2019 IEEE 58th Conference on Decision and Control (CDC)
Palais des Congrès et des Expositions Nice Acropolis
Nice, France, December 11-13, 2019

A two-sided iterative framework for model reduction of linear systems with quadratic output

Ion Victor Gosea and Athanasios C. Antoulas

Abstract— **We propose a model reduction procedure for approximating large-scale linear systems with quadratic output by means of lower dimensional systems with the same structure. The framework is based on an iteration that, at each step, computes left and right projections matrices (hence two-sided). This is done by means of solving two linear Sylvester equations.**

I. INTRODUCTION

The simulation, control and analysis of models characterizing dynamical time-dependent processes is of interest in many areas of science and industry. It is of course desirable for these tasks to be performed in a direct, automated way and as fast as possible. Sometimes, the degrees of freedom or dimension of the mathematical model that characterizes the technical process could be very large. This happens usually due to the need of having accurate spatial discretization schemes of the target domain.

In these cases, it is not feasible to use the large original models directly. Consequently, reduced order surrogate models are used instead.

Model order reduction (MOR) is a highly used methodology for which the practical application is reducing the computational complexity (in terms of time and memory) of large scale complex models in numerical simulations. The general goal of most of MOR methods is to construct a much smaller system with the same structure and similar response characteristics as the original. For an overview of the state of the art methodologies applicable to reducing linear systems, we refer the reader to [2], [3], [7]. Moreover, for extensions to nonlinear systems, see the surveys [4], [18].

Nonlinear dynamics is usually intrinsically present in most time-dependent processes. Consequently, the study, analysis, and modeling of nonlinear dynamical systems have received a lot of attention. To avoid applying linearization techniques (which induce an additional level of approximation and error generation), it is sometimes preferable to devise reduction methods that can be directly applied to the nonlinear system. MOR of nonlinear dynamical systems is still a challenging task. Nevertheless, considerable progress has been made in recent years for extending classical MOR methods (for linear systems) to reducing certain classes of structured mildly nonlinear systems such as bilinear or quadratic-bilinear (QB) systems.

The latter class represents an important category which typically arise from reformulating or lifting systems with more involved analytic nonlinearities. This can be done by means of the so-called McCormick's relaxation approach in [16]. By introducing auxiliary variables, one can recast the original nonlinear system as a QB system without performing any approximation. The first important contribution towards this approach was [13].

That is why, in recent years, MOR of QB systems has been intensively studied. We mention Krylov-based methods in [1], [6], balanced truncation in [8], $\mathcal{H}_2$-quasi-optimal approximation in [9], and data-driven methods (the Loewner framework) in [11].

The main focus of our research is studying MOR of linear systems with quadratic output (LQO). Although the nonlinearity is present in the state-output equation only, and not in the state equation, this class is appealing since it is at the boundary between linear and quadratic (nonlinear) systems. The classical approach for reducing such systems is to first rewrite the governing equations as an equivalent linear system with multiple outputs. Afterwards, one can apply basically any MOR method suitable for multiple output linear systems. This idea has been applied for balanced truncation in [19] and for Krylov-type methods in [20]. One downside of this approach is that it often produces systems with large number of outputs and hence, it is computationally expensive. In [17], a different approach is proposed. It is based on constructing a QB system, whose (single) linear output coincides with the quadratic output of the original LQO system. Then, again, any method for reducing QB systems can be applied. Typically, the bottleneck of such methods is represented by solving quadratic Lyapunov or Sylvester equations. More recently, a structure-preserving balanced truncation procedure was proposed in [10]. It is based on defining an appropriate algebraic observability Gramian and approximating the original LQO system directly by a low dimensional LQO system.

The method proposed in this work is based on an iterative interpolation procedure. The left and right subspaces are represented by matrices that are solutions of Sylvester equations. Hence, only linear equations need to be solved in our MOR approach. The inspiration for the proposed method came from the iterative rational Krylov algorithm, or in short IRKA, which was introduced in [14] for reducing linear systems. It was proven to be a very effective iterative procedure, which, upon convergence, yields a locally $\mathcal{H}_2$

This work was supported by the NSF Grant CNS-1701292

I. V. Gosea is with the Data-Driven Reduction and Identification Group, Max Planck Institute for Dynamics of Complex Technical Systems, Sandtorstrasse 1, 39106 Magdeburg, Germany, gosea@mpi-magdeburg.mpg.de

A. C. Antoulas is with the Department of Electrical and Computer Engineering, Rice University, Houston, USA, MPI Magdeburg, Germany and Baylor College of Medicine, Houston, USA aca@rice.edu

978-1-7281-1398-2/19/$31.00 ©2019 IEEE

optimal reduced system. Over the years, many extensions were proposed to extend IRKA to reducing (mildly) nonlinear classes of systems such as bilinear in [5], QB in [9] or switched in [12].

The rest of the paper is organized as follows: In Section II, some theoretical aspects of linear systems with quadratic output are presented, including the derivation of time-domain input-output mappings. Additionally, we propose a reformulation of the state-output equation that allows interpreting certain quantities in a familiar manner.

In Section III, Gramian matrices are introduced for linear systems with linear and quadratic outputs. Moreover, we propose a definition of the $\mathcal{H}_2$ norm for LQO systems. In Section IV, we introduce the interpolation-based iterative procedure for MOR of LQO system. We sketch an algorithm that computes, at each step, a pair of projection matrices by solving Sylvester equations. Finally, Section V presents a numerical example while Section VI concludes the paper.

II. Linear systems with quadratic output

In this paper, we study linear systems with quadratic output $\mathbf{\Sigma} : (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{M})$, for which the underlying dynamics is characterized by the following equations

$$\mathbf{\Sigma} : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{M}\mathbf{x}(t). \end{cases} \tag{1}$$

with $\mathbf{A}, \mathbf{M} \in \mathbb{R}^{n \times n}, \mathbf{B}, \mathbf{C}^T \in \mathbb{R}^n$. We are using the same notation for the quadratic term in the second equation in (1) (the state-output equation) as was previously used in [17], [19], [20]. Note that the linear term $\mathbf{C}\mathbf{x}(t)$ appears in this equation, while in the above mentioned contributions was not present.

