mental operations. In doing so, they have allowed for the presence or supply of minimal perceptual material or stimuli on which mental operations operate. We consider this as a second form of re-presentation. For example, researchers have used re-presentation relative to students' fractional reasoning and unit coordination activity (Hackenberg, 2010; Izsák et al., 2008; Steffe & Olive, 2010). In contexts where a bar, strip diagram, or segment was not provided, students often worked in environments in which the production of a bar was trivial (e.g., pushing a button). The authors' uses of re-presentation did not rest on regenerating the bar but instead the students' abilities to regenerate and anticipate the mental operations associated with partitioning that bar into parts. In our study, we consider students regenerating their partitioning activity in the context of us offering non-partitioned diagrams (e.g., circles, segments, bars, or orthogonal axes) to be consistent with this form of re-presentation. A student's re-presentation of their previously constructed partitioning activity involves them recognizing the supplied material to be relevant or similar to a prior context at first, and the partitioning operations must be self-generated by the student (e.g., reproducing decreasing amounts of change in height when the Ferris wheel or a circle is given, but without any material indicating partitions).

A third form of re-presentation relevant to the present study involves a student re-presenting mental operations via regenerating and *transforming* those operations from a previous experience to accommodate a novel context. As we illustrate above, a student can regenerate and transform their partitioning activity constructed in the Ferris wheel situation (Figure 2c) to bar pairs (Figure 3a) and different coordinate systems (Figure 3(b,c,d)). This form of re-presentation is more complex than the prior two forms due to having to transform the mental operations associated with partitioning in order to accommodate to differences in perceptual material.

With these forms of re-presentation introduced, we use "re-presentation" and "representation" in distinct ways from this point forward. We use "re-presentation" to refer to the enactment and regeneration of schemes and operations as defined above. We use "representation" in the canonical sense to refer to the modes of display and symbolization associated with the field of mathematics (e.g., graphs, inscriptions, and verbal statements). We also acknowledge it is common to use representation to refer to a general mental structure that has been abstracted and symbolized through its recurrent use, but we do not adopt this use in this article (e.g., Moore, 2014b).

Figurative and operative partitioning activity

Bringing together the notions of re-presentation and figurative thought, we continue to discuss how we operationalize the constructs of figurative and operative partitioning activity. We define *figurative partitioning activity* as an individual's partitioning activity that foregrounds particular states of their activity, thus not dissociating the results of their actions from the actions themselves. Consequently, when confronted with another context, the individual's re-presentation of their partitioning actions is constrained to repeating the same actions. That is, what they intend to partition is the perceptual material itself. Furthermore, the individual may require the presence of perceptual material permitting the same results through those actions. When the perceptual material constituting and resulting from the individual's prior partitioning actions is unavailable in the new context, they may perceive the new

Table 1. Figurative and operative partitioning activity.

	Property	Description
Figurative Partitioning Activity	Emulating actions of partitioning activity	A student re-presents partitioning activity by emulating the same partitioning actions to preserve the perceptual features of resulting partitions.
	Partitioning activity constrained to available perceptual material	A student recognizes and anticipates re-presenting partitioning activity only when perceptual material and resulting partitions are available.
Operative Partitioning Activity	Re-presenting and transforming partitioning activity	A student re-presents partitioning activity by transforming partitioning actions to accommodate context differences and justifies the invariant quantitative structure entailed by the resulting partitions.

context to be irrelevant to the prior and does not anticipate re-presenting their activity, *or* they cannot reproduce associated partitioning activity despite anticipating its potential relevance (see the second property in Table 1 and illustrations in the first theme of Results). Both cases are contraindications that the individual can enact the second form of re-presentation – re-presenting when non-partitioned material is provided.

In the event an individual does anticipate re-presenting due to perceiving perceptually similar elements in a new context, they may alternatively re-present their partitioning activity by repeating or emulating the prior actions to produce similar results, regardless of what an observer perceives to be differences in those two contexts (see the first property in Table 1 and illustrations in the second theme of Results). We consider this as a contraindication of the individual enacting the third form of re-presentation appropriately – re-presenting by transforming their partitioning actions to accommodate to the current context.

In comparison, *operative partitioning activity* refers to an individual's partitioning activity that foregrounds properties and transformations of coordinated mental operations (e.g., quantitative and covariational relationships) constructed and reflected upon in their partitioning experience. The individual abstracts the covariational relationships entailed by their partitioning actions enacted in a prior context and transforms their mental structures to account for the new context (i.e., the third form of re-presentation; see the third property in Table 1 and illustrations in the third theme of Results). What they partition is a varying quantity's magnitude displayed by perceptual material as opposed to the material itself. Due to its basis in quantitative and covariational reasoning (as opposed to perceptual features), operative partitioning activity is more likely to be re-presented in contexts without or with partial perceptual material given (i.e., the first and second forms of re-presentation).

We reiterate the "figure to ground" nature of figurative and operative meanings. Figurative partitioning activity can provide the conceptual foundation for the construction of operative partitioning activity as a student repeatedly engages in and reflects upon their actions, especially when repeating their actions leads to a perturbation due to an unanticipated result. Furthermore, partitioning activity that is operative in one context can become the figurative ground on which an individual acts to accommodate a novel context. For instance, an individual may engage in operative partitioning activity relative to a circular motion. They may then attempt to accommodate to a graphical context by emulating the actions of partitioning in the circle context. With respect to the new context, the individual's partitioning activity foregrounds particular states including perceptual features associated with that state and is thus figurative. Upon further activity and reflection, the individual can perceive the need to accommodate their actions to the quantitative organization of the graphical coordinate system, thus engendering operative partitioning activity. Such a phenomenon underscores how investigating an individual's activity across (what a researcher perceives to be) different contexts provides a researcher greater insights into the extent an individual's meanings foreground the coordination and transformation of mental operations versus particular perceptual material and activity.

Before continuing, we acknowledge that the relationship between students' partitioning activity and covariational reasoning is bidirectional. On one hand, students' constructed and abstracted covariational relationships can drive their partitioning activity. Abstracted covariational relationships can govern their partitioning actions and what results they anticipate those actions to produce. On the other hand, students' partitioning activity can either afford or constrain their construction of covarying quantities' amounts of change, which is the focus of our research design and analysis. Partitioning activity is necessary for students to construct and visualize quantities' accruals and amounts of change, thus providing the grounding for abstracted covariational relationships. During re-presentation in a new context, students may not perceive the same relationship due to failure of enacting partitioning activity, or they enact their activity in ways do not preserve particular covariational relationships. This is how figurative partitioning activity can *constrain* students' construction of quantitative and covariational relationships, because covariational relationships move to the background during such activity. When the relationships are persistently foregrounded and preserved in the students' (operative) partitioning activity, the partitioning activity *affords* the students' covariational reasoning, and vice versa.

Method

We take the epistemological stance of radical constructivism (Von Glasersfeld, 1995) and consider students' mathematical knowledge as "legitimate mathematics to the extent that we can find rational grounds for what students say and do" (Steffe & Thompson, 2000, p. 269). Because we view students' mathematics as entailing a rationality of its own, our research agenda seeks to generate and test hypotheses of the mathematics of students that culminate in viable explanations of what students say and do. Driven by these assumptions and goals, we adopted a *teaching experiment methodology* (Steffe & Thompson, 2000) to construct hypothetical models of students' mental actions and operations through on-going interactions with them. We specifically focused on our hypothesized mental actions and operations of students that we considered to be indications or contraindications of their figurative and operative partitioning activity. In the following sections, we discuss our methods, and we point the reader to more extensive discussions on the teaching experiment methodology in Cobb (2000), Lesh and Kelly (2000), Simon (2000), and Steffe and Thompson (2000).

