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ON TENSORING WITH THE STEINBERG REPRESENTATION

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Abstract. Let G be a simple, simply connected algebraic group over an algebraically closed field of prime characteristic $p > 0$. Recent work of Kildetoft and Nakano and of Sobaje has shown close connections between two long-standing conjectures of Donkin: one on tilting modules and the lifting of projective modules for Frobenius kernels of G and another on the existence of certain filtrations of G-modules. A key question related to these conjectures is whether the tensor product of the rth Steinberg module with a simple module with p^rth restricted highest weight admits a good filtration. In this paper we verify this statement (i) when $p \geq 2h - 4$ (h is the Coxeter number), (ii) for all rank two groups, (iii) for $p \geq 3$ when the simple module corresponds to a fundamental weight and (iv) for a number of cases when the rank is less than or equal to five.

1. Introduction

1.1. Representations and filtrations

Let G be a simple, simply connected algebraic group scheme over the algebraically closed field k of characteristic $p > 0$. Let X be the set of integral weights and

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X_+ denote the dominant integral weights (relative to a fixed choice of a Borel subgroup). For any $\lambda \in X_+$, one can construct a non-zero module $\nabla(\lambda) = \text{ind}_B^G \lambda$ and the Weyl module $\Delta(\lambda)$. The character of these modules is given by Weyl's character formula. The finite-dimensional simple modules $L(\lambda)$ are indexed by dominant integral weights X_+ and can be realized as the socle of $\nabla(\lambda)$ (and the head of $\Delta(\lambda)$).

A central idea in this area has been the concept of good and Weyl filtrations. A G-module admits a *good filtration* (resp. *Weyl filtration*) if and only if it admits a G-filtration with sections of the form $\nabla(\mu)$ (resp. $\Delta(\mu)$) where $\mu \in X_+$. Cohomological criteria have been proved by Donkin and Scott which give necessary and sufficient conditions for a module to admit a good filtration (resp. Weyl filtration). A module which admits both a good and Weyl filtration is called a *tilting module*. Ringel [Rin] and Donkin [Don2] proved that (i) for every $\lambda \in X_+$, there is an indecomposable tilting module $T(\lambda)$ of highest weight λ and (ii) every tilting module is a direct sum of these indecomposable tilting modules.

Determining the characters of simple modules and tilting modules remains a central problem. In 2013, Williamson [W] produced families of counterexamples to the Lusztig conjecture for $G = \text{SL}_n$ and showed that the Lusztig Character Formula (LCF) in this case cannot hold for any linear bound on p relative to h (the associated Coxeter number). It is now evident that the character formula for simple modules will be highly dependent on the prime p. Therefore, this makes the understanding of the behavior of various G-filtrations even more crucial. A new approach has been introduced by Riche and Williamson [RW] in which they conjecture (and prove for the general linear group) that the characters of tilting modules and simple modules are given by p-Kazhdan–Lusztig polynomials that are constructed using p-Kazhdan–Lusztig bases.

1.2. Donkin's conjectures

Let $\lambda \in X_+$ with unique decomposition $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$ (p^rth restricted weights) and $\lambda_1 \in X_+$. One can define $\nabla^{(p,r)}(\lambda) = L(\lambda_0) \otimes \nabla(\lambda_1)^{(r)}$ where (r) denotes the twisting of the module action by the rth Frobenius morphism. A G-module M has a *good (p, r)-filtration* if and only if M has a filtration with factors of the form $\nabla^{(p,r)}(\mu)$ for suitable $\mu \in X_+$. Let $\text{St}_r = L((p^r - 1)\rho)$ (which is also isomorphic to $\nabla((p^r - 1)\rho)$ and $\Delta((p^r - 1)\rho)$) be the rth Steinberg module, where ρ is the sum of the fundamental weights.

The following conjecture, introduced by Donkin at MSRI in 1990, interrelates good filtrations with good (p, r)-filtrations via the Steinberg module.

Conjecture 1.2.1. *Let M be a finite-dimensional G-module. Then M has a good (p, r)-filtration if and only if $\text{St}_r \otimes M$ has a good filtration.*

At the same meeting, Donkin presented another conjecture that realizes the injective hull, $Q_r(\lambda)$, for $L(\lambda)$ over G_r as a tilting module, where G_r denotes the rth Frobenius kernel of G.

Conjecture 1.2.2. *For all $\lambda \in X_r$, $T(2(p^r - 1)\rho + w_0\lambda)|_{G_r} = Q_r(\lambda)$ where w_0 denotes the long element in the Weyl group W.*

In exciting recent developments, it has been shown how these conjectures are related. Kildetoft and Nakano [KN] proved that Conjecture 1.2.2 implies the forward direction (i.e., the only if portion) of Conjecture 1.2.1 (which we will denote by "Conjecture 1.2.1($\Rightarrow$)"). Sobaje [So] has proved that Conjecture 1.2.1 implies Conjecture 1.2.2. It is well known that Conjecture 1.2.1($\Rightarrow$) is equivalent to $\mathrm{St}_r \otimes L(\lambda)$ having a good filtration for all $\lambda \in X_r$. Note that, since the module $\mathrm{St}_r \otimes L(\lambda)$ is contravariantly self-dual, having a good filtration is equivalent to $\mathrm{St}_r \otimes L(\lambda)$ being a tilting module. Combining these results, the following hierarchy of conjectures has now been established:

$$\text{Conjecture 1.2.1} \quad \Rightarrow \quad \text{Conjecture 1.2.2} \quad \Rightarrow \quad \text{Conjecture 1.2.1}(\Rightarrow).$$

Using different approaches, Andersen [And3] and later Kildetoft and Nakano [KN] verified Conjecture 1.2.1($\Rightarrow$) when $p \geq 2h - 2$.

In this paper we prove that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration in many new cases. The reader should note that the connections to these various conjectures are both useful and striking. For example, if one discovers an example when $\mathrm{St}_r \otimes L(\lambda)$ does not have a good filtration for some $\lambda \in X_r$ then Conjecture 1.2.2 would be false.

It also should be mentioned that the verification of Conjecture 1.2.2 would prove the 40-year-old Humphreys–Verma Conjecture about the existence of G-structures on injective indecomposable G_r-modules. Conjecture 1.2.2 holds for $p \geq 2h - 2$ and the proof under this bound entails locating one particular G-summand of $\mathrm{St}_r \otimes L(\lambda)$. It has become evident that, in order to prove either conjecture for all p, one needs to analyze all G-summands of $\mathrm{St}_r \otimes L(\lambda)$.

1.3. Outline

The paper is organized as follows. In Section 2, we summarize the basic definitions and fundamental results on good (resp. good (p, r)-) filtrations. The following section, Section 3, is devoted to developing sufficient conditions to guarantee that $\mathrm{St}_r \otimes M$ has a good filtration for a rational G-module M. These sufficient conditions involve the mysterious Frobenius contraction functor studied by Gros and Kaneda [GK] and Andersen [And4]. These results are used in Section 4 to prove that $\mathrm{St}_r \otimes L(\lambda)$ where $\lambda \in X_r$ has a good filtration for (i) $p \geq 2h - 4$ and (ii) for all rank two groups. The reader should note that Donkin's Tilting Module Conjecture (i.e., Conjecture 1.2.2) is not known for all rank 2 groups. Later in this section, $\mathrm{St}_r \otimes L(\lambda)$ is shown to have a good filtration when λ is a fundamental weight as long as one is not in the cases of E_7 and E_8 when $p = 2$.

Section 5 is devoted to verifying Conjecture 1.2.1($\Rightarrow$) for many cases when the rank of G is less than or equal to 5. In Section 6, we carefully analyze the type A_5, $p = 2$ situation and verify the conjecture using new and detailed information. This is an important case because it is indicative of the cases of fundamental weights for E_7 and E_8 when $p = 2$, where the conjecture is not yet verified. At the end of the paper in Section 7, we consider the question of whether $\mathrm{St}_r \otimes k[G_r]$ has a good filtration, where $k[G_r]$ is regarded as a G-module by the conjugation action.

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2. Preliminaries

2.1. Notation

Throughout this paper, the following basic notation will be used. See [Jan3] for a general overview of terminology.

(1) k: an algebraically closed field of characteristic $p > 0$.
(2) G: a simple, simply connected algebraic group scheme over k, defined over $\mathbb{F}_p$ (the assumption of G being simple is for convenience and the results easily generalize to G reductive).
(3) $\text{Dist}(G)$: the distribution algebra of G.
(4) $\mathfrak{g} = \text{Lie}(G)$: the Lie algebra of G.
(5) T: a maximal split torus in G.
(6) Φ: the corresponding (irreducible) root system associated to (G, T). When referring to short and long roots, when a root system has roots of only one length, all roots shall be considered as both short and long.
(7) $\Phi^{\pm}$: the positive (respectively, negative) roots.
(8) $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$: an ordering of the simple roots.
(9) B: the Borel subgroup containing T corresponding to the negative roots.
(10) U: the unipotent radical of B.
(11) $\mathbb{E}$: the Euclidean space spanned by Φ with inner product $\langle\, , \rangle$ normalized so that $\langle \alpha, \alpha \rangle = 2$ for $\alpha \in \Phi$ any short root.
(12) $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$: the coroot of $\alpha \in \Phi$.
(13) $X = X(T) = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_n$: the weight lattice, where the fundamental dominant weights $\varpi_i \in \mathbb{E}$ are defined by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$, $1 \leq i, j \leq n$.
(14) $X_+ = X(T)_+ = \mathbb{N}\varpi_1 + \cdots + \mathbb{N}\varpi_n$: the dominant weights.
(15) $X_r = X_r(T) = \{\lambda \in X(T)_+ : 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r, \ \forall \alpha \in S\}$: the set of p^r-restricted dominant weights.
(16) $F : G \to G$: the Frobenius morphism.
(17) $G_r = \ker F^r$: the rth Frobenius kernel of G. Similarly, B_r, T_r, and U_r denote the kernels of the restriction of F^r to B, T, and U respectively.
(18) Set $G^{(r)} = G/G_r$ and $B^{(r)} = B/B_r$.
(19) W: the Weyl group of Φ.
(20) w_0: the longest element of the Weyl group.
(21) ρ: the Weyl weight defined by $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$.
(22) α_0: the maximal short root.
(23) h: the Coxeter number of Φ, given by $h = \langle \rho, \alpha_0^\vee \rangle + 1$.
(24) $\leq$ on $X(T)$: a partial ordering of weights, for $\lambda, \mu \in X(T)$, $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a linear combination of simple roots with non-negative integral coefficients.
(25) $M^{(r)}$: the module obtained by composing the underlying representation for a rational G-module M with F^r.
(26) M^*: the k-linear dual module for a rational G-module M.
(27) $\lambda^* := -w_0\lambda$, $\lambda \in X$: the dual weight.
(28) ${}^\tau M$: the contravariant dual module, i.e., the dual module M^* of a rational G-module M with action composed with the anti-automorphism $\tau : G \to G$ that interchanges positive and negative root subgroups.

(29) $\nabla(\lambda) := \operatorname{ind}_B^G \lambda$, $\lambda \in X_+$: the induced module whose character is provided by Weyl's character formula.
(30) $\Delta(\lambda)$, $\lambda \in X_+$: the Weyl module of highest weight λ. Here $\Delta(\lambda) \cong \nabla(-w_0(\lambda))^*$.
(31) $L(\lambda)$, $\lambda \in X_+$: the simple finite-dimensional G-module with highest weight λ.
(32) $T(\lambda)$, $\lambda \in X_+$: the indecomposable finite-dimensional tilting G-module with highest weight λ.
(33) $\nabla^{(p,r)}(\lambda) := L(\lambda_0) \otimes \nabla(\lambda_1)^{(r)}$, $\lambda \in X_+$, where $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$ and $\lambda_1 \in X_+$.
(34) $\Delta^{(p,r)}(\lambda) := T(\widehat{\lambda_0}) \otimes \Delta(\lambda_1)^{(r)}$, $\lambda \in X_+$, where $\lambda = \lambda_0 + p^r\lambda_1$ with $\lambda_0 \in X_r$ and $\lambda_1 \in X_+$. Here $\widehat{\lambda_0} = 2(p^r-1)\rho + w_0\lambda_0$.
(35) $\mathrm{St}_r := L((p^r-1)\rho)$: the rth Steinberg module.
(36) $Q_r(\lambda)$, $\lambda \in X_r$: the injective hull (or equivalently, projective cover) of $L(\lambda)|_{G_r}$ as a G_r-module.

2.2. Important *G*-filtrations

Let M be a rational G-module. In this paper a *G-filtration* for M is an increasing sequence of G-submodules of M: $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$ such that $\cup_i M_i = M$. We now present the definition of a good filtration and a good (p,r)-filtration.

Definition 2.2.1. Let M be a G-module.