For simplicity of exposition, the single-input and single-output case is considered, but can be easily extend to multiple-input scenarios. Additionally, we assume that the matrix $\mathbf{A}$ is stable, i.e. it has all eigenvalues in the left-half complex plane.

Our goal is to construct a reduced order model (ROM) $\hat{\mathbf{\Sigma}} : (\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{M}})$ of order r with the same structure as the original system in (1), which is characterized by the following equations

$$\hat{\mathbf{\Sigma}} : \begin{cases} \dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t), \\ \hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{x}}^T(t)\hat{\mathbf{M}}\hat{\mathbf{x}}(t), \end{cases} \tag{2}$$

where $\hat{\mathbf{A}}, \hat{\mathbf{M}} \in \mathbb{R}^{r \times r}, \hat{\mathbf{B}}, \hat{\mathbf{C}} \in \mathbb{R}^r$, with $r \ll n$. The choice of the ROM should be such that, its output approximately matches that of the original system, i.e. $\mathbf{y}(t) \approx \hat{\mathbf{y}}(t)$ for all admissible inputs $\mathbf{u}(t)$.

A. Reformulation of the state-output equation

Introduce the vectorization operator $\mathrm{vec} : \mathbb{R}^{\mathrm{n \times n}} \to \mathbb{R}^{\mathrm{n}^2}$ as

$$\mathrm{vec}(\mathbf{X}) = \begin{bmatrix}(\mathbf{X}\mathbf{e}_1)^T & (\mathbf{X}\mathbf{e}_2)^T & \dots & (\mathbf{X}\mathbf{e}_n)^T\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\mathrm{n}^2}, \tag{3}$$

where $\mathbf{X} \in \mathbb{R}^{n \times n}$. Then, the following identity holds for any matrices $\mathbf{Y} \in \mathbb{R}^{m \times n}, \mathbf{Z} \in \mathbb{R}^{n \times p}$

$$\mathrm{vec}(\mathbf{Y}\mathbf{X}\mathbf{Z}) = (\mathbf{Z}^{\mathrm{T}} \otimes \mathbf{Y})\mathrm{vec}(\mathbf{X}), \tag{4}$$

where $\otimes$ represents the Kronecker product.

In what follows, we use $\mathbf{x}$ instead of $\mathbf{x}(t)$ to denote the time-dependent state variable.
Hence, by using (4), it follows that we can write

$$\mathrm{vec}(\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}) = (\mathbf{x}^{\mathrm{T}} \otimes \mathbf{x}^{\mathrm{T}})\mathrm{vec}(\mathbf{M}). \tag{5}$$

Now, since $\mathbf{x}^T\mathbf{M}\mathbf{x}$ is a scalar, use (5) and write

$$\begin{aligned} \mathbf{x}^T\mathbf{M}\mathbf{x} &= \mathrm{vec}(\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}) = (\mathbf{x}^{\mathrm{T}} \otimes \mathbf{x}^{\mathrm{T}})\mathrm{vec}(\mathbf{M}) \\ &= \big[(\mathbf{x}^T \otimes \mathbf{x}^T)\mathrm{vec}(\mathbf{M})\big]^T = \big[\mathrm{vec}(\mathbf{M})\big]^T(\mathbf{x} \otimes \mathbf{x}). \end{aligned} \tag{6}$$

Denote with $\mathbf{K} = \big[\mathrm{vec}(\mathbf{M})\big]^T$ where $\mathbf{K} \in \mathbb{R}^{1 \times n^2}$. By reformulating the quadratic term accordingly, we rewrite the second equation in (1) as follows

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{K}\Big(\mathbf{x}(t) \otimes \mathbf{x}(t)\Big). \tag{7}$$

The formulation of the quadratic term in (7) is the same as the term present in the differential state equations that correspond to QB systems (as in [6], [9], [11]). In what follows, we will sometimes use this formulation as well since most of the quantities of interest (Gramian matrices, time-domain and frequency-domain mappings etc.) can be easily extended from the QB case.

Note that, since the following result holds,

$$\mathbf{x}^T\mathbf{M}\mathbf{x} = \mathbf{x}^T\Big[(\mathbf{M} + \mathbf{M}^T)/2\Big]\mathbf{x},$$

one can assume that the matrix $\mathbf{M}$ is symmetric. Then, the following property can easily be proven

$$\mathbf{K}(\mathbf{v} \otimes \mathbf{w}) = \mathbf{K}(\mathbf{w} \otimes \mathbf{v}), \forall\ \mathbf{v}, \mathbf{w} \in \mathbb{R}^n. \tag{8}$$

Note that the following holds for all $\mathbf{v} \in \mathbb{R}^n$

$$\mathbf{K}\big(\mathbf{v} \otimes \mathbf{I}_n\big) = \mathbf{v}^T\mathbf{M}. \tag{9}$$

To prove the result in (9), first note that $\mathbf{v}^T\mathbf{M} \in \mathbb{R}^{1 \times n}$. Then, by using (4) and (8), we can write that

$$\begin{aligned} &\big(\mathbf{v}^T\mathbf{M}\big)^T = \mathrm{vec}(\mathbf{v}^T\mathbf{M}) = \big(\mathbf{I}_n \otimes \mathbf{v}^T\big)\mathrm{vec}(\mathbf{M}) = \big(\mathbf{I}_n \otimes \mathbf{v}^T\big)\mathbf{K}^T \\ &\Rightarrow \mathbf{v}^T\mathbf{M} = \Big(\big(\mathbf{I}_n \otimes \mathbf{v}^T\big)\mathbf{K}^T\Big)^T = \mathbf{K}\big(\mathbf{I}_n \otimes \mathbf{v}\big) = \mathbf{K}\big(\mathbf{v} \otimes \mathbf{I}_n\big). \end{aligned}$$