Subjects

We examined three pre-service teachers' (i.e., Lydia, Emma, and Brian) actions involved in reasoning with dynamic situations, quantities' magnitudes, and graphs. All three students were junior undergraduates enrolled in their first semester of a four-semester secondary mathematics education program at a large university in the southeast United States. They had completed at least two additional courses past an undergraduate calculus sequence at the time, and we chose them on a voluntary basis from a secondary mathematics content course that was paired with a pedagogy course. All study sessions were conducted independently and separately from the courses and program in which the students were enrolled. We chose these three students from the volunteer pool based on their written responses to an assessment at the beginning of the course, the test items of which were adapted from the Mathematical Meanings for Teaching secondary mathematics (MMTsm) instrument (Byerley & Thompson, 2017; Thompson et al., 2017; Thompson, 2016). The three students' responses indicated their ability to clearly communicate their thinking and demonstrated a wide range of understandings regarding covariation, function, graph, rate of change, and proportion.

Setting

We followed the teaching experiment methodology (Steffe & Thompson, 2000) to continually construct, test, and modify our hypothetical models of the three students' mathematical meanings. For each student, we conducted one pre-interview, nine to ten teaching sessions, and one post-interview over the course of a semester. Each session lasted for one to two hours.

Pre- and post- interviews

The pre- and post-interviews occurred prior and after the teaching sessions and followed the design of semi-structured clinical interviews (Clement, 2000). In these interviews, we aimed to gain insights into the students' mathematical meanings without providing interventions and guidance. Being aware that the interaction itself could influence their thinking, we tried our best to minimize this influence in our ways of interacting and focused on asking probing questions to elicit their thinking.

Teaching sessions

The second and the third teaching sessions involved all the three students working together on some tasks, which we term *group sessions*. All other teaching sessions involved only one participant at a time. At each session, the project principal investigator (the second author) served as the teacher-researcher (TR). After the student provided a brief review of the previous session, the TR typically started with asking the student to watch a dynamic situation demonstrated on a tablet device and describe what they observed in the situation. The TR then presented the prompts of the task and asked questions to generate and refine models of the student's thinking and engender changes in their ways of thinking in-the-moment. At least one other research team member was present as the witness-researchers (WR), who managed the video camera and asked additional probing questions when appropriate. Immediately after each session, the TR debriefed with the WR(s) to discuss their observation, form hypothesized models of the student's thinking (i.e., *ongoing conceptual analysis*; see Steffe and Thompson (2000) and Thompson (2008)), and design future sessions.

Data collection and analysis methods

We collected multiple sources of data to support our *retrospective conceptual analysis* (Steffe & Thompson, 2000; Thompson, 2008). We videotaped each session with two cameras (one captured a wide-angle view facing the student from the front and the other captured a focused view of the students' activities from above) and constructed annotated transcripts. We also used a screen recording program to capture the tablet device display including students' generated work. We digitized each student's work and the TR's and WRs' notes for use in retrospective analysis.

In conducting retrospective conceptual analysis, we started with initial themes based on our prior knowledge of students' quantitative and covariational reasoning (e.g., amounts of change, direction of change, and rate of change). Each research team member used these themes to guide their coding of the video data. For instance, some research team members coded for activity suggestive of amounts of change reasoning (as reported here), while others coded for activity suggestive of reasoning about frames of reference (Lee, Moore, & Tasova, 2019) or elapsed time (see Stalvey & Vidakovic, 2015; Thompson & Carlson, 2017). We then shared and compared our coding results as a group. Our iterative retrospective analysis efforts involved constructing hypothetical mental actions that viably explained each student's observable and audible behaviors. We continually searched the data for instances that the models could not account for, and we modified our models or attempted to explain developmental shifts in each student's meanings.

As an example of our retrospective analysis pertaining to the focus of this paper, we hypothesized that Lydia's repeated experiences of engaging in partitioning activity within and among various representational systems and situations might have supported her abstraction of quantitative meanings for amounts of change as her meanings for partitioning activity shifted from figurative to operative in nature. In reviewing the data, we first identified teaching episodes that offered us insights into Lydia's partitioning activity. With respect to each task, we identified instances of her initial engagement in partitioning activity and her subsequent re-presentation of that activity. Then, we identified a subset of instances that provided indications or contraindications of figurative and operative aspects of her partitioning activity. We paid particular attention to the interplay between Lydia's conceived covariational relationships in situations and her re-presentation of associated partitioning activity. We inferred her meanings as being figurative or operative based on the forms of re-presentation she engaged in and whether her actions foregrounded or backgrounded quantitative and covariational reasoning. For instance, when we observed her being able to enact the first or the second form of re-presentation quantitatively, we continued to look for instances where she engaged in the third form of re-presentation. If she could transform her actions to preserve quantities' meanings and covariation, we considered her partitioning activity as operative. Otherwise, an indication of figurative partitioning activity was her engaging in emulating prior partitioning actions. This indication was strengthened if we also identified a contraindication of operative partitioning activity, such as her encountering difficulty enacting the second form of re-presentation due to minor perceptual differences from a previous context or the absence of available perceptual material. We analyzed Emma and Brian's partitioning activity in a similar manner, with each student's activity also being compared and contrasted to each other.

Task design

We designed a series of tasks to gain insights into the students' covariational reasoning. In the following, we describe four related tasks: (1) Taking a Ride, (2) Which One?, (3) Circle, and (4) Blue-Red-Green.

Taking a ride

Taking a Ride (Figure 2a) included an animation of a Ferris wheel that is rotating in a counterclockwise direction (Desmos, n.d.). We designed this task to focus students on constructing the covariational relationship between the *height* of the green rider above the horizontal diameter of the wheel and its *arc length* traveled (the sine relationship). In students' first attempt on this task, we asked them to describe this relationship on the Ferris wheel situation and did not prompt them to produce a graph.

Which one?

After students' first attempt on Taking a Ride, we presented Which One? (Figure 4; also see [https://youtu.be/2pVVGl8eEr0]). This task included a simplified version of a Ferris wheel (left) with the position of a rider indicated by a dynamic point. The topmost bar (shown in blue, right) displayed the arc length the rider had traveled counterclockwise from the 3 o'clock position. Students could vary the length of this bar by dragging its endpoint or by clicking the "Vary" button, with the dynamic point on the circle moving correspondingly. We asked students to determine which of the six red bars, if any, accurately displays the rider's height above the

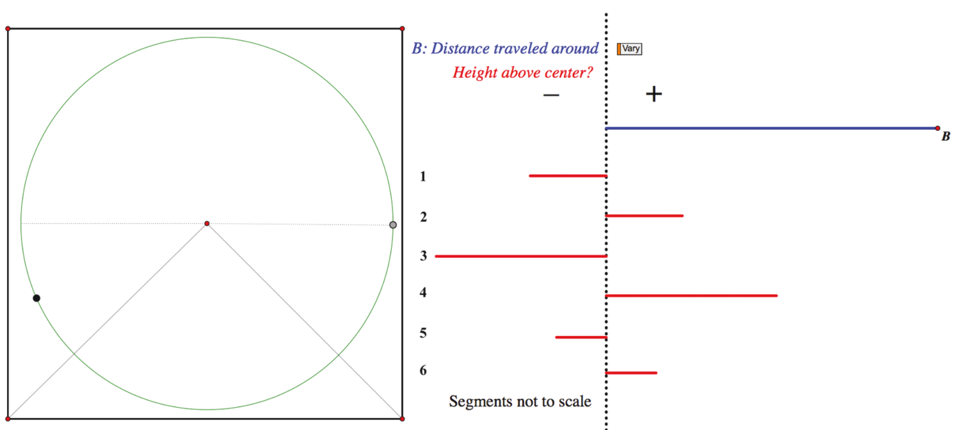


Figure 4. Which One? (numbering of segments is labeled for readers).

horizontal diameter as the rider's arc length varied. Bar 1 is a normative solution and bars 2–6 vary with either different directional variation (e.g., positive or negative; decreasing or increasing) or different rates (e.g., constant, increasing, or decreasing rate) than the normative solution. Similar to Taking a Ride, we did not prompt the students to construct a graph.

