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Counterexamples to the tilting and (p, r)-filtration conjectures

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In memory of Elena Galaktionova

Abstract. In this paper the authors produce a projective indecomposable module for the Frobenius kernel of a simple algebraic group in characteristic p that is not the restriction of an indecomposable tilting module. This yields a counterexample to Donkin's longstanding Tilting Module Conjecture. The authors also produce a Weyl module that does not admit a p-Weyl filtration. This answers an old question of Jantzen, and also provides a counterexample to the (p, r)-Filtration Conjecture.

1. Introduction

1.1. Let G be a semisimple, simply connected algebraic group over an algebraically closed field of characteristic $p > 0$ and let $\mathfrak{g}$ be its Lie algebra. Restricted representations for the Lie algebra $\mathfrak{g}$ are equivalent to representations for the first Frobenius kernel G_1. In the 1960s Curtis showed that the simple G_1-modules lift to simple modules for G. Later, Humphreys and Verma investigated the projective indecomposable G_1-modules. The expectation that these modules should also lift to G came to be known as the "Humphreys–Verma Conjecture." This was verified for $p \geq 2h - 2$ (where h is the Coxeter number) by work of Ballard [5] and Jantzen [11]. For over 50 years, it has been anticipated that the Humphreys–Verma Conjecture would hold for all p.

In 1990, Donkin presented a series of conjectures at MSRI. One of the conjectures, known as the Tilting Module Conjecture, states that a projective indecomposable module for G_r can be realized as an indecomposable tilting G-module (see Conjecture 2.2.2). Like the Humphreys–Verma Conjecture, the Tilting Module Conjecture holds for $p \geq 2h - 2$ with the hope of being valid for all p. Recently, the Tilting Module Conjecture has been shown to be related to another

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one of Donkin's conjectures involving good (p, r)-filtrations. A more detailed exposition with the connections is presented in Section 2.2.

The Tilting Module Conjecture has taken on additional importance following work by Achar, Makisumi, Riche and Williamson [1], who have shown that when $p > h$, the characters of indecomposable tilting modules can be given via p-Kazhdan-Lusztig polynomials, confirming a conjecture by Riche and Williamson [17]. When $p \geq 2h - 2$, the Tilting Module Conjecture then allows one to deduce the characters of simple G-modules. The authors of [1] credit Andersen with this observation.

1.2. The goal of this paper is to present counterexamples to the conjectures and questions stated in Section 2.2. In this subsection, let G be a simple algebraic group whose root system is of type G_2 and $p = 2$. In particular, we

(1.2.1) present a counterexample to the Tilting Module Conjecture – see Theorem 4.1.1;

(1.2.2) construct a counterexample to one direction of Donkin's Good (p, r)-Filtration Conjecture (i.e., Conjecture 2.2.3 $(\Leftarrow)$) – see Theorem 3.5.1 and Section 3.6;

(1.2.3) give an example of a costandard/induced module $\nabla(\lambda)$ that does not admit a good (p, r)-filtration – see Theorem 3.5.1.

Specifically, we demonstrate that there does not exist a good 2-filtration for the induced module $\nabla(2, 1)$.[1)] This gives a negative answer to an open question of Jantzen [11], and this module is also is a counterexample for (1.2.2). As a consequence of these results, we prove that the indecomposable tilting module $T(2, 2)$ is decomposable over the first Frobenius kernel of G. We present a proof of this fact that does not employ computer calculations, using instead information about extensions of simple G-modules of small highest weights.[2)]

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2. Preliminaries

2.1. Notation. The notation will follow the conventions in [6, Section 2.1], most of which follow those in [13] (though our notation for induced and Weyl modules follows the costandard and standard module conventions in highest weight category literature). Let G be a connected, semisimple algebraic group scheme defined over $\mathbb{F}_p$ and G_r be its rth Frobenius kernel.

Let X_+ denote the dominant weights for G, and let X_r be the p^r-restricted weights. For $\lambda \in X_+$, there are four fundamental classes of G-modules (each having highest weight λ): $L(\lambda)$ (simple), $\nabla(\lambda)$ (costandard/induced), $\Delta(\lambda)$ (standard/Weyl), and $T(\lambda)$ (indecomposable

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1) A major step in this process was a computation of a filtration of $\Delta(2, 1)$, obtained using Stephen Doty's WeylModule package for the software GAP [7, 19], that, when dualized, indicated that $\nabla(2, 1)$ could not have a good 2-filtration.