Finally, by multiplying the equality in (9) with $\mathbf{w} \in \mathbb{R}^n$ and by using classical properties of the Kronecker product, it follows that the following holds

$$\mathbf{K}\big(\mathbf{v} \otimes \mathbf{w}\big) = \mathbf{v}^T\mathbf{M}\mathbf{w}, \tag{10}$$

for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

B. Deriving time-domain input-output mappings

The solution of the differential equation in (1) is as follows

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau. \tag{11}$$

By assuming zero initial conditions $\mathbf{x}(0) = \mathbf{0}$, it follows that

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{u}(t-\tau)d\tau. \tag{12}$$

Then, by substituting the formula of the solution in (12) onto the state-output equation in (1), and by using the identity in (10), it follows that

$$\mathbf{y}(t) = \int_0^t \mathbf{C}e^{\mathbf{A}\tau}\mathbf{B}\mathbf{u}(t-\tau)d\tau + \mathbf{K}\Big[\int_0^t e^{\mathbf{A}\tau_1}\mathbf{B}\mathbf{u}(t-\tau_1)d\tau_1\Big]\otimes \Big[\int_0^t e^{\mathbf{A}\tau_2}\mathbf{B}\mathbf{u}(t-\tau_2)d\tau_2\Big] = \int_0^t \mathbf{C}e^{\mathbf{A}\tau}\mathbf{B}\mathbf{u}(t-\tau)d\tau + \int_0^t\int_0^t \Big[\mathbf{K}\big(e^{\mathbf{A}\tau_1}\mathbf{B}\otimes e^{\mathbf{A}\tau_2}\mathbf{B}\big)\Big]\Big[\mathbf{u}(t-\tau_1)\otimes\mathbf{u}(t-\tau_2)\Big]d\tau_1 d\tau_2$$

Let $\mathbf{h}(\tau_1)$ be the linear kernel, while $\overline{\mathbf{h}}(\tau_1,\tau_2)$ is the quadratic kernel with their exact definition as follows

$$\begin{cases} \mathbf{h}(\tau_1) = \mathbf{C}e^{\mathbf{A}\tau_1}\mathbf{B}, \\ \overline{\mathbf{h}}(\tau_1,\tau_2) = \mathbf{K}\big(e^{\mathbf{A}\tau_1}\mathbf{B}\otimes e^{\mathbf{A}\tau_2}\mathbf{B}\big). \end{cases} \tag{13}$$

Then, by following the above derivation, it follows that the input-output behavior of the LQO system can be written in terms of the two kernels as

$$\mathbf{y}(t) = \int_0^t \mathbf{h}(\tau)\mathbf{u}(t-\tau)d\tau + \int_0^t\int_0^t \overline{\mathbf{h}}(\tau_1,\tau_2)\Big[\mathbf{u}(t-\tau_1)\otimes\mathbf{u}(t-\tau_2)\Big]d\tau_1 d\tau_2. \tag{14}$$

III. Gramian matrices

Here, we introduce controllability and observability Gramians. First, we do so for the case of linear systems with linear output and then provide an extension for linear systems with quadratic output.

Assume that the matrix $\mathbf{A}$ is stable and that the original linear system is controllable and observable. Then, it follows that the Gramian matrices are unique, symmetric and positive definite. These matrices need to be computed (or only square root or low rank factors thereof) when applying the balanced truncation MOR method (see [2]).

A. Linear systems with linear output

Proceed with defining the Gramians for the simplified case when the output $\mathbf{y}$ is linear in terms of the variable $\mathbf{x}$, i.e. $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$. This can be viewed as a special case of the more general case treated in this work, i.e. by considering that $\mathbf{M} = \mathbf{0}$.

Introduce the input-to-state response of the system in (1) with $\mathbf{r}(t)$, where

$$\mathbf{r}(t) = e^{\mathbf{A}t}\mathbf{B}. \tag{15}$$

This quantity is independent of the observed output, i.e. depends only on how the control $\mathbf{u}$ enters the differential equation in (1).

We define the controllability infinite Gramian $\mathcal{P}\in\mathbb{R}^{n\times n}$ as follows

$$\mathcal{P} = \int_0^\infty \mathbf{r}(t)\mathbf{r}^T(t)dt = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}dt. \tag{16}$$

It follows that $\mathcal{P}$ satisfies the following Lyapunov equation

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{0}. \tag{17}$$

Additionally, denote the state-to-output response of the system in (1) (where $\mathbf{M} = 0$), with $\mathbf{o}(t)$. Then, one can write

$$\mathbf{o}(t) = \mathbf{C}e^{\mathbf{A}t}. \tag{18}$$

We define the observability infinite Gramian as

$$\mathcal{Q} = \int_0^\infty \mathbf{o}^T(t)\mathbf{o}(t)dt = \int_0^\infty e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}t}dt. \tag{19}$$

It follows that $\mathcal{Q}$ satisfies the following Lyapunov equation

$$\mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = \mathbf{0}. \tag{20}$$

B. Linear systems with quadratic output

We now consider the general case for which $\mathbf{M} \neq \mathbf{0}$. As previously stated, the input-to-state will not change by modifying the state-output equation. Hence, it follows that the controllability Gramian for this case (denoted with $\mathbf{P}$) coincides to that introduced in (16). Hence write $\mathbf{P} = \mathcal{P}$.