(a) M has a *good filtration* if and only if it has a G-filtration such that for each i, $M_{i+1}/M_i \cong \nabla(\lambda_i)$ where $\lambda_i \in X_+$.
(b) M has a *good (p,r)-filtration* if and only if it has a G-filtration such that for each i, $M_{i+1}/M_i \cong \nabla^{(p,r)}(\lambda_i)$ where $\lambda_i \in X_+$.
(c) If M has a *good $(p,1)$-filtration*, then we say that M has a good p-filtration.

2.3. Good filtrations: cohomological criterion

The following well-known result due to Donkin [Don1, Cor. 1.3] and Scott [Sc] (cf. [Jan3, Prop. II.4.16]) gives a very useful criterion to prove the existence of good filtrations.

Theorem 2.3.1. *Let M be a G-module with $\dim \operatorname{Hom}_G(\Delta(\lambda), M) < \infty$ for all $\lambda \in X_+$. The following are equivalent:*

(a) *M has a good filtration.*
(b) $\operatorname{Ext}^1_G(\Delta(\mu), M) = 0$ *for all $\mu \in X_+$.*
(c) $\operatorname{Ext}^n_G(\Delta(\mu), M) = 0$ *for all $\mu \in X_+$, $n \geq 1$.*

2.4. Good filtrations: tensoring with the Steinberg

Kildetoft and Nakano [KN, Thm. 9.2.3] gave necessary and sufficient conditions for $\mathrm{St}_r \otimes M$ to admit a good filtration (cf. [And3, Prop. 2.7]).

Theorem 2.4.1. *Let M be a G-module with $\dim \operatorname{Hom}_G(\Delta^{(p,r)}(\lambda), M) < \infty$ for all $\lambda \in X_+$. The following are equivalent:*

(a) $\mathrm{St}_r \otimes M$ *has a good filtration.*
(b) $\operatorname{Hom}_{G_r}(T(\widehat{\mu}), M)^{(-r)}$ *has a good filtration for all $\mu \in X_r$.*
(c) $\operatorname{Ext}^n_G(\Delta^{(p,r)}(\lambda), M) = 0$ *for all $\lambda \in X_+$, $n \geq 1$.*

(d) $\operatorname{Ext}^1_G(\Delta^{(p,r)}(\lambda), M) = 0$ *for all* $\lambda \in X_+$.

Observe that if Conjecture 1.2.1 holds then Theorem 2.4.1 would give cohomological criteria for a G-module M to admit a good (p,r)-filtration. Also note that, for a G-module M, the condition $\dim \operatorname{Hom}_G(\Delta^{(p,r)}(\lambda), M) < \infty$ for all $\lambda \in X_+$ is equivalent to $\dim \operatorname{Hom}_G(\Delta(\mu), \operatorname{St}_r \otimes M) < \infty$ for all $\mu \in X_+$.

3. Good filtrations on $\operatorname{St}_r \otimes M$

3.1. Frobenius contraction

We will first introduce an important class of functors via the rth-Steinberg module that sends G-modules to G/G_r-modules. For $\mu \in X_r$ and a rational G-module M, set

$$\mathcal{F}_\mu(M) = \operatorname{Hom}_{G_r}(k, \operatorname{St}_r \otimes \nabla(\mu) \otimes M) \cong \operatorname{Hom}_{G_r}(k, \operatorname{ind}_B^G(\operatorname{St}_r \otimes \mu \otimes M)), \qquad (1)$$

where the latter isomorphism follows from the tensor identity. The functor $\mathcal{F}_\mu$ is an exact functor from $\operatorname{Mod}(G) \to \operatorname{Mod}(G/G_r)$. Exactness of $\mathcal{F}_\mu$ follows from the fact that St_r is injective over G_r, and so $\operatorname{Ext}^i_{G_r}(k, \operatorname{St}_r \otimes \nabla(\mu) \otimes M) = 0$ for $i > 0$. We will call these functors *generalized Frobenius contraction functors.* When $\mu = (p^r - 1)\rho$ these functors were introduced by Gros and Kaneda [GK] and later investigated by Andersen [And4].

3.2. Applications to induced modules

The following theorem demonstrates that the functor $\mathcal{F}_\mu$ can be expressed in terms of induction from B/B_r to G/G_r. Note that $\operatorname{St}_r \cong \operatorname{ind}_{T_r}^{B_r}(-(p^r-1)\rho)$ as B_r-modules (cf. [Jan3, II.3.7 (4)]). From this along with the tensor identity and Frobenius reciprocity, we get the following isomorphism of B/B_r-modules (for $\mu \in X_+$ and a G-module M):

$$\begin{aligned}
\operatorname{Hom}_{B_r}(k, \operatorname{St}_r \otimes \mu \otimes M) &\cong \operatorname{Hom}_{B_r}(k, \operatorname{ind}_{T_r}^{B_r}(-(p^r-1)\rho) \otimes \mu \otimes M) \\
&\cong \operatorname{Hom}_{B_r}\left(k, \operatorname{ind}_{T_r}^{B_r}(-(p^r-1)\rho + \mu \otimes M)\right) \\
&\cong \operatorname{Hom}_{T_r}(k, \mu - (p^r-1)\rho \otimes M) \cong [\mu - (p^r-1)\rho \otimes M]^{T_r}.
\end{aligned}$$

Theorem 3.2.1. *Let* $\mu \in X_+$. *Then*

(a) $\mathcal{F}_\mu(M) = \operatorname{Hom}_{G_r}(k, \operatorname{ind}_B^G(\operatorname{St}_r \otimes \mu \otimes M)) \cong \operatorname{ind}_{B/B_r}^{G/G_r}\left([\mu - (p^r-1)\rho \otimes M]^{T_r}\right)$.

(b) $R^i \operatorname{ind}_{B/B_r}^{G/G_r}\left([\mu - (p^r-1)\rho \otimes M]^{T_r}\right) = 0$ *for* $i > 0$.

The B/B_r*-structures are given by the isomorphism*

$$[\mu - (p^r-1)\rho \otimes M]^{T_r} \cong \operatorname{Hom}_{B_r}(k, \operatorname{St}_r \otimes \mu \otimes M).$$

Proof. Consider the following isomorphic functors:

$$\begin{aligned}
\mathcal{F}_1(-) &= \left(\operatorname{ind}_B^G(-)\right)^{G_r}, \\
\mathcal{F}_2(-) &= \operatorname{ind}_{B/B_r}^{G/G_r}\left((-)^{B_r}\right).
\end{aligned}$$

As each arises as a composition, we obtain two spectral sequences, whose abutments agree (since the functors are isomorphic, cf. [AJ, Prop. 3.1]):

$$\hat{E}_2^{i,j} = \mathrm{Ext}^i_{G_r}(k, R^j\mathrm{ind}_B^G(\mathrm{St}_r \otimes \mu \otimes M)) \Rightarrow (R^{i+j}\mathcal{F}_1)(\mathrm{St}_r \otimes \mu \otimes M),$$
$$E_2^{i,j} = R^i\mathrm{ind}_{B/B_r}^{G/G_r}\mathrm{Ext}^j_{B_r}(\mathrm{St}_r, \mu \otimes M) \Rightarrow (R^{i+j}\mathcal{F}_2)(\mathrm{St}_r \otimes \mu \otimes M).$$

By the generalized tensor identity, $R^j\,\mathrm{ind}_B^G(\mathrm{St}_r \otimes \mu \otimes M) \cong \mathrm{St}_r \otimes R^j\,\mathrm{ind}_B^G(\mu \otimes M)$, which is injective over G_r, since St_r is. Hence, the first spectral sequence collapses. Precisely, $\hat{E}_2^{i,j} = 0$ for $i > 0$. One can then identify the abutment and combine this with the second spectral sequence to obtain

$$E_2^{i,j} = R^i\mathrm{ind}_{B/B_r}^{G/G_r}\mathrm{Ext}^j_{B_r}(\mathrm{St}_r, \mu \otimes M) \Rightarrow \mathrm{Hom}_{G_r}(k, R^{i+j}\mathrm{ind}_B^G(\mathrm{St}_r \otimes \mu \otimes M)).$$

This spectral sequence collapses and yields

$$R^i\mathrm{ind}_{B/B_r}^{G/G_r}\mathrm{Hom}_{B_r}(\mathrm{St}_r, \mu \otimes M) \cong \mathrm{Hom}_{G_r}(k, R^i\mathrm{ind}_B^G(\mathrm{St}_r \otimes \mu \otimes M)).$$

The statement of (a) follows by setting $i = 0$. From the generalized tensor identity and Kempf's Vanishing Theorem, one has

$$R^i\mathrm{ind}_B^G(\mathrm{St}_r \otimes \mu \otimes M) \cong [R^i\mathrm{ind}_B^G\ \mu] \otimes \mathrm{St}_r \otimes M = 0$$

when $i > 0$, which proves (b). □

From Theorem 3.2.1, it is interesting to note that, for any $\mu \in X_r$, the B/B_r-module

$$[\mu - (p^r - 1)\rho \otimes M]^{T_r} \cong \mathrm{Hom}_{B_r}(k, \mathrm{St}_r \otimes \mu \otimes M)$$

is acyclic with respect to the induction functor $\mathrm{ind}_{B^{(r)}}^{G^{(r)}}(-)$.

3.3. Good filtration criteria for $\mathrm{St}_r \otimes M$

Given $\mu \in X_r$, let $\mu_{(r)} := (p^r - 1)\rho - \mu \in X_r$. Note that the correspondence μ with $\mu_{(r)}$ gives a bijection on X_r. In particular, in Theorem 2.4.1(b), μ may be replaced with $\mu_{(r)}$.

Following [Jan3, II.10.4], let e_r be the functor that sends a rational G-module V to $\mathrm{e}_r V = \mathrm{Hom}_{G_r}(\mathrm{St}_r, V) \otimes St_r$, the summand of V whose composition factors are G_r-linked to the rth Steinberg module. The next result gives conditions on using the projection and generalized Frobenius contraction functors to ensure that $\mathrm{St}_r \otimes M$ has a good filtration.

Theorem 3.3.1. *Let M be a rational G-module with* $\dim \mathrm{Hom}_G(\Delta^{(p,r)}(\lambda), M) < \infty$ *for all $\lambda \in X_+$.*

(a) *If $e_r(L(\mu) \otimes M)$ has a good filtration for all $\mu \in X_r$ with the property that $\mu_{(r)}$ is G_r-linked to some composition factor of M, then $\mathrm{St}_r \otimes M$ has a good filtration.*

(b) *If $\mathcal{F}_\mu(M) = \mathrm{ind}_{B/B_r}^{G/G_r}\left([\mu - (p^r - 1)\rho \otimes M]^{T_r}\right)$ has a good filtration for all $\mu \in X_r$ with the property that $\mu_{(r)}$ is G_r-linked to some composition factor of M, where the B/B_r-structure is given by the isomorphism*

$$[\mu - (p^r - 1)\rho \otimes M]^{T_r} \cong \mathrm{Hom}_{B_r}(k, \mathrm{St}_r \otimes \mu \otimes M),$$

then $\mathrm{St}_r \otimes M$ has a good filtration.

Proof. For both parts, if it can be shown that $\mathrm{Hom}_{G_r}(T(\widehat{\mu_{(r)}}), M)$ has a G/G_r-good filtration for all $\mu \in X_r$, then the conclusion will follow from Theorem 2.4.1. Note that $\mathrm{Hom}_{G_r}(T(\widehat{\mu_{(r)}}), M) = 0$ unless $\mu_{(r)}$ is G_r-linked to some composition factor of M. Hence, one only has to consider weights μ that satisfy the above linkage condition.

(a) Suppose that $\mathrm{e}_r(L(\mu) \otimes M)$ has a good filtration. Since $\mathrm{St}_r \otimes \Delta(\sigma)^{(r)} \cong \Delta((p^r-1)\rho + p^r\sigma)$ (dual to [Jan3, Prop. II.3.19]),

$$\mathrm{Ext}^1_G(\mathrm{St}_r \otimes \Delta(\sigma)^{(r)}, L(\mu) \otimes M) = 0$$

for all $\mu \in X_r$ and $\sigma \in X_+$. The Lyndon–Hochschild–Serre spectral sequence

$$\begin{aligned} E_2^{i,j} &= \mathrm{Ext}^i_{G/G_r}(\Delta(\sigma)^{(r)}, \mathrm{Ext}^j_{G_r}(\mathrm{St}_r \otimes L(\mu^*), M)) \\ &\Rightarrow \mathrm{Ext}^{i+j}_G(\mathrm{St}_r \otimes \Delta(\sigma)^{(r)}, L(\mu) \otimes M) \end{aligned}$$

collapses because St_r is projective as a G_r-module and yields the isomorphism:

$$\mathrm{Ext}^1_G(\mathrm{St}_r \otimes \Delta(\sigma)^{(r)}, L(\mu) \otimes M) \cong \mathrm{Ext}^1_{G/G_r}(\Delta(\sigma)^{(r)}, \mathrm{Hom}_{G_r}(\mathrm{St}_r \otimes L(\mu^*), M)).$$

Therefore, $\mathrm{Hom}_{G_r}(\mathrm{St}_r \otimes L(\mu^*), M))$ has a good filtration as a G/G_r-module. The G/G_r-module $\mathrm{Hom}_{G_r}(T(\widehat{\mu_{(r)}}), M)$ is a direct summand of $\mathrm{Hom}_{G_r}(\mathrm{St}_r \otimes L(\mu^*), M)$, because $T(\widehat{\mu_{(r)}})$ is a G-direct summand of $\mathrm{St}_r \otimes L(\mu^*)$ by [Pil, Sect. 2, Cor. A]. Therefore, $\mathrm{Hom}_{G_r}(T(\widehat{\mu_{(r)}}), M)$ has a G/G_r-good filtration. It follows now by Theorem 2.4.1 that $\mathrm{St}_r \otimes M$ has a good filtration.