2) This fact was verified in another way by running Doty's GAP program to compute that the socle of $\Delta(2, 2)$ is isomorphic to $k \oplus L(0, 1)$. As $\Delta(2, 2)$ is a submodule of $T(2, 2)$, one concludes that the socle of $T(2, 2)$ has at least two factors over G_1, so that $T(2, 2)$ splits into at least two projective summands over G_1.

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tilting). A G-module M has a *good filtration* (resp. *Weyl filtration*) if and only if M has a filtration with factors of the form $\nabla(\mu)$ (resp. $\Delta(\mu)$) for suitable $\mu \in X_+$.

For $\lambda \in X_+$ with unique decomposition $\lambda = \lambda_0 + p^r \lambda_1$ with $\lambda_0 \in X_r$ and $\lambda_1 \in X_+$, define $\nabla^{(p,r)}(\lambda) = L(\lambda_0) \otimes \nabla(\lambda_1)^{(r)}$, where (r) denotes the twisting of the module action by the rth Frobenius morphism. Similarly, set $\Delta^{(p,r)}(\lambda) = L(\lambda_0) \otimes \Delta(\lambda_1)^{(r)}$. A G-module M has a *good* (p, r)*-filtration* (resp. *Weyl* (p, r)*-filtration*) if and only if M has a filtration with factors of the form $\nabla^{(p,r)}(\mu)$ (resp. $\Delta^{(p,r)}(\mu)$) for suitable $\mu \in X_+$. In the case when $r = 1$, we often refer to good $(p, 1)$-filtrations as good p-filtrations.

Let ρ be the sum of the fundamental weights and $\mathrm{St}_r = L((p^r - 1)\rho)$ (which is also isomorphic to $\nabla((p^r - 1)\rho)$ and $\Delta((p^r - 1)\rho)$) be the rth Steinberg module. For $\lambda \in X_r$, let $Q_r(\lambda)$ denote the projective cover (equivalently, injective hull) of $L(\lambda)$ as a G_r-module. If $\lambda \in X_r$, set $\hat{\lambda} = 2(p^r - 1)\rho + w_0\lambda$, where w_0 is the long element in the Weyl group W.

Let M be a finite-dimensional G-module, and let

$$M \supseteq \mathrm{rad}_G M \supseteq \mathrm{rad}_G^2 M \supseteq \cdots \supseteq \{0\}$$

be the radical series of M. Moreover, let

$$\{0\} \subseteq \mathrm{soc}_G M \subseteq \mathrm{soc}_G^2 M \subseteq \cdots \subseteq M$$

be the socle series for M. One can similarly define such filtrations for G_r-modules.

2.2. The conjectures. In the early 1970s Humphreys and Verma presented the following conjecture on the lifting of G-structures on the projective modules for G_r.

Conjecture 2.2.1. For $\lambda \in X_r$, the G_r-module structure on $Q_r(\lambda)$ can be lifted to G.

The conjecture was first verified by Ballard [5] for $p \geq 3h - 3$ and then by Jantzen [11] for $p \geq 2h - 2$, who further showed under this improved bound that the G-structure was unique up to isomorphism. Later, at a conference at MSRI in 1990, Donkin presented the following conjecture, predicting that a G-module structure on $Q_r(\lambda)$ arises from a specific tilting module which must be *the* G-module structure whenever uniqueness of G-structure holds.

Conjecture 2.2.2. For all $\lambda \in X_r$, $T(2(p^r - 1)\rho + w_0\lambda)|_{G_r} = Q_r(\lambda)$.

Conjecture 2.2.2 holds for $p \geq 2h - 2$ and the proof under this bound entails locating one particular G-summand of $\mathrm{St}_r \otimes L(\lambda)$. At the same conference at MSRI, another conjecture was introduced by Donkin that interrelates good filtrations with good (p, r)-filtrations via the Steinberg module.

Conjecture 2.2.3. A finite-dimensional G-module M has a good (p, r)-filtration if and only if $\mathrm{St}_r \otimes M$ has a good filtration.

We denote the two directions of the statement as follows:

- Conjecture 2.2.3 ($\Rightarrow$): If M has a good (p, r)-filtration, then $\mathrm{St}_r \otimes M$ has a good filtration.
- Conjecture 2.2.3 ($\Leftarrow$): If $\mathrm{St}_r \otimes M$ has a good filtration, then M has a good (p, r)-filtration.