Additionally, let the state-output equation in (1) be written as in (7). Then, by following the definition of the two input-output kernels in Section II-B, the state-to-output response of the system in (1) can be partitioned into two components (the linear and the quadratic) as

$$\mathbf{o}_1(t) = \mathbf{C}e^{\mathbf{A}t}, \quad \overline{\mathbf{o}}(t_1,t_2) = \mathbf{K}\Big(e^{\mathbf{A}t_1}\mathbf{B}\otimes e^{\mathbf{A}t_2}\Big). \tag{21}$$

We define the observability infinite Gramian corresponding to kernel $\mathbf{o}_2(t_1,t_2)$ as

$$\begin{aligned} \overline{\mathcal{Q}} &= \int_0^\infty\int_0^\infty \overline{\mathbf{o}}^T(t_1,t_2)\overline{\mathbf{o}}(t_1,t_2)dt_1dt_2 \\ &= \int_0^\infty\int_0^\infty \Big[\mathbf{K}\Big(e^{\mathbf{A}t_1}\mathbf{B}\otimes e^{\mathbf{A}t_2}\Big)\Big]^T \mathbf{K}\Big(e^{\mathbf{A}t_1}\mathbf{B}\otimes e^{\mathbf{A}t_2}\Big)dt_1dt_2 \\ &= \int_0^\infty\int_0^\infty (\mathbf{B}^T e^{\mathbf{A}^T t_1}\otimes e^{\mathbf{A}^T t_2})\mathbf{K}^T\mathbf{K}\big(e^{\mathbf{A}t_1}\mathbf{B}\otimes e^{\mathbf{A}t_2}\big)dt_1dt_2 \\ &= \int_0^\infty e^{\mathbf{A}^T t_2}\mathbf{U}e^{\mathbf{A}t_2}dt_2, \end{aligned} \tag{22}$$

where $\mathbf{U} = \int_0^\infty \big(\mathbf{I}_n\otimes\mathbf{B}^T e^{\mathbf{A}^T t_1}\big)\mathbf{K}^T\mathbf{K}\big(e^{\mathbf{A}t_1}\mathbf{B}\otimes\mathbf{I}_n\big)dt_1$. From (22), it follows that the matrix $\overline{\mathcal{Q}}$ satisfies the equation

$$\mathbf{A}^T\overline{\mathcal{Q}} + \overline{\mathcal{Q}}\mathbf{A} + \mathbf{U} = \mathbf{0}. \tag{23}$$

By choosing $\mathbf{v} = e^{\mathbf{A}t_1}\mathbf{B}$ in (9), it follows that $\mathbf{K}\big(e^{\mathbf{A}t_1}\mathbf{B}\otimes\mathbf{I}_n\big) = (e^{\mathbf{A}t_1}\mathbf{B})^T\mathbf{M}$. Then, by substituting this relation into the definition of matrix $\mathbf{U}$, it follows that

$$\begin{aligned} \mathbf{U} &= \int_0^\infty \mathbf{M}^T e^{\mathbf{A}t_1}\mathbf{B}(e^{\mathbf{A}t_1}\mathbf{B})^T\mathbf{M}dt_1 \\ &= \mathbf{M}^T\Big[\int_0^\infty e^{\mathbf{A}t_1}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t_1}dt_1\Big]\mathbf{M} = \mathbf{M}^T\mathbf{P}\mathbf{M}. \end{aligned} \tag{24}$$

From (23) and (24), it follows that the matrix $\overline{\mathcal{Q}}$ satisfies the following Lyapunov equation

$$\mathbf{A}^T\overline{\mathcal{Q}} + \overline{\mathcal{Q}}\mathbf{A} + \mathbf{M}^T\mathbf{P}\mathbf{M} = \mathbf{0}. \tag{25}$$

Let $\mathbf{Q} = \mathcal{Q} + \overline{\mathcal{Q}}$ be the observability Gramian of the LQO system in (1). By adding the equations in (20) and in (25), it follows that the Gramian $\mathbf{Q}$ satisfies the following Lyapunov equation

$$\mathbf{A}^T\mathbf{Q}+\mathbf{Q}\mathbf{A}+\mathbf{C}^T\mathbf{C}+\mathbf{M}^T\mathbf{P}\mathbf{M}=\mathbf{0}. \tag{26}$$

Note that equation (26) corresponds to equation (3.6) in [10], provided that the linear output term is neglected, i.e. $\mathbf{C}=\mathbf{0}$.

C. The $\mathcal{H}_2$ norm

For linear systems with linear output ($\mathbf{M}=\mathbf{0}$), it is known that the $\mathcal{H}_2$ norm of the system can be written in terms of the linear kernel $\mathbf{h}(\tau)=\mathbf{C}e^{\mathbf{A}\tau}\mathbf{B}$, as

$$\|\mathbf{\Sigma}\|_{\mathcal{H}_2}^2=\int_0^\infty \mathbf{h}(\tau)\mathbf{h}^T(\tau)d\tau. \tag{27}$$

Then, it directly follows from the definitions introduced in (16) and (19), that the $\mathcal{H}_2$ norm can be written in terms of the Gramians $\mathcal{P}$ and $\mathcal{Q}$ as

$$\|\mathbf{\Sigma}\|_{\mathcal{H}_2}^2=\mathbf{B}^T\mathcal{Q}\mathbf{B}=\mathbf{C}\mathcal{P}\mathbf{C}^T. \tag{28}$$

Next, based on the definitions introduced in Section III-B, one can consequently define the $\mathcal{H}_2$ norm of the LQO system in (1), as follows

$$\begin{aligned}\|\mathbf{\Sigma}\|_{\mathcal{H}_2}^2&=\int_0^\infty \mathbf{h}(\tau)\mathbf{h}^T(\tau)d\tau\\&+\int_0^\infty\int_0^\infty \overline{\mathbf{h}}(\tau_1,\tau_2)\overline{\mathbf{h}}^T(\tau_1,\tau_2)d\tau_1 d\tau_2\\&=\mathbf{B}^T\mathcal{Q}\mathbf{B}+\mathbf{B}^T\overline{\mathcal{Q}}\mathbf{B}.\end{aligned} \tag{29}$$

Note that equation (29) corresponds to equation (4.3) proposed in [10], provided that the linear output term is neglected, i.e. $\mathbf{C}=\mathbf{0}$. In this case, $\mathbf{h}(\tau)=0,\ \forall\tau\geqslant 0$.

