



Geometry and interior nodal sets of Steklov eigenfunctions

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Abstract

We investigate the geometric properties of Steklov eigenfunctions in smooth manifolds. We derive the refined doubling estimates and Bernstein's inequalities. For the real analytic manifolds, we are able to obtain the sharp upper bound for the measure of interior nodal sets.

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1 Introduction

In this paper, we address the geometric properties and interior nodal sets of Steklov eigenfunctions

$$\begin{cases} \Delta_g e_\lambda(x) = 0, & x \in \mathcal{M}, \\ \frac{\partial e_\lambda}{\partial \nu}(x) = \lambda e_\lambda(x), & x \in \partial \mathcal{M}, \end{cases} \quad (1.1)$$

where ν is a unit outward normal on $\partial \mathcal{M}$. Assume that $(\mathcal{M}, g)$ is a n -dimensional smooth, connected and compact manifold with smooth boundary $\partial \mathcal{M}$, where $n \geq 2$. The Steklov eigenfunctions were first studied by Steklov in 1902 for bounded domains in the plane. It is also regarded as eigenfunctions of the Dirichlet-to-Neumann map, which is a first order homogeneous, self-adjoint and elliptic pseudodifferential operator. The spectrum λ_j of Steklov eigenvalue problem consists of an infinite increasing sequence with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3, \dots, \text{ and } \lim_{j \rightarrow \infty} \lambda_j = \infty.$$

The eigenfunctions $\{e_{\lambda_j}\}$ form an orthonormal basis such that

$$e_{\lambda_j} \in C^\infty(\mathcal{M}), \quad \int_{\partial \mathcal{M}} e_{\lambda_j} e_{\lambda_k} dV_g = \delta_j^k.$$

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Recently, the study of nodal geometry of eigenfunctions has been attracting much attention. Estimating the Hausdorff measure of nodal sets has always been an important subject concerning the nodal geometry of eigenfunctions. The celebrated problem about nodal sets centers around the famous Yau's conjecture for smooth manifolds. Let ϕ_λ be eigenfunctions of

$$-\Delta_g \phi_\lambda = \lambda^2 \phi_\lambda \quad (1.2)$$

on compact manifolds $(\mathcal{M}, g)$ without boundary, Yau conjectured that the upper and lower bounds of nodal sets of eigenfunctions in (1.2) satisfy

$$c\lambda \leq H^{n-1}(\{x \in \mathcal{M} | \phi_\lambda(x) = 0\}) \leq C\lambda \quad (1.3)$$

where C, c depend only on the manifold $\mathcal{M}$. The conjecture is shown to be true for real analytic manifolds by Donnelly-Fefferman in [8,10]. Lin [19] also proved the upper bound for the analytic manifolds using a different approach. Note that we use λ^2 for the eigenvalue of Laplacian eigenfunctions to reflect the order of the elliptic operator, since the Dirichlet-to-Neumann map is a first order elliptic pseudodifferential operator and Laplace operator is a second order elliptic operator.

Let us briefly review the recent literature concerning the progress of Yau's conjecture on nodal sets of Laplacian eigenfunctions (1.2). For the conjecture (1.3) on the measure of nodal sets on smooth manifolds, there are important breakthrough made by Logunov and Malinnikova [20,22] and [21] in recent years. For the upper bound of nodal sets on two dimensional manifolds, Logunov and Malinnikova [22] showed that $H^1(\{x \in \mathcal{M} | \phi_\lambda(x) = 0\}) \leq C\lambda^{\frac{3}{2}-\epsilon}$, which slightly improves the upper bound $C\lambda^{\frac{3}{2}}$ by Donnelly and Fefferman [9] and Dong [7]. For higher dimensions $n \geq 3$ on smooth manifolds, Logunov in [20] obtained a polynomial upper bound $H^{n-1}(\{x \in \mathcal{M} | \phi_\lambda(x) = 0\}) \leq C\lambda^\beta$ for some β depending on the dimension n . The polynomial upper bound improves the exponential upper bound $H^{n-1}(\{x \in \mathcal{M} | \phi_\lambda(x) = 0\}) \leq C\lambda^{C\lambda}$ derived by Hardt and Simon [17]. For the lower bound, Logunov [21] completely solved the Yau's conjecture and obtained the sharp lower bound in (1.3). For $n = 2$, such sharp lower bound was obtained earlier by Brüning [4]. This sharp lower bound improves a polynomial lower bound obtained early by Colding and Minicozzi [6], Sogge and Zelditch [30,31]. See also other polynomial lower bounds by different methods, e.g. [18,25,28] and other related results on nodal sets and geometric properties of Laplacian eigenfunctions, e.g. [5,14,16,24]. For detailed account about this subject, interested readers may refer to the book [15] and surveys [23,33].

For the Steklov eigenfunctions, by the maximum principle, there exist nodal sets in $\mathcal{M}$ and those sets intersect the boundary $\partial\mathcal{M}$ transversally. It is interesting to ask Yau's type questions about the Hausdorff measure of nodal sets of Steklov eigenfunctions on the boundary and interior of the manifolds, respectively. The natural and corresponding conjecture for Steklov eigenfunctions should state as

$$c\lambda \leq H^{n-2}(\{x \in \partial\mathcal{M} | e_\lambda(x) = 0\}) \leq C\lambda, \quad (1.4)$$

$$c\lambda \leq H^{n-1}(\{x \in \mathcal{M} | e_\lambda(x) = 0\}) \leq C\lambda. \quad (1.5)$$

See also the survey by Girouard and Polterovich in [13] about these open questions.

Recently, much work has been devoted to the bounds of nodal sets of Steklov eigenfunctions on the boundary

$$Z_\lambda = \{x \in \partial\mathcal{M} | e_\lambda(x) = 0\}. \quad (1.6)$$

The study of (1.6) was initiated by Bellova and Lin [3] who proved the $H^{n-2}(Z_\lambda) \leq C\lambda^6$ with C depending only on $\mathcal{M}$, if $\mathcal{M}$ is an analytic manifold. By microlocal analysis argument, Zelditch [34] was able to improve their results and gave the optimal upper bound $H^{n-2}(Z_\lambda) \leq C\lambda$ for analytic manifolds. For the smooth manifold $\mathcal{M}$, Wang and the author in [32] established a lower bound

$$H^{n-2}(Z_\lambda) \geq C\lambda^{\frac{4-n}{2}} \quad (1.7)$$

by considering the fact that the Steklov eigenfunctions are eigenfunctions of a first order elliptic pseudodifferential operator. The polynomial lower bound (1.7) is the Steklov analogue of the lower bounds of nodal sets for Laplacian eigenfunctions (1.2) obtained in [6] and [31].

Concerning about the bounds of interior nodal sets of eigenfunctions,

$$\mathcal{N}_\lambda = \{x \in \mathcal{M} | e_\lambda(x) = 0\}$$

Sogge, Wang and the author [29] obtained a lower bound for interior nodal sets

$$H^{n-1}(\mathcal{N}_\lambda) \geq C\lambda^{\frac{2-n}{2}}$$

for smooth manifolds $\mathcal{M}$. The measure of nodal sets is more clear on surfaces. In [36], the author was able to obtain an upper bound for the measure of interior nodal sets

$$H^1(\mathcal{N}_\lambda) \leq C\lambda^{\frac{3}{2}}.$$

On surfaces, the singular set $\mathcal{S}_\lambda = \{x \in \mathcal{M} | e_\lambda = 0, \nabla e_\lambda = 0\}$ is a finite set that consists of isolated points on the nodal curves. It was also shown that $H^0(\mathcal{S}_\lambda) \leq C\lambda^2$ in [36]. Recently, Polterovich, Sher and Toth [27] verified Yau's type conjecture for upper and lower bounds in (1.5) for the real-analytic Riemannian surfaces $\mathcal{M}$. Georgiev and Roy-fortin [12] obtained polynomial upper bounds for interior nodal sets on smooth manifolds. There are still many challenges for the study of Steklov eigenfunctions. For instance, it is well-known that the Laplacian eigenfunctions in (1.2) are so dense that there are nodal sets in each geodesic ball with radius $C\lambda^{-1}$. This fundamental result is crucial to derive the lower bounds of nodal sets for Laplacian eigenfunctions (1.2) in [8] and [4]. For the Steklov eigenfunctions, it is unknown whether such density results remain true on the boundary and interior of the manifold, which cause difficulties in studying the Steklov eigenfunctions.

An interesting topic in the study of eigenfunction is called as the doubling inequality. Doubling inequality plays an important role in deriving strong unique continuation property, the vanishing order of eigenfunctions and obtaining the measure of nodal sets, see e.g. [8, 10]. The doubling inequality for Laplacian eigenfunctions (1.2)

$$\int_{\mathbb{B}(p, 2r)} e_\lambda^2 \leq e^{C\lambda} \int_{\mathbb{B}(p, r)} e_\lambda^2 \quad (1.8)$$

is derived using Carleman estimates in [8] for $0 < r < r_0$, where $\mathbb{B}(p, c)$ denotes as a ball in $\mathcal{M}$ centered at p with radius c , and C, r_0 depends on $\mathcal{M}$. For the Steklov eigenfunctions on $\partial\mathcal{M}$, the author has obtained a similar type of doubling inequality on the boundary $\partial\mathcal{M}$ and derived that the sharp vanishing order is less than $C\lambda$ on the boundary $\partial\mathcal{M}$ in [35]. For Steklov eigenfunctions in $\mathcal{M}$, we were also able to get the doubling inequality as (1.8) in [36]. For the Laplacian eigenfunctions (1.2), a refined doubling inequality

$$\int_{\mathbb{B}(p, (1+\frac{1}{\lambda})r)} e_\lambda^2 \leq C \int_{\mathbb{B}(p, r)} e_\lambda^2 \quad (1.9)$$

was derived in [11] by some stronger Carleman estimates. The refined doubling inequality also leads to Bernstein's gradient inequalities for Laplacian eigenfunctions. The first goal in this note is to study a refined version doubling inequality for the Steklov eigenfunctions and its applications.