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Conjecture 2.2.3 $(\Rightarrow)$ is equivalent to $\mathrm{St}_r \otimes L(\lambda)$ being a tilting module for all $\lambda \in X_r$. Andersen [2] first verified Conjecture 2.2.3 $(\Rightarrow)$ when $p \geq 2h-2$. Later, Kildetoft and Nakano [14] supplied two additional proofs for this bound using different methods. In recent work, the authors of this paper lowered the bound to $p \geq 2h-4$ (cf. [6]). For rank 2 groups (including G_2), Conjecture 2.2.3 $(\Rightarrow)$ was proved for all p in [14] and [6].

There are also strong relationships, established by Kildetoft and Nakano [14] and also by Sobaje [18], between these conjectures given by the following hierarchy of implications:

$$\text{Conjecture 2.2.3} \implies \text{Conjecture 2.2.2} \implies \text{Conjecture 2.2.3}\,(\Rightarrow).$$

While we will provide counterexamples to Conjecture 2.2.2 and the full Conjecture 2.2.3, we remark that Conjecture 2.2.3 $(\Rightarrow)$ may still hold for all p. A special case of Conjecture 2.2.3 $(\Leftarrow)$ was earlier posed by Jantzen [11].

Question 2.2.1. For $\lambda \in X_+$, does $\nabla(\lambda)$ admit a good (p, r)-filtration?

Parshall and Scott affirmatively answered the aforementioned question if $p \geq 2h-2$ and the Lusztig Conjecture holds for the given prime and group [16]. Recently, Andersen [3] has shown this for $p \geq (h-2)h$.

3. Weyl modules and good (p, r)-filtrations for type G_2

3.1. Simple and projective modules. Assume throughout this section that the root system of G is of type G_2 and that the prime $p = 2$. We follow the Bourbaki ordering of the simple roots: α_1 is the short root and α_2 is the long root. For $a, b \in \mathbb{Z}$, we denote by (a, b) the weight $a\varpi_1 + b\varpi_2$, where ϖ_1 and ϖ_2 are the fundamental dominant weights. The characters of the simple modules were determined by Jantzen in [10]. The set of restricted weights is

$$X_1 = \{(0,0), (1,0), (0,1), (1,1)\}.$$

Let $\mathrm{St} = \mathrm{St}_1$ denote the first Steinberg module $L(1,1)$. The module

$$L(0,1) \cong \nabla(0,1) \cong \Delta(0,1)$$

is the 14-dimensional adjoint representation. Among the four costandard G-modules of restricted highest weight, only $\nabla(1,0)$ is not simple, and we have that $\nabla(1,0)/L(1,0) \cong k$. Every simple G-module is self-dual, and the weight lattice and root lattice coincide.

Since the characters of the simple G-modules of restricted highest weight are known here, it is possible to compute directly the dimensions of the projective indecomposable G_1-modules. We recall in Table 1 some of the information provided by Humphreys in [9, 18.4, Table 4].

λ	$\dim L(\lambda)$	$\dim Q_1(\lambda)$
$(0,0)$	1	$36 \cdot 64$
$(1,0)$	6	$12 \cdot 64$
$(0,1)$	14	$6 \cdot 64$
$(1,1)$	64	64

Table 1. Dimensions of simple and projective G_1-modules.

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3.2. Ext^1-calculations. In our analysis of the structure of the Weyl modules we will need the following Ext^1-calculations that appear in Dowd and Sin [8, Lemma 3.3], part (c) of which dates back to work of Jantzen [12].

Proposition 3.2.1. *One has the following isomorphisms as G-modules:*

(a) $\mathrm{Ext}^1_{G_1}(L(1,0), L(0,1)) = 0$,
(b) $\mathrm{Ext}^1_{G_1}(L(0,1), L(0,1)) = 0$,
(c) $\mathrm{Ext}^1_{G_1}(k, L(0,1)) \cong \nabla(1,0)^{(1)}$.