Circle

We designed two versions of the Circle task: Circle A (Figure 5a; also see [https://youtu.be/vRGJ5psVhs4]) and Circle B (Figure 5b; also see [https://youtu.be/4REU4nXH7Ic]). For both tasks, we simplified the Ferris wheel to a circle to minimize the influence of the perceptual features of the wheel (e.g., the spokes on the wheel) and focus the students' attention on the displayed magnitudes of the two quantities. For Circle A, we asked students to graph the relationship between the *horizontal distance* and the *arc length* associated with a dynamic point (the cosine relationship). We used this task in the group sessions and we asked each student to recall their thinking in a subsequent individual session. For Circle B, we asked students to graph the relationship between the *height* and the *arc length* in two different coordinate systems: one with arc length on the horizontal axis and height on the vertical axis and the other with the alternative axes orientation. We used this task to engage students in exploring the sine-inverse sine relationship.

Blue-red-green

Blue-Red-Green (Figure 6a; also see [https://youtu.be/SznQQnwtNyM]) included three vertically-oriented bars that varied simultaneously. The three bars entailed the same variations and relationships as the three segments shown in Figure 6b (see [https://youtu.be/GLLsiyPWSGw]; Point B is draggable), with their colors matching each other. With respect to the blue bar (arc length) increasing, the red bar (sine) increases at a decreasing rate, and the green bar (versine) increases at an increasing rate. We asked students to describe how any of the two bars varied simultaneously and to construct graphs that display the paired relationships. We intentionally chose the red and green quantities (sine and versine) in an attempt to engage students in reasoning with quantities' magnitudes that entailed the *same* directional change but *different* rates of change with respect to the blue (i.e., the red *increases* at a *decreasing* rate, and the green *increases* at an *increasing* rate with respect to the blue). In students' initial attempt on the task, we presented the interface shown in Figure 6a but not Figure 6b. We intended to explore their reasoning with quantities' magnitudes independent of the circle context. After sufficient exploration, we presented students the animation shown in Figure 6b and asked them if the variations of the three segments matched the variations of the three bars in Figure 6a.

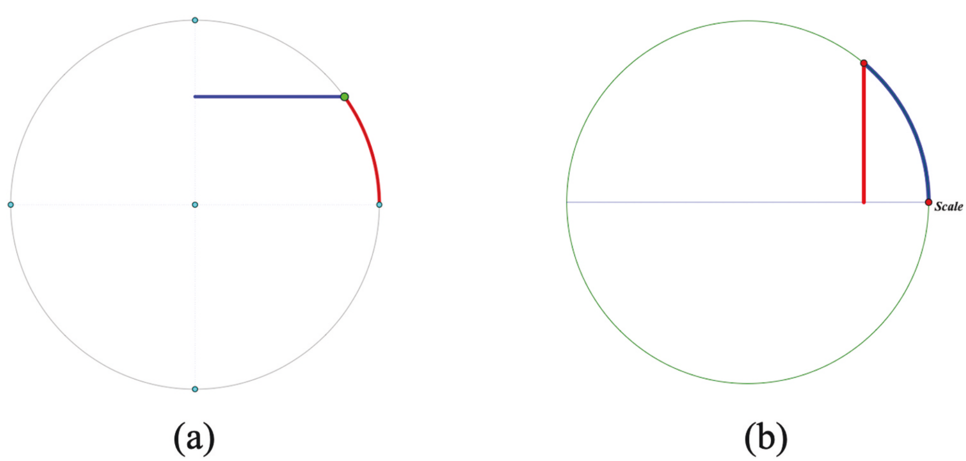


Figure 5. (a) Circle A and (b) Circle B.

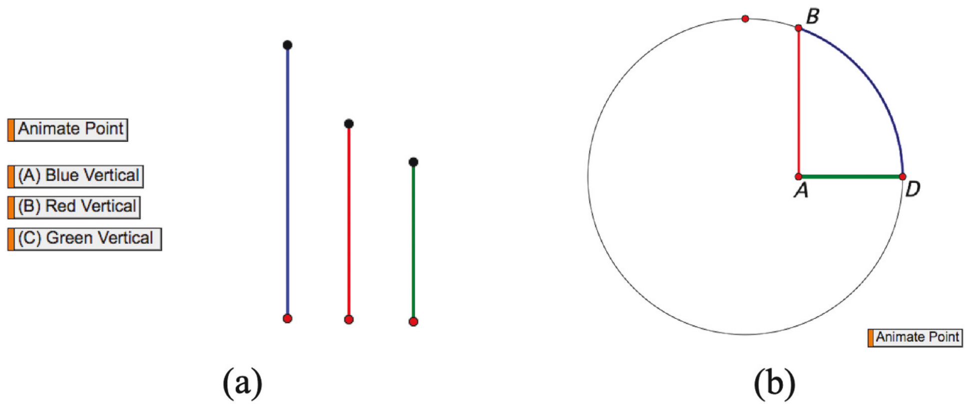


Figure 6. (a) Blue-Red-Green, and (b) the quantitative meanings of the three bars with respect to a circle.

Task design principles

We draw attention to a few common task design principles, with additional design decisions reported elsewhere (Stevens, Paoletti, Moore, Liang, & Hardison, 2017). First, we designed the tasks to entail what we perceive to be perceptual material displaying the quantities' magnitudes without providing numerical values (unless introduced by students). This decision afforded us understandings of students' images of quantities' covariation in the context of such material. Second, we designed a multitude of contexts that entailed what we perceive to be perceptual differences but similar or identical covariational relationships. By supplying different perceptual material (e.g., a Ferris wheel, circles, arcs, horizontal and vertical segments, and parallel bars) associated with a dynamic situation and asking students to construct coordinate graphs in multiple axes orientations, we were afforded inferences as to whether the students could recognize and re-present (mostly in the second and the third forms of re-presentation) across different perceptual material or if their activity was constrained to emulating particular actions on available or similar perceptual material. Third, although we focused a majority of the tasks on the sine relationship, we included tasks involving other (but similar) relationships (e.g., cosine in Circle and versine in Blue-Red-Green) to compare students' partitioning activity across different covariational relationships. This allowed us to gain insights into the ways in which students coordinated different quantities with respect to a similar situation.

Results

We illustrate the distinction between figurative and operative partitioning activity (as summarized in Table 1) with data from the three students. We analyze their partitioning activity by focusing on their attempts to re-present that activity as they reasoned among various representational systems and situations. We use data from Lydia in each subsection due to her activity suggesting evolutions in her meanings. We also include data from Emma and Brian to provide additional examples.

Partitioning activity constrained to available perceptual material

In Week 1, we worked with Lydia on Taking a Ride (Figure 2a). She initially described, "the arc length has increased to this [*drawing a red arc on the first quarter circumference in Figure 7a*] while the distance from the center has increased to that [*drawing a vertical red segment from the top position to the center of the wheel*]." Eventually, with much effort, Lydia made use of the spokes of the Ferris wheel to partition traveled distance equally (Figure 7a), constructed the corresponding height at each state (Figure 7b), and constructed successive amounts of change in height (Figure 7c). Noticing the blue segments (in Figure 7c) decreased in magnitude, Lydia concluded, "as the arc length is increasing,

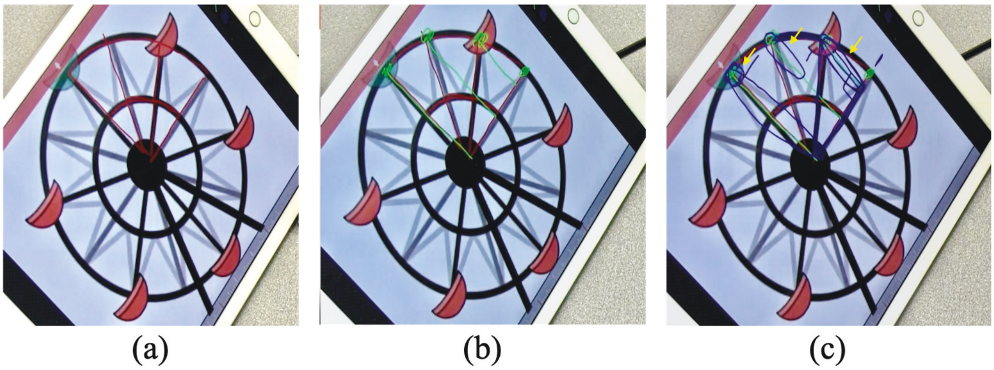


Figure 7. Lydia (a) used the four spokes to partition distance traveled into three equal increments, (b) identified height of the green rider in each successive state, and (c) identified successive amounts of change in height.