Since the observability Gramian can be split $\mathbf{Q}=\mathcal{Q}+\overline{\mathcal{Q}}$, it follows that the identity in (29) can be rewritten as

$$\|\mathbf{\Sigma}\|_{\mathcal{H}_2}^2=\mathbf{B}^T\mathbf{Q}\mathbf{B}. \tag{30}$$

IV. Interpolation based iterative procedure

In this section, we introduce the proposed procedure for reducing LQO systems. It is an iterative procedure that interpolates, at each step, input-output mappings in frequency domain corresponding to the original system.

A. Frequency-domain input-output mappings

For linear systems (with linear output), the transfer function represents a system invariant quantity that relates the input and output of the system. It can be explicitly written in terms of the system matrices as follows

$$\mathbf{H}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}=\mathbf{C}\mathbf{\Phi}(s)\mathbf{B}, \tag{31}$$

where $\mathbf{\Phi}(s)=(s\mathbf{I}-\mathbf{A})^{-1}$. Note that $\mathbf{H}(s)$ can be obtained by applying the Laplace transform to the linear kernel $\mathbf{h}(t)$ in (13). One can also apply the (generalized multivariate) Laplace transform to the quadratic kernel in (13). Then, write the definition of the quadratic transfer functions as

$$\begin{aligned}\overline{\mathbf{H}}(s_1,s_2)&=\mathbf{K}\big[(s_1\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\otimes(s_2\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\big]\\&=\mathbf{K}\big[\mathbf{\Phi}(s_1)\mathbf{B}\otimes\mathbf{\Phi}(s_2)\mathbf{B}\big].\end{aligned} \tag{32}$$

B. An interpolation framework

In general, let $\{\mu_1,\mu_2,\ldots,\mu_{2k}\}$ and $\{\lambda_1,\lambda_2,\ldots,\lambda_{2k}\}$ be sets of left and respectively, right interpolation points with $\mu_i,\lambda_i\in\mathbb{C}$. The interpolation grids are constructed as follows

$$\begin{aligned}&\text{Left grid}: \boldsymbol{\mu}=[\{\mu_1\},\{\lambda_1,\mu_2\},\ldots,\{\mu_{2k-1}\},\{\lambda_{2k-1},\mu_{2k}\}],\\&\text{Right grid}: \boldsymbol{\lambda}=[\{\lambda_1\},\{\lambda_2\},\ldots,\{\lambda_{2k}\}].\end{aligned} \tag{33}$$

Introduce the generalized reachability matrix $\mathcal{R}\in\mathbb{C}^{n\times 2k}$ corresponding to grid $\boldsymbol{\lambda}$ in (33) as follows

$$\mathcal{R}=\begin{bmatrix}\mathbf{\Phi}(\lambda_1)\mathbf{B} & \mathbf{\Phi}(\lambda_2)\mathbf{B} & \cdots & \mathbf{\Phi}(\lambda_{2k-1})\mathbf{B} & \mathbf{\Phi}(\lambda_{2k})\mathbf{B}\end{bmatrix}, \tag{34}$$

and the generalized observability matrix $\mathcal{O}\in\mathbb{C}^{2k\times n}$ corresponding to grid $\boldsymbol{\mu}$ in (33) as

$$\mathcal{O}=\begin{bmatrix}\mathbf{C}\mathbf{\Phi}(\mu_1)\\ \mathbf{K}\big[\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\mu_2)\big]\\ \vdots\\ \mathbf{C}\mathbf{\Phi}(\mu_{2k-1})\\ \mathbf{K}\big[\mathbf{\Phi}(\lambda_{2k-1})\mathbf{B}\otimes\mathbf{\Phi}(\mu_{2k})\big]\end{bmatrix}=\begin{bmatrix}\mathbf{C}\mathbf{\Phi}(\mu_1)\\ [\mathbf{\Phi}(\lambda_1)\mathbf{B}]^T\mathbf{M}\mathbf{\Phi}(\mu_2)\\ \vdots\\ \mathbf{C}\mathbf{\Phi}(\mu_{2k-1})\\ [\mathbf{\Phi}(\lambda_{2k-1})\mathbf{B}]^T\mathbf{M}\mathbf{\Phi}(\lambda_{2k})\end{bmatrix}. \tag{35}$$

Note that, the following relations hold and hence relate the entries of matrices $\mathcal{R}$ and $\mathcal{O}$ to the transfer functions in Section IV-A

$$\begin{aligned}\mathbf{C}\mathcal{R}&=\begin{bmatrix}\mathbf{H}(\lambda_1) & \mathbf{H}(\lambda_2) & \cdots & \mathbf{H}(\lambda_{2k-1} & \mathbf{H}(\lambda_{2k})\end{bmatrix},\\(\mathcal{O}\mathbf{B})^T&=\big[\mathbf{H}(\mu_1)\quad \overline{\mathbf{H}}(\lambda_1,\mu_2)\quad\cdots\\&\qquad\qquad \mathbf{H}(\mu_{2k-1})\quad \overline{\mathbf{H}}(\lambda_{2k-1},\mu_{2k})\big].\end{aligned} \tag{36}$$

Also, it follows by construction that the matrix $\mathcal{R}$ satisfies the following Sylvester equation

$$\mathbf{A}\mathcal{R}+\mathcal{R}(-\mathbf{\Lambda})+\mathbf{B}\mathbf{r}=\mathbf{0}, \tag{37}$$