3.3. Decomposition of $\mathrm{St} \otimes L(\lambda)$, $\lambda \in X_1$. Recall that St is projective over the first Frobenius kernel G_1. Hence, for $\lambda \in X_1$, $\mathrm{St} \otimes L(\lambda)$ is also projective over G_1. As the highest weight of $\mathrm{St} \otimes L(\lambda)$ is $\rho + \lambda = 2\rho - (\rho - \lambda)$, which is the same as that of $Q_1(\rho - \lambda)$, the module $Q_1(\rho - \lambda)$ is necessarily a G_1-summand of $\mathrm{St} \otimes L(\lambda)$. The following proposition gives a precise decomposition of $\mathrm{St} \otimes L(\lambda)$ for each $\lambda \in X_1$ (we note that this information can be found elsewhere, for example it is in [4, Chapter 3]).

Proposition 3.3.1. *We have the following decompositions into projective indecomposable modules over G_1:*

(a) $\mathrm{St} \otimes k \cong \mathrm{St}$,
(b) $\mathrm{St} \otimes L(1,0) \cong Q_1(0,1)$,
(c) $\mathrm{St} \otimes L(0,1) \cong Q_1(1,0) \oplus \mathrm{St}^{\oplus 2}$,
(d) $\mathrm{St} \otimes \mathrm{St} \cong Q_1(0,0) \oplus Q_1(0,1)^{\oplus 2} \oplus \mathrm{St}^{\oplus 16}$.

Proof. The first isomorphism is immediate, and the second follows by the module dimensions given in Table 1. Recall that $\mathrm{St} \cong \mathrm{ind}_{B_1}^{G_1}(p-1)\rho$. Moreover, St is the projective cover of $(p-1)\rho$ as a B_1-module. Hence, for any G-module M, one obtains via Frobenius reciprocity

$$\mathrm{Hom}_{G_1}(\mathrm{St}, \mathrm{St} \otimes M) \cong \mathrm{Hom}_{B_1}(\mathrm{St}, (p-1)\rho \otimes M) \cong M^{T_1},$$

where T_1 is the Frobenius kernel of the maximal torus T. Now the weight 0 appears twice in $L(0,1)$, so that $\mathrm{St}^{\oplus 2} \subseteq \mathrm{St} \otimes L(0,1)$. There is also an embedding of $L(1,0)$ into $\mathrm{St} \otimes L(0,1)$. The dimensions in Table 1 then imply that (c) holds.

Finally, the G_1-socle of $\mathrm{St} \otimes \mathrm{St}$ is determined by all $L(\lambda)^{T_1}$ for $\lambda \in X_1$. Using a table of weights for G-modules (see for example [15]) and the fact that $\mathrm{St} \otimes \mathrm{St}$ is a tilting module, one finds that

$$\mathrm{soc}_{G_1}(\mathrm{St} \otimes \mathrm{St}) \cong k \oplus L(0,1)^{\oplus 2} \oplus (\mathrm{St} \otimes T(1,0)^{(1)})^{\oplus 2},$$

when viewed as a G-module. Note that

$$\mathrm{St} \otimes T(1,0)^{(1)} \cong \mathrm{St}^{\oplus 8}$$

as a G_1-module, proving (d). □

For $\lambda \in X_1$, we know that $\mathrm{St} \otimes L(\lambda)$ is a tilting module [14] of highest weight $\rho + \lambda$. Hence, the indecomposable tilting module $T(\rho + \lambda)$ embeds in $\mathrm{St} \otimes L(\lambda)$. Furthermore, the G_1-Steinberg block component of any G-module splits off as a summand over G. Thus we conclude from Proposition 3.3.1 the following theorem.

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Theorem 3.3.1. *Over G_1 there are isomorphisms*

(a) $T(1,1) \cong \mathrm{St}$,

(b) $T(2,1) \cong Q_1(0,1)$,

(c) $T(1,2) \cong Q_1(1,0)$.

One can show that these are the unique G-structures on these modules, by showing that any G-structure on $Q_1(1,0)$ or on $Q_1(0,1)$ must admit a good filtration (a more detailed explanation of this will be provided in a forthcoming paper).

3.4. There exists a surjective homomorphism of G-modules

$$T(2,1) \twoheadrightarrow \nabla(2,1).$$

Since $T(2,1) \cong Q_1(0,1)$, $L(0,1)$ is its unique semisimple quotient over G_1, and therefore the same holds over G since every simple G-module is semisimple over G_1. These facts are then true of its homomorphic image $\nabla(2,1)$. That is,

$$\operatorname{rad}_{G_1} \nabla(2,1) = \operatorname{rad}_G \nabla(2,1),$$
$$\nabla(2,1)/\operatorname{rad}_G \nabla(2,1) \cong L(0,1).$$

Since $T(2,1) \cong Q_1(0,1)$ as a G_1-module, the G_1-socle of $T(2,1)$ is $L(0,1)$.