Theorem 1 *For the Steklov eigenfunctions in (1.1) in the smooth manifold $\mathcal{M}$, there hold*
(A): *a refined doubling inequality*

$$\int_{\mathbb{B}(p, (1+\frac{1}{\lambda})r)} e_{\lambda}^2 \leq C \int_{\mathbb{B}(p, r)} e_{\lambda}^2,$$

(B): L^2 -Bernstein's inequality

$$\int_{\mathbb{B}(p, r)} |\nabla e_{\lambda}|^2 \leq \frac{C\lambda^2}{r^2} \int_{\mathbb{B}(p, r)} e_{\lambda}^2,$$

(C) L^{∞} -Bernstein's inequality

$$\max_{\mathbb{B}(p, r)} |\nabla e_{\lambda}| \leq \frac{C\lambda^{\frac{n+2}{2}}}{r} \max_{\mathbb{B}(p, r)} |e_{\lambda}|$$

for $\mathbb{B}(p, (1 + \frac{1}{\lambda})r) \subset \mathcal{M}$ and $0 < r < r_0$, where r_0 depends on $\mathcal{M}$.

Our second goal is to obtain the optimal upper bound of interior nodal sets of Steklov eigenfunctions for real analytic manifolds. Our work extends the optimal upper bound in [27] to real analytic manifolds in any dimensions, which proves the upper bound of Yau's type conjecture for interior nodal sets in (1.5).

Theorem 2 *Let $\mathcal{M}$ be the real analytic compact and connected manifold with analytic boundary. There exists a positive constant C depending on $\mathcal{M}$ such that,*

$$H^{n-1}(\mathcal{N}_{\lambda}) \leq C\lambda$$

for the Steklov eigenfunctions.

The outline of the paper is as follows. In Sect. 2, we reduce the Steklov eigenvalue problem into an equivalent elliptic equation without boundary. Then we obtain the refined doubling inequality and show Theorem 1. Section 3 is devoted to the upper bound of interior nodal sets for real analytic manifolds. Section 4 is the "Appendix" which provides the proof of some arguments for the Carleman estimates. The letter c , C , C_i denote generic positive constants and do not depend on λ . They may vary in different lines and sections. In the paper, since we study the asymptotic properties for eigenfunctions, we assume that λ is sufficiently large.

2 Refined doubling inequality

In this section, we will establish a stronger Carleman estimate than that in [35]. We will transform the Steklov eigenvalue problem into a second order elliptic equation on a boundaryless manifold. The eigenvalue λ will be reflected in the coefficients of the elliptic equation.

To make the Steklov eigenvalue problem into an elliptic equation, adapting the ideas in [3], we choose an auxiliary function involving the distance function. Let $d(x) = \text{dist}\{x, \partial\mathcal{M}\}$ be the distance function from $x \in \mathcal{M}$ to the boundary $\partial\mathcal{M}$. If $\mathcal{M}$ is smooth, $d(x)$ is smooth

in the small neighborhood $\mathcal{M}_\rho^0$ of $\partial\mathcal{M}$ in $\mathcal{M}$, where $\mathcal{M}_\rho^0 = \{x \in \mathcal{M} | \text{dist}\{x, \partial\mathcal{M}\} \leq \rho\}$. By the partition of unity, we extend $d(x)$ in a smooth manner by introducing

$$\varrho(x) = \begin{cases} d(x) & x \in \mathcal{M}_\rho^0, \\ l(x) & x \in \mathcal{M} \setminus \mathcal{M}_\rho^0, \end{cases}$$

where the smooth function $l(x)$ is a smooth extension of $d(x)$ in $\mathcal{M} \setminus \mathcal{M}_\rho^0$. Therefore, the extended function $\varrho(x)$ is a smooth function in $\mathcal{M}$. We consider an auxiliary function

$$u(x) = e_\lambda \exp\{\lambda \varrho(x)\}.$$

Then the new function $u(x)$ satisfies

$$\begin{cases} \Delta_g u + b(x) \cdot \nabla_g u + q(x)u = 0 & \text{in } \mathcal{M}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{M} \end{cases} \quad (2.1)$$

with

$$\begin{cases} b(x) = -2\lambda \nabla_g \varrho(x), \\ q(x) = \lambda^2 |\nabla_g \varrho(x)|^2 - \lambda \Delta_g \varrho(x). \end{cases} \quad (2.2)$$

In order to construct a boundaryless model, we attach two copies of $\mathcal{M}$ along the boundary and consider a double manifold $\overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}$. Then induced metric g' of g on the double manifold $\overline{\mathcal{M}}$ is Lipschitz. We consider a canonical involutive isometry $\mathcal{F} : \overline{\mathcal{M}} \rightarrow \overline{\mathcal{M}}$ which interchanges the two copies of $\mathcal{M}$. In this sense, the function $u(x)$ can be extended to the double manifold by a even extension as $\overline{\mathcal{M}}$ by $u \circ \mathcal{F} = u$. Thus, $u(x)$ satisfies

$$\Delta_{g'} u + \bar{b}(x) \cdot \nabla_{g'} u + \bar{q}(x)u = 0 \quad \text{in } \overline{\mathcal{M}}. \quad (2.3)$$

Note that the new metric g' is Lipschitz metric. From the assumptions in (2.2) and the even extension, it follows that

$$\begin{cases} \|\bar{b}\|_{W^{1,\infty}(\overline{\mathcal{M}})} \leq C\lambda, \\ \|\bar{q}\|_{W^{1,\infty}(\overline{\mathcal{M}})} \leq C\lambda^2. \end{cases} \quad (2.4)$$

By a standard regularity argument for dealing with Lipschitz metrics in [10] and [1], we have established some quantitative Carleman inequality in [35] for the general second order elliptic equation (2.3). See also e.g. [2] for similar estimates for smooth manifolds. The quantitative Carleman estimate inequality is stated as follows.

Lemma 1 *There exist positive constants ϵ_0 and C such that for any $u \in C_0^\infty(\mathbb{B}(p, \epsilon_0) \setminus \mathbb{B}(p, \epsilon_1))$, and $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$, one has*

$$\int r^4 e^{2\beta\psi(r)} |\Delta_{g'} u + \bar{b} \cdot \nabla_{g'} u + \bar{q}u|^2 d\text{vol} \geq C\beta^3 \int r^\epsilon e^{2\beta\psi(r)} u^2 d\text{vol}, \quad (2.5)$$

where $\psi(r) = -\ln r(x) + r^\epsilon(x)$, $r(x)$ is the geodesic distance from x to p , and $0 < \epsilon < 1$ is some fixed constant.

Since some arguments are used in the proof of Proposition 2 later, we include the major arguments of the proof of Lemma 1 in the ‘‘Appendix’’. By the Carleman estimates in Lemma 1, we can derive a Hadamard’s three-ball theorem. Based on a propagation of smallness argument, we have obtained the following doubling inequality in $\overline{\mathcal{M}}$ in [36].

Proposition 1 *There exist positive constants r_0 and C depending only on $\overline{\mathcal{M}}$ such that for any $0 < r < r_0$ and any $p \in \overline{\mathcal{M}}$, there holds*

$$\|u\|_{L^2(\mathbb{B}(p, 2r))} \leq e^{C\lambda} \|u\|_{L^2(\mathbb{B}(p, r))} \quad (2.6)$$

for any solutions of (2.3).

From the proposition, it is easy to see that the doubling inequality for Steklov eigenfunctions as (1.8) holds in $\mathcal{M}$ if $\mathbb{B}(p, 2r) \subset \mathcal{M}$, since $\varrho(x)$ is a bounded function. By standard elliptic estimates, the L^∞ norm of doubling inequality

$$\|u\|_{L^\infty(\mathbb{B}(p, 2r))} \leq e^{C\lambda} \|u\|_{L^\infty(\mathbb{B}(p, r))} \quad (2.7)$$

holds, which also implies that

$$\|e_\lambda\|_{L^\infty(\mathbb{B}(p, 2r))} \leq e^{C\lambda} \|e_\lambda\|_{L^\infty(\mathbb{B}(p, r))}.$$

Next we will establish a stronger Carleman inequality than that in Lemma 1 with weight function $\exp\{\beta\psi(x)\}$ following from [11], where the function ψ satisfies some convexity properties. Choosing a fixed number ϵ such that $0 < \epsilon < 1$ and $T_0 < 0$, we define the function ϕ on $(-\infty, T_0]$ by $\phi(t) = t - e^{\epsilon t}$. If $|T_0|$ is sufficiently large, the function $\phi(t)$ satisfies the following properties

$$1 - \epsilon e^{\epsilon T_0} \leq \phi'(t) \leq 1, \quad (2.8)$$

$$\lim_{t \rightarrow -\infty} \frac{-\phi''(t)}{e^t} = +\infty. \quad (2.9)$$

Let $\psi(x) = -\phi(\ln r(x))$, where $r(x) = d(x, p)$ is geodesic distance between x and p . The stronger Carleman estimate is stated as follows.