(b) By duality and Theorem 3.2.1, we have

$$\begin{aligned} \mathrm{Hom}_{G_r}(\mathrm{St}_r \otimes \Delta(\mu^*), M) &\cong \mathrm{Hom}_{G_r}(k, \mathrm{St}_r \otimes \nabla(\mu) \otimes M) \\ &\cong \mathrm{ind}_{B/B_r}^{G/G_r}\left([\mu - (p^r-1)\rho \otimes M]^{T_r}\right). \end{aligned}$$

Following [Pil, Sect. 2, Cor. A] (see [So, Rem. 4.1.4]) , $T(\widehat{\mu_{(r)}})$ is also a G-direct summand of $\mathrm{St}_r \otimes \Delta(\mu^*)$ for $\mu \in X_r$. So, if $\mathrm{ind}_{B/B_r}^{G/G_r}\left([\mu - (p^r-1)\rho, \otimes M]^{T_r}\right)$ has a good filtration as a G/G_r-module for all $\mu \in X_r$, then, by Theorem 2.4.1, $\mathrm{St}_r \otimes M$ has a good filtration. □

Note that in part (a) of the previous theorem the module $L(\mu)$ could be replaced by any of the following: $\nabla(\mu)$, $\Delta(\mu)$ or $T(\mu)$.

4. Applications: tensoring with simple modules

4.1. Reduction to $r = 1$

In this section we will apply the results from the previous section to verify cases when $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration. In order to do so, the following result of Kildetoft and Nakano shows that it suffices to focus on the case when $r = 1$.

Theorem 4.1.1 ([KN, Prop. 4.4.1]). *If* $\mathrm{St}_1 \otimes L(\lambda)$ *has a good filtration for all* $\lambda \in X_1(T)$*, then* $\mathrm{St}_r \otimes L(\lambda)$ *has a good filtration for all* $\lambda \in X_r$*,* $r \geq 1$*.*

4.2. General bound on p

In [KN, Thm. 5.3.1, Thm. 9.4.1] [And3, Prop. 2.1], it was shown that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for $p \geq 2h-2$. This bound agrees with the present state of Donkin's Tilting Module Conjecture. The following result lowers the general bound for Conjecture 1.2.1($\Rightarrow$) to hold. This fact will be later used for our analysis of low rank groups.

Theorem 4.2.1. *Let $\lambda \in X_r$ and $p \geq 2h-4$. Then $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration.*

Proof. By Theorem 4.1.1, it suffices to prove this for $r = 1$, and we will do so by using the characterization given in Theorem 3.3.1(a) applied to $M = L(\lambda)$. Suppose that $\mu, \lambda \in X_1(T)$ and that $\mathrm{St}_1 \otimes L(\gamma)^{(1)}$ is a composition factor of $L(\mu) \otimes L(\lambda)$. Since $(p-1)\rho + p\gamma \leq \mu + \lambda$,

$$\begin{aligned} \langle (p-1)\rho + p\gamma, \alpha_0^\vee \rangle &\leq \langle \mu + \lambda, \alpha_0^\vee \rangle \\ &\leq 2(p-1)(h-1). \end{aligned}$$

Thus

$$\langle \gamma, \alpha_0^\vee \rangle \leq \frac{(p-1)}{p}(h-1) < h-1.$$

From this, we have

$$\langle \gamma + \rho, \alpha_0^\vee \rangle < 2(h-1),$$

so that

$$\langle \gamma + \rho, \alpha_0^\vee \rangle \leq 2h-3.$$

If $p \geq 2h-3$, then γ is contained in the closure of the fundamental alcove, and $L(\gamma) \cong \nabla(\gamma)$. This proves the result for $p \geq 2h-3$. The case when $p = 2h-4$ only occurs if $p = 2$ and $h = 3$. But this result (indeed, the Tilting Module Conjecture as well) is known to hold for SL_3 in characteristic 2, as the four restricted simple modules are all tilting in this case. Therefore, the result holds when $p \geq 2h-4$. $\square$

4.3. General bound on λ

One can also give a general upper bound on λ that will ensure that tensoring the rth-Steinberg with a simple G_r-module will have a good filtration.

Proposition 4.3.1. *If $\lambda \in X_+$ and $\langle \lambda, \alpha_0^\vee \rangle \leq 2p^r - 1$, then $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration.*

Proof. We work again with the characterization in Theorem 3.3.1(a). Suppose that $\mathrm{St}_r \otimes L(\gamma)^{(r)}$ is a composition factor of $L(\mu) \otimes L(\lambda)$ for some $\mu \in X_r$. One has

$$\langle \mu, \alpha_0^\vee \rangle \leq \langle (p^r - 1)\rho, \alpha_0^\vee \rangle,$$

so it follows that

$$p^r \langle \gamma, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle \leq 2p^r - 1.$$

Thus

$$\langle \gamma, \alpha_0^\vee \rangle \leq 2 - 1/p^r < 2,$$

forcing γ to be minuscule and therefore $L(\gamma) \cong \nabla(\gamma)$. $\square$

4.4. Rank 2 groups

The following theorem completes work on rank two groups initiated in [KN, Sect. 8].

Theorem 4.4.1. *Assume the Lie rank of G is less than or equal to two. Then* $\mathrm{St}_r \otimes L(\lambda)$ *has a good filtration for all* $\lambda \in X_r$.

Proof. Again, we use Theorem 4.1.1 to reduce to the case that $r = 1$. In [KN, 8.5], the claim was shown in all cases except for when Φ is of type G_2 and $p = 7$, so we consider this case. Here $h = 6$ and $\alpha_0^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee$. Set $\lambda = a\varpi_1 + b\varpi_2$ with $0 \leq a, b \leq 6$. We may assume that $\lambda \neq 6\rho$, so at least one of a or b is strictly less than 6.

We make use of Theorem 3.3.1(a). Suppose now that $\mu \in X_1(T)$ is such that $\mu_{(1)} = (p-1)\rho - \mu$ is G_1-linked to λ. Our goal is to show that, if $\mathrm{St}_1 \otimes L(\gamma)^{(1)}$ is a composition factor of $L(\mu) \otimes L(\lambda)$, then $L(\gamma) = \nabla(\gamma)$. Now

$$7\langle \gamma, \alpha_0^\vee \rangle \leq 2a + 3b \leq 10 + 18 = 28,$$

with equality occurring only if $\lambda = 5\varpi_1 + 6\varpi_2$. So $\langle \gamma, \alpha_0^\vee \rangle \leq 4$. Let $\gamma = c\varpi_1 + d\varpi_2$ with $0 \leq c, d \leq 6$. Then $2c + 3d = \langle \gamma, \alpha_0^\vee \rangle \leq 4$. So $c \leq 2$, $d \leq 1$, and at most one of c or d may be non-zero.

Case I. $\gamma = 0$: Then $L(\gamma) = L(0) = k = \nabla(0)$.

Case II. $\gamma = \varpi_1$: This lies in closure of the bottom alcove, and so $L(\varpi_1) = \nabla(\varpi_1)$.

Case III. $\gamma = \varpi_2$: Similarly, this does not lie in the bottom alcove, however, there is nothing lower linked to it, and so $L(\varpi_2) = \nabla(\varpi_2)$.

Case IV. $\gamma = 2\varpi_1 = 2\alpha_0$: Note that this is the same weight observed in [KN, 8.5.4] to be problematic. Here $L(2\varpi_1) \neq \nabla(2\varpi_1)$. However, this situation occurs only if $\mu = (p-1)\rho$ and $\lambda = 5\varpi_1 + 6\varpi_2$. This case can be dismissed because $\mu_{(1)} = 0$ is not G_1-linked to λ. □

Although we rely on [KN, Sect. 8.2] to remove most of the cases, the results in this paper could have been used in other type G_2 cases and lead to very short proofs for the other rank ≤ 2 groups. For example, Theorem 4.2.1 (and its proof) establish the result for SL_2 and SL_3 in all characteristics. For type B_2, we have $h = 4$, so that the result holds for all $p \geq 4$ by Theorem 4.2.1, leaving only $p = 2, 3$ to check. If $p = 2$ and $\lambda \in X_1(T)$, then $\langle \lambda, \alpha_0^\vee \rangle \leq 3 = 2p - 1$. If $p = 3$ and $\lambda \in X_1(T)$ is not the Steinberg weight (for which the result is clear), then $\langle \lambda, \alpha_0^\vee \rangle \leq 5 = 2p - 1$. Thus, in both of these cases, the result holds by Proposition 4.3.1.

4.5. Fundamental weights

We now consider the case of a restricted irreducible G-module where the highest weight is a fundamental weight. To do this, we need to extend the usual partial order on weights to a partial ordering $\leq_{\mathbb{Q}}$ relative to the rational numbers. For $\mu, \lambda \in X$, we say that $\mu \leq_{\mathbb{Q}} \lambda$ if

$$\lambda - \mu = \sum_{\alpha \in S} q_\alpha \alpha$$

for $q_\alpha \in \mathbb{Q}$ with $q_\alpha \geq 0$. Note that $\mu \leq \lambda$ implies $\mu \leq_{\mathbb{Q}} \lambda$.

Theorem 4.5.1. *Let the Lie rank of G be n. Then*

(a) $\mathrm{St}_r \otimes L(\varpi_j)$ *has a good filtration for* $j = 1, 2, \ldots, n$ *and* $r \geq 2$.
(b) $\mathrm{St}_1 \otimes L(\varpi_j)$ *has a good filtration for* $j = 1, 2, \ldots, n$, *except possibly when* $p = 2$ *and* $\Phi = \mathsf{E}_7,\ \mathsf{E}_8$.
(c) *In the case when* $p = 2$ *and* $\Phi = \mathsf{E}_7$ *or* E_8,
   (i) $\mathrm{St}_1 \otimes L(\varpi_j)$ *has a good filtration for* $j \neq 4$ *when* $\Phi = \mathsf{E}_7$.
   (ii) $\mathrm{St}_1 \otimes L(\varpi_j)$ *has a good filtration for* $j \neq 4, 5$ *when* $\Phi = \mathsf{E}_8$.

Proof. We want to consider $\operatorname{ind}_{B/B_r}^{G/G_r}\left([\mu - (p^r-1)\rho \otimes L(\varpi_j)]^{T_r}\right)$ for $\mu \in X_r$ and show it has a good filtration. This will occur if all the dominant weights in $[\mu - (p^r-1)\rho \otimes L(\varpi_j)]^{T_r}$ are of the form $p^r\delta$ with the property that $\nabla(\delta) = L(\delta)$.

Let $p^r\delta$ be a dominant weight of $[\mu - (p^r-1)\rho \otimes L(\varpi_j)]^{T_r}$. Then $p^r\delta = \mu - (p^r-1)\rho + \gamma$ where γ is a weight of $L(\varpi_j)$. Since $\mu \leq_{\mathbb{Q}} (p^r-1)\rho$, it follows that

$$p^r\delta \leq_{\mathbb{Q}} \gamma. \tag{2}$$

By taking the inner product with $\alpha_0^\vee$, one obtains

$$0 \leq \langle \delta, \alpha_0^\vee \rangle \leq \frac{1}{p^r}\langle \varpi_j, \alpha_0^\vee \rangle. \tag{3}$$

Set $h(j, r, p) = \langle \varpi_j, \alpha_0^\vee \rangle / p^r$.

For types A_n, B_n, C_n, D_n, E_6 and G_2, one has $h(j, r, p) < 2$ for all j, r. In these cases this implies that δ is either zero or minuscule and $\nabla(\delta) = L(\delta)$.

For type F_4, one can repeat this argument, but replacing $\alpha_0^\vee$ with the coroot of the highest root $\alpha_h^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee$. In this case, one obtains $h(j, r, p) = \langle \varpi_j, \alpha_h^\vee \rangle / p^r < 2$. This implies that $\delta = 0$ or ω_4 (with $p = 2, 3$). The only solution to $p^r\omega_4 \leq_{\mathbb{Q}} \omega_j$ occurs when $p = 2$, $r = 1$ and $j = 2$. However, $\nabla(\omega_4) = L(\omega_4)$ when $p = 2$ (cf. [Jan2, p. 299]).

