A quaternion-based approach to construct quaternary periodic complementary pairs

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***Abstract*—Two arrays form a periodic complementary pair if the sum of their periodic autocorrelations is a delta function. Finding such pairs, however, is challenging for large arrays whose entries are constrained to a small alphabet. One such alphabet is the quaternary set which contains the complex fourth roots of unity. In this paper, we propose a technique to construct periodic complementary pairs defined over the quaternary set using perfect quaternion arrays. We show how new pairs of quaternary sequences, matrices, and four-dimensional arrays that satisfy a periodic complementary property can be constructed with our method.**

I. Introduction

A periodic complementary pair (PCP) is a collection of two arrays whose periodic autocorrelation sums up to a delta function. An array can refer to a sequence, matrix, or a tensor. For example, a sequence or vector of length M is a one-dimensional array of size M, while an $M \times N$ matrix is a two-dimensional array of size $M \times N$. PCPs are arrays with special periodic autocorrelation properties that find applications in coded aperture imaging [1], communications [2], and radar [3]. PCPs are different than other complementary sequence constructions like Golay pairs [4], where the notion of complementary applies to aperiodic correlations.

Prior work has considered the design of PCPs over a binary alphabet [5]–[7]. The constructions in [5]–[7] are for sequences, i.e., 1D arrays, over $\{1, -1\}$ or $\{1, e^{\mathrm{i}\theta}\}$ where $\mathrm{i} = \sqrt{-1}$ is the standard unit imaginary number. A class of two-dimensional arrays in $\{-1, 1\}$ which are PCPs was derived in [8]. PCPs over binary alphabets, however, do not exist for every array size. For example, it was shown in [9] that binary PCP sequences of length 18 do not exist. Relaxing the binary constraint on the alphabet to a quaternary one provides additional flexibility to design new PCPs.

Sequences over $\mathcal{C} = \{1, \mathrm{i}, -1, -\mathrm{i}\}$, called the quaternary alphabet, which form PCPs were proposed in [10]–[12]. The quaternary PCPs in [10]–[12] were generated using the Gray method, inverse Gray method with interleaving, or the product method. An extensive survey on the length of sequences for which a quaternary PCP exists can be found in [10]. The literature on quaternary PCPs, however, is limited to sequences. The existence and construction of two-dimensional or multi-dimensional quaternary PCPs has not been studied to the best of our knowledge. One application of 2D quaternary PCPs is in dual polarized planar antenna arrays [13] equipped with two-bit phase shifters. Quaternary PCPs when applied to such systems result in quasi-omnidirectional beams which are useful for initial access in millimeter wave systems.

In this paper, we construct new one-, two-, and four-dimensional quaternary PCPs by leveraging perfect quaternion arrays [14]. It is important to note that the entries of the PCPs constructed in this paper contain elements in $\mathcal{C}$ which are complex numbers. The construction, however, is derived using quaternion algebra which is different from standard algebra over complex numbers [15]. In Section III, we show how a perfect quaternion array can be decomposed into a PCP. We use this decomposition in Section IV to show that perfect arrays over the basic unit quaternions [14], [16]–[18] can be mapped to quaternary PCPs of the same size. A Matlab implementation of the quaternary PCPs derived in this paper is available on our GitHub page [19].

Notation: $\mathbf{A}$ is a matrix, $\mathbf{a}$ is a column vector and a, A denote scalars. $\mathbf{A}^*$ and a^* denote the complex conjugate of $\mathbf{A}$ and a. We use $\mathbf{1}$ to denote an all-ones matrix. $A(k, \ell)$ denotes the entry of $\mathbf{A}$ in the k^{th} row and the ℓ^{th} column. The indices k and ℓ start from 0. $\|\mathbf{A}\|_{\mathrm{F}}$ is the Frobenius norm of $\mathbf{A}$. $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}$ denote the set of real, complex, and quaternion numbers. $\langle \cdot \rangle_N$ denotes the modulo N operation.

II. Preliminaries

Quaternions are a generalization of complex numbers which have one real component and three imaginary components [15]. We define i, j, and k as the fundamental quaternion units. These units satisfy

$$\begin{aligned} &\mathsf{i}^2 = -1, && \mathsf{j}^2 = -1, && \mathsf{k}^2 = -1, \\ &\mathsf{ij} = \mathsf{k}, && \mathsf{jk} = \mathsf{i}, && \mathsf{ki} = \mathsf{j}, \\ &\mathsf{ji} = -\mathsf{k}, && \mathsf{kj} = -\mathsf{i}, && \mathsf{ik} = -\mathsf{j}. \end{aligned} \tag{1}$$

Any quaternion $q \in \mathbb{Q}$ can be expressed as [15]

$$q = q_1 + q_2\mathsf{i} + q_3\mathsf{j} + q_4\mathsf{k}, \tag{2}$$

where q_1, q_2, q_3, $q_4 \in \mathbb{R}$. Complex numbers are a special instance of quaternions, i.e., $\mathbb{C} = \{q \in \mathbb{Q} : q_3 = 0, q_4 = 0\}$. We use the properties in (1) and the distributive law, to express the quaternion q in terms of two complex numbers as

$$q = \underbrace{(q_1 + q_2\mathsf{i})}_{q_{\mathrm{h}}} + \underbrace{(q_3 + q_4\mathsf{i})}_{q_{\mathrm{v}}}\mathsf{j}. \tag{3}$$

The quaternion q can be written as $q = q_{\mathrm{h}} + q_{\mathrm{v}}\mathsf{j}$.