Proposition 2 *There exist positive constants h , C_0 and C such that for any $u \in C_0^\infty(\mathbb{B}(p, h) \setminus \mathbb{B}(p, \delta))$, and $\beta > C_0(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$, one has*

$$\begin{aligned} \int_{\mathbb{B}(p, h)} r^4 e^{2\beta\psi} |\Delta u + \bar{b} \cdot \nabla u + \bar{q}u|^2 dvol &\geq C\beta^3 \int_{\mathbb{B}(p, h)} r^\epsilon e^{2\beta\psi} u^2 dvol \\ &+ C\beta^4 \int_{\mathbb{B}(p, \delta(1+\frac{C}{\beta}))} e^{2\beta\psi} u^2 dvol. \end{aligned} \quad (2.10)$$

Proof By the standard argument in dealing with Lipschitz Riemannian manifold in [10] and [1], using a conformal change, we can still use polar geodesic coordinates (r, ω) . The change only results in the change of C in the norm estimates of coefficient functions in (2.4). The geodesic balls are comparable under the conformal change of the different metrics. For simplicity, we still keep the notations in (2.3). We introduce the polar geodesic coordinates (r, ω) near p . Following the Einstein notation, for any $v \in C^\infty$, we denote the Laplace-Beltrami operator as

$$r^2 \Delta v = r^2 \partial_r^2 v + r^2 \left(\partial_r \ln(\sqrt{\gamma}) + \frac{n-1}{r} \right) \partial_r v + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j v),$$

where $\partial_i = \frac{\partial}{\partial \omega_i}$ and $\gamma_{ij}(r, \omega)$ is a metric on S^{n-1} , $\gamma = \det(\gamma_{ij})$. One can check that, for r small enough,

$$\begin{cases} \partial_r(\gamma_{ij}) \leq C(\gamma_{ij}) \text{ in term of tensors,} \\ |\partial_r(\gamma)| \leq C, \\ C^{-1} \leq \gamma \leq C, \end{cases} \quad (2.11)$$

where C depends on $\mathcal{M}$. Set a new coordinate as $\ln r = t$. Using this new coordinate,

$$e^{2t} \Delta v = \partial_t^2 v + (n - 2 + \partial_t \ln \sqrt{\gamma}) \partial_t v + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j v) \quad (2.12)$$

and

$$e^{2t} \bar{b} = e^{2t} \bar{b}_t \partial_t + e^{2t} \bar{b}_i \partial_i.$$

Since u is supported in a small neighborhood, u is supported in $(-\infty, T_0) \times S^{n-1}$ with $T_0 < 0$ and $|T_0|$ large enough. Under this new coordinate, the condition (2.11) becomes

$$\begin{cases} \partial_t (\gamma_{ij}) \leq C e^t (\gamma_{ij}) & \text{in term of tensors,} \\ |\partial_t (\gamma)| \leq C e^t, \\ C^{-1} \leq \gamma \leq C. \end{cases} \quad (2.13)$$

Let

$$u = e^{-\beta \psi(x)} v.$$

Define the conjugate operator,

$$\begin{aligned} \mathcal{L}_\beta(v) &= r^2 e^{\beta \psi(x)} \Delta (e^{-\beta \psi(x)} v) + r^2 e^{\beta \psi(x)} \bar{b} \cdot \nabla (e^{-\beta \psi(x)} v) + r^2 \bar{q} v \\ &= e^{2t} e^{-\beta \phi(t)} \Delta (e^{\beta \phi(t)} v) + e^{2t} e^{-\beta \phi(t)} \bar{b} \cdot \nabla (e^{\beta \phi(t)} v) + e^{2t} \bar{q} v. \end{aligned} \quad (2.14)$$

From (2.12), straightforward calculations show that

$$\begin{aligned} \mathcal{L}_\beta(v) &= \partial_t^2 v + (2\beta \phi' + e^{2t} \bar{b}_t + (n - 2) + \partial_t \ln \sqrt{\gamma}) \partial_t v + e^{2t} \bar{b}_i \partial_i v \\ &\quad + (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + \beta \phi'' + (n - 2) \beta \phi' + \beta \partial_t \ln \sqrt{\gamma} \phi') v + \Delta_\omega v + e^{2t} \bar{q} v, \end{aligned} \quad (2.15)$$

where

$$\Delta_\omega v = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j v).$$

We will work in the following L^2 norm

$$\|v\|_\phi^2 = \int_{(-\infty, T_0] \times S^{n-1}} |v|^2 \sqrt{\gamma} \phi'^{-3} dt d\omega,$$

where $d\omega$ is measure on S^{n-1} . By the triangle inequality, we have

$$\|\mathcal{L}_\beta(v)\|_\phi^2 \geq \frac{1}{2} \mathcal{A} - \mathcal{B},$$

where

$$\begin{aligned} \mathcal{A} &= \|\partial_t^2 v + \Delta_\omega v + (2\beta \phi' + e^{2t} \bar{b}_t) \partial_t v + e^{2t} \bar{b}_i \partial_i v \\ &\quad + (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + (n - 2) \beta \phi' + e^{2t} \bar{q}) v\|_\phi^2 \end{aligned} \quad (2.16)$$

and

$$\mathcal{B} = \|\beta \phi'' v + \beta \partial_t \ln \sqrt{\gamma} \phi' v + (n - 2) \partial_t v + \partial_t \ln \sqrt{\gamma} \partial_t v\|_\phi^2. \quad (2.17)$$

By integration by parts argument, we can absorb $\mathcal{B}$ into $\mathcal{A}$. It holds that

$$\|\mathcal{L}_\beta(v)\|_\phi^2 \geq \frac{1}{4} \mathcal{A}. \quad (2.18)$$

We can also obtain a lower bound for $\mathcal{A}$,

$$\begin{aligned} C\mathcal{A} \geq & \beta^3 \int |\phi''| |v|^2 \phi'^{-3} \sqrt{\gamma} dt d\omega + \beta \int |\phi''| |D_\omega v|^2 \phi'^{-3} \sqrt{\gamma} dt d\omega \\ & + \beta \int |\partial_t v|^2 \phi'^{-3} \sqrt{\gamma} dt d\omega, \end{aligned} \quad (2.19)$$

where $|D_\omega v|^2$ stands for

$$|D_\omega v|^2 = \gamma^{ij} \partial_i v \partial_j v.$$

For the completeness of the presentation, we include the proof of (2.18) and (2.19) in the “Appendix”.

We also want to find another refined lower bound for $\mathcal{A}$. We write $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4, \quad (2.20)$$

where

$$\mathcal{A}_1 = \|\partial_t^2 v + (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q})v + \Delta_\omega v\|_\phi^2$$

and

$$\mathcal{A}_2 = \|(2\beta \phi' + e^{2t} \bar{b}_t) \partial_t v + e^{2t} \bar{b}_i \partial_i v + \beta g v\|_\phi^2$$

and

$$\begin{aligned} \mathcal{A}_3 = & 2 < \partial_t^2 v + (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q})v + \Delta_\omega v - \beta g v, \\ & (2\beta \phi' + e^{2t} \bar{b}_t) \partial_t v + e^{2t} \bar{b}_i \partial_i v >_\phi \end{aligned}$$

and

$$\mathcal{A}_4 = -\beta^2 \|g v\|_\phi^2,$$

and $g(t)$ is a function to be determined. We continue to break $\mathcal{A}_3$ down as

$$\mathcal{A}_3 = \mathcal{I}_1 + \mathcal{I}_2, \quad (2.21)$$

where

$$\begin{aligned} \mathcal{I}_1 = & 2 < \partial_t^2 v + (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q})v + \Delta_\omega v, \\ & (2\beta \phi' + e^{2t} \bar{b}_t) \partial_t v + e^{2t} \bar{b}_i \partial_i v >_\phi \end{aligned}$$

and

$$\mathcal{I}_2 = -2 < \beta g v, (2\beta \phi' + e^{2t} \bar{b}_t) \partial_t v + e^{2t} \bar{b}_i \partial_i v >_\phi.$$

Performing the integration by part arguments shows that

$$\begin{aligned} \mathcal{I}_1 \geq & 3\beta \int |\phi''| |D_\omega v|^2 \phi'^{-3} \sqrt{\gamma} dt d\omega - c\beta^3 \int e^t |v|^2 \phi'^{-3} \sqrt{\gamma} dt d\omega \\ & - c\beta \int |\phi''| |\partial_t v|^2 \phi'^{-3} \sqrt{\gamma} dt d\omega - c\beta^2 \int |\phi''| |v|^2 \phi'^{-3} \sqrt{\gamma} dt d\omega. \end{aligned} \quad (2.22)$$

From (2.19) and (2.22), it follows that

$$\mathcal{I}_1 + C'\mathcal{A} \geq 0$$

for some positive constant C' . That is,

$$\mathcal{I}_1 \geq -C' \mathcal{A}. \quad (2.23)$$

We compute $\mathcal{I}_2$. Applying the integrating by parts gives that

$$\begin{aligned} \mathcal{I}_2 = & \int \beta g'(2\beta\phi' + e^{2t}\bar{b}_t)v^2\phi'^{-3}\sqrt{\gamma} dt d\omega \\ & + \int \beta g(2\beta\phi'' + 2e^{2t}\bar{b}_t + e^{2t}\partial_t\bar{b}_t)v^2\phi'^{-3}\sqrt{\gamma} dt d\omega \\ & - \int 3\beta g(2\beta\phi' + e^{2t}\bar{b}_t)\phi'^{-1}\phi''v^2\phi'^{-3}\sqrt{\gamma} dt d\omega \\ & + \int \beta g(2\beta\phi' + e^{2t}\bar{b}_t)\partial_t \ln \sqrt{\gamma} v^2\phi'^{-3}\sqrt{\gamma} dt d\omega \\ & + \int \beta g e^{2t}(\partial_i\bar{b}_i + \bar{b}_i\partial_i \ln \sqrt{\gamma})v^2\phi'^{-3}\sqrt{\gamma} dt d\omega. \end{aligned}$$

Combining terms in the later identity yields that

$$\begin{aligned} \mathcal{I}_2 = & \beta^2 \int \left\{ g'(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}) + g\left(2\phi'' + \frac{2e^{2t}\bar{b}_t + e^{2t}\partial_t\bar{b}_t}{\beta} - 6\phi'' - 3\frac{e^{2t}\bar{b}_t\phi'^{-1}\phi''}{\beta} \right. \right. \\ & \left. \left. + (2\phi' + \frac{e^{2t}\bar{b}_t}{\beta})\partial_t \ln \sqrt{\gamma} + \frac{e^{2t}\partial_i\bar{b}_i + e^{2t}\bar{b}_i\partial_i \ln \sqrt{\gamma}}{\beta} \right) \right\} v^2\phi'^{-3}\sqrt{\gamma} dt d\omega. \end{aligned} \quad (2.24)$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are nonnegative, from (2.20), (2.21) and (2.23), we have

$$\begin{aligned} \mathcal{A} & \geq \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{A}_4 \\ & \geq -C' \mathcal{A} + \mathcal{I}_2 + \mathcal{A}_4. \end{aligned}$$

