Jonah Botvinick-Greenhouse is a junior mathematics major at Amherst College in Massachusetts, and **Troy Shinbrot** is a professor in the department of biomedical engineering at Rutgers University in New Jersey.

Juggling dynamics

Jonah Botvinick-Greenhouse and Troy Shinbrot

With complex throwing patterns of multiple objects, jugglers seemingly defy human limits of reaction time and throwing accuracy.

In 1993 Claude Shannon, founder of information theory, wrote a popular analysis of juggling, and he even accompanied the article with a working model of a juggling robot. Building such a robot—in fact, juggling at all—is remarkable, because it seems to require faster reaction times than most of us can muster. Speed jugglers can achieve nearly 500 catches in a minute, a rate that allows just 120 ms per catch. Yet typical human reaction times are 250 ms, and even experts in high-speed sports such as tennis take 200 ms to adjust their responses.

So how do jugglers with reaction times no better than 200 ms catch balls every 120 ms? In part, multitasking may allow multiple balls to be processed simultaneously, though how that is done with 11 balls—the Guinness world record—is far from clear. And in part, balls are not thrown to random locations, so each ball need not be tracked and caught independently. Indeed, up to five balls can be juggled while the juggler is blindfolded. Jugglers rely on making accurate throws and predictions of where the balls will travel. The accuracy required is a measure of how unstable—and thus how difficult—a particular juggling pattern is.

Showers and cascades

Figure 1 shows two five-ball juggling patterns above plots that define their sensitivities to deviations in throw speed and angle. On the left is the most common pattern, the cascade, in which each hand catches and throws balls to equal heights across the body's centerline. Hand motions in the cascade are left–right antisymmetric—that is, 180° out of phase. It is an amusing exercise to prove what all jugglers know—that only an odd number of balls can be juggled in the cascade. It's impossible to juggle an even number of balls in a cascade without breaking the antisymmetry—for example, by throwing with both hands simultaneously or by throwing balls to different heights.

The shower pattern (shown on the right), by contrast, lacks symmetry and can be performed with either even or odd numbers of balls. Although the pattern is dynamically simpler, it's also less stable—and thus more difficult—than the cascade. To appreciate why, consider the sensitivity plots (figure 1b) for both cascade and shower.

Both panels show the results of simulated parabolic trajectories under gravity. Figure 2a illustrates the ideal positions: The trajectories define throws that leave the exact center of the throwing hand and land precisely in the center of the catching hand. The hands move in ellipses, centered 50 cm apart and 180° out of phase, with horizontal radius 10 cm and vertical radius 5 cm. In simulations we add Gaussian deviations in initial speed v_0 and angle θ_0 to the throws and record how rates of

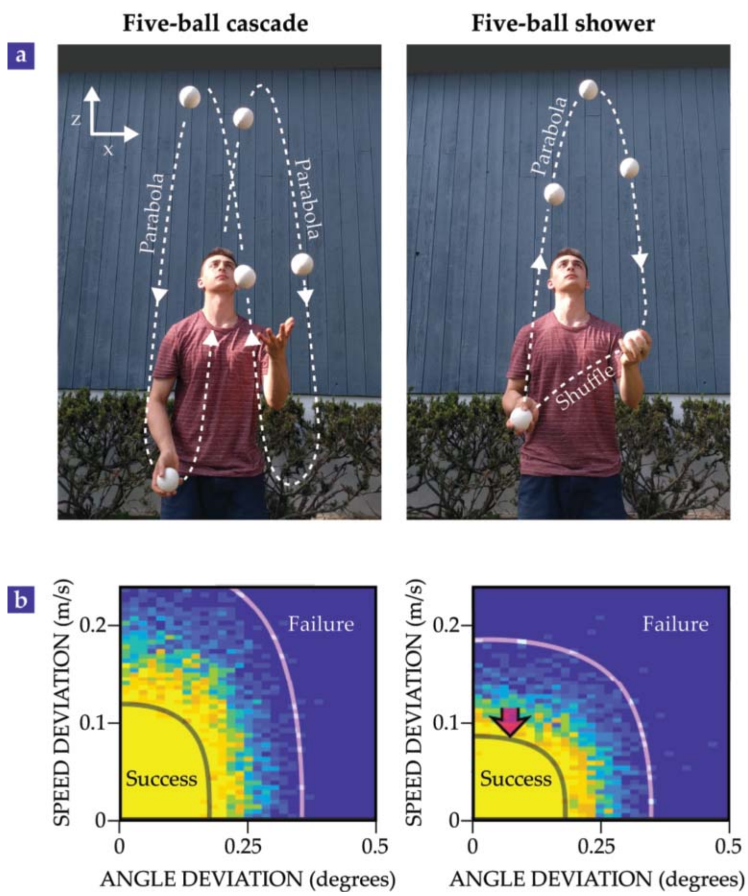

FIGURE 1. TRAJECTORIES AND SENSITIVITIES. (a) Two common juggling patterns are shown, each with five balls thrown to nearly identical heights. In the cascade (left), the five balls follow two parabolic trajectories, whereas in the shower (right), they follow one. **(b)** The seemingly more complicated cascade pattern is significantly less sensitive to deviations in speed than the shower, as indicated by the red arrow. Gray and red lines delimit the bounds within which juggling will always be a success and beyond which it will always be a failure.