We now want to compute the second layer of the radical series of $\nabla(2,1)$. This will be accomplished by calculating the second socle layer of $T(2,1)$ using the Ext^1-results of Proposition 3.2.1.

Proposition 3.4.1. *There exist the following isomorphisms of G-modules:*

(a) $\operatorname{soc}^2_{G_1} T(2,1)/\operatorname{soc}_{G_1} T(2,1) \cong \nabla(1,0)^{(1)}$,

(b) $\operatorname{soc}^2_G T(2,1)/\operatorname{soc}_G T(2,1) \cong L(1,0)^{(1)}$,

(c) $\operatorname{rad}_G \nabla(2,1)/\operatorname{rad}^2_G \nabla(2,1) \cong L(1,0)^{(1)}$.

Proof. (a) and (b) For $\lambda \in X_1$, one has isomorphisms

$$\begin{aligned}\operatorname{Hom}_{G_1}(L(\lambda), T(2,1)/L(0,1)) &\cong \operatorname{Hom}_{G_1}(L(\lambda), Q_1(0,1)/L(0,1)) \\ &\cong \operatorname{Ext}^1_{G_1}(L(\lambda), L(0,1)),\end{aligned}$$

where the first isomorphism holds since $T(2,1) \cong Q_1(0,1)$, and the second comes from degree shifting in cohomology. Proposition 3.2.1 then establishes that

$$\operatorname{soc}^2_{G_1} T(2,1)/\operatorname{soc}_{G_1} T(2,1)$$

is 7-dimensional and is trivial as a G_1-module. Considering this, as a G-module, its only possible composition factors are k and $L(1,0)^{(1)}$. Since k does not extend $L(0,1)$ nontrivially over G, we conclude that

$$\operatorname{soc}^2_G T(2,1)/\operatorname{soc}_G T(2,1) \cong L(1,0)^{(1)},$$

and that

$$\operatorname{soc}^2_{G_1} T(2,1)/\operatorname{soc}_{G_1} T(2,1) \cong \nabla(1,0)^{(1)}$$

(which agrees with the G-module structure in Proposition 3.2.1; this extended argument is included to be precise on the inference of G-module structure).

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(c) Every tilting G-module and every simple G-module is self-dual, and

$$\Delta(2,1)^* \cong \nabla(2,1),$$

so we will work in the dual situation. We have that $\Delta(2,1) \subseteq T(2,1)$, therefore

$$\mathrm{soc}^2_G \Delta(2,1)/\mathrm{soc}_G \Delta(2,1) \subseteq \mathrm{soc}^2_G T(2,1)/\mathrm{soc}_G T(2,1) \cong L(1,0)^{(1)}.$$

But, $\mathrm{soc}^2_G \Delta(2,1)/\mathrm{soc}_G \Delta(2,1) \neq 0$, therefore $\mathrm{soc}^2_G \Delta(2,1)/\mathrm{soc}_G \Delta(2,1) \cong L(1,0)^{(1)}$.

Finally, one has

$$\mathrm{rad}_G \nabla(2,1)/\mathrm{rad}^2_G \nabla(2,1) \cong (\mathrm{soc}^2_G \Delta(2,1)/\mathrm{soc}_G \Delta(2,1))^* \cong L(1,0)^{(1)}. \qquad \square$$

3.5. This following example answers Question 2.2.1 in the negative, and it is also a counterexample to Conjecture 2.2.3 ($\Leftarrow$), since $\mathrm{St} \otimes \nabla(2,1)$ has a good filtration.