By (2.24), we have a lower bound of $\mathcal{A}$ as

$$\begin{aligned} C\mathcal{A} \geq & \beta^2 \int \left\{ \left[g'\left(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}\right) + g\left(\frac{2e^{2t}\bar{b}_t + e^{2t}\partial_t\bar{b}_t}{\beta} - 4\phi'' - 3\frac{e^{2t}\bar{b}_t\phi'^{-1}\phi''}{\beta} \right. \right. \right. \\ & \left. \left. + \left(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}\right)\partial_t \ln \sqrt{\gamma} + \frac{e^{2t}\partial_i\bar{b}_i + e^{2t}\bar{b}_i\partial_i \ln \sqrt{\gamma}}{\beta} \right) \right] - g^2 \right\} v^2\phi'^{-3}\sqrt{\gamma} dt d\omega. \end{aligned} \quad (2.25)$$

From the assumption (2.8), we know ϕ' is close to 1 as $|T_0|$ is sufficiently large. By the assumption of $\bar{b}$ and the condition $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$, it is clear that $|\frac{e^{2t}\bar{b}_t}{\beta}|$ is small. Thus, the condition

$$2\phi' + \frac{e^{2t}\bar{b}_t}{\beta} > 0$$

holds. Let

$$\begin{aligned} g'\left(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}\right) + g\left(\frac{2e^{2t}\bar{b}_t + e^{2t}\partial_t\bar{b}_t}{\beta} - 4\phi'' - 3\frac{e^{2t}\bar{b}_t\phi'^{-1}\phi''}{\beta} + \left(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}\right)\partial_t \ln \sqrt{\gamma} \right. \\ \left. + \frac{e^{2t}\partial_i\bar{b}_i + e^{2t}\bar{b}_i\partial_i \ln \sqrt{\gamma}}{\beta} \right) - g^2 = \beta^2 \left(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta}\right) \varphi(\beta(t - t_*)), \end{aligned} \quad (2.26)$$

where $\varphi(t) = 0$ for $t \geq 0$, $\varphi(t) > 0$ for $t < 0$, and $|t_*|$ is an arbitrary large number with $t_* < 0$. We attempt to solve (2.26) with $g = 0$ for $t \geq t_*$. Making the change of rescale, we have

$$g = \beta G, \quad z = \beta(t - t_*).$$

Then (2.26) is transformed into an equation of the form

$$\begin{cases} \frac{\partial G}{\partial z} = H_1(z) + H_2(z)G + H_3(z)G^2, \\ G(0) = 0, \end{cases}$$

with H_1 , H_2 and H_3 are uniformly bounded in C^2 . Standard existence theorem from ordinary differential equations shows a solution to (2.26) for $-C_1 \leq \beta(t - t_*) \leq 0$ with a fixed small positive constant C_1 . Then (2.26) can be solved for $\frac{-C_1}{\beta} + t_* \leq t \leq t_*$. If we assume that $\text{supp } v \subset \{\frac{-C_1}{\beta} + t_* \leq t \leq T_0\}$ with $T_0 < 0$, then (2.25) implies that

$$CA \geq \beta^4 \int \left(2\phi' + \frac{e^{2t}\bar{b}_t}{\beta} \right) \varphi(\beta(t - t_*)) v^2 \phi'^{-3} \sqrt{\gamma} dt d\omega. \quad (2.27)$$

There exist $0 < -T_0 < C_2 < C_3 < C_1$ such that

$$\varphi(z) > C_4 \quad \text{for } -C_3 < z < -C_2$$

and C_4 depends on C_2, C_3 . It follows from the last inequality that

$$CA \geq C_4 \beta^4 \int_{t_* - \frac{C_3}{\beta} < t < t_* - \frac{C_2}{\beta}} v^2 \phi'^{-3} \sqrt{\gamma} dt d\omega. \quad (2.28)$$

Since $r = e^t$ and recall that $u = e^{-\beta\psi(x)}v$, the previous estimates yield that

$$A \geq C_5 \beta^4 \int_{t_* - \frac{C_3}{\beta} < \ln r < t_* - \frac{C_2}{\beta}} e^{2\beta\psi(x)} u^2 r^{-1} \phi'^{-3} \sqrt{\gamma} dr d\omega. \quad (2.29)$$

Set $e^{t_*} = r_*$. If $r_* \exp\{-\frac{C_3}{\beta}\} < r < r_* \exp\{-\frac{C_2}{\beta}\}$, there exist positive constants C_6 and C_7 such that $r_*(1 - \frac{C_6}{\beta}) < r < r_*(1 - \frac{C_7}{\beta})$. Recall the estimates (2.18), it follows that

$$\|\mathcal{L}_\beta(v)\|_\phi^2 \geq C_5 \beta^4 \int_{r_*(1 - \frac{C_6}{\beta}) < r < r_*(1 - \frac{C_7}{\beta})} e^{2\beta\psi(x)} u^2 r^{-1} \phi'^{-3} \sqrt{\gamma} dr d\omega. \quad (2.30)$$

Note that ϕ' is close to 1, we have

$$\|\mathcal{L}_\beta(v)\|^2 \geq C_5 \beta^4 \int_{r_*(1 - \frac{C_6}{\beta}) < r < r_*(1 - \frac{C_7}{\beta})} e^{2\beta\psi(x)} u^2 d\text{vol} \quad (2.31)$$

by a constant change of the value of β . Since $u \in C_0^\infty(\mathbb{B}(p, h) \setminus \mathbb{B}(p, \delta))$, choosing $r_* = \frac{\delta}{1 - \frac{C_6}{\beta}}$, we have

$$\|r^2 e^{\beta\psi} |\Delta u + \bar{b} \cdot \nabla u + \bar{q}u|\|^2 \geq C_5 \beta^4 \int_{\delta < r < \delta(1 + \frac{C_8}{\beta})} e^{2\beta\psi(x)} u^2 d\text{vol}. \quad (2.32)$$

From Lemma 1, we have established that

$$\|r^2 e^{\beta\psi} |\Delta u + \bar{b} \cdot \nabla u + \bar{q}u|\| \geq C_9 \beta^{\frac{3}{2}} \|r^{\frac{\epsilon}{2}} e^{\beta\psi} u\|. \quad (2.33)$$

Combining those two Carleman inequalities (2.32) and (2.33) yields that

$$\begin{aligned} \int_{\mathbb{B}(p,h)} r^4 e^{2\beta\psi} |\Delta u + \bar{b} \cdot \nabla u + \bar{q}u|^2 d\text{vol} &\geq C\beta^3 \int_{\mathbb{B}(p,h)} r^\epsilon e^{2\beta\psi} u^2 d\text{vol} \\ &\quad + C\beta^4 \int_{\mathbb{B}\left(p,\delta\left(1+\frac{C_8}{\beta}\right)\right)} e^{2\beta\psi} u^2 d\text{vol} \end{aligned} \quad (2.34)$$

for $u \in C_0^\infty(\mathbb{B}(p,h) \setminus \mathbb{B}(p,\delta))$ and $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}})$. $\square$

With aid of the Carleman estimates (2.34), we are in the position to give the proof of Theorem 1. The refined doubling inequality and Bernstein's inequalities have been obtained for Laplacian eigenfunctions in [11].

Proof of Theorem 1 We introduce a cut-off function $\theta(x) \in C_0^\infty(\mathbb{B}(p,h) \setminus \mathbb{B}(p,\delta))$ satisfying the following properties:

- (i): $\theta = 1$ in $\mathbb{B}(p, \frac{h}{2}) \setminus \mathbb{B}(p, \delta + \frac{C_8}{10\beta})$,
- (ii): $|\nabla\theta| \leq \frac{C\beta}{\delta}$, $|\Delta\theta| \leq \frac{C\beta^2}{\delta^2}$ in $\mathbb{B}(p, \delta + \frac{C_8}{10\beta})$,
- (iii): $|\nabla\theta| \leq C$ in $\mathbb{B}(p,h) \setminus \mathbb{B}(p, \frac{h}{2})$.

Let $w(x) = \theta(x)u(x)$. Since u satisfies

$$\Delta u + \bar{b} \cdot \nabla u + \bar{q}u = 0,$$

then w satisfies

$$\Delta w + \bar{b} \cdot \nabla w + \bar{q}w = \Delta\theta u + 2\nabla\theta \cdot \nabla u + \bar{b} \cdot \nabla\theta u.$$

Substituting w into the left hand side of the stronger inequality (2.34) and calculating its integrals gives that

$$\begin{aligned} &\int_{(\mathbb{B}(p,h) \setminus \mathbb{B}(p, \frac{h}{2})) \cup (\mathbb{B}(p, \delta + \frac{C_8}{10\beta}) \setminus \mathbb{B}(p, \delta))} r^4 e^{2\beta\psi} |\Delta\theta u + 2\nabla\theta \cdot \nabla u + \bar{b} \cdot \nabla\theta u|^2 \\ &\leq C\beta^2 \int_{\mathbb{B}(p,h) \setminus \mathbb{B}(p, \frac{h}{2})} r^4 e^{2\beta\psi} (u^2 + |\nabla u|^2) \\ &\quad + C \int_{\mathbb{B}(p, \delta + \frac{C_8}{10\beta}) \setminus \mathbb{B}(p, \delta)} r^4 e^{2\beta\psi} \left(\frac{\beta^4}{\delta^4} u^2 + \frac{\beta^2}{\delta^2} |\nabla u|^2 + \frac{\beta^4}{\delta^2} u^2 \right), \end{aligned}$$

where we have used the assumption for $\bar{b}$ and $\bar{q}$ in (2.4) and the assumption $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}})^{1/2}$.