$\mathbf{r}=[1\ 1\ \cdots\ 1]\in\mathbb{R}^{1\times 2k}$ and $\mathbf{\Lambda}=\operatorname{diag}(\lambda_1,\ldots,\lambda_{2k})\in\mathbb{R}^{2k\times 2k}$. Additionally, also by construction, one can show that the matrix $\mathcal{O}$ satisfies the following equation

$$\mathcal{O}\mathbf{A}+\sum_{i=1}^{k}\mathbf{e}_{2i}\mathbf{K}(\mathcal{R}_{2i-1}\otimes\mathbf{I}_n)+\boldsymbol{\ell}\mathbf{C}=\mathbf{\Omega}\mathcal{O}, \tag{38}$$

where $\mathbf{e}_i$ is the i^{th} unit vector of length $2k$, i.e. $\mathbf{e}_i(h)=1$, for $h=i$ and $\mathbf{e}_i(h)=0$, for $h\neq i$. Additionally, $\boldsymbol{\ell}\in\mathbb{R}^{2k}$ is a vector with $\boldsymbol{\ell}=\sum_{i=1}^{k}\mathbf{e}_{2i-1}$ and $\mathbf{\Omega}=\operatorname{diag}(\mu_1,\ldots,\mu_{2k})\in\mathbb{R}^{2k\times 2k}$.

Next, rewrite the term $\mathbf{e}_{2i}\mathbf{K}(\mathcal{R}_{2i-1}\otimes\mathbf{I}_n)$ in (38) by using the identity in (9) as

$$\mathbf{e}_{2i}\mathbf{K}(\mathcal{R}_{2i-1}\otimes\mathbf{I}_n)=\mathcal{R}_{2i-1}^*\mathbf{M}=\mathbf{e}_{2i-1}^*\mathcal{R}^T\mathbf{M}. \tag{39}$$

By substituting (39) into (38), we can write that

$$\mathcal{O}\mathbf{A}+\sum_{i=1}^{k}\mathbf{e}_{2i}\mathbf{e}_{2i-1}^*\mathcal{R}^*\mathbf{M}+\boldsymbol{\ell}\mathbf{C}=\mathbf{\Omega}\mathcal{O}. \tag{40}$$

By applying the complex transposition of (40), it follows that

$$\mathbf{A}^*\mathcal{O}^*+\mathcal{O}^*(-\mathbf{\Omega})+\mathbf{M}^*\mathcal{R}\mathbf{Z}^*+\mathbf{C}^*\boldsymbol{\ell}^*=\mathbf{0}, \tag{41}$$

where $\mathbf{Z}^*=\sum_{i=1}^{k}\mathbf{e}_{2i-1}\mathbf{e}_{2i}^*\in\mathbb{R}^{2k\times 2k}$ is a block diagonal matrix containing k Jordan blocks $\mathbf{J}_2=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$.

The matrices $\mathcal{R},\ \mathcal{O}^*\in\mathbb{C}^{n\times 2r}$ will be used as projectors for the initialization step of the algorithm presented in the following section.

C. The proposed algorithm

We propose an iterative algorithm that computes projection matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n \times 2k}$ by solving (linear) Sylvester equations related to those introduced in (37) and (41):

$$\begin{aligned} \mathbf{A}\mathbf{X} + \mathbf{X}\hat{\mathbf{A}} + \mathbf{B}\hat{\mathbf{B}}^* = \mathbf{0}, \\ \mathbf{A}^*\mathbf{Y} + \mathbf{Y}\hat{\mathbf{A}} + \mathbf{M}^*\mathbf{X}\hat{\mathbf{M}} + \mathbf{C}^*\hat{\mathbf{C}} = \mathbf{0}. \end{aligned} \tag{42}$$

In the previous section it was shown that the solution of such equations are linked with certain sample values of the transfer functions corresponding to the original LQO system.

In general it is not clear exactly how to choose the interpolation points, i.e. the sets $\{\mu_1, \mu_2, \ldots, \mu_{2k}\}$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_{2k}\}$. Hence, we propose a solution to this problem by means of iteration.

We start with an initial guess of the points $\mu_i, \lambda_i \in \mathbb{C}$ for $i \in \{1, 2, \ldots, 2k\}$. Based on this selection, we compute the projection matrices for the initialization step. More precisely, for the right side take $\mathbf{X} = \mathcal{R}$ which satisfies equation (37), and respectively for the left side $\mathbf{Y} = \mathcal{O}^*$, that satisfies (41). Hence, start with the following reduced order matrices

$$\begin{aligned} \hat{\mathbf{A}} = (\mathcal{O}\mathcal{R})^{-1}\mathcal{O}\mathbf{A}\mathcal{R}, \quad \hat{\mathbf{B}} = (\mathcal{O}\mathcal{R})^{-1}\mathcal{O}\mathbf{B}, \\ \hat{\mathbf{C}} = \mathbf{C}\mathcal{R}, \quad \hat{\mathbf{M}} = \mathcal{R}^*\mathbf{M}\mathcal{R}. \end{aligned} \tag{43}$$

The iterative procedure is repeated until the eigenvalues of matrix $\hat{\mathbf{A}}$ are constant (the deviation with respect to the previous step does not exceed a certain tolerance value $\epsilon > 0$). The interpolation points can be found in the last step of the algorithm (when convergence is reached), i.e. as the eigenvalues of the matrix $-\hat{\mathbf{A}}$.

Similar to the case when IRKA is applied, in this case the reduced order model satisfies specific interpolation conditions. Optimality conditions can also be derived with respect to minimizing the $\mathcal{H}_2$ norm of the error system $\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}$.

Algorithm 1 Iterative two-sided MOR approach based on solving Sylvester equations

---

Input: $\mathbf{A} \in \mathbb{R}^{n \times n}$ stable matrix, $\mathbf{B}$, $\mathbf{C}^* \in \mathbb{R}^n$, $\mathbf{M} \in \mathbb{R}^{n \times n}$ symmetric matrix and initial choice of $\hat{\mathbf{A}}, \hat{\mathbf{M}} \in \mathbb{C}^{2k \times 2k}$, $\hat{\mathbf{B}}, \hat{\mathbf{C}}^* \in \mathbb{C}^{2k}$ as in (43).