[the] vertical distance from the center is increasing, but the value that we're increasing by is decreasing." Suggesting she was excited that she had identified this relationship, she explained with enthusiasm, "I just discovered this by myself." We inferred that her partitioning actions were novel to her, and these actions afforded her initial construction of quantities' amounts of change.

Then, we presented Which One? (Figure 4), and after some explorations, Lydia claimed she desired to choose a red bar that is moving at a *constant rate*; recall that immediately prior on Taking a Ride she identified the vertical distance was increasing by decreasing amounts. She eliminated four bars and had difficulty deciding which of the other two was moving constantly. She then oriented the bars vertically and placed each inside the circle. She confirmed that the length of one of the bars (the normatively correct solution) matched the height of the dynamic point on a circle at a few different states. From our perspective, this bar did not vary at a constant rate with respect to the blue bar, and thus the TR asked Lydia if her chosen bar entailed the amounts of change relationship constructed in the initial Taking a Ride task. She responded:

Lydia: Not really . . . Um, I don't know. [*laughs*] Because that was just like something that I had seen for the first time, so I don't know if that will like show in every other case . . . Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.

TR: So is what we're looking at right now different than what we were looking at with the Ferris wheel?

Lydia: No. It's – No . . . Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don't see it here, and so –

TR: And you "don't see it here," you mean you don't see it in that red segment?

Lydia: Yes.

It is noteworthy that Lydia described height increasing by decreasing amounts as a "theory" to be tested in Which One? Despite her having identified that the red bar worked point-wise with respect to the traversed arc length. Her knowing that the red bar worked for each state did not imply by necessity that the red and blue bars existed in a covariational relationship equivalent to that between height and arc in Taking a Ride. Our explanation is that her understandings of amounts of change were rooted in carrying out particular partitioning actions and creating perceptually available incremental curves and vertical segments for comparison in Taking a Ride (Figure 7c). Hence, when provided with what was to her novel perceptual material in Which One?, in which the spokes of a Ferris wheel were not perceptually available and instead several bars were varying continuously, she did not anticipate re-presenting amounts of change. As she engaged in point-wise checking the red bar at different states, she might also encounter difficulty with holding in mind the red bar associated with a prior state to compare it to that in a current state, and thus she was unable to mentally re-present the successive, decreasing changes in height. We thus claim that her partitioning activity constructed in Taking a Ride was figurative such that it constrained her from conceiving the equivalent covariational relationship in Which One?.

As the session continued, Lydia recognized the equivalence and relevance of the two contexts, and she desired to "see" her previous partitioning activity; namely, her partitioning activity in Taking a Ride became a figurative ground on which she attempted to operate here. However, as she progressed she was unable to re-present that activity ("I saw that difference in the Ferris wheel, but I don't see it here"). This is a contraindication that Lydia was able to enact the second form of re-presentation. Then, the researchers intervened by using pens to denote amounts of change of the red bar that correspond to three states of the animiation (Figure 8). As further evidence of our above claims, Lydia immediately recognized the relationship and responded with enthusiasm that her "theory" held true.

Together, we describe Lydia's present partitioning activity as being figurative due to her ability to recognize amounts of change when the perceptual material was given, but not anticipating re-presenting or regenerating the changes when the perceptual material was absent. Her conceived invariance of partitioning activity necessitated carrying out actions on available segments in order to produce perceptually available results. We did not observe her anticipating and transforming her prior partitioning activity to accommodate the novel features of Which One?, but she could recognize and relate the results of such activity when provided.

Emulating actions of partitioning activity

Lydia's activity

During the group sessions, Lydia worked with Emma and Brian on Taking a Ride (Figure 2a) and Circle A (Figure 5a). They produced graphs to re-present their constructed relationships regarding these two tasks (i.e., sine and cosine), and Lydia primarily observed the other two

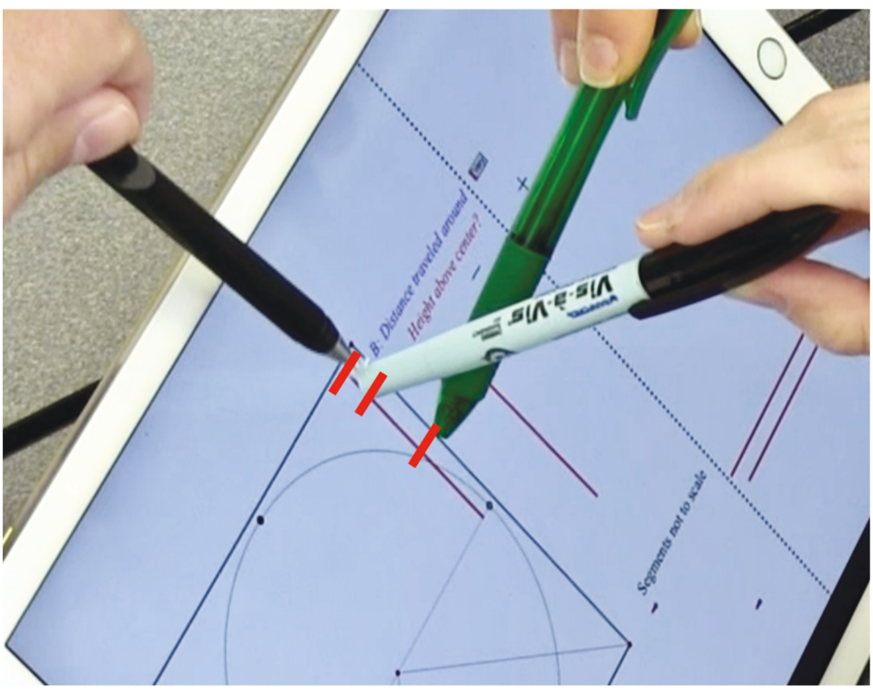


Figure 8. The researchers were assisting Lydia to identify amounts of change in height on a red segment.

students. The TR began the next individual session by asking Lydia what she recalled from those sessions. She drew a quarter of a circle and discussed the relationship between arc length and horizontal distance:

> "as the arc length is increasing in the first quadrant that our X distance is decreasing [*drawing the black horizontal segments within the circle from bottom to top in Figure 9a*], and then, distance will decrease more in the same amount of space. So like from here to here [*highlighting the bottom blue arc*], then we'll say these are the same arc length [*highlighting the top blue arc*] . . . so we're going to take this point here [*marking a point at the top of the far-right pink segment*] and then drag it down [*drawing the far-right pink segment*], we've only lost this much [*highlighting the shorter yellow segment*]. And then from here [*drawing the middle pink segment*] to here [*tracing the far-left pink segment*] we lost this distance [*highlighting the longer yellow segment*], but we're saying those are the same arc length [*pointing to the two blue arcs*], so it's a lot more distance."