Substituting w into the right hand side of (2.34) and taking the later inequality into consideration yields that

$$\begin{aligned} &C \int_{\mathbb{B}(p,h) \setminus \mathbb{B}(p, \frac{h}{2})} r^4 e^{2\beta\psi} (u^2 + |\nabla u|^2) + C \int_{\mathbb{B}(p, \delta + \frac{C_8}{10\beta}) \setminus \mathbb{B}(p, \delta)} r^4 e^{2\beta\psi} \left(\frac{\beta^2}{\delta^4} u^2 + \frac{1}{\delta^2} |\nabla u|^2 \right) \\ &\geq \beta \int_{\mathbb{B}(p, \frac{h}{2}) \setminus \mathbb{B}(p, \delta + \frac{C_8}{10\beta})} r^\epsilon e^{2\beta\psi} u^2 + \beta^2 \int_{\mathbb{B}(p, \delta + \frac{C_8}{\beta}) \setminus \mathbb{B}(p, \delta + \frac{C_8}{10\beta})} e^{2\beta\psi} u^2. \end{aligned} \quad (2.35)$$

Using the fact that ψ is a decreasing function and the standard elliptic estimates, we have

$$\begin{aligned} \int_{\mathbb{B}(p,h) \setminus \mathbb{B}(p, \frac{h}{2})} r^4 e^{2\beta\psi} |\nabla u|^2 &\leq C h^4 e^{2\beta\psi(\frac{h}{2})} \int_{\mathbb{B}(p,h) \setminus \mathbb{B}(p, \frac{h}{2})} |\nabla u|^2 \\ &\leq C \lambda^2 h^2 e^{2\beta\psi(\frac{h}{2})} \int_{\mathbb{B}(p, \frac{5h}{4}) \setminus \mathbb{B}(p, \frac{h}{4})} u^2. \end{aligned} \quad (2.36)$$

Thus,

$$C \int_{\mathbb{B}(p,h) \setminus \mathbb{B}(p, \frac{h}{2})} r^4 e^{2\beta\psi} (u^2 + |\nabla u|^2) \leq C \lambda^2 h^2 e^{2\beta\psi(\frac{h}{2})} \int_{\mathbb{B}(p, \frac{5h}{4}) \setminus \mathbb{B}(p, \frac{h}{4})} u^2. \quad (2.37)$$

By the decreasing property of ψ , we have

$$\begin{aligned} \beta \int_{\mathbb{B}(p, \frac{h}{2}) \setminus \mathbb{B}(p, \delta + \frac{C\delta}{10\beta})} r^\epsilon e^{2\beta\psi} u^2 &\geq \beta \int_{\mathbb{B}(p, \frac{h}{10}) \setminus \mathbb{B}(p, \frac{h}{20})} r^\epsilon e^{2\beta\psi} u^2 \\ &\geq \beta h^\epsilon e^{2\beta\psi(\frac{h}{10})} \int_{\mathbb{B}(p, \frac{h}{10}) \setminus \mathbb{B}(p, \frac{h}{20})} u^2. \end{aligned} \quad (2.38)$$

From the doubling inequality in [36], we learn that

$$e^{C\lambda} \int_{\mathbb{B}(p, \frac{h}{10}) \setminus \mathbb{B}(p, \frac{h}{20})} u^2 \geq \int_{\mathbb{B}(p, \frac{5h}{4}) \setminus \mathbb{B}(p, \frac{h}{4})} u^2 \quad (2.39)$$

for some C depending on $\mathcal{M}$. If we choose $\beta > C_0\lambda$ for some large constant C_0 , from (2.37) and (2.38), we arrive at

$$\beta \int_{\mathbb{B}(p, \frac{h}{2}) \setminus \mathbb{B}(p, \delta + \frac{C\delta}{10\beta})} r^\epsilon e^{2\beta\psi} u^2 \geq C \int_{\mathbb{B}(p,h) \setminus \mathbb{B}(p, \frac{h}{2})} r^4 e^{2\beta\psi} (|\nabla u|^2 + u^2). \quad (2.40)$$

The combination of (2.35) and (2.40) yields that

$$\int_{\mathbb{B}(p, \delta + \frac{C\delta}{10\beta}) \setminus \mathbb{B}(p, \delta)} r^4 e^{2\beta\psi} \left(\frac{\beta^2}{\delta^4} u^2 + \frac{1}{\delta^2} |\nabla u|^2 \right) \geq \beta^2 \int_{\mathbb{B}(p, \delta + \frac{C\delta}{\beta}) \setminus \mathbb{B}(p, \delta + \frac{C\delta}{10\beta})} e^{2\beta\psi} u^2. \quad (2.41)$$

We continue to simplify the last inequality,

$$\begin{aligned} (\delta + \frac{C\delta}{10\beta})^4 e^{2\beta\psi(\delta)} \frac{\beta^2}{\delta^4} \int_{\mathbb{B}(p, \delta + \frac{C\delta}{10\beta}) \setminus \mathbb{B}(p, \delta)} u^2 &+ (\delta + \frac{C\delta}{10\beta})^4 e^{2\beta\psi(\delta)} \frac{1}{\delta^2} \int_{\mathbb{B}(p, \delta + \frac{C\delta}{10\beta}) \setminus \mathbb{B}(p, \delta)} |\nabla u|^2 \\ &\geq \beta^2 e^{2\beta\psi(\delta + \frac{C\delta}{\beta})} \int_{\mathbb{B}(p, \delta + \frac{C\delta}{\beta}) \setminus \mathbb{B}(p, \delta + \frac{C\delta}{10\beta})} u^2. \end{aligned} \quad (2.42)$$

From the explicit form of $\psi(x)$, there exists some small positive constant c such that

$$\exp\{2\beta\psi(\delta + \frac{C\delta}{\beta}) - 2\beta\psi(\delta)\} > c$$

for β large enough. Thus,

$$\frac{\beta^2}{\delta^2} \int_{\mathbb{B}(p, \delta + \frac{C\delta}{10\beta}) \setminus \mathbb{B}(p, \delta)} u^2 + \int_{\mathbb{B}(p, \delta + \frac{C\delta}{10\beta}) \setminus \mathbb{B}(p, \delta)} |\nabla u|^2 \geq c \frac{\beta^2}{\delta^2} \int_{\mathbb{B}(p, \delta + \frac{C\delta}{\beta}) \setminus \mathbb{B}(p, \delta + \frac{C\delta}{10\beta})} u^2. \quad (2.43)$$

Let

$$\frac{C\delta}{10\beta} \leq \lambda^{-1}.$$

Since u satisfies (2.3), standard elliptic theory yields that

$$|\nabla u(x)|^2 \leq C \left(\frac{\beta}{\delta} \right)^{n+2} \int_{y \in \mathbb{B}\left(x, \frac{C\delta}{10\beta}\right)} u^2(y) dy. \quad (2.44)$$

We integrate last inequality for $x \in \mathbb{B}\left(p, \delta + \frac{C\delta}{10\beta}\right) \setminus \mathbb{B}(p, \delta)$. It follows that

$$\begin{aligned} \int_{\mathbb{B}\left(p, \delta + \frac{C\delta}{10\beta}\right) \setminus \mathbb{B}(p, \delta)} |\nabla u|^2 &\leq C \left(\frac{\beta}{\delta} \right)^{n+2} \int_{\{x \in \mathbb{B}\left(p, \delta + \frac{C\delta}{10\beta}\right) \setminus \mathbb{B}(p, \delta), y \in \mathbb{B}\left(x, \frac{C\delta}{10\beta}\right)\}} u^2(y) dy dx \\ &\leq C \frac{\beta^2}{\delta^2} \int_{y \in \mathbb{B}\left(p, \delta + \frac{C\delta}{5\beta}\right) \setminus \mathbb{B}\left(p, \delta - \frac{C\delta}{10\beta}\right)} u^2(y) dy, \end{aligned} \quad (2.45)$$

where we have changed the order of integration in the last inequality. Substituting last inequality into (2.43) gives that

$$\int_{\mathbb{B}\left(p, \delta + \frac{C\delta}{5\beta}\right) \setminus \mathbb{B}\left(p, \delta - \frac{C\delta}{10\beta}\right)} u^2 \geq \int_{\mathbb{B}\left(p, \delta + \frac{C\delta}{\beta}\right) \setminus \mathbb{B}\left(p, \delta + \frac{C\delta}{10\beta}\right)} u^2. \quad (2.46)$$

Recall that $u(x) = e_\lambda(x) \exp\{\lambda \varrho(x)\}$. Let

$$\varrho(x_0) = \max_{\mathbb{B}\left(p, \delta + \frac{C\delta}{5\beta}\right) \setminus \mathbb{B}\left(p, \delta - \frac{C\delta}{10\beta}\right)} \varrho(x), \quad \varrho(x_1) = \min_{\mathbb{B}\left(p, \delta + \frac{C\delta}{\beta}\right) \setminus \mathbb{B}\left(p, \delta + \frac{C\delta}{10\beta}\right)} \varrho(x).$$

Then

$$\lambda |\varrho(x_0) - \varrho(x_1)| \leq C \max_{\overline{\mathcal{M}}} |\nabla \varrho(x)| \delta,$$

since $\beta \geq C_0 \lambda$. Furthermore, thanks to the fact that $\nabla \varrho(x)$ is a bounded function in $\overline{\mathcal{M}}$, we have

$$C \int_{\mathbb{B}\left(p, \delta + \frac{C\delta}{5\beta}\right) \setminus \mathbb{B}\left(p, \delta - \frac{C\delta}{10\beta}\right)} e_\lambda^2 \geq \int_{\mathbb{B}\left(p, \delta + \frac{C\delta}{\beta}\right) \setminus \mathbb{B}\left(p, \delta + \frac{C\delta}{10\beta}\right)} e_\lambda^2. \quad (2.47)$$

Adding $\int_{\mathbb{B}\left(p, \delta + \frac{C\delta}{10\beta}\right)} e_\lambda^2$ to both sides of (2.47) yields that

$$C \int_{\mathbb{B}\left(p, \delta + \frac{C\delta}{5\beta}\right)} e_\lambda^2 \geq \int_{\mathbb{B}\left(p, \delta + \frac{C\delta}{\beta}\right)} e_\lambda^2. \quad (2.48)$$

If we replace $\delta = \frac{\delta'}{1 + \frac{C}{5\beta}}$, we get

$$C \int_{\mathbb{B}(p, \delta')} e_\lambda^2 \geq \int_{\mathbb{B}\left(p, \delta' + \frac{C\delta'}{\beta}\right)} e_\lambda^2. \quad (2.49)$$

Since we can choose $\beta = C_0 \lambda$, by finite number of iteration, we arrive at

$$\int_{\mathbb{B}(p, \delta)} e_\lambda^2 \geq C \int_{\mathbb{B}\left(p, \delta\left(1 + \frac{1}{\lambda}\right)\right)} e_\lambda^2. \quad (2.50)$$

This completes conclusion (A) in Theorem 1. Next we show the L^2 -Bernstein's inequality. By the standard elliptic estimates,

$$|\nabla e_\lambda(x)|^2 \leq \frac{C}{r^{2+n}} \int_{\mathbb{B}(x,r)} e_\lambda^2(y) dy \quad (2.51)$$

if $\lambda r \leq 1$ and $\mathbb{B}(x, r) \subset \mathcal{M}$. Choosing $r = \frac{\delta}{\lambda}$ and integrating over $x \in \mathbb{B}(p, \delta)$,

$$\begin{aligned} \int_{\mathbb{B}(p,\delta)} |\nabla e_\lambda(x)|^2 dx &\leq \frac{C}{r^{2+n}} \int_{\{y \in \mathbb{B}(x,r), x \in \mathbb{B}(p,\delta)\}} e_\lambda^2(y) dy dx \\ &\leq \frac{C}{r^2} \int_{\mathbb{B}(p,\delta+r)} e_\lambda^2(x) dx, \end{aligned} \quad (2.52)$$

where we have changed the order of integration in last inequality. Application of (2.50) yields that

$$\int_{\mathbb{B}(p,\delta)} |\nabla e_\lambda(x)|^2 dx \leq \frac{C\lambda^2}{\delta^2} \int_{\mathbb{B}(p,\delta)} e_\lambda^2(x) dx. \quad (2.53)$$

Thus, we arrive at the conclusion (B).