Theorem 3.5.1. *The module $\nabla(2,1)$ for the group G of type G_2 does not have a good 2-filtration.*

Proof. Suppose that

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \nabla(2,1)$$

is a good 2-filtration. In view of the structure of the radical series of $\nabla(2,1)$,

$$F_n/F_{n-1} \cong L(0,1) \quad \text{and} \quad F_{n-1}/F_{n-2} \cong \nabla(\mu)^{(1)},$$

with $L(1,0)$ being the G-head of $\nabla(\mu)$. Since $2\mu \leq (2,1)$ under the usual partial ordering of weights, we have

$$2\langle \mu, \alpha_0^\vee \rangle \leq \langle (2,1), \alpha_0^\vee \rangle = 7,$$

where α_0 denotes the maximal short root. Therefore,

$$\langle \mu, \alpha_0^\vee \rangle \leq 3,$$

implying that $\mu \in \{(0,0),(1,0),(0,1)\}$. But $L(1,0)$ is not in the head of $\nabla(\mu)$ for any of these choices of μ, therefore no such filtration on $\nabla(2,1)$ is possible. $\square$

Remark 3.5.1. H. H. Andersen has pointed out to us that the module $\nabla(0,2)$ is uniserial, and that its top two layers are the same as those of $\nabla(2,1)$, so that this module also fails to have a good 2-filtration.

3.6. The lack of a good 2-filtration leads to other interesting phenomena which will factor into our proof that the Tilting Module Conjecture does not hold.

Proposition 3.6.1. *For the group G of type G_2 with $p = 2$, the module $\mathrm{St} \otimes \mathrm{rad}_G \nabla(2,1)$ does not have a good filtration.*

Proof. It suffices to show that the Steinberg block component of this module does not admit a good filtration. Any composition factor of $\mathrm{St} \otimes \mathrm{rad}_G \nabla(2,1)$ that lies within the Steinberg block has the form $\mathrm{St} \otimes L(\mu)^{(1)}$. Further, for any such composition factor, we have $2\mu \leq (2,1)$, and as in the previous proof one has $\mu \in \{(0,0),(1,0),(0,1)\}$. Since $L(1,0)^{(1)}$

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is the head of $\mathrm{rad}_G\,\nabla(2,1)$, $\mathrm{St}\otimes L(1,0)^{(1)}$ must appear in the head of (the Steinberg block of) $\mathrm{St}\otimes \mathrm{rad}_G\,\nabla(2,1)$. But we again reason as in the proof above. If the Steinberg block of $\mathrm{St}\otimes \mathrm{rad}_G\,\nabla(2,1)$ has a good filtration, then there is some $\nabla(\mu)$ such that $L(1,0)$ is the head of $\nabla(\mu)$ and $\mathrm{St}\otimes\nabla(\mu)^{(1)}$ is a subquotient of $\mathrm{St}\otimes \mathrm{rad}_G\,\nabla(2,1)$. But no such subquotient is possible with the limitations on μ. □

3.7. Conjecture 2.2.3 (⇐): Minimal counterexample. The module $\mathrm{St}\otimes\nabla(2,1)$ has a good filtration, and none of its ∇-quotients map onto $L(3,1)\cong \mathrm{St}\otimes L(1,0)^{(1)}$. It was observed earlier that two copies of St are contained in $\mathrm{St}\otimes L(0,1)$. Therefore, it follows that one of these copies nontrivially extends the composition factor $\mathrm{St}\otimes L(1,0)^{(1)}$ in $\mathrm{St}\otimes \mathrm{rad}_G\,\nabla(2,1)$ that comes from

$$\mathrm{St}\otimes[\mathrm{rad}_G\,\nabla(2,1)/\mathrm{rad}_G^2\,\nabla(2,1)].$$

Now define the G-module M via the short exact sequence

$$0\to \mathrm{rad}_G^2\,T(2,1)\to T(2,1)\to M\to 0. \tag{3.7.1}$$

Then the non-split sequences

$$0\to \mathrm{rad}_G^2\,\nabla(2,1)\to \nabla(2,1)\to M\to 0$$

and

$$0\to L(1,0)^{(1)}\to M\to L(0,1)\to 0$$

are immediate consequences of Proposition 3.4.1.