Lydia was describing that as the arc length increased by equal amounts, the horizontal distance decreased by increasing amounts. Her activity appeared compatible with that from the previous sessions, which we characterized as the first form of re-presentation. Then, the TR asked her how such activity related to graphing the relevant relationship (i.e., a *cosine* graph), attempting to engage her in the third form of re-presentation. Lydia drew a graph (Figure 9b; what we perceive to be a *sine* graph) and explained how she conceived the graph as entailing the same properties as what she discussed in Figure 9a (we use the same color to match components that she perceived to be related):

> "As we go up in arc length [*highlighting the left blue curve in Figure 9b*], that distance is decreasing [*drawing the black horizontal segments from bottom to top*], and so we see that here [*drawing the far-left pink segment*] is like this [*highlighting the left yellow segment*], and then [*highlighting the right blue curve and drawing the middle and far-right pink segments*], here is this [*drawing the right yellow segment*]. So that's the same conclusion we had gotten from the circle, so then we can say that this circle relates to this graph."

Lydia's partitioning activity across the circle and the graph included: (a) drawing horizontal segments emanating from the circle and curve (see the black segments), (b) tracing arcs and curves from lower end points to higher end points (denoted in blue), (c) drawing vertical segments from the end points produced by the arcs or curves to a horizontal segment or line (denoted in pink), and (d) drawing horizontal segments between two pink segments (denoted in yellow) and comparing their lengths.


(a)

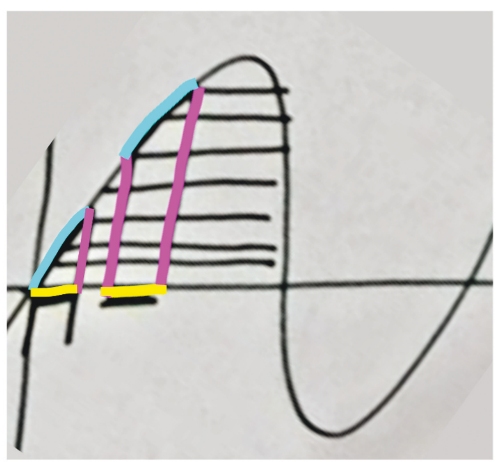
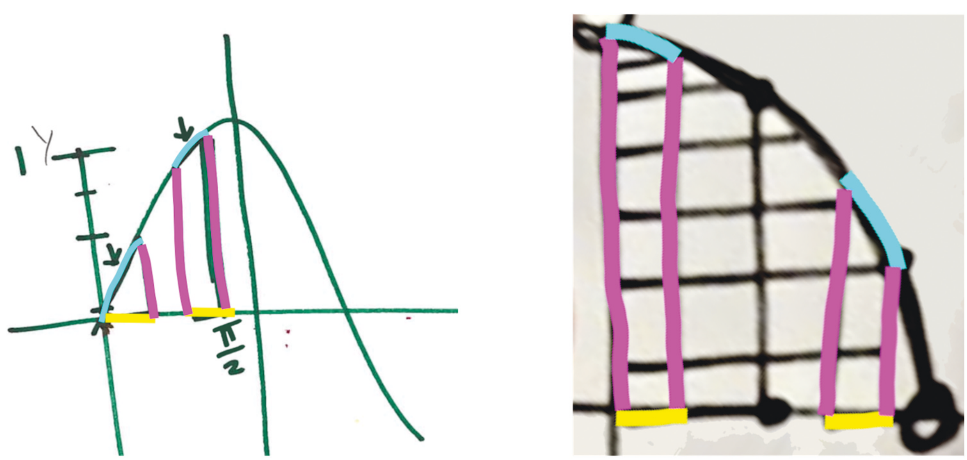

(b)

Figure 9. Lydia's partitioning activity on the (a) circle and (b) graph for re-presenting the covariational relationship between arc length and horizontal distance.

Although Lydia's partitioning activity did entail some operative schemes (e.g., making quantitative comparisons between the yellow segments), we characterize her activity as being figurative due to repeating actions in order to produce similar perceptual results. This suggested that she did not dissociate the mental and physical aspects of her activity (e.g., *partitioning* along *something curved, constructing* segments in a *vertical* and *horizontal* orientations, and *comparing* segments *horizontally*). We specifically note that she constructed partitions along her graph to refer to changes in "arc length" (see the blue curves in Figure 9b) and did not maintain a fixed reference point for the black horizontal segments (see Figure 9b). These were contraindications that her activity was operative, because constructing and committing to frames of reference are key aspects of operative, quantitative activity (Joshua et al., 2015; Lee et al., 2019). Her figurative partitioning activity constrained her from conceiving relationships quantitatively in the graphical context. An indication of operative partitioning activity would involve her constructing a conventional cosine graph, producing equal partitions along the horizontal axis (arc length), and identifying corresponding decrements along the vertical axis (horizontal distance).

As the session moved forward, Lydia's work provided additional evidence that her partitioning activity was figurative. She drew a similar graph (Figure 10a) in order to discuss the relationship between "height" and "arc length".[1] Her activity included tracing from two equal horizontal segments (denoted in yellow, Figure 10a), drawing vertical segments up to the curve (denoted in pink, Figure 10a), and tracing two corresponding curves (denoted in blue, Figure 10a). She compared the lengths of these curves and concluded the increases in height decreased for equal changes in arc length. When transitioning to a circle (Figure 10b), she traced two equal, horizontal segments (denoted in yellow), drew vertical segments (denoted in pink), and traced and compared two arcs (denoted with blue). Suggestive of continued figurative partitioning activity, Lydia was carrying out the same sequence of actions on her graph and circle, the elements of which entailed similar perceptual results. We note that she partitioned along the horizontal diameter of the circle to refer to amounts of change in "arc length" and constructed and compared lengths of the curves and arcs to refer to changes in "height"; these were contraindications that her partitioning activity was operative (see Figure 3b for an illustration of operative partitioning activity related to this context).



(a) (b)

Figure 10. Lydia's partitioning activity on her (a) new graph and (b) the circle for re-presenting the covariational relationship between arc length and height.

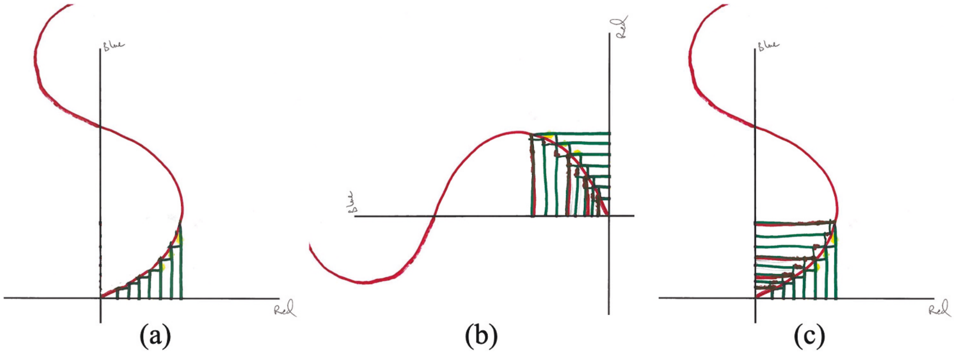


Figure 11. Brian's (a) first attempt on partitioning activity, (b) second attempt on partitioning activity (with paper rotated), and (c) final graph (with paper rotated back).

Brian's activity

We illustrate another example of figurative partitioning activity based on Brian's activity in Circle B (Figure 5b) where he explored the relationship between height (in red) and arc length (in blue). He first carried out partitioning activity on the circle and concluded that as the blue increases by equal amounts, the red increases by less and less. Then, the TR asked him to construct a graph in a given Cartesian coordinate system with "blue" labeled on the vertical axis and "red" labeled on the horizontal axis. He constructed the graph shown in Figure 11a and subsequently attempted to justify how his graph displayed the same relationship as that of the circle:

Brian: I believe we could just do very similar to what we did here [*drawing the seven vertical segments in Figure 11a*], try to break it up into equal as we can, and we can see that [*drawing the small horizontal segments in Figure 11a*], there's almost no height change for this first arc length [*pointing to the first two partitions*], and then as it gets bigger, the height is getting larger [*tracing along the red curve from left to right*].