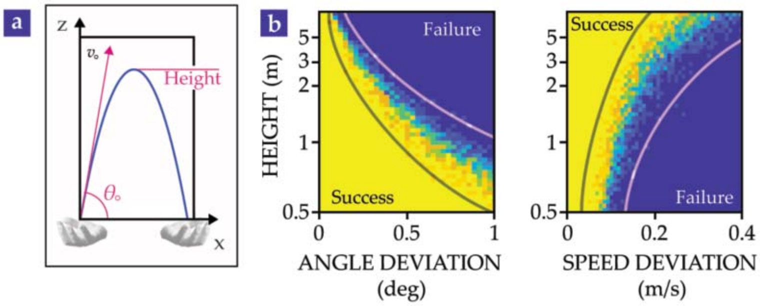

FIGURE 2. AN IDEAL THROW AND CATCH (a) with initial speed v_0 and angle θ_0. **(b)** Successful and failed catches occur in yellow and dark blue regions, respectively, for five-ball cascade patterns that include deviations in throw angle (left) and throw speed (right). At large heights, a juggler can tolerate large deviations in speed but little deviation in angle, whereas at low heights, the opposite is true.

catch success depend on throw deviations. Those rates are determined by counting the number of failures from 15 trials. Each trial comprises 50 throws for each combination of v_0 and θ_0 in the plots. Any throw in a trial that misses a catching hand 10 cm in diameter counts as a failure.

The take-home message of figure 1 is that the success region is larger for the cascade than for the shower. Cascade patterns can evidently tolerate greater variability in throw speed than can the shower, with balls thrown to the same height. For that reason, balls in a shower are typically thrown higher than those in a cascade. The higher throws provide more time between catches, but at the expense of requiring tighter control over the angle, as we'll see.

For throws to the same height, the two patterns are indistinguishable in sensitivity to throw angle. That feature makes sense in terms of the dynamics of the problem. The trajectories of both patterns are parabolic and nearly identical, so changing the throw angle produces the same displacement of a catch from its expected location in both.

In the shower pattern, each ball travels first through a parabola and then through a quick shuffle, whereas in the cascade, each ball must travel through two parabolas to return to its starting point. So hands must move nearly twice as rapidly in the shower as in the cascade, which makes catches in the shower much more sensitive to timing.

The different effects of throw angle and speed can be quantified by plotting sensitivities of throws to independent deviations in the two parameters, shown in figure 2b for a five-ball cascade. Notice that throws to greater heights permit less angle deviation but more speed deviation than throws to lower heights. That phenomenon occurs because errors in throw angle produce variations in catch location that worsen as throws become higher.

Initial speed, on the other hand, chiefly affects the timing of the catch. And because greater heights are produced at lower hand frequencies, higher throws provide more tolerance to time variation. So for high, nearly vertical throws, it's speed, not angle, that largely determines the time between throw and catch. The effects—particularly for higher throws—are that throw angle determines the location of a catch, whereas throw speed determines its timing. This distinguishes juggling from darts and other throwing games in which targeting accuracy does not depend on the relative timing between two hands.

The distinction between angle and speed is complicated by so-called siteswaps, juggling patterns defined by the order in which successive balls are manipulated, often resulting in throws to varying heights. When balls are thrown many meters high, throw angles must be accurate to within 0.1 degree. That's more than an order of magnitude tighter than is achievable by world-class athletes, and the time between successive catches is shorter than human reaction times. For an analysis of sensitivity plots for several siteswaps, see the online supplement.

Keep your eyes on the flies

It's unclear how jugglers achieve accuracy and response times beyond apparent physiological limits. But two experiments offer clues to how people can successfully juggle despite these limitations. In 2004 researchers presented evidence that both human and monkey brains can compute trajectories using an internal representation of equations of motion. According to that picture, jugglers keep track of the locations of their balls—and so effectively extend their reaction times—using dynamical prediction. Just as an outfielder predicts where a fly ball will land, a juggler predicts trajectories from how balls are thrown.

A second clue lies in the well-known but poorly understood phenomenon of muscle memory: A practiced sequence of movements can be recalled and repeated (see PHYSICS TODAY, November 2018, page 16). Human physiologists have long known that the brain's motor cortex contains somatotopic maps such that when particular locations are briefly stimulated electrically, specific muscles contract. In 2002 neuroscientist Michael Graziano and his colleagues showed that by prolonging the duration of stimulation, they could produce—at least in monkeys—complex, coordinated motions involving multiple muscles. They concluded that the brain appears to have developed a clear-cut and hierarchical way to encode complex tasks.

Complex tasks like juggling can be successfully performed without understanding the physiology behind motor control, although a deeper understanding is both intriguing and useful. Feedback-controlled robots can juggle a small number of balls by tracking them at 60 Hz. Physiology has apparently found ways to do the same, but with many more balls tracked at the slower rate of just 5 Hz. Unraveling the secrets behind how our nervous systems pull that off may pave the way for more dexterous robots.

Additional resources

▶ C. E. Shannon, *Claude Elwood Shannon: Collected Papers*, Wiley (1993).

▶ M. Venkadesan, L. Mahadevan, "Optimal strategies for throwing accurately," *R. Soc. Open Sci.* **4**, 170136 (2017).

▶ D. E. Angelaki et al., "Neurons compute internal models of the physical laws of motion," *Nature* **430**, 560 (2004).

▶ D. M. Wolpert et al., "An internal model for sensorimotor integration," *Science* **269**, 1880 (1995).

▶ M. S. Graziano, C. S. Taylor, T. Moore, "Complex movements evoked by microstimulation of precentral cortex," *Neuron* **34**, 841 (2002). PT