We continue to obtain L^∞ version of Bernstein's inequality. For $x \in \mathbb{B}(p, \delta)$, choosing $r = \frac{\delta}{\lambda}$, the refined doubling inequality (2.50) and (2.51) yield that

$$\begin{aligned} |\nabla e_\lambda(x)|^2 &\leq \frac{C}{r^{2+n}} \int_{\mathbb{B}(x,r)} e_\lambda^2 \leq \frac{C}{r^{2+n}} \int_{\mathbb{B}(p,\delta+r)} e_\lambda^2 \\ &\leq \frac{C}{r^{2+n}} \int_{\mathbb{B}(p,\delta)} e_\lambda^2 \\ &\leq \frac{C}{r^{2+n}} \delta^n \max_{\mathbb{B}(p,\delta)} e_\lambda^2. \end{aligned} \quad (2.54)$$

Therefore,

$$|\nabla e_\lambda(x)| \leq \frac{C\lambda^{\frac{n+2}{2}}}{\delta} \max_{\mathbb{B}(p,\delta)} |e_\lambda| \quad (2.55)$$

for any $x \in \mathbb{B}(p, \delta)$. The conclusion (C) in Theorem 1 is arrived. $\square$

3 Upper bound of nodal sets of Steklov eigenfunctions

In this section, we will prove the optimal upper bound for the interior nodal sets of the Steklov eigenfunctions. Assume that $\mathcal{M}$ is a real analytic Riemannian manifold with boundary. We first estimate the measure of nodal sets in the neighborhood close to boundary, then show the upper bound of nodal sets away from the boundary $\partial\mathcal{M}$. Since $\mathcal{M}$ is a real analytic Riemannian manifold with boundary, we may embed $\mathcal{M} \subset \mathcal{M}_1$ as a relatively compact subset, where $\mathcal{M}_1$ is an open real analytic Riemannian manifold. The real analytic Riemannian manifold $\mathcal{M}$ and $\mathcal{M}_1$ are of the same dimension. We analytically extend the eigenfunction e_λ in $\mathcal{M}_1$. Denote the neighborhood of the boundary $\partial\mathcal{M}$ as $\mathcal{M}_r = \{x \in \mathcal{M}_1 | \text{dist}\{x, \mathcal{M}\} \leq r\}$. To do the analytic continuation across the boundary, we want to get rid of λ on the boundary. We introduce the following lifting argument. Let

$$\hat{v}(x, t) = e^{\lambda t} e_\lambda(x).$$

Then $\hat{v}(x, t)$ satisfies the equation

$$\begin{cases} \Delta_g \hat{v} + \partial_t^2 \hat{v} - \lambda^2 \hat{v} = 0 & \text{in } \mathcal{M} \times (-\infty, -\infty), \\ \frac{\partial \hat{v}}{\partial \nu} - \frac{\partial \hat{v}}{\partial t} = 0 & \text{on } \partial \mathcal{M} \times (-\infty, -\infty). \end{cases} \quad (3.1)$$

We also want to get rid of λ in the equation. Choose

$$v(x, t, s) = e^{i\lambda s} \hat{v}(x, t).$$

Then we get that

$$\begin{cases} \Delta_g v + \partial_t^2 v + \partial_s^2 v = 0 & \text{in } \mathcal{M} \times (-\infty, -\infty) \times (-\infty, -\infty), \\ \frac{\partial v}{\partial \nu} - \frac{\partial v}{\partial t} = 0 & \text{on } \partial \mathcal{M} \times (-\infty, -\infty) \times (-\infty, -\infty). \end{cases} \quad (3.2)$$

We can see that (3.2) is as a uniform elliptic equation with oblique boundary conditions. We introduce the cubes with unequal side-length as

$$\Omega_{R,\rho} = \{(x, t, s) \in \mathbb{R}^{n+2} \mid |x_i| < R \text{ when } i < n, |x_n| < \rho R, |t| < R, |s| < R\}$$

and half-cube

$$\Omega_{R,\rho}^+ = \{(x, t, s) \in \mathbb{R}^{n+2} \mid |x_i| < R \text{ when } i < n, 0 \leq x_n < \rho R, |t| < R, |s| < R\}.$$

Choose any point $p \in \partial \mathcal{M}$, using Fermi coordinates and rescaling arguments, we may consider the function $v(x, t, s)$ locally in the cube centered at origin with the flatten boundary. Hence, $v(x, t, s)$ satisfies the following equation locally

$$\begin{cases} \Delta_g v + \partial_t^2 v + \partial_s^2 v = 0 & \text{in } \Omega_{2,1}^+, \\ \frac{\partial v}{\partial x_n} - \frac{\partial v}{\partial t} = 0 & \text{on } \Omega_{2,1}^+ \cap \{x_n = 0\}. \end{cases} \quad (3.3)$$

Thanks to the analyticity results in [26], we can extend $v(x, t, s)$ to the region $\Omega_{1,\rho}$, where $\rho > 0$ depends only on $\mathcal{M}$. Furthermore, the growth of the extended v is controlled as

$$\|v\|_{L^\infty(\Omega_{1,\rho})} \leq C \|v\|_{L^\infty(\Omega_{2,1}^+)}, \quad (3.4)$$

where C depends only on $\mathcal{M}$. By compactness of the manifold and the uniqueness of the analytic continuation, it follows that

$$-\Delta e_\lambda(x) = 0 \quad \text{in } \widehat{\mathcal{M}}_1, \quad (3.5)$$

where $\widehat{\mathcal{M}}_1 = \{x \in \mathcal{M}_1 \mid \text{dist}\{x, \mathcal{M}\} \leq \rho\}$. Recall the definition of $v(x, t, s) = e^{\lambda t} e^{\lambda i s} e_\lambda(x)$, it follows from (3.4) that

$$\|e_\lambda\|_{L^\infty(\mathbb{B}_\rho)} \leq e^{C\lambda} \|e_\lambda\|_{L^\infty(\mathbb{B}_{2,1}^+)}. \quad (3.6)$$

From the Proposition 1 and the even extension of u , the following doubling inequality holds in half balls as

$$\|u\|_{L^\infty(\mathbb{B}_{2r}^+)} \leq e^{C\lambda} \|u\|_{L^\infty(\mathbb{B}_r^+)}$$

for $0 < r < r_0$, where r_0 depends only $\mathcal{M}$. From the relations of u and e_λ , we obtain the following doubling inequality

$$\|e_\lambda\|_{L^\infty(\mathbb{B}_{2r}^+)} \leq e^{C\lambda} \|e_\lambda\|_{L^\infty(\mathbb{B}_r^+)}. \quad (3.7)$$

Iterating the doubling inequality (3.7) in the half balls by finite number of steps, we can show that

$$\begin{aligned}\|e_\lambda\|_{L^\infty(\mathbb{B}_\rho)} &\leq e^{C\lambda}\|e_\lambda\|_{L^\infty(\mathbb{B}_{\frac{\rho}{2}}^+)} \\ &\leq e^{C\lambda}\|e_\lambda\|_{L^\infty(\mathbb{B}_{\frac{\rho}{2}})}.\end{aligned}\quad (3.8)$$

By rescaling arguments, it also implies that

$$\|e_\lambda\|_{L^\infty(\mathbb{B}_{2r})} \leq e^{C\lambda}\|e_\lambda\|_{L^\infty(\mathbb{B}_r)} \quad (3.9)$$

for any $r \leq \frac{\rho}{2}$ with $\mathbb{B}_{2r} \subset \widehat{\mathcal{M}}_1$ and C depending only on $\mathcal{M}$.

To get the upper bounds of nodal sets for the Steklov eigenfunctions, we need to extend $e_\lambda(x)$ locally as a holomorphic function in $\mathbb{C}^n$. Applying elliptic estimates for e_λ in (3.5) in a ball $\mathbb{B}(p, r) \subset \widehat{\mathcal{M}}_1$, we have

$$\left| \frac{D^\alpha e_\lambda(p)}{\alpha!} \right| \leq C_1^{|\alpha|} r^{-|\alpha|} \|e_\lambda\|_{L^\infty}, \quad (3.10)$$

where α is a multi-index and $C_1 > 1$ depends on $\mathcal{M}$. By translation, we still consider the point p as the origin. Summing up a geometric series, we can extend $e_\lambda(x)$ to be a holomorphic function $e_\lambda(z)$ with $z \in \mathbb{C}^n$ to have

$$\sup_{|z| \leq \frac{r}{2C_1}} |e_\lambda(z)| \leq C_2 \sup_{|x| \leq r} |e_\lambda(x)| \quad (3.11)$$

with $C_2 > 1$.

Iterating the doubling inequality (3.9) finitely many times, by the rescaling arguments, we obtain that

$$\sup_{|z| \leq 2r} |e_\lambda(z)| \leq e^{C_3\lambda} \sup_{|x| \leq r} |e_\lambda(x)| \quad (3.12)$$

for $0 < r < \rho_0$ with ρ_0 depending on $\mathcal{M}$ and C_3 depends on $\mathcal{M}$.