TR: And the change in height is getting –

Brian: Oh, it's growing. Oh, crap. [*pausing*] That's not [*pausing*] Well, maybe it – I'm trying to think. Maybe it would be this way [*turning Figure 11a for 90 degrees in a counterclockwise direction*]. Maybe we'd be drawing the bars [*gesturing vertical partitions*], because this way [*drawing new partitions on the paper with the rotated orientation; see Figure 11b*] That would make my statement true.

TR: So what's going on there [*turning the graph back to the original orientation; see Figure 11c*]?

Brian: So I don't know exactly why, but when I did it from the X, when I was doing the height [*tracing along the vertical partitions in Figure 11c*], um, what I said was wrong or this is representing something I didn't say, but when I did the width [*tracing along the horizontal partitions*], that it represents what I was saying, that it grows fast and then it slows down [*tracing along the red curve from the origin*]. Um, [*pausing*] I don't really know why that is, why I did it from that way.

We inferred that Brain's activity stemmed from him emulating the sequence of partitioning actions he had enacted in a previous task involving a canonical axis orientation (i.e., arc length or blue on the horizontal axis). That sequence of actions included constructing vertical segment to partition the horizontal axis equally and then constructing partitions in those segments to illustrate their accumulation in terms of accruals, each of which he repeated here. His activity was a contraindication of him enacting the third form of re-presentation quantitatively. He was, however, perturbed by the unanticipated results of his current partitioning activity. In his attempt to reconcile his perturbation, he rotated his graph. He could have proceeded as he intended in its rotated orientation, but he instead drew in segments that intersected with the red curve at the same locations as his previous partitions. Although he perceived those new partitions as consistent with his goals of showing equal changes in blue and decreasing changes in red, he ended the interaction perturbed due to being unable to conceive his partitioning activity in one orientation (Figure 11b) as related to that in the alternative orientation (Figures 11a and 11c).

We note that Brian's activity differs from Lydia's in that Lydia repeated her partitioning activity across the circle and graph, while Brian repeated his actions in graphs across different coordinate orientations. Their activities were compatible in that both foregrounded emulating their partitioning activity, and neither of them preserved the quantities' covariational relationship from our perspective. Hence, as with Lydia, we characterize Brain's partitioning activity as being dominated by physical fragments of partitioning including perceptual orientations rather than abstracted covariational relationships, and thus it was figurative. Moreover, his activity was a contraindication of him transforming his previously constructed partitioning activity in a way that accommodated and committed to the frames of reference he had established.

Re-presenting and transforming partitioning activity

Lydia's activity

As the teaching experiment proceeded, we provided Lydia with additional opportunities to engage in partitioning activity as it related to other dynamic situations. In Week 8, her activity suggested that she could re-present partitioning activity across multiple representations while accounting for perceptual differences in those contexts, indicating her thinking had become operative. When working on Blue-Red-Green (Figure 6a), she watched the animation and claimed that as the blue bar increased at a constant rate, the red bar increased at a decreasing rate. She then constructed a graph, carried out partitioning activity (Figure 12a), and concluded from the orange segments that, "because that amount of change is getting smaller and smaller, it's increasing at a decreasing rate". Here, Lydia's image of the blue and red bars' covariation afforded her construction of the graph and the subsequent partitioning activity.

The TR then asked Lydia to draw a picture of the two bars on a paper and demonstrate how she would manipulate the blue and red bars in ways consistent with the covariational relationship re-presented on her graph. Lydia first drew two vertical arrows and a collection of horizontal segments to indicate landmarks of equal increments (Figure 12b). She then discussed how she imagined each bar first increasing to the respective bottom star symbol, then to the middle, then to the top. She explained that she intentionally placed the three stars with respect to the black arrow at equal partitions but not the red arrow on the right because "the red is

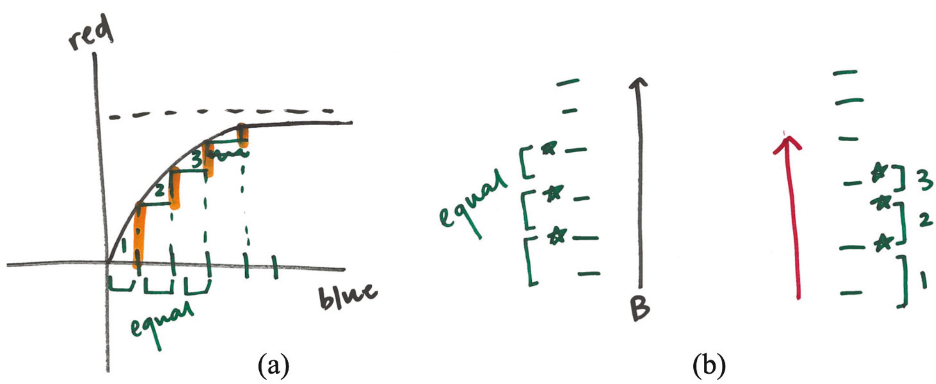


Figure 12. (a) Lydia's partitioning activity on her graph for re-presenting the relationship between the red and the blue bars and (b) her re-presentation of her partitioning activity on the red and blue bars.

already like started to slow down, so then it hasn't reached the next partition, so then we can see they're not traveling at equal paces. It's not reaching the next partition when the black is." The TR then asked her to return to her graph and talk about how her two drawings were related. She labeled "equal," "1", "2," and "3" on both drawings to indicate how they corresponded to each other (Figure 12(a,b)). In contrast to her work on Which One?, Lydia could conceptualize these two continuously varying bars as varying by successive increments and re-present their covariations on these bars. She could also enact the third form of re-presentation by transforming her partitioning activity between the graphical context and the bar context such that they both entailed the same quantitative relationship and structure, despite their perceptual differences. We also contrast her activity here with her activity we present in the previous section, in which she emulated her partitioning actions on the circles and graphs without maintaining the same quantitative relationship and structure from our perspective.

Observing that Lydia could re-present her partitioning activity on the blue and red bars, the TR hypothesized she could re-present similar activity regarding the blue and the green bars. To test this hypothesis, the TR presented her another version of the sketch where the same three bars were presented, but they were not growing and shrinking continuously; rather, the endpoint of each bar was draggable so that Lydia could drag any chosen endpoint while the other bars would animate correspondingly. Manipulating the blue bar to increase by equal increments, Lydia claimed that there was "hardly a change in green" for an initial increment in the blue and there was a "decent jump in the value of green" for a subsequent equal increment in the blue (see [https://youtu.be/J-uoB2r97WY] for an illustration of her activity). She concluded that the green increased at an increasing rate as the blue increased at a constant rate.[2]

Turning to a graphical representation, the TR asked Lydia to create a graph on a non-conventional coordinate system[3] with the horizontal axis labeled as green and the vertical axis labeled as blue (left and up defined as positive; see Figure 13a). By doing so, we attempted to gain insights into the extent her meanings were tied to the perceptual features of her previous graphs. In response, Lydia created a drawing shown in Figure 13a where she first drew equally-spaced partitions on the horizontal axis, then drew partitions with decreasing amounts of space on the vertical axis (starting from the origin), and finally produced a graph by uniting the corresponding partitions. She explained that this graph displayed the *blue bar increased at a decreasing rate with respect to the green* and that it was equivalent to her prior conclusion that *the green bar increased at an increasing rate with respect to the blue*. The following excerpt shows her justification of the equivalence of the two statements:

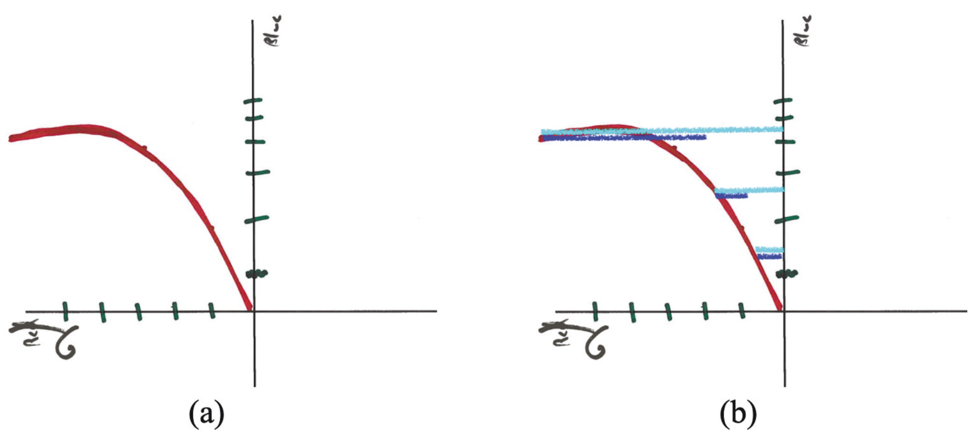


Figure 13. (a) Lydia's constructed graph for re-presenting the relationship between the green and the blue (with equal partitions on the green), and (b) our annotation of her mentally re-presented partitioning activity (with equal partitions on the blue).

> So say we were to partition the blue into equal increments [*motioning her hand as if she was drawing the light blue segments in Figure 13b*], we would be able to clearly see the change in green will become larger and larger as we increase [*tracing along the longest dark blue segment*].

The novel coordinate system did not constrain Lydia from re-presenting invariant covariational relationship of the two quantities. She could transform her partitioning activity on the vertically-oriented blue and green bars to accommodate the novel coordinate system. Moreover, she was able to *mentally* envision equal partitions along the vertical axis and anticipate successive, horizontally-oriented increments increasing in size (Figure 13b). We inferred Lydia's partitioning activity was operative because her re-presentation did not rely on her physical enactment of specific actions to visualize the partitions; rather, she could mentally regenerate the partitions that were coordinated with the relationship she intended to re-present. Her ability to re-present two ways of partitioning on the same graph and conceive them as being equivalent and compatible with each other suggested her activity foregrounding quantitative and covariational relationships so that perceptual properties were subordinate to those relationships.

Approaching the end of this session, we presented Lydia with the circle shown in Figure 6b and asked her to determine if the variation of the green segment within the circle corresponded to that of the green bar. She explained that in order for the circle segments to have the same relationships as the bars, "the change in the arc length is like, when it changes 1 unit, then like the change in green is very small, but then as the green value increases, the change is also increasing." Again, Lydia's activity involved her taking the covariational relationship as given and anticipating re-presenting partitioning activity without physically carrying it out to visualize the increments. She further dragged Point B for six successive equal increments and perceived that the green segment did increase by increasing amounts.

Emma's activity

As another example of operative partitioning activity, we draw on Emma's response when considering Circle B. She initially created the graph shown in Figure 14a and concluded that "B is increasing at a decreasing rate[4]". In response, the TR created Figure 14b in order to determine if Emma continued to consider her claim as viable; interactions with Lydia and Brian indicated that a student with figurative partitioning meanings would either be constrained to making claims based on the perceptual features of available partitions or requiring emulating previous partitioning activity (e.g., claiming a quantity increases at a decreasing rate only when it is partitioned into successive decreasing increments). With respect to Figure 14b, Emma first claimed "B is increasing at a constant rate", and then emphasized that "but at the same time, you can use it to say that B would be increasing at a decreasing rate". She continued to say:

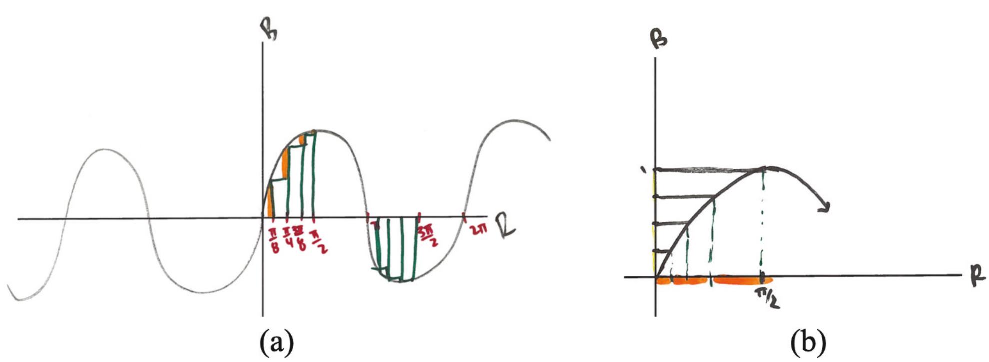


Figure 14. (a) Emma's graph with equal changes in R and (b) the TR's graph with equal changes in B.

> Like if I wanted to make these radians change in equal increments [*pointing to the orange partitions along the "R" axis*], I would have a bigger change [in] the first pi over 8 [*pointing to the first orange segment*], I would have, then my changes would get smaller and smaller [*pointing to the vertical increments along the "B" axis*], because like if this, if this like little segment [*pointing to the first orange segment*] was stretched out to be like in more equal, I guess, like if you divided these links between four of them [*pointing to the orange segments*], the ones with the smaller links [*pointing to the first orange segment*] would span out a larger B length [*pointing to the first vertical increment*] than the ones that are bigger [*pointing to the last orange segment*] when they got smaller, they would now span out a smaller B length [*pointing to the last vertical increment*].

Emma conceived that in order to achieve successive and equivalent increases in R, the first increase of B would necessarily be larger than the displayed partition while the culminating partition of B would necessarily be smaller than the displayed partition. Importantly, at no time did Emma physically carry out partitioning activity to produce the partitions she envisioned. She took the result of transforming the displayed partitions as a given such that those transformations would regenerate her previously constructed structure. We take this as indicating her partitioning activity was operative due to her ability to mentally re-present partitions that were perceptually unavailable, in addition to her transforming the displayed partitions in order to preserve a previously constructed structure. Consequently, she could conceive both graphs as displaying the same relationship. This is an indication of Emma's meaning foregrounding quantities' covariational relationship rather than the perceptual features of the partitions.

Discussion

We summarize our findings about the three students' partitioning activity and relate them to extant theories of covariational reasoning. We also discuss how our findings afford a reconceptualization of "mathematical concepts" through the lens of re-presentation and figurative and operative thought.

What do we learn from Lydia, Emma, and Brian?

The three students' responses suggest two distinctive meanings for their partitioning activity: figurative and operative partitioning activity. Characterizing a student's meanings in terms of figurative or operative partitioning activity is significant in that it adds nuances to Carlson et al.'s (2002) and Thompson and Carlson's (2017) covariation frameworks. Carlson et al. (2002) specified that Level 3 Covariational Reasoning involved students "coordinating the amount of change in one variable with changes in another variable[5]" (p. 358). Our findings suggest that a student's meanings for partitioning activity, including the extent it is restricted to carrying out particular actions and their results, have important implications for the student's meanings for amounts of change. For example, Lydia's partitioning activity was initially figurative because it involved her seeking to emulate particular actions in a particular order across various contexts. Furthermore, her activity was constrained to physically enacting those actions to produce perceptually available partitions. When confronted with a context in which these perceptual elements were absent or carrying out the same actions failed (e.g., Which One), she struggled to regenerate her partitioning activity and construct equivalent quantitative and covariational properties, including amounts of change.

In contrast, operative partitioning activity is more generative to various mathematical situations and ideas when compared to the figurative ones. For example, operative partitioning activity allowed Lydia and Emma to re-present their activity such that they could conceive of equivalent covariational relationships in novel contexts where the perceptual material associated with a prior construction is absent or different.