In the case when $\Phi = \mathsf{E}_7$, one has $h(j, r, p) < 2$ unless $j = 4$, $r = 1$ and $p = 2$. For $\Phi = E_8$, one has $h(j, r, p) < 2$ unless

(i) $j = \varpi_3$, $r = 1$, $p = 2$;
(ii) $j = \varpi_6$, $r = 1$, $p = 2$;
(iii) $j = \varpi_5$, $r = 1$, $p = 2$;
(iv) $j = \varpi_4$, $r = 1$, $p = 2$;
(v) $j = \varpi_4$, $r = 1$, $p = 3$.

For type E_8, the root lattice and the weight lattice coincide, so in this case

$$p^r\delta \leq \gamma \leq \varpi_j.$$

Suppose $\delta \neq 0$. Using [UGA, Fig. 3], in cases (i), (ii), (v) it follows that $\delta = \varpi_8$, and one has $\nabla(\delta) = L(\delta)$ by [GGN, Thm. 1.1]. One is left with cases (iii), (iv) for type E_8. □

5. Higher rank cases

5.1. The strategy

In this section, we consider the question of whether $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for some higher rank groups over small primes. Here the results are less complete, and we will focus on the $r = 1$ situation. For those cases where we can show that

$$\mathrm{St}_1 \otimes L(\lambda) \text{ has a good filtration for all } \lambda \in X_1, \tag{4}$$

it will then follow from Theorem 4.1.1 that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for all $\lambda \in X_r$ for $r > 1$.

Note that $L((p-1)\rho) \simeq \mathrm{St}_1$, and the claim (4) always holds for this particular weight. For a given $\lambda \in X_1$, we may therefore assume throughout that $\lambda \neq (p-1)\rho$. We again make use of Theorem 3.3.1(a). Suppose that $\mathrm{St}_1 \otimes L(\gamma)^{(1)}$ is a composition factor of $L(\mu) \otimes L(\lambda)$ for $\gamma \in X_+$ and $\mu \in X_1$. Our goal is to show that either no such γ and μ exist or that $L(\gamma) = \nabla(\gamma)$. Recall the notation $\mu_{(1)} := (p-1)\rho - \mu$. As in the proof of Theorem 4.2.1, we must have

$$p\gamma \leq \lambda - \mu_{(1)}, \tag{5}$$

from which we may conclude that

$$p\langle \gamma, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle \quad \text{and} \quad p\langle \gamma, \tilde{\alpha}^\vee \rangle \leq \langle \lambda, \tilde{\alpha}^\vee \rangle, \tag{6}$$

where $\tilde{\alpha}$ denotes the highest root. These inequalities are often sufficient to eliminate options, but further reductions can also be made by noting that we only have to consider those μ with $\mu_{(1)}$ being G_1-linked to λ.

5.2. Use of computer algebra systems and online data

In the following, due to the complexity of some calculations, we make repeated use of the computer algebra systems LiE [LiE] and Magma [Magma]. The systems are used to

(a) find the formal characters of tensor products of induced modules (LiE),
(b) find the weight space multiplicities of induced modules (LiE),
(c) find the W-orbits of certain weights (LiE, Magma).

The last item allows for the calculation of G- and G_1-linkage classes, either by hand or via computer. In addition, we make use of Frank Lübeck's online tables of weight multiplicities of small degree representations in defining characteristic [L].

5.3. Rank 3 groups

For rank three groups, the claim (4) holds in almost all cases (cf. also [KN, §8.3]).

Theorem 5.3.1. *Let G be of type A_3, B_3, or C_3 and $\lambda \in X_1$. Then $\mathrm{St}_1 \otimes L(\lambda)$ has a good filtration except possibly for the following cases:*

- *Type B_3 with $p = 7$: $\lambda = (6,5,5), (6,4,5), (6,5,4), (5,5,5), (5,5,4), (5,5,4), (5,4,5), (4,5,5), (4,5,4)$, or $(3,5,5)$,*
- *Type C_3 with $p = 3$: $\lambda = (2,1,2)$ or $(2,2,1)$,*
- *Type C_3 with $p = 7$: $\lambda = (6,5,5), (6,4,5), (6,5,4), (5,5,5)$, or $(4,5,5)$.*

Proof. The case of type A_3 was shown by Kildetoft and Nakano [KN, §8.3], but also follows from our previous results. By Theorem 4.2.1, we are reduced to $p = 2$ or 3. But those cases follow from Proposition 4.3.1.

For type B_3, $h = 6$. By Theorem 4.2.1, we are done if $p > 7$. Note that $\alpha_0^\vee = 2\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$ and $\tilde{\alpha}^\vee = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$, from which we see that $\langle \rho, \alpha_0^\vee \rangle = 5$ and $\langle \rho, \tilde{\alpha}^\vee \rangle = 4$.

For $p = 2$, Proposition 4.3.1 reduces us to $\lambda = \varpi_1 + \varpi_2$. The only γ and μ satisfying (5) are $\gamma = \varpi_1$ and $\mu = \rho$ (so $L(\mu) = \mathrm{St}_1$). We will show that $\mathrm{St}_1 \otimes L(\varpi_1)^{(1)}$ does not appear as a composition factor of $\mathrm{St}_1 \otimes L(\varpi_1 + \varpi_2)$. First, note that $\mathrm{St}_1 \otimes \nabla(\varpi_1)^{(1)}$ appears exactly once as a factor in a good filtration of $\mathrm{St}_1 \otimes \nabla(\varpi_1 + \varpi_2)$ [LiE]. Moreover, $\rho + 2\varpi_1$ is the unique highest weight in the Steinberg linkage class inside this tensor product. That implies that $\mathrm{St}_1 \otimes L(\varpi_1)^{(1)}$ appears exactly once as a composition factor of $\mathrm{St} \otimes \nabla(\varpi_1 + \varpi_2)$. On the other hand, Lübeck's calculations [L] show that the weight $2\varpi_1$ does not appear in the simple module $L(\varpi_1 + \varpi_2)$. It does, however, appear with multiplicity one in $\nabla(\varpi_1 + \varpi_2)$. This implies that $L(\varpi_1)^{(1)}$ is a composition factor of $\nabla(\varpi_1 + \varpi_2)$. Hence, $\mathrm{St}_1 \otimes L(\varpi_1)^{(1)}$ is a composition factor of $\mathrm{St}_1 \otimes (\nabla(\varpi_1 + \varpi_2)/L(\varpi_1 + \varpi_2))$ and not of $\mathrm{St}_1 \otimes L(\varpi_1 + \varpi_2)$.

For $p = 3$, applying both inequalities in (6), we are reduced to the following options for γ: ϖ_1, ϖ_2, ϖ_3, $2\varpi_3$, or $\varpi_1 + \varpi_3$. Applying (5) to $\gamma = 2\varpi_3$, we would need $6\varpi_3 \leq \lambda - \mu_{(1)} <_{\mathbb{Q}} 2\rho$, which fails to hold. That leaves us with $\gamma = \varpi_i$ or $\varpi_1 + \varpi_3$. For $p \neq 2$, each $\nabla(\varpi_i)$ is simple (cf. [Jan3, II.8.21]). Further, $\nabla(\varpi_1 + \varpi_3)$ is also known to be simple, except when $p = 7$ (cf. [GGN]).

For $p = 5$, (6) reduces us to $\gamma = \varpi_1$, ϖ_2, ϖ_3, $2\varpi_3$, $3\varpi_3$, $\varpi_1 + \varpi_3$, or $\varpi_2 + \varpi_3$. From (5), one has $5\gamma \leq \lambda - \delta <_{\mathbb{Q}} 4\rho$. This fails to hold for $\gamma = 3\varpi_3$. While it is true that $5(\varpi_2 + \varpi_3) <_{\mathbb{Q}} 4\rho$, there is no $\lambda \neq 4\rho$ with $5(\varpi_2 + \varpi_3) \leq \lambda - \mu_{(1)}$. So this reduces us to $\gamma = \varpi_i$, $\varpi_1 + \varpi_3$, or $2\varpi_3$. As noted above, each $\nabla(\varpi_i)$ and $\nabla(\varpi_1 + \varpi_3)$ is simple. Lastly, by explicit dimension computations of Lübeck [L], $\nabla(2\varpi_3)$ is simple for all odd primes.

For $p = 7$, from (6), we are reduced to the following options for γ: ϖ_1, ϖ_2, ϖ_3, $2\varpi_1$, $2\varpi_3$, $3\varpi_3$, $\varpi_1 + \varpi_2$, $\varpi_1 + \varpi_3$, $\varpi_2 + \varpi_3$, $\varpi_1 + 2\varpi_3$. Using known facts and dimension computations of Lübeck [L], the only cases where $\nabla(\gamma)$ is not simple are $\gamma = 2\varpi_1$ or $\varpi_1 + \varpi_3$. Both can satisfy (5). In particular, for $\gamma = 2\varpi_1$, we can have $\lambda - \mu_{(1)} = 6\rho - \varpi_1$, $6\rho - \varpi_2$, or $6\rho - 2\varpi_3$. One can check that, in each case, λ is not G_1-linked to $\mu_{(1)}$. So this leaves only the second case of $\gamma = \varpi_1 + \varpi_3$, which admits a large number of options for λ (and $\mu_{(1)}$). However G_1-linkage holds only in the following cases:

λ	(6,5,5)	(6,4,5)	(6,5,4)	(5,5,5)	(5,5,4)	(5,5,4)
$\mu_{(1)}$	(3,0,0)	(1,1,0)	(2,0,1)	(0,0,0)	(0,0,1)	(1,0,1)

λ	(5,5,5)	(5,4,5)	(4,5,5)	(4,5,4)	(3,5,5)
$\mu_{(1)}$	(2,0,0)	(0,1,0)	(1,0,0)	(0,0,1)	(0,0,0)

For type C_3, we are again done if $p > 7$. In this case $\alpha_0^\vee = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee$ and $\tilde{\alpha}^\vee = \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$, from which we see that $\langle \rho, \alpha_0^\vee \rangle = 5$ and $\langle \rho, \tilde{\alpha}^\vee \rangle = 3$.

For $p = 2$, Proposition 4.3.1 reduces us to $\lambda = \varpi_2 + \varpi_3$. The only weight γ satisfying (5) is $\gamma = \varpi_1$, which is miniscule. Hence, $\nabla(\varpi_1)$ is simple.

For $p = 3$, (6) reduces us to $\gamma = \varpi_i$ for some $1 \leq i \leq 3$. Again, $\nabla(\varpi_1)$ is simple. Premet and Suprunenko [PrSu] showed that $\nabla(\varpi_2)$ is simple if and only if $p \neq 3$ and $\nabla(\varpi_3)$ is simple if and only if $p \neq 2$. So we are reduced to the case $\gamma = \varpi_2$, which does satisfy (5) for many values of $\lambda - \mu_{(1)}$. The following table summarizes the possible cases where λ is G_1-linked to $\mu_{(1)}$.

λ	(2,1,2)	(2,1,2)	(2,2,1)	(2,0,2)
$\mu_{(1)}$	(0,0,0)	(0,1,0)	(1,0,0)	(0,0,0)

With some further investigation, we can eliminate two of those four options. For $\nu \in X_1$, set $Z_1(\nu) := \operatorname{coind}_{B_1^+T}^{G_1T}(\nu)$. First, suppose that $\lambda = (2, 0, 2)$ and $\mu_{(1)} = (0, 0, 0)$ (the last case). Then $\mu = (p-1)\rho$ and $L(\mu) \cong \mathrm{St}_1$. We are reduced to showing that $\mathrm{e}_1(\mathrm{St}_1 \otimes L(\lambda))$ has a good filtration or, equivalently, that $N := \mathrm{Hom}_{G_1}(\mathrm{St}_1, \mathrm{St}_1 \otimes L(\lambda))^{(-1)}$ has a good filtration. This can only fail if $L(\varpi_2)$ is a composition factor of N. The weight-space multiplicity of $5\varpi_2$ in St_1 is one. It follows that $Z_1(2\rho + 5\varpi_2) = Z_1(2(p-1)\rho - \lambda + p\varpi_2)$ appears exactly once as a section of the G_1T-module $\mathrm{St}_1 \otimes \mathrm{St}_1$. Moreover, the only weights in $\mathrm{St}_1 \otimes \mathrm{St}_1$ that are higher than $2(p-1)\rho - \lambda + p\varpi_2 = 4\rho - \alpha_1 - \alpha_3$ are 4ρ, $4\rho - \alpha_1$, and $4\rho - \alpha_3$. The weight $2(p-1)\rho - \lambda + p\varpi_2 = 4\rho - \alpha_1 - \alpha_3$ is not strongly linked to any of these because the reflections $s_{\alpha_1,p}$ and $s_{\alpha_3,p}$ are elements of its stabilizer in the affine Weyl group. Hence, $2(p-1)\rho - \lambda + p\varpi_2$ is a maximal weight inside $\mathrm{St}_1 \otimes \mathrm{St}_1$. It follows that $Q_1(\lambda + p\varpi_2)$ appears exactly once as a summand of the G_1T-module $\mathrm{St}_1 \otimes \mathrm{St}_1$ and that $L(\varpi_2)$ appears exactly once as a composition factor of N.