From weight considerations, Proposition 3.3.1, and Theorem 3.3.1, it follows that

$$\mathrm{St}\otimes M\cong T(1,2)\oplus S,$$

where S is the summand containing all composition factors in the G_1-Steinberg block of $\mathrm{St}\otimes M$. We know that S contains $\mathrm{St}\otimes L(1,0)^{(1)}$ once as a composition factor and the Steinberg module twice. No other composition factors occur. As a consequence of previous discussion, one of the Steinberg factors must sit on top of $\mathrm{St}\otimes L(1,0)^{(1)}$. Hence, together the two composition factors form a module that is isomorphic to $\mathrm{St}\otimes\nabla(1,0)^{(1)}\cong\nabla(3,1)$. In conclusion,

$$\begin{aligned}\mathrm{St}\otimes M&\cong T(1,2)\oplus(\mathrm{St}\otimes\nabla(1,0)^{(1)})\oplus \mathrm{St}\\ &\cong T(1,2)\oplus\nabla(3,1)\oplus\mathrm{St},\end{aligned}$$

which has a good filtration. Then this proves the following:

Proposition 3.7.1. *Let M be the module defined in* (3.7.1).

(a) $\mathrm{St}\otimes M$ *has a good filtration.*

(b) $\mathrm{Hom}_G(\mathrm{St},\mathrm{St}\otimes M)=k$.

The module M has composition factors $L(0,1)$ and $L(1,0)^{(1)}$. Since

$$L(1,0)^{(1)}\ncong\nabla(1,0)^{(1)},$$

we see that M does not have a good 2-filtration, even though $\mathrm{St}\otimes M$ has a good filtration. One could then consider M as a minimal counterexample to Conjecture 2.2.3 (⇐), as it has only two composition factors.

Indeed, in the general context of a semisimple G and arbitrary prime p, a counterexample with only one composition factor is not possible. For example, if for some $\lambda=\lambda_0+p\lambda_1$,

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with $\lambda_0 \in X_1$ and $\lambda_1 \in X_+$, the module $\mathrm{St} \otimes L(\lambda_0) \otimes L(\lambda_1)^{(1)}$ has a good filtration, then it must be tilting. But then $\mathrm{St} \otimes L(\lambda_0) \otimes T((p-1)\rho - \lambda_0) \otimes L(\lambda_1)^{(1)}$ is tilting, and since St is a summand of $L(\lambda_0) \otimes T((p-1)\rho - \lambda_0)$, we have that $\mathrm{St} \otimes \mathrm{St} \otimes L(\lambda_1)^{(1)}$ is also tilting, and then that $\mathrm{St}^{\otimes 3} \otimes L(\lambda_1)^{(1)}$ is tilting. But St is a summand of $\mathrm{St}^{\otimes 3}$, so that $\mathrm{St} \otimes L(\lambda_1)^{(1)}$ is tilting, and we conclude that $L(\lambda_1) \cong \nabla(\lambda_1) \cong T(\lambda_1)$. Consequently, $L(\lambda_0) \otimes L(\lambda_1)^{(1)}$ is a good p-filtration module.

4. On the Tilting Module Conjecture

4.1. We continue to assume that G has a root system of type G_2 and the prime $p = 2$. The fact that $\mathrm{St} \otimes \mathrm{rad}_G \nabla(2,1)$ does not have a good filtration guarantees that the Tilting Module Conjecture does not hold in this case. This essentially follows from [18, Theorem 5.1.1], but here we will give a simple self-contained proof of this fact using the results already established in this paper.

Theorem 4.1.1. *The Tilting Module Conjecture does not hold for groups of type* G_2 *and* $p = 2$*.*

Proof. Assume that the Tilting Module Conjecture holds, so that $T(2,2)|_{G_1} \cong Q_1(0,0)$. From the G-module structure of the G_1-socle of $\mathrm{St} \otimes \mathrm{St}$, as observed in the proof of Proposition 3.3.1 part (d), and Theorem 3.3.1, one then concludes that (as G-modules)

$$\mathrm{St} \otimes \mathrm{St} \cong T(2,2) \oplus T(2,1)^{\oplus 2} \oplus T(3,1)^{\oplus 2}. \tag{4.1.1}$$