One potential explanation for Lydia's later success in re-presenting partitioning activity among various contexts is her repeated experience of partitioning throughout the teaching experiment. We intentionally prompted her to switch back and forth among what we perceived to be various representational systems to re-present her activity, which included circle situations (e.g., Taking a Ride, Which One, and Circle), dynamic bars (e.g., Which One and Blue-Red-Green), and graphs with different axes orientations (e.g., Figures 11 and 13). Such experience supported her reflection on her partitioning activity that led to her construction of operative meanings for amounts of change. Toward the end, she could sustain quantitative meanings across different representations and re-present and transform her partitioning activity to perceive invariant relationships among perceptually different representations.

Our incorporation of the notion of re-presentation is critical for us to distinguish students' partitioning activity. A hallmark of figurative partitioning activity is that a student conceives their activity in terms of carrying out partitioning actions on perceptual material, thus likely failing to re-present their activity in the absence of that material or encountering difficulty re-presenting the activity in other contexts. Operative partitioning activity is featured by a student conceiving their activity in terms of quantities' relationships, which affords their transformation and re-presentation of their activity to accommodate novel contexts.

How can we reconceptualize "sophisticated mathematical concept"?

A broader implication of these students' activity is that students' ability to recognize amounts of change when perceptual material is given does not imply that they can re-present these changes when the perceptual material is absent. Such a distinction regarding students' meanings for amounts of change motivates us to reconsider: what do we mean by "a student has constructed a concept of amounts of change"?

von Glasersfeld (1982) defined a *concept* as "any structure that has been abstracted from the process of experiential construction as recurrently usable … must be stable enough to be re-presented in the absence of perceptual 'input'" (p. 194). Characterizing students' partitioning activity as we have enables us to extend and apply this definition in the context of students' reasoning about relationships between covarying quantities. When a student abstracts their partitioning activity so that it is not tied to the presence of perceptual material or particular figurative features (i.e., physical actions or perceptual entailments), thus mentally anticipating the transformation and re-presentation of such, we consider that they have constructed a *concept* of amounts of change. This conceptualization of mathematical concepts aligns with Moore and Silverman's (2015) *abstracted quantitative structure* – a mental structure of related quantities a student has internalized *as if* it is independent of specific perceptual material so that they can re-present this structure to accommodate novel contexts permitting the associated quantitative operations. Our conceptualization also echoes the features of *anticipatory* or *participatory* stages of conceptual learning as Tzur, Simon, and colleagues (Simon et al., 2016; Tzur & Simon, 2004) have described. An individual's operative partitioning activity is anticipatory in that it involves the individual taking the covariational relationship as given and anticipating transformation of the partitioning activity in re-presentation. An individual's figurative partitioning activity can be participatory when it is constrained to carrying out partitioning actions in order to conceive the covariational relationship (e.g., Lydia's activity in Which One?). However, our findings also suggest that figurative partitioning activity can be anticipatory in the context of re-presentation. This occurs when an individual can take the covariational relationship as given and anticipate the results of partitioning, but the resulting partitions do not align with their quantitative referents due to perceptual differences in contexts (e.g., Lydia and Brian's emulating partitioning activity). We thus conclude that anticipation is a necessary but not sufficient condition for operative partitioning activity.

Our conceptualization of mathematical concepts has several implications for mathematics educators and researchers. The immediate, although not novel (see Goldenberg, 1995; Lobato & Bowers, 2000; Montiel et al., 2008; Moore, Paoletti, & Musgrave, 2013; Moschkovich et al., 1993; Oehrtman et al., 2008; Thompson, 1994b) implication is the necessity of providing students sufficient opportunities to experience and organize various situations and representations in order to support their construction of a concept. Also, this notion of concept requires us to be more careful when making claims about students' covariational reasoning and related mathematical meanings (e.g., function and rate of change) without first gaining insights into their activities among a variety of contexts, including their ability to re-present their actions. Students' ability to carry out particular actions in one context does not imply that their meanings are operative or quantitative, or that they have constructed a concept related to those actions. In our previous work (e.g., Moore, 2014a), we have made such claims without developing what we now view as convincing evidence for them.

Any framing of concept sophistication or any differentiation in meanings invites the question, "How does such a concept develop or transition in meanings occur?" Because the figurative and operative distinctions emerged from our analysis, we did not conduct the study with the detailed focus necessary (see Simon et al. (2010)) to articulate an explanatory mechanism for the shifts in Lydia's partitioning activity. Although limited in our data, we note one potential explanation for her shifts compatible with our theoretical framing is that of Piaget's *reflective abstraction* (Simon et al., 2004; Piaget, 2001; Von Glasersfeld, 1995), including his distinctions based on the locus of attention (e.g., results of actions or the actions themselves) and level of consciousness. We call for continued explorations into the processes by which students construct a mathematical concept through reflecting upon their actions and abstracting quantitative relationships and structures as they engage in a variety of re-presentational activities.

Closing remarks

Our findings highlight the significance of explaining students' mathematical meanings through the lens of figurative and operative thought. Such a distinction is not only internally viable in one student's activity but also across students' activities. Future researchers can continue to investigate students' meanings for amounts of change and, more broadly, quantities' covariation, by detailing the figurative or operative grounds for their meanings. For example, our research design did not allow us to understand the extent students conceptualize an accumulated magnitude *varying* from one partition to another (i.e., *chunky continuous variation* (Castillo-Garsow, Johnson, & Moore, 2013; Thompson & Carlson, 2017)); we could only conclude that they at least conceived a partition as a marker for a momentary state of a quantity's continuous variation. Other researchers can design specific tasks to explore this nuance, including the affordances of our distinction between figurative and operative thought in explaining students' chunky and smooth continuous covariation. Researchers can also test and confirm the generalizability of ourdistinction in terms of explaining students' mathematical meanings regarding other topical areas and grade levels (e.g., children's figurative and operative counting scheme (Steffe, 1991; Steffe & Olive, 2010)), including content areas that require (e.g., unit coordination, fractions, proportionality, geometric dilation and similarity) or do not require students' engagement with partitioning activity.

A final remark is that our distinction regarding students' partitioning activity are not to be interpreted as developmental stages, although they do imply a hierarchy of reasoning (Von Glasersfeld & Kelley, 1982). Making such claims requires research focused on several students' reasoning and posing *how* such reasoning develops and transitions over time, including how particular ways of reasoning may or may not form epistemological obstacles. We do not accomplish those goals here. Our intention is to characterize distinctive mathematical meanings of students and the extent such a distinction is explanatory for undergraduates. We call for other researchers to investigate students' partitioning activity with additional populations including the extent that our distinction does provide markers in students' mathematical development.

Notes

1. Lydia conceived her graph as displaying *both* the relationship between arc length and horizontal distance *and* the relationship between arc length and height. See Stevens and Moore (2017) for further details of Lydia's all-encompassing meaning for graphs.

2. We inferred that by increasing, decreasing, or constant "rate", Lydia was referring to a quantity's increasing, decreasing, or constant *amounts of change* with respect to implicit, experiential time.
3. This was not Lydia's first attempt to produce graphs in non-conventional Cartesian coordinate systems. In Week 3–7, we provided her with ample activities to create and compare graphs in coordinate systems with various axis orientations. See Lee et al. (20192019) for further details.
4. Emma was recalling her activity in a previous session where she worked on Circle B. Here, she mismemorized the colors of the two segments in Circle B. She used B to refer to height and R to refer to arc length, thinking height was in blue and arc length was in red.
5. We interpret Level 3 in Carlson et al.'s (2002) framework as loosely corresponding to Thompson and Carlson's (2017) chunky continuous covariation (i.e., a person "envision[ing] changes in one variable's value as happening simultaneously with changes in another variable's value"(p. 440), and envisioning both variables varying by intervals.)

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