By the same argument we can conclude that the induced module $\nabla(2(p-1)\rho - \lambda + p\varpi_2)$ also appears exactly once as a section in the good filtration of the G-module $\mathrm{St}_1 \otimes \mathrm{St}_1$. The fact that $2(p-1)\rho - \lambda + p\varpi_2$ is a maximal weight in $\mathrm{St}_1 \otimes \mathrm{St}_1$ says that the tilting module $T(2(p-1)\rho - \lambda + p\varpi_2) \cong T(2(p-1)\rho - \lambda) \otimes T(\varpi_2)^{(1)}$ is a summand of the tilting module $\mathrm{St}_1 \otimes \mathrm{St}_1$. This implies that the tilting module $T(\varpi_2)$ is a summand of N. Since the multiplicity of $L(\varpi_2)$ in N is one, it appears inside this summand. All other composition factors $L(\eta)$ of N satisfy $L(\eta) \cong \nabla(\eta)$. One concludes that N is tilting, and thus has a good filtration, eliminating the weight $(2, 0, 2)$ from the above list.

Consider now the second case in the above list: $\lambda = (2, 1, 2)$ and $\mu_{(1)} = (0, 1, 0)$. Then $\mu = (p-1)\rho - \mu_{(1)} = \lambda$. An argument similar to the preceding case shows that $\mathrm{e}_1(T(\lambda) \otimes L(\lambda))$ has a good filtration, which eliminates this case as well, leaving only the following unknown cases:

λ	(2,1,2)	(2,2,1)
$\mu_{(1)}$	(0,0,0)	(1,0,0)

For $p = 5$, (6) reduces us to the following options for γ: ϖ_1, ϖ_2, ϖ_3, $2\varpi_1$, $\varpi_1 + \varpi_2$, or $\varpi_1 + \varpi_3$. Here we know the simplicity of $\nabla(\varpi_i)$ for each i. From dimension computations of Lübeck [L], $\nabla(2\varpi_1)$ is simple for $p > 2$, $\nabla(\varpi_1 + \varpi_2)$ is simple if and only if $p \neq 3$ or 7, and $\nabla(\varpi_1 + \varpi_3)$ is simple if and only if $p > 3$. In particular, all are simple for $p = 5$.

For $p = 7$, (6) reduces us to the following options for γ: ϖ_1, ϖ_2, ϖ_3, $2\varpi_1$, $2\varpi_2$, $2\varpi_3$, $\varpi_1+\varpi_2$, $\varpi_1+\varpi_3$, $\varpi_2+\varpi_3$. From previous discussions, the only cases where $\nabla(\gamma)$ is not simple are $\gamma = 2\varpi_2, \varpi_1+\varpi_2$. Both can satisfy (5). For $\gamma = 2\varpi_2$, there are three options for $\lambda : 6\varpi_1 + 5\varpi_2 + 6\varpi_3$ (with $\mu_{(1)} = 0$); $4\varpi_1 + 6\varpi_2 + 6\varpi_3$ (with $\mu_{(1)} = 0$); and $5\varpi_1+6\varpi_2+6\varpi_3$ (with $\mu_{(1)} = \varpi_1$). However, one can directly check that in each case λ and $\mu_{(1)}$ are not G_1-linked. In the second case ($\gamma = \varpi_1 + \varpi_2$), there are numerous values of λ that satisfy (5), however G_1-linkage holds only in the following cases:

λ	(6,5,5)	(6,4,5)	(6,5,4)	(5,5,5)	(4,5,5)
$\mu_{(1)}$	(0,0,0)	(0,1,0)	(0,0,1)	(1,0,0)	(0,0,0)

□

5.4. Rank 4 groups

In types A_4 and D_4, the claim also holds in almost all cases. While potentially problematic weights are not listed explicitly in the following theorem, some information is provided in the proof.

Theorem 5.4.1. *Assume G is of type A_4 or D_4 and $\lambda \in X_1$. Then $\mathrm{St}_1 \otimes L(\lambda)$ has a good filtration except possibly for the following cases:*

- *Type A_4 with $p = 5$,*
- *Type D_4 with $p = 7$.*

Proof. We first consider type A_4, where $h = 5$. By Theorem 4.2.1, we are done for $p > 5$. For $p = 2$, the result follows from Proposition 4.3.1.

For $p = 3$, one could again eliminate many λ via Proposition 4.3.1. However, we more directly focus on the weight γ. First, (6) reduces us to $\gamma = \varpi_i + \varpi_j$ for $i, j \in \{1, 2, 3, 4\}$. Of those, the only weights potentially satisfying (5) are $\gamma = \varpi_1 + \varpi_3$, $\varpi_2 + \varpi_4$, or $\varpi_1 + \varpi_4$. However, in each case $\nabla(\gamma)$ is simple, as can be seen by using Jantzen's algorithm [Jan1, Satz 9] (cf. also [Jan3, II.8.21]) for checking the simplicity of a standard induced module in type A_n.

For $p = 5$, using (6), (5), and Jantzen's algorithm for simplicity, one can reduce the problem to just one possible value of γ: $\varpi_1 + \varpi_4 = \alpha_0$. We have the following values of λ and $\mu_{(1)}$ which are G_1-linked and satisfy (5).

λ	(4,3,3,4)	(4,3,3,3)	(3,3,3,4)	(4,3,2,4)	(4,2,3,4)
$\mu_{(1)}$	(2,0,0,2)	(2,0,0,1)	(1,0,0,2)	(1,1,0,1)	(1,0,1,1)

λ	(3,3,2,4)	(3,2,3,4)	(4,3,2,3)	(4,2,3,3)	(3,3,3,3)
$\mu_{(1)}$	(0,1,0,1)	(0,0,1,1)	(1,1,0,0)	(1,0,1,0)	(1,0,0,1)

λ	(2,3,3,4)	(4,3,3,2)	(4,3,1,4)	(4,1,3,4)	(4,2,2,4)
$\mu_{(1)}$	(0,0,0,2)	(2,0,0,0)	(0,2,0,0)	(0,0,2,0)	(0,1,1,0)

λ	(3,3,3,3)	(2,3,3,3)	(3,2,3,3)	(3,3,2,3)	(3,3,3,2)	(2,3,3,2)
$\mu_{(1)}$	(0,0,0,0)	(0,0,0,1)	(0,0,1,0)	(0,1,0,0)	(1,0,0,0)	(0,0,0,0)

For type D_4, $h = 6$. By Theorem 4.2.1, we are done if $p > 7$. We have $\alpha_0^\vee = \tilde{\alpha}^\vee = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee$, and so $\langle \rho, \alpha_0^\vee \rangle = 5$.

For $p = 2$, using (6) and (5), we are reduced to $\gamma = \varpi_1$, ϖ_3, or ϖ_4. But all those weights are miniscule, giving a simple $\nabla(\gamma)$.

For $p = 3$, many values of γ satisfy (6). Using dimension computations of Lübeck [L], one finds that the only cases where $\nabla(\gamma)$ is not simple are as follows: $3\varpi_1$, $3\varpi_3$, $3\varpi_4$, $\varpi_1 + \varpi_2$, $\varpi_2 + \varpi_3$, $\varpi_2 + \varpi_4$, and $\varpi_1 + \varpi_3 + \varpi_4$. By direct verification, none of these can satisfy (5).

For $p = 5$, similarly, (6) and dimension computations of Lübeck [L] reduce us to the following options for γ: $3\varpi_i$, $i \in \{1, 3, 4\}$; $4\varpi_i$, $i \in \{1, 3, 4\}$; $2\varpi_i + \varpi_j$, $i, j \in \{1, 3, 4\}$, $i \neq j$; $3\varpi_i + \varpi_j$, $i, j \in \{1, 3, 4\}$, $i \neq j$; $2\varpi_i + 2\varpi_j$, $i, j \in \{1, 3, 4\}$, $i \neq j$; $\varpi_i + 2\varpi_2$, $i \in \{1, 3, 4\}$; and $\varpi_2 + \varpi_i + \varpi_j$, $i, j \in \{1, 3, 4\}$, $i \neq j$. One then checks that (5) fails to hold in all cases.

For $p = 7$, as above, (6) and Lübeck's computations reduce us to the following options for γ: $\varpi_i + \varpi_2$, $i \in \{1, 3, 4\}$, $2\varpi_2$. Unfortunately, (5) can hold here. In the case $\gamma = 2\varpi_2$, the only values of λ and $\mu_{(1)}$ that work are as follows:

λ	(6,5,6,6)	(4,6,6,6)	(6,6,4,6)	(6,6,6,4)
$\mu_{(1)}$	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)

λ	(5,6,6,6)	(6,6,5,6)	(6,6,6,5)
$\mu_{(1)}$	(1,0,0,0)	(0,0,1,0)	(0,0,0,1)

One can check that in each case λ is not G_1-linked to $\mu_{(1)}$. So that case is eliminated.

In the first (symmetric) cases for γ, there are options for λ and $\mu_{(1)}$ where linkage holds. For example, for $\gamma = \varpi_1 + \varpi_2$, one has the following cases where G_1-linkage holds between λ and $\mu_{(1)}$:

λ	(6,4,5,5)	(4,5,5,5)	(5,5,5,5)	(6,5,5,5)
$\mu_{(1)}$	(0,1,0,0)	(0,0,0,0)	(1,0,0,0)	(2,0,0,0)

Similar cases would exist for $\mu_1 = \varpi_2 + \varpi_3$ and $\mu_1 = \varpi_2 + \varpi_4$. Additional cases may also exist, as a complete list has not been computed. □

For types B_4 and C_4, $h = 8$, and the claim holds for $p > 7$. No investigation has been made for small primes.

For type F_4, $h = 12$, and we are done if $p > 17$. We make some observations for $p = 2$. We have $\alpha_0^\vee = 2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee$ and $\tilde{\alpha}^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee$. So $\langle \rho, \alpha_0^\vee \rangle = 11$ and $\langle \rho, \tilde{\alpha}^\vee \rangle = 8$. The inequalities in (6) force $\langle \gamma, \alpha_0^\vee \rangle \leq 5$ and $\langle \gamma, \tilde{\alpha}^\vee \rangle \leq 3$. One has the following options: $\gamma = \varpi_1$, ϖ_2, ϖ_3, ϖ_4, $2\varpi_4$, $\varpi_1 + \varpi_4$, or $\varpi_3 + \varpi_4$. The case $\gamma = \varpi_3 + \varpi_4$ may be eliminated as it does not satisfy (5), and the case case $\gamma = \varpi_4$ may be eliminated as $\nabla(\varpi_4)$ is simple (for $p = 2$) by [Jan2]. One finds the following possibilities where (5) holds and λ is G_1-linked to $\mu_{(1)}$:

λ	$\mu_{(1)}$
(1,1,0,1)	(0,0,0,0), (0,0,1,0), (0,0,0,1), (0,0,0,2)
(1,1,1,0)	(0,0,0,0), (1,0,0,0), (0,0,1,0), (0,0,0,1), (1,0,0,1)
(0,1,1,0)	(0,0,0,1)
(1,1,0,0)	(0,0,0,1)

5.5. Type A_5

For type A_5, $h = 6$. By Theorem 4.2.1, we are done for $p > 7$. In contrast to the smaller rank cases, even for $p = 2$, our earlier methods do not completely resolve the issue. Using (6), (5), and Jantzen's simplicity algorithm, one is reduced to one case: $\lambda = \varpi_1 + \varpi_2 + \varpi_4 + \varpi_5$ with $\mu_{(1)} = 0$ and $\gamma = \varpi_1 + \varpi_5$. Note that λ is indeed G_1-linked to the zero weight. However, through an intricate analysis of the modules involved, in Section 6, we are able to address this case. See Theorem 6.4.4. For $p = 3$, 5, or 7, there will be many more options for λ that cannot be dealt with by the above methods.