1: **while**(change in $\sigma(\hat{\mathbf{A}}) > 0$) **do**

2: Solve the Sylvester equation for $\mathbf{X} \in \mathbb{C}^{n \times 2k}$:

$$\mathbf{A}\mathbf{X} + \mathbf{X}\hat{\mathbf{A}} + \mathbf{B}\hat{\mathbf{B}}^* = \mathbf{0}.$$

3: Solve the Sylvester equation for $\mathbf{Y} \in \mathbb{C}^{n \times 2k}$:

$$\mathbf{A}^*\mathbf{Y} + \mathbf{Y}\hat{\mathbf{A}} + \mathbf{M}^*\mathbf{X}\hat{\mathbf{M}} + \mathbf{C}^*\hat{\mathbf{C}} = \mathbf{0}.$$

4: Perform orthogonalization of the solution matrices:

$$\mathbf{V} = \text{orth}(\mathbf{X}), \ \ \mathbf{W} = \text{orth}(\mathbf{Y}).$$

5: Compute reduced-order matrices

$$\begin{aligned} \hat{\mathbf{A}} = (\mathbf{W}^*\mathbf{V})^{-1}\mathbf{W}^*\mathbf{A}\mathbf{V}, \quad \hat{\mathbf{B}} = (\mathbf{W}^*\mathbf{V})^{-1}\mathbf{W}^*\mathbf{B}, \\ \hat{\mathbf{C}} = \mathbf{C}\mathbf{V}, \quad \hat{\mathbf{M}} = \mathbf{V}^*\mathbf{M}\mathbf{V}. \end{aligned}$$

6: **end while**.

Output: Reduced order matrices $\hat{\mathbf{A}}, \hat{\mathbf{M}} \in \mathbb{C}^{2k \times 2k}$, $\hat{\mathbf{B}}, \hat{\mathbf{C}}^* \in \mathbb{C}^{2k}$.

V. Numerical example

In this section, we analyze the classical benchmark example of the international space station (ISS) 1R (Russian service module) from [15]. The original system characterizing the dynamics is a linear system (with linear output) of order $n = 270$ with 3 inputs and 3 outputs.

We modify this model such that the observed output is quadratic (with respect to state variable). We do this by introducing a matrix $\mathbf{M}$ in the state-output equation, as in (1). Take $\mathbf{M} = 2\mathbf{\Omega}_1$, where $\mathbf{\Omega}_1$ is a matrix composed of ones. Additionally, we select only the first input and the second output of the original ISS model. We approximate this large-scale LQO system of order $n = 270$ with a reduced-order LQO system of order $2k = 28$.

In Fig. 1, we depict the magnitude of the first transfer function of the LQO system evaluated in the interval $[10^{-2}, 10^3]$ rad/sec.

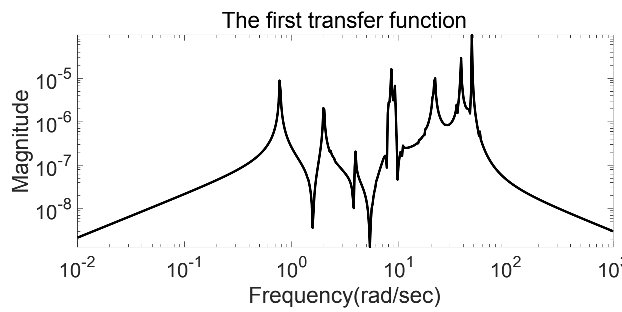

Fig. 1. Frequency response

Next, choose 56 left and right interpolation points as logarithmically spaced inside the interval $[10^{-1}, 10^2]$. Although not necessarily needed, we decided to choose real points in order to avoid complex arithmetics. Based on the procedure included in Algorithm 1, we compute a reduced-order LQO system and perform a time-domain simulation from 0 to 2 seconds (with $\Delta_t = 10^{-4}$). A classical forward Euler scheme has been used for approximating the derivative.

In Fig. 2, we depict the observed outputs of the original and reduced-order model. Note that the curve corresponding to $\hat{\mathbf{y}}(t)$ matches that of the original quadratic output $\mathbf{y}(t)$. Hence, conclude that the reduced system accurately reproduces the response of the original system.

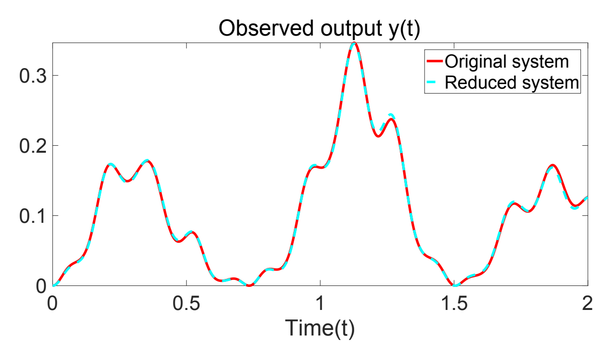

Fig. 2. Time-domain simulations: observed outputs

In the the time-domain simulation that was performed, the control input was chosen as $\mathbf{u}(t) = \cos(4t)$.

Additionally, the deviation between the observed outputs of the two systems is depicted in Fig. 3.

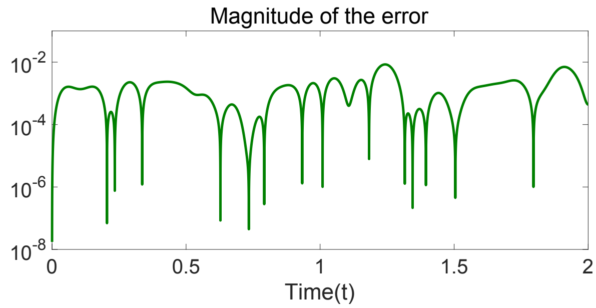

Fig. 3. The magnitude of the error between the two outputs

Finally, since the proposed procedure is iterative, we next analyze some convergence properties. The stopping tolerance has chosen to be $\epsilon = 10^{-10}$. This resulted in a number of 25 steps performed by Algorithm 1 (until the deviation between the eigenvalues of $\hat{\mathbf{A}}$ dropped below ϵ). In Fig. 4, we show how the eigenvalue deviation varied with the number of iterations.

