ABSTRACTED QUANTITATIVE STRUCTURES

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Research on quantitative and covariational reasoning has emerged as a critical area of study in recent decades. We extend this body of research by introducing the construct of an abstracted quantitative structure. In addition to introducing the construct, we illustrate it by presenting empirical examples of student actions. We close with implications for research and teaching.

Keywords: Cognition, Algebra and Algebraic Thinking, Calculus, and Learning Theory

Steffe and Thompson enacted and sustained research programs that have characterized students' (and teachers') mathematical development in terms of their conceiving and reasoning about measurable or countable attributes (see Steffe & Olive, 2010; Thompson & Carlson, 2017). Thompson (1990, 2011) formalized such reasoning about measurable attributes into a system of mental actions and operations he termed *quantitative reasoning.* In this paper, we extend this work by introducing the construct *abstracted quantitative structure*: a system of quantitative relationships a person has interiorized to the extent they can operate as if it is independent of specific figurative material (i.e., representation free).

Background

Thompson (2011) defined quantitative reasoning as the mental operations involved in conceiving a situation in terms of measurable attributes, called *quantities*, and relationships between those attributes, called quantitative relationships. One form of quantitative reasoning involves constructing relationships between two quantities that vary in tandem, or covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Carlson et al. (2002), Confrey and Smith (1995), Ellis (2011), Johnson (2015a, 2015b), and Thompson and Carlson (2017) are a few researchers who have detailed frameworks and particular mental actions associated with students' covariational reasoning. We narrow our focus to that of Carlson et al. (2002), who identified five mental actions associated with covariational reasoning. A critical mental action of their framework, especially for differentiating between various function classes, is to compare *amounts of change* to draw inferences about quantities' covariation (Figure 1).

Piagetian notions of *figurative* and *operative* thought (Piaget, 2001; Steffe, 1991; Thompson, 1985) differentiate between thought based in and constrained to figurative material (e.g., perceptual objects and sensorimotor actions)—termed *figurative thought*—and thought in which figurative material is subordinate to logico-mathematical operations and possibly their transformations—termed *operative thought*. Quantitative and covariational reasoning are examples of operative thought due to their basis in logico-mathematical operations (Steffe & Olive, 2010). Moore, Stevens, Paoletti, Hobson, and Liang (online) illustrated figurative graphing meanings in which prospective secondary teachers were constrained to constructing graphs with particular perceptual features (e.g., drawing a graph solely left-to-right) even when they acknowledged those features did not enable them to viably represent a conceived relationship. In contrast, Moore et al. (online) described that a prospective secondary teacher's graphing meaning is operative when perceptual and sensorimotor features of their graphing actions are persistently dominated by the mental operations associated with quantitative and covariational reasoning.

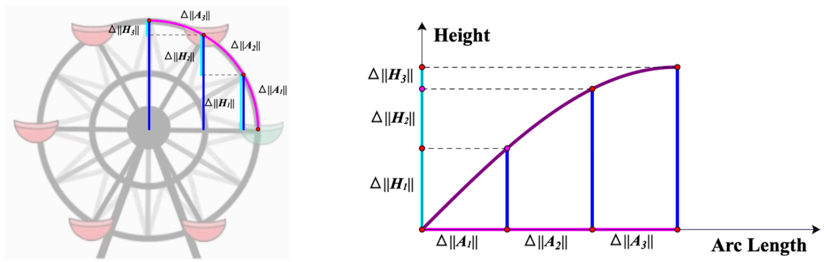

Figure 1. For equal increases in arc length (counterclockwise direction from the 3 o'clock position), height increases by decreasing amounts.

Abstracted Quantitative Structure

We define an abstracted quantitative structure as a system of quantitative relationships a person has interiorized to the extent he or she can operate as if it is independent of specific figurative material. An abstracted quantitative structure can also be re-presented to accommodate novel contexts or situations permitting the associated quantitative operations. Our notion of an abstracted quantitative structure is an extension of von Glasersfeld's (1982) definition of concept to the context of quantitative and covariational reasoning. von Glaserfeld defined a *concept* as, "any structure that has been abstracted from the process of experiential construction as recurrently usable…must be stable enough to be re-presented in the absence of perceptual 'input'" (p. 194). An abstracted quantitative structure entails both of these features; an abstracted quantitative structure is recurrently usable beyond the initial experiential construction and it can be re-presented in the absence of available perceptual (or figurative) material.