5.6. Summary

For $\lambda \in X_1$, $\mathrm{St}_1 \otimes L(\lambda)$ has a good filtration in the following cases:

- Type A_n: $p > 2n - 3$.
  - Type A_2: all primes.
  - Type A_3: all primes.
  - Type A_4: $p \neq 5$.
  - Type A_5: $p \neq 3, 5, 7$.
- Type B_n: $p > 4n - 5$.
  - Type B_2 (equivalently C_2): all primes.
  - Type B_3: $p \neq 7$.
    - $p = 7$ case, all except $\lambda = (6,5,5), (6,4,5), (6,5,4), (5,5,5),$ $(5,5,4), (5,4,5), (4,5,5), (4,5,4), (3,5,5)$.
- Type C_n: $p > 4n - 5$.
  - Type C_3: $p \neq 3, 7$.
    - $p = 3$ case, all except $\lambda = (2,1,2)$ or $(2,2,1)$.
    - $p = 7$ case, all except $\lambda = (6,5,5), (6,4,5), (6,5,4), (5,5,5), (4,5,5)$.
- Type D_n: $p \geq 4n - 9$.
  - Type D_4: $p \neq 7$.
- Type E_6: $p > 19$.
- Type E_7: $p > 31$.
- Type E_8: $p > 53$.
- Type F_4: $p > 19$
  - $p = 2$ case, reduced to $\lambda = (1,1,0,1), (1,1,1,0), (0,1,1,0), (1,1,0,0)$.
- Type G_2: all primes.

6. A detailed analysis in characteristic 2

In this section we investigate two very similar situations in which a proof that $\mathrm{St}_1 \otimes L(\lambda)$ has a good filtration is beyond the reach of our earlier arguments. In particular, basic weight combinatorics are not conclusive, and thus it becomes necessary to better understand the submodule structure of a tensor product of G-modules. We show in one of the two cases that we are able to verify that $\mathrm{St}_1 \otimes L(\lambda)$ does have a good filtration, which allows us to conclude that Conjecture 1.2.1 ($\Rightarrow$) holds. That this holds in such a nontrivial setting could be viewed as the strongest evidence yet for its truth in arbitrary characteristic. However, if this is indeed true, one will need to find the underlying reason why it holds in situations similar to those considered here.

6.1. Composition factors of the form $\mathrm{St}_1 \otimes L(\alpha_0)$

Unless otherwise noted, we assume throughout this section that $p = 2$. Further, we assume that (i) $G = \mathrm{SL}_6$ (type A_5) and $\lambda = \varpi_1 + \varpi_2 + \varpi_4 + \varpi_5$ or (ii) G is of type E_7 and $\lambda = \varpi_4$. In either case, $2\alpha_0$ appears as a weight in $L(\lambda)$. For SL_6, $\alpha_0 = \varpi_1 + \varpi_5$, and one verifies the multiplicity of $2\alpha_0$ directly from the tables provided by [L]. For E_7, $L(\lambda)$ has dimension outside the range for modules appearing in these tables. This can be overcome by observing here that $\alpha_0 = \varpi_1$, and that

$$\lambda - 2\alpha_0 = 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7.$$

We may then work over the Levi subgroup $L_J \leq \mathrm{E}_7$ with $J = \{\alpha_2, \dots, \alpha_7\}$, a group of type D_6, and observe that the multiplicity of $2\varpi_1$ in $L(\varpi_4)$ (for E_7) is the same as the multiplicity of 0 in the D_6-module whose highest weight is the fundamental weight corresponding to the triality node in the Dynkin diagram. According to the orientation of the Dynkin diagrams in [L], this is the module $L(\varpi_3)$ for the group D_6. This module has dimension 364 (for $p = 2$), and the 0 weight occurs with multiplicity 4.

Since $\rho + 2\alpha_0$ is a highest weight, belonging to the Steinberg linkage class, inside $\mathrm{St}_1 \otimes L(\lambda)$, the multiplicity of $\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}$ as a composition factor of $\mathrm{St}_1 \otimes L(\lambda)$ equals the multiplicity of $2\alpha_0$ as a weight in $L(\lambda)$. At the same time, $\Delta(\alpha_0) \cong \mathfrak{g}$ (the adjoint representation), and $L(\alpha_0) \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, with $\mathfrak{z}(\mathfrak{g})$ denoting the one-dimensional center of $\mathfrak{g}$. Since $L(\alpha_0)^{(1)} \subsetneq \nabla(\alpha_0)^{(1)}$, by [Jan3, Prop. II.3.19],

$$\mathrm{St}_1 \otimes L(\alpha_0)^{(1)} \subsetneq \mathrm{St}_1 \otimes \nabla(\alpha_0)^{(1)} \cong \nabla(\rho + 2\alpha_0),$$

and so the composition factor $\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}$ does not have a good filtration. We note that G is simply-laced, so that α_0 is the highest root.

For $G = \mathrm{SL}_6$, the other dominant weights γ such that 2γ is a weight of $\Delta(\lambda)$ are

$$\gamma = 0, \varpi_3.$$

For G of type E_7, they are

$$\gamma = 0, \varpi_7.$$

For $G = \mathrm{SL}_6$, we have $L(\varpi_3) \cong \nabla(\varpi_3)$, and for G of type E_7, we have $L(\varpi_7) \cong \nabla(\varpi_7)$. Neither module (for the given G) extends nor can be extended by $L(\alpha_0)$. Furthermore, we have in both cases that

$$\mathrm{Ext}^1_G(k, L(\alpha_0)) \cong \mathrm{Ext}^1_G(L(\alpha_0), k) \cong k.$$

These one-dimensional extension groups are accounted for by the indecomposable modules $\Delta(\alpha_0)$ and $\nabla(\alpha_0)$. One may further check by standard long exact sequence computations that

$$\begin{aligned} \operatorname{Ext}^1_G(\nabla(\alpha_0), L(\alpha_0)) = 0; \quad & \operatorname{Ext}^1_G(L(\alpha_0), \nabla(\alpha_0)) = 0; \\ \operatorname{Ext}^1_G(k, \nabla(\alpha_0)) = 0; \quad & \operatorname{Ext}^1_G(\nabla(\alpha_0), k) \cong k. \end{aligned}$$

Applying the contravariant dual τ-functor, (28) in Section 2.1, which interchanges Weyl and induced modules, while preserving simple (and tilting) modules, we obtain all extensions involving $\Delta(\alpha_0)$, k, and $L(\alpha_0)$. Summarizing, the collection of indecomposable G-modules having composition factors coming from the collection $\{k, L(\alpha_0)\}$ are (up to isomorphism)

$$\{k,\ L(\alpha_0),\ \nabla(\alpha_0),\ \Delta(\alpha_0),\ T(\alpha_0)\}.$$

The structure of the tilting module $T(\alpha_0)$ is given by the exact sequence

$$0 \to k \to T(\alpha_0) \to \nabla(\alpha_0) \to 0.$$

Via the equivalence of categories between G-mod and its Steinberg block, it follows that an indecomposable summand of $\mathrm{St}_1 \otimes L(\lambda)$ that contains $\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}$ as a composition factor must be isomorphic to one of the following:

$$\{\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}, \mathrm{St}_1 \otimes \nabla(\alpha_0)^{(1)}, \mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)}, \mathrm{St}_1 \otimes T(\alpha_0)^{(1)}\}.$$

Note that, if we instead work with $\mathrm{St}_1 \otimes \Delta(\lambda)$, the only possibilities from this list are the two involving the Weyl module or the tilting module. One can also make the deduction about the module structures above by working with the truncated category obtained by looking at the full subcategory of rational G-modules having composition factors with highest weight less than or equal to α_0 (and linked to α_0). This category has finite representation type.

Lemma 6.1.1. *Assume $p = 2$. Let G be of type A_5 with $\lambda = \varpi_1 + \varpi_2 + \varpi_4 + \varpi_5$ or G be of type E_7 with $\lambda = \varpi_4$. The summands of $\mathrm{St}_1 \otimes L(\lambda)$ containing the composition factor $\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}$ all have a good filtration if and only if*

$$\operatorname{Hom}_G(\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}, \mathrm{St}_1 \otimes L(\lambda)) = 0.$$

Proof. By the previous discussion, if $\operatorname{Hom}_G(\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}, \mathrm{St}_1 \otimes L(\lambda)) = 0$, then the only relevant summands that can appear in $\mathrm{St}_1 \otimes L(\lambda)$ are of the form $\mathrm{St}_1 \otimes T(\alpha_0)^{(1)}$ and $\mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)}$, thus these summands have a Weyl filtration. Now, $\mathrm{St}_1 \otimes L(\lambda)$ is τ-invariant. Further, if M is a summand of $\mathrm{St}_1 \otimes L(\lambda)$, then ${}^{\tau}M$ is isomorphic to a summand of $\mathrm{St}_1 \otimes L(\lambda)$ having the same composition factors. It follows then that a summand containing the factor $\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}$ also has a good filtration.

The converse is established by the reverse implication of each step in the argument. □

6.2. Analysis of tensor products

In order to analyze this situation further, we need to establish some basic facts about tensor products of modules. Note that the results in this subsection hold under our general assumption on G, and Lemmas 6.2.1 and 6.2.2 hold for arbitrary primes p. Recall that for a B-module M, there is an "evaluation map" $\varepsilon_M : \mathrm{ind}_B^G M \to M$ which induces one direction of the Frobenius reciprocity bijection [Jan3, I.3.4].

Lemma 6.2.1. *Let M be a B-module and N, N' be G-modules. Composition with the B-module homomorphism $\varepsilon_M \otimes \mathrm{id}$ defines a bijection*

$$\mathrm{Hom}_G(N', (\mathrm{ind}_B^G M) \otimes N) \xrightarrow{\sim} \mathrm{Hom}_B(N', M \otimes N).$$

Proof. As recalled above, composition with $\varepsilon_{M\otimes N}$ defines a bijection

$$\mathrm{Hom}_G(N', \mathrm{ind}_B^G(M \otimes N)) \xrightarrow{\sim} \mathrm{Hom}_B(N', M \otimes N).$$

The "tensor identity" in [Jan3, I.3.6] is established by a canonical isomorphism

$$\mathrm{ind}_B^G(M \otimes N) \xrightarrow{\sim} (\mathrm{ind}_B^G M) \otimes N.$$

This isomorphism is specified via canonical embeddings

$$\mathrm{ind}_B^G(M \otimes N) \hookrightarrow M \otimes N \otimes k[G] \hookleftarrow (\mathrm{ind}_B^G M) \otimes N,$$

together with an automorphism of $M \otimes N \otimes k[G]$ that sends the embedding on the left isomorphically onto the embedding on the right. Now, the morphisms $\varepsilon_M \otimes \mathrm{id}$ and $\varepsilon_{M\otimes N}$ both come from these embeddings, by restricting the map

$$\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon_G : M \otimes N \otimes k[G] \to M \otimes N \otimes k,$$

where ε_G is the counit map on $k[G]$, to each embedded subgroup. This proves the claim. □

Lemma 6.2.2. *Let $\mu, \lambda \in X_+$, and let v_μ and z_λ denote highest weight vectors of the modules $\Delta(\mu)$ and $L(\lambda)$ respectively. Let M be any G-module. If*

$$\phi : \Delta(\mu) \to M \otimes L(\lambda)$$

is a non-zero homomorphism of G-modules, then there is some $0 \neq m \in M$ such that

$$\phi(v_\mu) = (m \otimes z_\lambda) + y, \quad \text{with } y \in M \otimes \left(\sum_{\sigma<\lambda} L(\lambda)_\sigma\right).$$

Proof. There is a canonical inclusion

$$\mathrm{Hom}_G(\Delta(\mu), M \otimes L(\lambda)) \hookrightarrow \mathrm{Hom}_G(\Delta(\mu), M \otimes \nabla(\lambda)).$$

By Lemma 6.2.1, the B-module homomorphism

$$M \otimes \nabla(\lambda) \xrightarrow{\mathrm{id}\otimes\varepsilon_\lambda} M \otimes \lambda$$

induces a bijection

$$\mathrm{Hom}_G(\Delta(\mu), M \otimes \nabla(\lambda)) \cong \mathrm{Hom}_B(\Delta(\mu), M \otimes \lambda).$$

As a B-module, $\Delta(\mu)$ is generated by v_μ, thus a non-zero B-homomorphism from $\Delta(\mu)$ to any B-module must send v_μ to a non-zero element. We have a vector space decomposition

$$M \otimes \nabla(\lambda) = M \otimes z_\lambda + M \otimes \left(\sum_{\sigma<\lambda} \nabla(\lambda)_\sigma\right).$$

The result then follows by noting that $\mathrm{id} \otimes \varepsilon_\lambda$ sends

$$M \otimes \left(\sum_{\sigma<\lambda} \nabla(\lambda)_\sigma\right) \to 0. \qquad \square$$

Lemma 6.2.3. *Assume $p = 2$. Let $w_{\rho+2\alpha_0}$ be a highest weight vector of the module $\mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)}$ and w_ρ be a maximal vector generating the simple submodule $\mathrm{St}_1 \otimes k \leq \mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)}$. There is some $X \in \mathrm{Dist}(U)$ of weight $-2\alpha_0$ such that*

$$X.w_{\rho+2\alpha_0} = w_\rho.$$

The comultiplication of X in $\mathrm{Dist}(U)$ is given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X_i' \otimes X_i'',$$

where for each i

$$-2\alpha_0 < \mathrm{wt}(X_i'), \mathrm{wt}(X_i'') < 0.$$

Proof. The Weyl module $\Delta(\rho + 2\alpha_0) \cong \mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)}$ is generated over B, and over U, by any highest weight vector. The same is true over $\mathrm{Dist}(B)$ and $\mathrm{Dist}(U)$, thus there is some $X \in \mathrm{Dist}(U)$ that gives the required action. Moreover, it is clear that we can choose X to be a T-weight vector of weight $-2\alpha_0$ (indeed, any X such that $X.w_{\rho+2\alpha_0} = w_\rho$ will be a sum of weight vectors, and any elements in the sum not having weight $-2\alpha_0$ must then act as zero, so we can modify X by subtracting off if necessary such terms).