We need a lemma concerning the growth of a complex analytic function with the number of zeros. See e.g. [8] and [15].

Lemma 2 Suppose $f : \mathbf{B}(0, 1) \subset \mathbb{C} \rightarrow \mathbb{C}$ is an analytic function satisfying

$$f(0) = 1 \quad \text{and} \quad \sup_{\mathbf{B}(0, 1)} |f| \leq 2^N$$

for some positive constant N . Then for any $r \in (0, 1)$, there holds

$$\sharp\{z \in \mathbf{B}(0, r) : f(z) = 0\} \leq cN$$

where c depends on r . Especially, for $r = \frac{1}{2}$, there holds

$$\sharp\{z \in \mathbf{B}(0, 1/2) : f(z) = 0\} \leq N.$$

We are ready to show the upper bound of interior nodal sets of Steklov eigenfunctions based on doubling inequality and the growth control lemma, see the pioneering work in [8] and [19].

Proof of Theorem 2 We first prove the nodal sets in a neighborhood $\mathcal{M}_{\frac{\rho}{4}}$. By rescaling and translation, we can argue on scales of order one. Let $p \in \mathbb{B}_{1/4}$ be the point where the maximum of $|e_\lambda|$ in $\mathbb{B}_{1/4}$ is attained. For each direction $\omega \in S^{n-1}$, set $\hat{e}_\omega(z) = e_\lambda(p + z\omega)$ in $z \in \mathbf{B}(0, 1) \subset \mathbb{C}$. Denoted by $N(\omega) = \#\{z \in \mathbf{B}(0, 1/2) \subset \mathbb{C} \mid \hat{e}_\omega(z) = 0\}$. By the doubling property (3.12) and the Lemma 2, we have

$$\begin{aligned} & \#\{x \in \mathbb{B}(p, 1/2) \mid x - p \text{ is parallel to } \omega \text{ and } e_\lambda(x) = 0\} \\ & \leq \#\{z \in \mathbf{B}(0, 1/2) \subset \mathbb{C} \mid \hat{e}_\omega(z) = 0\} \\ & = N(\omega) \leq C\lambda. \end{aligned} \quad (3.13)$$

With aid of integral geometry estimates, it implies that

$$\begin{aligned} H^{n-1}\{x \in \mathbb{B}(p, 1/2) \mid e_\lambda(x) = 0\} & \leq c(n) \int_{S^{n-1}} N(\omega) d\omega \\ & \leq \int_{S^{n-1}} C\lambda d\omega = C\lambda. \end{aligned} \quad (3.14)$$

Therefore, we have

$$H^{n-1}\{x \in \mathbb{B}(0, 1/4) \mid e_\lambda(x) = 0\} \leq C\lambda. \quad (3.15)$$

By covering the compact manifold $\mathcal{M}_{\frac{\rho}{4}} \subset \widehat{\mathcal{M}}_1$ by a finite number of coordinate charts, we arrive at

$$H^{n-1}\{x \in \mathcal{M}_{\frac{\rho}{4}} \mid e_\lambda(x) = 0\} \leq C\lambda. \quad (3.16)$$

Next we deal with the measure of nodal sets in $\mathcal{M} \setminus \mathcal{M}_{\frac{\rho}{4}}$. We have obtained the doubling inequality (2.7) in the interior of the manifold. Since $u(x) = e_\lambda(x) \exp\{\lambda \varrho(x)\}$ and $-\hat{C} < \varrho(x) \leq \hat{C}$ for some constant $\hat{C}$ depending on $\mathcal{M}$, it is true that

$$\|e_\lambda\|_{L^\infty(\mathbb{B}(p, 2r))} \leq e^{C\lambda} \|e_\lambda\|_{L^\infty(\mathbb{B}(p, r))} \quad (3.17)$$

holds for $p \in \mathcal{M} \setminus \mathcal{M}_{\frac{\rho}{4}}$ and $0 < r \leq \rho_0 \leq \frac{\rho}{4}$. We similarly extend $e_\lambda(x)$ locally as a holomorphic function in $\mathbb{C}^n$. Since $e_\lambda(x)$ is harmonic in $\mathcal{M} \setminus \mathcal{M}_{\frac{\rho}{4}}$, applying elliptic estimates in a small ball $\mathbb{B}(p, r)$, we have

$$\left| \frac{D^\alpha e_\lambda(p)}{\alpha!} \right| \leq C_4^{|\alpha|} r^{-|\alpha|} \|e_\lambda\|_{L^\infty}, \quad (3.18)$$

where $C_4 > 1$ depends only on $\mathcal{M}$. We consider the point p as the origin as well. Summing up a geometric series, we can extend $e_\lambda(x)$ to be a holomorphic function $e_\lambda(z)$ with $z \in \mathbb{C}^n$. Moreover, we have

$$\sup_{|z| \leq \frac{r}{2C_4}} |e_\lambda(z)| \leq C_5 \sup_{|x| \leq r} |e_\lambda(x)| \quad (3.19)$$

with $C_5 > 1$.

Thanks to the doubling inequality (3.17), by finite steps of iterations, we obtain that

$$\sup_{|z| \leq \frac{r}{2C_4}} |e_\lambda(z)| \leq e^{C_6\lambda} \sup_{|x| \leq \frac{r}{4C_4}} |e_\lambda(x)| \quad (3.20)$$

with C_6 depends on $\mathcal{M}$. In particular, by rescaling arguments,

$$\sup_{|z| \leq 2r} |e_\lambda(z)| \leq e^{C\lambda} \sup_{|x| \leq r} |e_\lambda(x)| \quad (3.21)$$

holds for $0 < r < \frac{\rho_0}{2}$ with ρ_0 depending on $\mathcal{M}$. Using the same arguments as obtaining the nodal sets in the neighborhood of the boundary, we take advantage of lemma 2 and the inequality (3.21). By rescaling and translation, we can argue on scales of order one. Let $p \in \mathbb{B}_{1/4}$ be the point where the maximum of $|e_\lambda|$ in $\mathbb{B}_{1/4}$ is achieved. For each direction $\omega \in S^{n-1}$, set $e_\lambda^\omega(z) = e_\lambda(p + z\omega)$ in $z \in \mathbf{B}(0, 1) \subset \mathbb{C}$. From the doubling property (3.21) and the lemma 2 above, we have

$$\begin{aligned} \sharp\{x \in \mathbb{B}(p, 1/2) \mid x - p \text{ is parallel to } \omega \text{ and } e_\lambda(x) = 0\} \\ \leq \sharp\{z \in \mathbf{B}(0, 1/2) \subset \mathbb{C} \mid e_\lambda^\omega(z) = 0\} \\ = N(\omega) \leq C\lambda. \end{aligned} \quad (3.22)$$

Thanks to the integral geometry estimates, we get

$$\begin{aligned} H^{n-1}\{x \in \mathbb{B}(p, 1/2) \mid e_\lambda(x) = 0\} &\leq c(n) \int_{S^{n-1}} N(\omega) d\omega \\ &\leq \int_{S^{n-1}} C\lambda d\omega = C\lambda. \end{aligned} \quad (3.23)$$

Thus, we obtain

$$H^{n-1}\{x \in \mathbb{B}(0, 1/4) \mid e_\lambda(x) = 0\} \leq C\lambda. \quad (3.24)$$

Using the finite number of coordinate charts to cover the compact manifold $\mathcal{M} \setminus \mathcal{M}_{\frac{\rho}{4}}$, we obtain

$$H^{n-1}\{x \in \mathcal{M} \setminus \mathcal{M}_{\frac{\rho}{4}} \mid e_\lambda(x) = 0\} \leq C\lambda. \quad (3.25)$$

Together with (3.16) and (3.25), we arrive at the conclusion in Theorem 2. $\square$

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4 Appendix

In this section, we provide the proof of Lemma 1 and some arguments stated in the proof Proposition 2. Recall that

$$\|\mathcal{L}_\beta(v)\|_\phi^2 \geq \frac{1}{2}\mathcal{A} - \mathcal{B},$$

where

$$\begin{aligned} \mathcal{L}_\beta(v) &= \partial_t^2 v + (2\beta\phi' + e^{2t}\bar{b}_t + (n-2) + \partial_t \ln \sqrt{\gamma})\partial_t v + e^{2t}\bar{b}_i\partial_i v \\ &\quad + (\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + \beta\phi'' + (n-2)\beta\phi' + \beta\partial_t \ln \sqrt{\gamma}\phi')v + \Delta_\omega v + e^{2t}\bar{q}. \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \mathcal{A} &= \|\partial_t^2 v + \Delta_\omega v + (2\beta\phi' + e^{2t}\bar{b}_t)\partial_t v + e^{2t}\bar{b}_i\partial_i v \\ &\quad + (\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q})v\|_\phi^2 \end{aligned} \quad (4.2)$$

and

$$\mathcal{B} = \|\beta\phi''v + \beta\partial_t \ln \sqrt{\gamma}\phi'v + (n-2)\partial_t v + \partial_t \ln \sqrt{\gamma}\partial_t v\|_\phi^2. \quad (4.3)$$

Modifying the arguments in [2] and [35], we can obtain the following lemma, which verifies the proof of (2.18) and (2.19) in Proposition 2.