In particular, the tilting module $T(2,1)$ appears twice in the tensor product $\mathrm{St} \otimes \mathrm{St}$. Let M be the quotient of $T(2,1)$ from Proposition 3.7.1. Then we have that

$$2 \leq \dim \mathrm{Hom}_G(\mathrm{St} \otimes \mathrm{St}, M) = \dim \mathrm{Hom}_G(\mathrm{St}, M \otimes \mathrm{St}),$$

a contradiction to part (b) of Proposition 3.7.1. □

4.2. The socle of $T(2,2)$. There are two copies of $L(0,1)$ in the G-socle of $\mathrm{St} \otimes \mathrm{St}$, but we have now established that $T(2,1)$ occurs as a summand of $\mathrm{St} \otimes \mathrm{St}$ at most once (i.e., the decomposition in (4.1.1) *fails* to hold). Looking again at Theorem 3.3.1, it follows that $L(0,1)$ must appear as a submodule of $T(2,2)$. This fact has been independently confirmed by Doty's program [7, 19], which has computed more precisely that

$$k \oplus L(0,1) \cong \mathrm{soc}_G \Delta(2,2) \subseteq T(2,2).$$

We note that, whenever $T(\hat{\lambda}) = Q_1(\lambda)$ as a G_1-module for $\lambda \in X_1$, then $\mathrm{soc}_G \Delta(\hat{\lambda})$ must be simple and isomorphic to $L(\lambda)$.

4.3. The Humphreys–Verma Conjecture. Although $T(2,2)$ is not a lift of $Q_1(0,0)$, it is still possible that $Q_1(0,0)$ has some other G-module structure, so the Humphreys–Verma Conjecture remains open for now. Nevertheless, it is significant that even if there is some G-structure, it will not occur as a G-submodule of $\mathrm{St} \otimes \mathrm{St}$ (though it could appear as a subquotient). This defies the long held expectation, going back to early work by Humphreys and Verma, that a G-structure should occur in precisely this way.

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References

[1] *P. N. Achar*, *S. Makisumi*, *S. Riche* and *G. Williamson*, Koszul duality for Kac–Moody groups and characters of tilting modules, J. Amer. Math. Soc. **32** (2019), no. 1, 261–310.

[2] *H. H. Andersen*, p-filtrations and the Steinberg module, J. Algebra **244** (2001), no. 2, 664–683.

[3] *H. H. Andersen*, p-filtrations of dual Weyl modules, Adv. Math. **352** (2019), 231–245.

[4] *M. F. Anwar*, Representations and cohomology of algebraic groups, Ph.D. Thesis, University of York, 2011.

[5] *J. W. Ballard*, Injective modules for restricted enveloping algebras, Math. Z. **163** (1978), no. 1, 57–63.

[6] *C. P. Bendel*, *D. K. Nakano*, *C. Pillen* and *P. Sobaje*, On tensoring with the Steinberg representation, Transformation Groups (2019), DOI 10.1007/s00031-019-09530-x.

[7] *S. Doty*, WeylModules, a GAP package, Version 1.1, 2009, `http://doty.math.luc.edu/weylmodules`.

[8] *M. F. Dowd* and *P. Sin*, On representations of algebraic groups in characteristic two, Comm. Algebra **24** (1996), no. 8, 2597–2686.

[9] *J. E. Humphreys*, Modular representations of finite groups of Lie type, London Math. Soc. Lecture Note Ser. **326**, Cambridge University Press, Cambridge 2006.

[10] *J. C. Jantzen*, Darstellungen halbeinfacher Gruppen und kontravariante Formen, J. reine angew. Math. **290** (1977), 117–141.

[11] *J. C. Jantzen*, Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne, J. reine angew. Math. **317** (1980), 157–199.

[12] *J. C. Jantzen*, First cohomology groups for classical Lie algebras, in: Representation theory of finite groups and finite-dimensional algebras (Bielefeld 1991), Progr. Math. **95**, Birkhäuser, Basel (1991), 289–315.

[13] *J. C. Jantzen*, Representations of algebraic groups, 2nd ed., Math. Surveys Monogr. **107**, American Mathematical Society, Providence 2003.

[14] *T. Kildetoft* and *D. K. Nakano*, On good (p, r)-filtrations for rational G-modules, J. Algebra **423** (2015), 702–725.

[15] *F. Lübeck*, Tables of weight multiplicities, `http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/WMSmall/index.html`.

[16] *B. J. Parshall* and *L. L. Scott*, On p-filtrations of Weyl modules, J. Lond. Math. Soc. (2) **91** (2015), no. 1, 127–158.

[17] *S. Riche* and *G. Williamson*, Tilting modules and the p-canonical basis, Astérisque (2018), no. 397.

[18] *P. Sobaje*, On (p, r)-filtrations and tilting modules, Proc. Amer. Math. Soc. **146** (2018), no. 5, 1951–1961.

[19] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.8.10*, 2018, `https://www.gap-system.org`.

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