The augmentation ideal of $\mathrm{Dist}(U)$ is the vector subspace spanned by all T-weight vectors of weight $\neq 0$, hence X is in this ideal. The claim about $\Delta(X)$ then follows from a general fact about the comultiplication of elements in the augmentation ideal of a Hopf algebra, together with the fact that the terms in $\Delta(X)$ must have total weight $-2\alpha_0$. $\square$

6.3. Reductions

Returning to our special assumptions on G, p, and λ, we now give a series of reductions toward proving that any summand of $\mathrm{St}_1 \otimes L(\lambda)$ containing $\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}$ as a composition factor is tilting.

Reduction 1: By Lemma 6.1.1, this is equivalent to showing that $\mathrm{St}_1 \otimes L(\alpha_0)^{(1)}$ does not appear as a submodule of $\mathrm{St}_1 \otimes L(\lambda)$.

Reduction 2: This is equivalent to showing that any maximal vector in $\mathrm{St}_1 \otimes L(\lambda)$ of weight $\rho + 2\alpha_0$ generates a submodule isomorphic to $\mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)}$.

Reduction 3: By Lemma 6.2.3, this is equivalent to showing that if $v_{\rho+2\alpha_0}$ is any such maximal vector of $\mathrm{St}_1 \otimes L(\lambda)$ and $X \in \mathrm{Dist}(U)$ is chosen as in the lemma, then $X.v_{\rho+2\alpha_0} \neq 0$.

6.4. Type A_5

For this subsection, we restrict ourselves to the case $G = \mathrm{SL}_6$ and $\lambda = \varpi_1 + \varpi_2 + \varpi_4 + \varpi_5$. Fix, for each positive root β, the usual negative and positive Chevalley basis elements f_β and e_β of $\mathfrak{g}$ coming from the natural representation, and let $h_\beta = [e_\beta, f_\beta]$. We view these as elements inside $\mathrm{Dist}(G)$.

We have $[e_{\alpha_i}, f_{\alpha_j}] = 0$ if $i \neq j$. Because $p = 2$, we also have

$$[h_{\alpha_i}, e_{\alpha_j}] = e_{\alpha_j} \quad \text{and} \quad [h_{\alpha_i}, f_{\alpha_j}] = f_{\alpha_j} \ \text{ if } |i-j| = 1,$$

otherwise

$$[h_{\alpha_i}, e_{\alpha_j}] = 0 = [h_{\alpha_i}, f_{\alpha_j}].$$

One computes that

$$\lambda - 2\alpha_0 = \alpha_2 + \alpha_3 + \alpha_4.$$

Let $J = \{\alpha_2, \alpha_3, \alpha_4\}$. By [Jan3, II.2.11] (see [GGN, Prop. 3.3] for a more thorough discussion), the L_J-Weyl module $\Delta_J(\lambda)$ is a direct summand of $\Delta(\lambda)$, considered as an L_J-module. More specifically

$$\Delta_J(\lambda) \cong \sum_{\lambda - \mu \in \mathbb{Z}J} \Delta(\lambda)_\mu,$$

with the L_J-complement to $\Delta_J(\lambda)$ consisting of the sum of the remaining weight spaces. If we further restrict our attention to the weight spaces from $2\alpha_0$ to λ, it follows (by weight considerations) that there is an isomorphism of B^+-modules

$$\sum_{\mu \geq 2\alpha_0} \Delta(\lambda)_\mu \cong \sum_{\mu \geq 2\alpha_0} \Delta_J(\lambda)_\mu.$$

The derived subgroup (L_J, L_J) is isomorphic to SL_4, the restriction of $\Delta_J(\lambda)$ to (L_J, L_J) is the adjoint representation, and $2\alpha_0$ restricts to the zero weight for $T \cap (L_J, L_J)$. Using the structure of $\mathrm{Lie}(\mathrm{SL}_4)$, one readily computes the B^+ structure of $\sum_{\mu \geq 2\alpha_0} \Delta(\lambda)_\mu$.

In particular, given a maximal weight vector z_λ of $\Delta(\lambda)$, the following is a T-basis for $\Delta(\lambda)_{2\alpha_0}$:

$$\{f_{\alpha_2}f_{\alpha_3}f_{\alpha_4}.z_\lambda,\ \ f_{\alpha_4}f_{\alpha_3}f_{\alpha_2}.z_\lambda,\ \ f_{\alpha_3}f_{\alpha_4}f_{\alpha_2}.z_\lambda\}.$$

We also have $(f_{\alpha_2}f_{\alpha_3}f_{\alpha_4}.z_\lambda + f_{\alpha_4}f_{\alpha_3}f_{\alpha_2}.z_\lambda)$ as a U^+-fixed vector, and

$$L(\lambda)_{2\alpha_0} \cong \Delta(\lambda)_{2\alpha_0}/(f_{\alpha_2}f_{\alpha_3}f_{\alpha_4}.z_\lambda + f_{\alpha_4}f_{\alpha_3}f_{\alpha_2}.z_\lambda). \tag{7}$$

Let w_ρ be a highest weight vector in St_1. Any T-basis $\{y_i\}$ for $\mathrm{Dist}(U_1)$ yields a T-basis $\{y_i.w_\rho\}$ for St_1. While a standard choice is to take a PBW-basis after choosing an ordering of roots, one can alternatively take a basis consisting of products of the various f_{α_i} for α_i a simple root. While it is in general harder to list all such basis elements in this manner, we will find it more convenient in our limited consideration.

Lemma 6.4.1. *Assume G is SL_6 and $p=2$. Let w_ρ be a a highest weight vector in St_1. Let $\alpha_{i_1},\dots,\alpha_{i_m}$ be an ordered collection of simple roots without repetition (so $1 \le m \le 5$). For any simple root α,*

$$e_\alpha f_{\alpha_{i_1}}\cdots f_{\alpha_{i_m}}.w_\rho = 0$$

if $\alpha_{i_j} \neq \alpha$ for all i_j. Otherwise, if some $\alpha_{i_j} = \alpha$, then

$$e_\alpha f_{\alpha_{i_1}}\cdots f_{\alpha_{i_m}}.w_\rho = (s+1)f_{\alpha_{i_1}}\cdots f_{\alpha_{i_{j-1}}}f_{\alpha_{i_{j+1}}}\cdots f_{\alpha_{i_m}}.w_\rho,$$

where s is the number of i_k such that $k > j$ and $|i_j - i_k| = 1$.

Proof. If no $\alpha_{i_j} = \alpha$, then e_α commutes past each $f_{\alpha_{i_j}}$ in $\mathrm{Dist}(G_1)$, and since e_α annihilates w_ρ, the first statement follows.

Otherwise, if $\alpha = \alpha_{i_j}$ for some j, then e_α commutes past each $f_{\alpha_{i_\ell}}$, $\ell < j$. One then applies the commutation relations above. We have

$$\begin{aligned} e_\alpha f_{\alpha_{i_1}}\cdots f_{\alpha_{i_m}}.w_\rho &= f_{\alpha_{i_1}}\cdots f_{\alpha_{i_{j-1}}}e_\alpha f_{\alpha_{i_j}}f_{\alpha_{i_{j+1}}}\cdots f_{\alpha_{i_m}}.w_\rho \\ &= f_{\alpha_{i_1}}\cdots f_{\alpha_{i_{j-1}}}(f_{\alpha_{i_j}}e_\alpha + h_{\alpha_{i_j}})f_{\alpha_{i_{j+1}}}\cdots f_{\alpha_{i_m}}.w_\rho \\ &= f_{\alpha_{i_1}}\cdots f_{\alpha_{i_{j-1}}}f_{\alpha_{i_j}}e_\alpha f_{\alpha_{i_{j+1}}}\cdots f_{\alpha_{i_m}}.w_\rho \\ &\quad + f_{\alpha_{i_1}}\cdots f_{\alpha_{i_{j-1}}}h_{\alpha_{i_j}}f_{\alpha_{i_{j+1}}}\cdots f_{\alpha_{i_m}}.w_\rho \\ &= f_{\alpha_{i_1}}\cdots f_{\alpha_{i_{j-1}}}h_{\alpha_{i_j}}f_{\alpha_{i_{j+1}}}\cdots f_{\alpha_{i_m}}.w_\rho, \end{aligned}$$

since the first term is seen to be zero, by commuting the e_α past the remaining terms. We now use the fact that

$$h_{\alpha_{i_j}}f_{\alpha_{i_k}} = f_{\alpha_{i_k}}h_{\alpha_{i_j}} + f_{\alpha_{i_k}} = f_{\alpha_{i_k}}(h_{\alpha_{i_j}} + 1)$$

if $|i_j - i_k| = 1$, otherwise

$$h_{\alpha_{i_j}}f_{\alpha_{i_k}} = f_{\alpha_{i_k}}h_{\alpha_{i_j}}.$$

Repeatedly applying this we obtain

$$e_\alpha f_{\alpha_{i_1}}\cdots f_{\alpha_{i_m}}.w_\rho = f_{\alpha_{i_1}}\cdots f_{\alpha_{i_{j-1}}}f_{\alpha_{i_{j+1}}}\cdots f_{\alpha_{i_m}}(h_{\alpha_{i_j}} + s).w_\rho,$$

and as $h_{\alpha_{i_j}}.w_\rho = w_\rho$, the result follows. $\square$

Lemma 6.4.2. *Assume G is SL_6 and $p = 2$. The following vectors form a basis of maximal vectors in $\mathrm{St}_1 \otimes \Delta(\lambda)$ having weight $\rho + 2\alpha_0$:*

$$\begin{aligned}
\mathbf{v_1} &= w_\rho \otimes f_{\alpha_3} f_{\alpha_2} f_{\alpha_4}.z_\lambda + f_{\alpha_2}.w_\rho \otimes f_{\alpha_3} f_{\alpha_4}.z_\lambda + f_{\alpha_4}.w_\rho \otimes f_{\alpha_3} f_{\alpha_2}.z_\lambda \\
&\quad + f_{\alpha_2} f_{\alpha_3}.w_\rho \otimes f_{\alpha_4}.z_\lambda + f_{\alpha_4} f_{\alpha_3}.w_\rho \otimes f_{\alpha_2}.z_\lambda \\
&\quad + (f_{\alpha_2} f_{\alpha_3} f_{\alpha_4} + f_{\alpha_4} f_{\alpha_3} f_{\alpha_2}).w_\rho \otimes z_\lambda, \\
\mathbf{v_2} &= w_\rho \otimes f_{\alpha_2} f_{\alpha_3} f_{\alpha_4}.z_\lambda + f_{\alpha_3}.w_\rho \otimes f_{\alpha_2} f_{\alpha_4}.z_\lambda + f_{\alpha_3} f_{\alpha_2}.w_\rho \otimes f_{\alpha_4}.z_\lambda \\
&\quad + f_{\alpha_3} f_{\alpha_4}.w_\rho \otimes f_{\alpha_2}.z_\lambda + f_{\alpha_3} f_{\alpha_2} f_{\alpha_4}.w_\rho \otimes z_\lambda, \\
\mathbf{v_3} &= w_\rho \otimes (f_{\alpha_2} f_{\alpha_3} f_{\alpha_4} + f_{\alpha_4} f_{\alpha_3} f_{\alpha_2}).z_\lambda.
\end{aligned}$$

Proof. The elements

$$f_{\alpha_3} f_{\alpha_2} f_{\alpha_4}.z_\lambda, \quad f_{\alpha_2} f_{\alpha_3} f_{\alpha_4}.z_\lambda, \quad \text{and} \quad (f_{\alpha_2} f_{\alpha_3} f_{\alpha_4} + f_{\alpha_4} f_{\alpha_3} f_{\alpha_2}).z_\lambda$$

are linearly independent in $\Delta(\lambda)$. From this it follows that the three different sums of simple tensors listed above are linearly independent in $\mathrm{St}_1 \otimes \Delta(\lambda)$.