An abstracted quantitative structure can accommodate novel situations through another process of experiential construction within the context of figurative material not previously experienced in the context of using that structure. This action is a hallmark of operative thought due to the action entailing an individual using the operations of their quantitative structure to accommodate novel quantities and associated figurative material. It is in this way that the quantitative structure of an abstracted quantitative structure is abstract; mathematical properties are understood as not tied to any particular two quantities and associated figurative material.

Indications and Contraindications

The way we have defined abstracted quantitative structure presents an inherent problem in characterizing a student as having abstracted a quantitative structure: it is impossible to investigate a student's reasoning in every situation in which an abstracted quantitative structure could be relevant. For this reason, we find it necessary to discuss a student's actions in terms of indications and contraindications of her or him having constructed an abstracted quantitative structure. What follows are examples that we use to illustrate contraindications or indications of students having constructed abstracted quantitative structures.

Lydia and re-presenting. Lydia was a prospective secondary teacher in a teaching experiment focused on trigonometric relationships and re-presentation (Liang & Moore, 2018). Lydia initially engaged in a task in which she constructed incremental changes compatible with those presented in Figure 1 (left). We took her actions to indicate her reasoning quantitatively and presented her the *Which One?* task. The task (Figure 2, left) presented Lydia with numerous red segments that varied in tandem as the user varied a horizontal (blue) segment, which represented the rider's arc length traveled along the circle. We designed only one red segment to covary with the blue segment in a way compatible with the vertical height and arc length of the rider. Lydia's task was to choose which, if any, of the red segments represented that relationship. Lydia chose the correct red segment by re-orienting it vertically and checking whether the heights matched pointwise within the displayed circle (Figure 2, middle). We then asked her if the chosen segment and blue segment entailed the same covariational relationship constructed in her previous activity involving the Ferris wheel (see Figure 1, left) (from Liang & Moore, 2018):

Lydia: Not really…Um, I don't know. [*laughs*] Because that was just like something that I had seen for the first time, so I don't know if that will like show in every other case…Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.

TR: So is what we're looking at right now different than what we were looking at with the Ferris wheel?

Lydia: No. It's – No…Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don't see it here, and so –

TR: And by you "don't see it here," you mean you don't see it in that red segment?

Lydia: Yes.

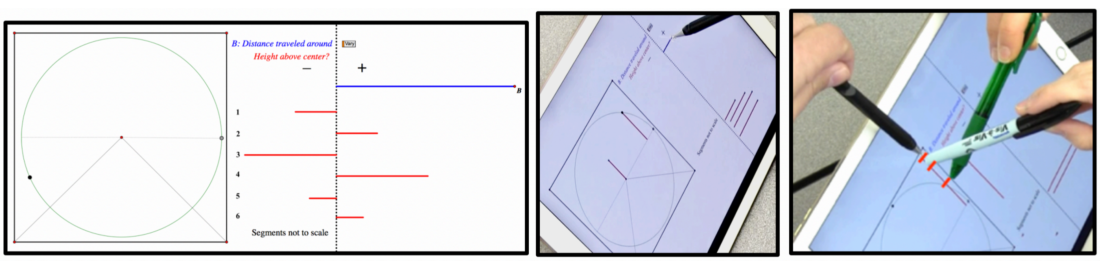

Figure 2. (left) Which One?, (middle) Lydia checking segment pointwise, and (right) Lydia attempting to re-present a quantitative structure.

Lydia's actions are a contraindication of her having constructed an abstracted quantitative structure during her previous activity. Specifically, she could not re-present the activity she engaged in with respect to the Ferris wheel situation (Figure 1, left) when provided with what was to her novel figurative material in the *Which One?* task. As a further contraindication of her having constructed an abstracted quantitative structure, it was only after much subsequent teacher-researcher guiding and their introducing perceptual material using their pens (Figure 2, right) that she was able to conceive the red and blue segments' covariation as compatible with the relationship she had constructed in the Ferris wheel situation.