Lemma 3 *There holds that*

$$\begin{aligned} \|\mathcal{L}_\beta(v)\|_\phi^2 &\geq \frac{1}{4}\mathcal{A} \\ &\geq C\beta^3 \int |\phi''||v|^2\phi'^{-3}\sqrt{\gamma} dt d\omega + C\beta \int |\phi''||D_\omega v|^2\phi'^{-3}\sqrt{\gamma} dt d\omega \\ &\quad + C\beta \int |\partial_t v|^2\phi'^{-3}\sqrt{\gamma} dt d\omega. \end{aligned} \quad (4.4)$$

Proof We decompose $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}'_1 + \mathcal{A}'_2 + \mathcal{A}'_3,$$

where

$$\mathcal{A}'_1 = \|\partial_t^2 v + (\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q})v + \Delta_\omega v\|_\phi^2$$

and

$$\mathcal{A}'_2 = \|(2\beta\phi' + e^{2t}\bar{b}_t)\partial_t v + e^{2t}\bar{b}_t\partial_i v\|_\phi^2$$

and

$$\begin{aligned} \mathcal{A}'_3 = 2 &< \|\partial_t^2 v + (\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q})v + \Delta_\omega v, \\ &\quad (2\beta\phi' + e^{2t}\bar{b}_t)\partial_t v + e^{2t}\bar{b}_t\partial_i v\|_\phi. \end{aligned}$$

We first compute $\mathcal{A}'_1$. Let $\hat{\alpha}$ be some small positive constant. Recall that $\phi(t) = t - e^{\epsilon t}$. Since $|\phi''| \leq 1$ and β is large enough, it is true that

$$\mathcal{A}'_1 \geq \frac{\hat{\alpha}}{\beta}\mathcal{A}''_1, \quad (4.5)$$

where $\mathcal{A}''_1$ is given by

$$\mathcal{A}''_1 = \left\| \sqrt{|\phi''|} [\partial_t^2 v + (\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q})v + \Delta_\omega v] \right\|_\phi^2.$$

We split $\mathcal{A}''_1$ into three parts:

$$\mathcal{A}''_1 = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3, \quad (4.6)$$

where

$$\mathcal{K}_1 = \left\| \sqrt{|\phi''|} (\partial_t^2 v + \Delta_\omega v) \right\|_\phi^2$$

and

$$\mathcal{K}_2 = \left\| \sqrt{|\phi''|} (\beta^2\phi'^2 + \beta\phi'\bar{b}_t e^{2t} + (n-2)\beta\phi' + e^{2t}\bar{q})v \right\|_\phi^2$$

and

$$\mathcal{K}_3 = 2 \left\langle |\phi''| (\partial_t^2 v + \Delta_\omega v), (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q}) v \right\rangle_\phi.$$

The expression $\mathcal{K}_1$ is considered to be a nonnegative term. We estimate $\mathcal{K}_2$. By the triangle inequality,

$$\mathcal{K}_2 \geq \beta^4 \left\| \sqrt{|\phi''|} \phi' v \right\|_\phi^2 - \left\| \sqrt{|\phi''|} (\beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q}) v \right\|_\phi^2. \quad (4.7)$$

Using the fact that $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$, we have

$$\begin{aligned} \left\| \sqrt{|\phi''|} (\beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q}) v \right\|_\phi^2 &\leq C\beta^4 \left\| \sqrt{|\phi''|} e^t v \right\|_\phi^2 \\ &\quad + C\beta^2 \left\| \sqrt{|\phi''|} v \right\|_\phi^2. \end{aligned} \quad (4.8)$$

Since t is close to negative infinity and then ϕ' is close to 1, from (4.7) and (4.8), we obtain that

$$\mathcal{K}_2 \geq C\beta^4 \left\| \sqrt{|\phi''|} v \right\|_\phi^2, \quad (4.9)$$

where we also used the fact that ϕ' is close to 1. We derive a lower bound for $\mathcal{K}_3$. Integration by parts shows that

$$\begin{aligned} \mathcal{K}_3 &= -2 \int |\phi''| |\partial_t v|^2 (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q}) \phi'^{-3} \sqrt{\gamma} dt d\omega \\ &\quad - 2 \int \partial_t v v \partial_t [|\phi''| (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q}) \phi'^{-3} \sqrt{\gamma}] dt d\omega \\ &\quad - 2 \int |\phi''| |D_\omega v|^2 (\beta^2 \phi'^2 + \beta \phi' \bar{b}_t e^{2t} + (n-2)\beta \phi' + e^{2t} \bar{q}) \phi'^{-3} \sqrt{\gamma} dt d\omega \\ &\quad - 2 \int \beta |\phi''| \phi' \gamma^{ij} \partial_i v \partial_j \bar{b}_t e^{2t} \phi'^{-3} \sqrt{\gamma} dt d\omega \\ &\quad - 2 \int |\phi''| \gamma^{ij} \partial_i v \partial_j \bar{q} e^{2t} v \phi'^{-3} \sqrt{\gamma} dt d\omega. \end{aligned} \quad (4.10)$$

By the Cauchy-Schwartz inequality and the condition that $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$, we arrive at

$$\mathcal{K}_3 \geq -C\beta^2 \int |\phi''| (|\partial_t v|^2 + |D_\omega v|^2 + v^2) \phi'^{-3} \sqrt{\gamma} dt d\omega. \quad (4.11)$$

Since $\mathcal{K}_1$ is nonnegative, the combination of (4.6), (4.9) and (4.11) yields that

$$\begin{aligned} \mathcal{A}_1'' &\geq C\beta^4 \left\| \sqrt{|\phi''|} v \right\|_\phi^2 - C\beta^2 \left\| \sqrt{|\phi''|} \partial_t v \right\|_\phi^2 \\ &\quad - C\beta^2 \left\| \sqrt{|\phi''|} |D_\omega v| \right\|_\phi^2. \end{aligned} \quad (4.12)$$

From (4.5), it follows that

$$\begin{aligned} \mathcal{A}'_1 &\geq C\hat{\alpha}\beta^3 \left\| \sqrt{|\phi''|}v \right\|_\phi^2 - C\hat{\alpha}\beta \left\| \sqrt{|\phi''|}\partial_t v \right\|_\phi^2 \\ &\quad - C\hat{\alpha}\beta \left\| \sqrt{|\phi''|}|D_\omega v| \right\|_\phi^2. \end{aligned} \quad (4.13)$$

Recall that

$$\mathcal{A}'_2 = \|(2\beta\phi' + e^{2t}\bar{b}_t)\partial_t v + e^{2t}\bar{b}_i\partial_i v\|_\phi^2.$$

By the triangle inequality, one has

$$\mathcal{A}'_2 \geq 2\beta^2 \|\phi'\partial_t v\|_\phi^2 - \|e^{2t}\bar{b}_t\partial_t v + e^{2t}\bar{b}_i\partial_i v\|_\phi^2.$$

It is obvious that

$$\mathcal{A}'_2 \geq \frac{1}{\beta} \mathcal{A}'_2.$$

From the assumption that $\beta > C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$, we obtain that

$$\begin{aligned} \mathcal{A}'_2 &\geq C\beta \|\phi'\partial_t v\|_\phi^2 - C\beta \|e^t\partial_t v\|_\phi^2 - C\beta \|e^t|D_\omega v|\|_\phi^2 \\ &\geq C\beta \|\phi'\partial_t v\|_\phi^2 - C\beta \|e^t|D_\omega v|\|_\phi^2. \end{aligned} \quad (4.14)$$

For the inner product $\mathcal{A}'_3$, using the arguments of integration by parts, since $e^t \ll 1$ as $t < T_0$ and $|T_0|$ is large enough, we can show a lower bound of $\mathcal{A}'_3$,

$$\begin{aligned} \mathcal{A}'_3 &\geq C\beta \left\| \sqrt{|\phi''|}|D_\omega v| \right\|_\phi^2 - C\beta^3 \|e^t v\|_\phi^2 - C\beta \left\| \sqrt{|\phi''|}\partial_t v \right\|_\phi^2 \\ &\quad - C\beta^2 \left\| \sqrt{|\phi''|}v \right\|_\phi^2. \end{aligned} \quad (4.15)$$

Recall that $\mathcal{A} = \mathcal{A}'_1 + \mathcal{A}'_2 + \mathcal{A}'_3$. From (4.13), (4.14) and (4.15), it follows that

$$\begin{aligned} \mathcal{A} &\geq C\hat{\alpha}\beta^3 \int |\phi''|v^2\phi'^{-3}\sqrt{\gamma}dtd\omega + C\beta \int |\partial_t v|^2\phi'^{-3}\sqrt{\gamma} \\ &\quad + C\beta \int |\phi''||D_\omega v|^2\phi'^{-3}\sqrt{\gamma}dtd\omega - C\beta^2 \int |\phi''|v^2\phi'^{-3}\sqrt{\gamma}dtd\omega \\ &\quad - C\beta^3 \int e^{2t}v^2\phi'^{-3}\sqrt{\gamma}dtd\omega - C\beta \int |\phi''||\partial_t v|^2\phi'^{-3}\sqrt{\gamma}dtd\omega \\ &\quad - C\hat{\alpha}\beta \int |\phi''||D_\omega v|^2\phi'^{-3}\sqrt{\gamma}dtd\omega - C\beta \int e^{2t}|D_\omega v|^2\phi'^{-3}\sqrt{\gamma}dtd\omega. \end{aligned} \quad (4.16)$$

If we choose $\hat{\alpha}$ to be appropriately small and take the fact $|\phi''| > e^t$ into account, we obtain that

$$\begin{aligned} C\mathcal{A} &\geq \beta^3 \int |\phi''||v|^2\phi'^{-3}\sqrt{\gamma}dtd\omega + \beta \int |\phi''||D_\omega v|^2\phi'^{-3}\sqrt{\gamma}dtd\omega \\ &\quad + \beta \int |\partial_t v|^2\phi'^{-3}\sqrt{\gamma}dtd\omega. \end{aligned} \quad (4.17)$$

Now we show $\mathcal{B}$ can be absorbed into $\mathcal{A}$ for large $|T_0|$ and large β . Since

$$|\partial_t \ln \sqrt{\gamma}| \leq C e^t \leq |\phi''|,$$

then

$$\begin{aligned} \mathcal{B} &= \|\beta \phi'' v + \beta \partial_t \ln \sqrt{\gamma} \phi' v + (n-2) \partial_t v + \partial_t \ln \sqrt{\gamma} \partial_t v\|_{\phi}^2 \\ &\leq \beta^2 \int |\phi''| v^2 \phi'^{-3} \sqrt{\gamma} dt d\omega + C \int |\partial_t v|^2 e^{2t} \phi'^{-3} \sqrt{\gamma} dt d\omega. \end{aligned} \quad (4.18)$$

Thus, the right hand side of (4.18) can be incorporated by the right hand side of (4.17). Hence the proof of the lemma is arrived. $\square$

Proof of Lemma 1 If we recall that $u = e^{-\beta\varphi(x)}v$, the proof of Lemma 3 just implies Lemma 1 stated in Sect. 2. $\square$

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