To verify maximality, since $(\rho + \lambda) - (\rho + 2\alpha_0) = \alpha_2 + \alpha_3 + \alpha_4$, we need only check that each e_{α_i}, $2 \leq i \leq 4$, annihilates these elements, where the action on a simple tensor is via $e_{\alpha_i} \otimes 1 + 1 \otimes e_{\alpha_i}$. The verification for $\mathbf{v_3}$ is immediate as it is annihilated both by $e_{\alpha_i} \otimes 1$ and by $1 \otimes e_{\alpha_i}$. For $\mathbf{v_1}, \mathbf{v_2}$, one applies Lemma 6.4.1 to see that the sum $e_{\alpha_i} \otimes 1 + 1 \otimes e_{\alpha_i}$ annihilates each vector. □

Fix a surjective G-module homomorphism $f : \Delta(\lambda) \to L(\lambda)$. Over L_J, f restricts to a surjective homomorphism $\Delta_J(\lambda) \to L_J(\lambda)$. By (7), it follows that

$$(\ker f) \cap \Delta(\lambda)_{2\alpha_0} = \mathrm{Span}\{f_{\alpha_2} f_{\alpha_3} f_{\alpha_4}.z_\lambda + f_{\alpha_4} f_{\alpha_3} f_{\alpha_2}.z_\lambda\}.$$

We obtain from f a surjective G-module homomorphism

$$\mathrm{id} \otimes f : \mathrm{St}_1 \otimes \Delta(\lambda) \to \mathrm{St}_1 \otimes L(\lambda).$$

On a simple tensor of the form $w_\rho \otimes v$ we have $(\mathrm{id} \otimes f)(w_\rho \otimes v) = w_\rho \otimes f(v)$. Given the definition of f on $\Delta(\lambda)_{2\alpha_0}$, it follows that $(\mathrm{id} \otimes f)(\mathbf{v_3}) = 0$, and that $(\mathrm{id} \otimes f)(\mathbf{v_1})$ and $(\mathrm{id} \otimes f)(\mathbf{v_2})$ are linearly independent.

Lemma 6.4.3. *Assume G is SL_6 and $p = 2$. Let $X \in \mathrm{Dist}(U)$ be as in Lemma 6.2.3 and $\mathbf{v_i}$ be as in Lemma 6.4.2. Then, in $\mathrm{St}_1 \otimes L(\lambda)$,*

$$X.(\mathrm{id} \otimes f)(\mathbf{v_1}) \neq 0, \quad X.(\mathrm{id} \otimes f)(\mathbf{v_2}) \neq 0,$$

and these elements are linearly independent. Therefore, $(\mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)})^{\oplus 2} \hookrightarrow \mathrm{St}_1 \otimes L(\lambda)$.

Proof. The module $\mathrm{St}_1 \otimes \Delta(\lambda)$, being the tensor product of two Weyl modules, has a Weyl filtration. Therefore, the vectors $\mathbf{v_1}, \mathbf{v_2}$, and $\mathbf{v_3}$ each generate copies of $\mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)}$. It follows then that $X.\mathbf{v_i} \neq 0$, for $1 \leq i \leq 3$. Further, since the $\mathbf{v_i}$ are linearly independent, the elements $X.\mathbf{v_i}$ must be also (the $\mathbf{v_i}$ must together generate a submodule isomorphic to $(\mathrm{St}_1 \otimes \Delta(\alpha_0)^{(1)})^{\oplus 3}$). Thus each $X.\mathbf{v_i}$

is a maximal vector in $\mathrm{St}_1 \otimes \Delta(\lambda)$ of weight ρ (lying within a copy of St_1 inside the G-socle of $\mathrm{St}_1 \otimes \Delta(\lambda)$). By Lemma 6.2.2, it follows that each $X.\mathbf{v_i}$, expressed as a sum of simple tensors in $\mathrm{St}_1 \otimes \Delta(\lambda)$, necessarily involves a term of the form $w_\rho \otimes v_0$, with $0 \neq v_0 \in \Delta(\lambda)_0$. On the other hand, multiplying the expression for the comultiplication of X in Lemma 6.2.3 against the explicit computations of the $\mathbf{v_i}$ above, for every $X.\mathbf{v_i}$ to contain a term of this form it must hold that

$$w_\rho \otimes X f_{\alpha_3} f_{\alpha_2} f_{\alpha_4}.z_\lambda \neq 0, \quad w_\rho \otimes X f_{\alpha_2} f_{\alpha_3} f_{\alpha_4}.z_\lambda \neq 0,$$
$$w_\rho \otimes X(f_{\alpha_2} f_{\alpha_3} f_{\alpha_4} + f_{\alpha_4} f_{\alpha_3} f_{\alpha_2}).z_\lambda \neq 0,$$

and that they are linearly independent elements in $\mathrm{St}_1 \otimes \Delta(\lambda)$ (this last fact follows by considering the argument just stated applied to $X.(a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3})$ for all $(a, b, c) \neq (0, 0, 0)$). This implies that the collection

$$\{X f_{\alpha_3} f_{\alpha_2} f_{\alpha_4}.z_\lambda,\ \ X f_{\alpha_2} f_{\alpha_3} f_{\alpha_4}.z_\lambda,\ \ X(f_{\alpha_2} f_{\alpha_3} f_{\alpha_4} + f_{\alpha_4} f_{\alpha_3} f_{\alpha_2}).z_\lambda\}$$

is linearly independent in $\Delta(\lambda)$.

There is a non-zero B-module (composite) homomorphism

$$\mathrm{St}_1 \to \Delta(\lambda) \otimes \nabla(\rho - \lambda) \to \Delta(\lambda) \otimes (\rho - \lambda)$$

which restricts to a surjective U-module homomorphism $\mathrm{St}_1 \to \Delta(\lambda)$ (this holds more generally). Thus any element in $\mathrm{Dist}(U)$ that does not annihilate z_λ cannot annihilate w_ρ. It follows that

$$\{X f_{\alpha_3} f_{\alpha_2} f_{\alpha_4}.w_\rho,\ \ X f_{\alpha_2} f_{\alpha_3} f_{\alpha_4}.w_\rho,\ \ X(f_{\alpha_2} f_{\alpha_3} f_{\alpha_4} + f_{\alpha_4} f_{\alpha_3} f_{\alpha_2}).w_\rho\}$$

is linearly independent in St_1. In particular

$$X f_{\alpha_3} f_{\alpha_2} f_{\alpha_4}.w_\rho \otimes z_\lambda \quad \text{and} \quad X(f_{\alpha_2} f_{\alpha_3} f_{\alpha_4} + f_{\alpha_4} f_{\alpha_3} f_{\alpha_2}).w_\rho \otimes z_\lambda$$

and linearly independent in $\mathrm{St}_1 \otimes \Delta(\lambda)$.

Since $f(z_\lambda) \neq 0$, it follows that

$$(\mathrm{id} \otimes f)(X.\mathbf{v_1}) \quad \text{and} \quad (\mathrm{id} \otimes f)(X.\mathbf{v_2})$$

are linearly independent vectors in $\mathrm{St}_1 \otimes L(\lambda)$, proving the claim. $\square$

By our reductions in Section 6.3, it follows that $\mathrm{St}_1 \otimes L(\lambda)$ has a good filtration. As discussed in Section 5.5, this case was the only remaining one to check for SL_6, which proves the following.

Theorem 6.4.4. *Assume $G = \mathrm{SL}_6$ and $p = 2$. Then*

(a) $\mathrm{St}_r \otimes L(\lambda)$ *has a good filtration for all* $\lambda \in X_r$, $r \geq 1$.
(b) *In this case Conjecture* 1.2.1($\Rightarrow$) *holds.*

7. Tensoring with $k[G_r]$

7.1. Filtrations on $k[G]$

Recall that given any affine scheme X over k and an algebraic action of G on X, the coordinate ring $k[X]$ becomes a G-module by the action $g.f(x) = f(g^{-1}.x)$ (such actions are intended to hold functorially, i.e., for every k-algebra A, every $g \in G(A), x \in X(A)$, and $f \in A[X_A]$). In particular, $k[G]$ becomes a G-module via the conjugation action of G on itself. Further, since G_r is a normal subgroup scheme of G, one similarly obtains a G-module structure on $k[G_r]$.

There is a $(G \times G)$-action on G via $(g_1, g_2).h = g_1 h g_2^{-1}$. If we take the diagonal embedding of G into $(G \times G)$, then the restriction of this action is the conjugation action of G on itself. We recall the following result due to Donkin and Koppinen [Jan3, Prop. II.4.20] (see also [Jan3, Rem. II.4.21]).

Proposition 7.1.1. *As a $(G \times G)$-module, $k[G]$ has a good filtration. The good filtration factors are of the form*

$$\nabla(\lambda) \otimes \nabla(\lambda^*), \quad \lambda \in X_+,$$

each occuring with multiplicity one. In particular, $k[G]$, as a G-module under the action induced by the conjugation action of G, has a good filtration.

7.2. Filtrations on $\mathrm{St}_r \otimes k[G_r]$

In consideration of this result and Conjecture 1.2.1, we ask the following question that has been answered in the affirmative when p is good by Donkin [Don3].

Question 7.2.1. For each $r \geq 1$, does $\mathrm{St}_r \otimes k[G_r]$ have a good filtration?

We conclude by making the following observation. Let I_ε denote the augmentation ideal of $k[G]$. Under the conjugation action of G, this ideal is a G-submodule of $k[G]$, with G-module complement spanned by the subalgebra $k \cdot 1 \leq k[G]$. It follows then that I_ε has a good filtration over G. For each $r \geq 1$, let M_r denote the subset of $k[G]$ defined as

$$M_r = \{x^{p^r} \mid x \in I_\varepsilon\}.$$

Since k is algebraically closed and of characteristic p, this is a subspace of $k[G]$, and indeed a submodule for G. Further, the ideal defining the closed subgroup scheme G_r is the ideal generated by M_r. Thus, the algebra homomorphism $k[G] \to k[G_r]$ factors (as G-modules) through $k[G]/M_r$.

Theorem 7.2.2. *The G-module $\mathrm{St}_r \otimes (k[G]/M_r)$ has a good filtration.*

Proof. If X and Y are any two affine schemes with algebraic G-actions, and if $f : X \to Y$ is a G-equivariant homomorphism of schemes, then the comorphism $f^* : k[Y] \to k[X]$ is a homomorphism of G-modules (with the induced G-action on coordinate rings as above).

Now let $F : G \to G$ be the Frobenius morphism arising from a chosen split $\mathbb{F}_p$-structure on G, and let F^r denote its r-th iterate. Consider $(F^r)^* : k[G] \to k[G]$. We claim that

$$(F^r)^*(I_\varepsilon) = M_r.$$

To see this, let $\mathbb{F}_p[G]$ denote the chosen $\mathbb{F}_p$-structure of $k[G]$ and $I_{\varepsilon,\mathbb{F}_p}$ denote the augmentation ideal of this Hopf algebra. The map $(F^r)^*$ then arises (using this chosen $\mathbb{F}_p$-structure) as simply the p^r-th power map on $\mathbb{F}_p[G]$. If we fix an $\mathbb{F}_p$-basis $\{x_i\}$ for $I_{\varepsilon,\mathbb{F}_p}$, then $\{x_i \otimes_{\mathbb{F}_p} 1\}$ is a k-basis for I_ε. We have that

$$(F^r)^*(x_i \otimes_{\mathbb{F}_p} 1) = {x_i}^{p^r} \otimes_{\mathbb{F}_p} 1 = (x_i \otimes_{\mathbb{F}_p} 1)^{p^r}.$$

From this the claim follows. Further, since G is reduced, $(F^r)^*$ is injective, and so $(F^r)^*$ maps I_ε bijectively to M_r.

Now, the homomorphism $F^r : G \to G$ is G-equivariant, if G acts on the target via F^r. Therefore, the comorphism $(F^r)^*$ is a G-module homomorphism. It follows then that, as a G-module, $M_r \cong {I_\varepsilon}^{(r)}$. Applying [Jan3, II.3.19], it follows that $\mathrm{St}_r \otimes M_r$ has a good filtration. Since $\mathrm{St}_r \otimes k[G]$ also has a good filtration, it then follows from [Jan3, II.4.17] (which is a consequence of Theorem 2.3.1) that $\mathrm{St}_r \otimes (k[G]/M_r)$ has a good filtration. □

For each $\lambda \in X_+$, $\nabla(\lambda)^{(r)} \subseteq \nabla(p^r\lambda)$. In particular,

$$\nabla(\lambda)^{(r)} \otimes \nabla(\lambda^*)^{(r)} \subseteq \nabla(p^r\lambda) \otimes \nabla(p^r\lambda^*).$$

Note that the proof of the preceding result shows that there is a submodule of $k[G]$, namely M_r, having a filtration over G with sections of the form $\nabla(\lambda)^{(r)} \otimes \nabla(\lambda^*)^{(r)}$, $0 \neq \lambda \in X_+$.

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