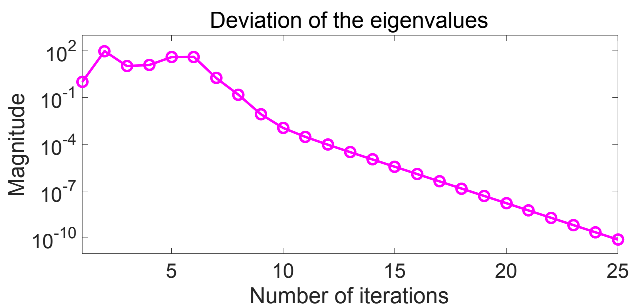

Fig. 4. The eigenvalue offset from one step to another

VI. Conclusion

In this work, we proposed a novel model order reduction procedure for approximating linear systems with quadratic output. The method is based on an iteration. At each step, one needs two solve two Sylvester equations in order to compute the left and right projection matrices. The method has been successfully applied to a benchmark example. The numerical simulations that were performed showed the potential of the new proposed method. Further research topics and developments include (i) deriving explicit interpolation-based optimality conditions (similar to the ones in [14]), (ii) study the convergence properties of the proposed method (related to the choice of the starting points), and (iii) extending the procedure for systems with general polynomial output and with mild nonlinearities (quadratic or bilinear) appearing in the state equation as well.

VII. Acknowledgments

The first author would like to thank Dr. Igor Pontes Duff for useful discussions on the topic of MOR for LQO systems.

References

[1] Ahmad, M., Benner, P., Jaimoukha, I.: Krylov subspace methods for model reduction of quadratic bilinear systems. IET Control Theory Appl. **10**, 2010–2018 (2016)

[2] Antoulas, A.C.: Approximation of Large-Scale Dynamical Systems. SIAM, Philadelphia, US (2005)

[3] Antoulas, A.C., Beattie, C.A., Gugercin, S.: Interpolatory model reduction of large-scale dynamical systems, pp. 3–58. Efficient Modeling and Control of Large-Scale Systems, Series: Computational Science & Engineering. Springer-Verlag (2010)

[4] Baur, U., Benner, P., Feng, L.: Model order reduction for linear and nonlinear systems: a system-theoretic perspective. Arch. Comput. Methods Eng. **21**(4), 331–358 (2014)

[5] Benner, P., Breiten, T.: Interpolation-based $\mathcal{H}_2$-model reduction of bilinear control systems. SIAM J. Matrix Anal. Appl. **33**, 859–885 (2012)

[6] Benner, P., Breiten, T.: Two-sided projection methods for nonlinear model order reduction. SIAM J. Sci. Comput. **37**, B239–B260 (2015)

[7] Benner, P., Cohen, A., Ohlberger, M., Willcox, K., (eds.): Model Reduction and Approximation. SIAM, Philadelphia, US (2017)

[8] Benner, P., Goyal, P.: Balanced truncation model order reduction for quadratic-bilinear control systems. Tech. rep., `https://arxiv.org/pdf/1705.00160.pdf` (April, 2017)

[9] Benner, P., Goyal, P., Gugercin, S.: $\mathcal{H}_2$-quasi-optimal model order reduction for quadratic-bilinear control systems. SIAM J. Matrix Anal. Appl. **39**(2), 983–1032 (2018)

[10] Benner, P., Goyal, P., Pontes Duff, I.: Gramians, energy functionals and balanced truncation for linear dynamical systems with quadratic outputs. Tech. rep., `https://arxiv.org/pdf/1909.04597.pdf` (September, 2019)

[11] Gosea, I.V., Antoulas, A.C.: Data-driven model order reduction of quadratic-bilinear systems. Numer. Linear Algebra Appl. **25**(6) (2018)

[12] Gosea, I.V., Antoulas, A.C.: On the $\mathcal{H}_2$ norm and iterative model order reduction of linear switched systems. In: Proceedings of the European Control Conference, Limassol, Cyprus (2018)

[13] Gu, C.: QLMOR: A projection-based nonlinear model order reduction approach using quadratic-linear representation of nonlinear systems. IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems **30**(9), 1307–1320 (2011)

[14] Gugercin, S., Antoulas, A., Beattie, C.: $\mathcal{H}_2$ model reduction for large-scale linear dynamical systems. SIAM J. Matrix Anal. Appl. **30**, 609–638 (2008)

[15] Gugercin, S., Antoulas, A.C., Bedrossian, N.: Approximation of the international space station 1r and 12a models. In: Proceedings of the 40th Conference on Decision and Control, Orlando, USA (2001)

[16] McCormick, G.P.: Computability of global solutions to factorable nonconvex programs: Part i - convex underestimating problems. Math. Program. **10**(1), 147–175 (1976)

[17] Pulch, R., Narayan, A.: Balanced truncation for model order reduction of linear dynamical systems with quadratic outputs. Tech. rep., `https://arxiv.org/pdf/1709.06677.pdf` (September, 2017)

[18] Scarciotti, G., Astolfi, A.: Nonlinear model reduction by moment matching, pp. 224–409. Foundations and Trends in Systems and Control 4(3-4) (2017)

[19] Van Beeumen, R., Meerbergen, K.: Model reduction by balanced truncation of linear systems with a quadratic output, pp. 2033–2036. T.E. Simons, G. Psihoyios, Ch. Tsitouras (eds.), International Conference on Numerical Analysis and Applied Mathematics (ICNAAM) (2010)

[20] Van Beeumen, R., Van Nimmen, K., Lombaert, G., Meerbergen, K.: Two-sided projection methods for nonlinear model order reduction. Int. J. Numer. Meth. Engng. **91**, 229–248 (2012)