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We now discuss basic operations over quaternions. The product of two quaternions p and q is [15]

$$\begin{aligned} pq = &(p_1q_1 - p_2q_2 - p_3q_3 - p_4q_4) \\ &+ (q_1p_2 + p_1q_2 - q_3p_4 + p_3q_4)\mathsf{i} \\ &+ (q_1p_3 + p_1q_3 + q_2p_4 - p_2q_4)\mathsf{j} \\ &+ (q_1p_4 + p_1q_4 - q_2p_3 + p_2q_3)\mathsf{k}. \end{aligned} \quad (4)$$

Multiplication over quaternions is non-commutative. For example, it can be observed from (1) that $\mathsf{ij} \neq \mathsf{ji}$. It is important to note, however, that $\alpha q = q\alpha$ for $\alpha \in \mathbb{R}$ and $q \in \mathbb{Q}$. The complex conjugate of q is [15]

$$q^* = q_1 - q_2\mathsf{i} - q_3\mathsf{j} - q_4\mathsf{k}. \quad (5)$$

The conjugates corresponding to the product and sum of two quaternions can be expressed as

$$(pq)^* = q^*p^* \text{ and} \quad (6)$$
$$(p+q)^* = p^* + q^*. \quad (7)$$

The properties in (4)$-$(7) naturally extend to matrices.

Now, we define matrices over quaternions and the periodic autocorrelation of a quaternion matrix. Consider a quaternion matrix $\mathbf{A} \in \mathbb{Q}^{M\times N}$. Similar to the representation in (3), we use $\mathbf{A}_{\mathrm{h}} \in \mathbb{C}^{M\times N}$ and $\mathbf{A}_{\mathrm{v}} \in \mathbb{C}^{M\times N}$ to denote the complex components of $\mathbf{A}$ such that

$$\mathbf{A} = \mathbf{A}_{\mathrm{h}} + \mathbf{A}_{\mathrm{v}}\mathsf{j}. \quad (8)$$

The non-commutative nature of quaternion multiplication leads to a different left and right periodic correlation [16]. In this paper, we focus on the right periodic correlation. For two matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{Q}^{M\times N}$, we define the conjugate-free periodic cross correlation as the matrix $\mathbf{X}\star\mathbf{Y} \in \mathbb{Q}^{M\times N}$. The $(m,n)^{\mathrm{th}}$ entry of $\mathbf{X}\star\mathbf{Y}$ is

$$(\mathbf{X}\star\mathbf{Y})_{m,n} = \sum_{k=0}^{M-1}\sum_{\ell=0}^{N-1} X(k,\ell)Y(\langle k+m\rangle_M, \langle \ell+n\rangle_N). \quad (9)$$

We use $\mathbf{R_A}$ to denote the 2D-periodic autocorrelation of $\mathbf{A}$. The $(m,n)^{\mathrm{th}}$ entry of $\mathbf{R_A} \in \mathbb{Q}^{M\times N}$ is [16]

$$R_{\mathbf{A}}(m,n) = \sum_{k=0}^{M-1}\sum_{\ell=0}^{N-1} A(k,\ell)A^*(\langle k+m\rangle_M, \langle \ell+n\rangle_N). \quad (10)$$

It can be observed that $\mathbf{R_A} = \mathbf{A}\star\mathbf{A}^*$. For the special case when $\mathbf{A}$ is a complex matrix, i.e., when $\mathbf{A}$ does not have j and k components, the right periodic autocorrelation $\mathbf{R_A}$ is the common 2D-periodic autocorrelation of $\mathbf{A}$.

We now define periodic complementary matrix pairs over the complex numbers. We use $\boldsymbol{\delta}$ to denote a unit delta matrix of size $M\times N$, i.e., $\delta(0,0) = 1$ and $\delta(m,n) = 0\,\forall\,(m,n) \neq (0,0)$. Two matrices $\mathbf{X} \in \mathbb{C}^{M\times N}$ and $\mathbf{Y} \in \mathbb{C}^{M\times N}$ form a PCP if [8]

$$R_{\mathbf{X}}(m,n) + R_{\mathbf{Y}}(m,n) = 0\ \forall (m,n) \neq (0,0). \quad (11)$$

Equivalently, $\mathbf{R_X} + \mathbf{R_Y} = 2MN\boldsymbol{\delta}$ for a PCP with $\|\mathbf{X}\|_{\mathrm{F}} = \sqrt{MN}$ and $\|\mathbf{Y}\|_{\mathrm{F}} = \sqrt{MN}$. A trivial PCP is $\mathbf{X} = \sqrt{MN}\boldsymbol{\delta}$ and $\mathbf{Y} = \sqrt{MN}\boldsymbol{\delta}$. Finding PCPs with entries in $\{1,-1\}$ or $\mathcal{C}$, however, is challenging when the size of the matrices is large. In this paper, we propose a technique to construct new PCPs whose entries are in $\mathcal{C}$.