Noli and the inverse sine relationship. We draw on a prospective teacher's response to the graphs in Figure 3 (left and middle). We presented a graph consistent with Figure 3 (left) to Noli, a prospective teacher, as hypothetical student work and asked whether the graph represents the inverse sine (or arcsine) function. Noli identified that Figure 3 (left) can be thought of as representing the inverse sine function by considering the vertical axis the input of the function (see Figure 3, right). In response, we presented Figure 3 (middle) and explained that a hypothetical student claimed Figure 3 (middle) represents the inverse sine function rather than Figure 3 (left). Noli claimed that both graphs could represent the inverse sine function (completed work in Figure 3, right):

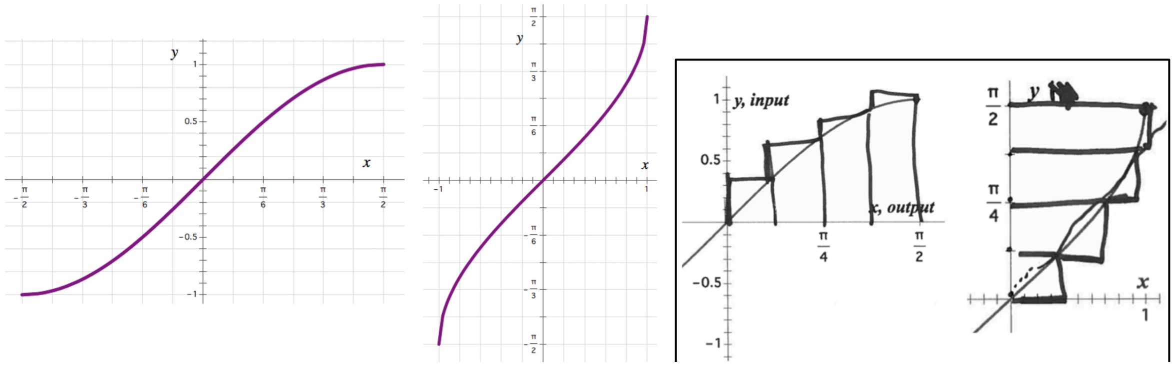

Figure 3. Graphs of (left) $x = \sin^{-1}(y)$, (middle) $y = \sin^{-1}(x)$, and (right) Noli's work.

Noli: [*Noli has identified that Figure 3, left and middle, represent $x = \sin^{-1}(y)$ and $y = \sin^{-1}(x)$, respectively*] They're both representing the same thing just considering their outputs and inputs differently [*referring to axes*]…So it's like here [*referring to Figure 3, middle, $y > 0$*], with equal changes of angle measures [*denoting equal changes along the vertical axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]…here [*referring to Figure 3, left, $x > 0$*] it's doing the exact same thing. With equal changes of angle measures [*denoting equal changes along the horizontal axis*] my vertical distance is increasing at a decreasing rate [*tracing graph*]…this one looks like it's concave up [*referring to Figure 3, middle from $0 < x < 1$*] and this one concave down [*referring to Figure 3, left from $0 < x < \pi/2$*], it's still showing the same thing.

We interpret Noli's actions as indicating her having constructed an abstracted quantitative structure that she associates with the "inverse sine function…sine function." Noli understood each graph as representing equivalent covariational properties despite their differences in shape; she understood that both "concave up" and "concave down" graphs represent one quantity increasing by decreasing amounts for equal successive variations in the other quantity. Furthermore, she flexibly moved between re-presentations of this covariational relationship, adjusting for the alternative coordinate orientations, which is a contraindication of her reasoning being dominated by figurative aspects of thought.

Implications

The construct of an abstracted quantitative structure provides a specified criterion for claims about students' and teachers' quantitative and covariational reasoning. In our previous work (e.g., Moore, 2014), we made claims about a student constructing a particular function or relationship based on her or his activity in one, or at most two, contexts. If we frame the construction of a function or relationship in terms of an abstracted quantitative structure, then our evidence within those previous studies is insufficient to make such claims. Making such claims requires studying a student's actions in a variety of contexts in which her or his actions can provide indications or contraindications of such a structure. Likewise, and transitioning our focus to teaching, it is speculative at best to claim one has taught a function or relationship concept if one has not explored their students' reasoning in more than one context and relationship.

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