III. Connection between perfect quaternion arrays and complex periodic complementary pairs

In this section, we show that every perfect quaternion array (PQA) can be decomposed into a PCP with complex entries in $\mathbb{C}$. A matrix $\mathbf{A} \in \mathbb{Q}^{M\times N}$ is a PQA if its right periodic autocorrelation is a delta function, i.e.,

$$R_{\mathbf{A}}(m,n) = 0\ \forall (m,n) \neq (0,0). \quad (12)$$

This property can be expressed as $\mathbf{R_A} = MN\boldsymbol{\delta}$ when $\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{MN}$. An example of a 2×2 PQA is [18]

$$\mathbf{D} = \begin{pmatrix} 1 & \mathsf{i} \\ \mathsf{j} & \mathsf{k} \end{pmatrix}. \quad (13)$$

Quaternion matrices that are PQAs were investigated in [14], [16]–[18]. To the best of our knowledge, prior work has not studied the connection between PQAs and PCPs.

We first express the autocorrelation $\mathbf{R_A}$ as a function of the complex matrices $\mathbf{A}_{\mathrm{h}}$ and $\mathbf{A}_{\mathrm{v}}$ in Lemma 1. The result in Lemma 1 is then used in Theorem 2 to derive PCPs.

Lemma 1. *For any quaternion matrix* $\mathbf{A}$*,*

$$\mathbf{R_A} = \mathbf{R}_{\mathbf{A}_{\mathrm{h}}} + \mathbf{R}_{\mathbf{A}_{\mathrm{v}}} + [\mathbf{A}_{\mathrm{v}}\star\mathbf{A}_{\mathrm{h}} - \mathbf{A}_{\mathrm{h}}\star\mathbf{A}_{\mathrm{v}}]\mathsf{j}. \quad (14)$$

Proof. We use the complex decomposition in (8) to write $\mathbf{R_A} = (\mathbf{A}_{\mathrm{h}} + \mathbf{A}_{\mathrm{v}}\mathsf{j})\star(\mathbf{A}_{\mathrm{h}} + \mathbf{A}_{\mathrm{v}}\mathsf{j})^*$. Using the distributive law and (6), the autocorrelation can be simplified to

$$\mathbf{R_A} = \mathbf{A}_{\mathrm{h}}\star\mathbf{A}_{\mathrm{h}}^* + \mathbf{A}_{\mathrm{v}}\mathsf{j}\star\mathsf{j}^*\mathbf{A}_{\mathrm{v}}^* + \mathbf{A}_{\mathrm{h}}\star\mathsf{j}^*\mathbf{A}_{\mathrm{v}}^* + \mathbf{A}_{\mathrm{v}}\mathsf{j}\star\mathbf{A}_{\mathrm{h}}^*. \quad (15)$$

The first summand in (15) is $\mathbf{R}_{\mathbf{A}_{\mathrm{h}}}$. The second summand in (15) is simplified using (9) and the property that $\mathsf{jj}^* = 1$. The simplification results in $\mathbf{A}_{\mathrm{v}}\mathsf{j}\star\mathsf{j}^*\mathbf{A}_{\mathrm{v}}^* = \mathbf{R}_{\mathbf{A}_{\mathrm{v}}}$. Therefore, $\mathbf{R_A} = \mathbf{R}_{\mathbf{A}_{\mathrm{h}}} + \mathbf{R}_{\mathbf{A}_{\mathrm{v}}} + \mathbf{A}_{\mathrm{h}}\star\mathsf{j}^*\mathbf{A}_{\mathrm{v}}^* + \mathbf{A}_{\mathrm{v}}\mathsf{j}\star\mathbf{A}_{\mathrm{h}}^*$.

We now show that the sum of the third and the fourth summands in (15) is $[\mathbf{A}_{\mathrm{v}}\star\mathbf{A}_{\mathrm{h}} - \mathbf{A}_{\mathrm{h}}\star\mathbf{A}_{\mathrm{v}}]\mathsf{j}$. We define $\alpha_{m,n}$ as the $(m,n)^{\mathrm{th}}$ element of $\mathbf{A}_{\mathrm{h}}\star\mathsf{j}^*\mathbf{A}_{\mathrm{v}}^* + \mathbf{A}_{\mathrm{v}}\mathsf{j}\star\mathbf{A}_{\mathrm{h}}$. Then,

$$\begin{aligned}\alpha_{m,n} = \sum_{k=0}^{M-1}\sum_{k=0}^{M-1} \big[&A_{\mathrm{h}}(k,\ell)\mathsf{j}^*A_{\mathrm{v}}^*(\langle k+m\rangle_M, \langle \ell+n\rangle_N) \\ &+ A_{\mathrm{v}}(k,\ell)\mathsf{j}A_{\mathrm{h}}^*(\langle k+m\rangle_M, \langle \ell+n\rangle_N)\big].\end{aligned} \quad (16)$$

To simplify $\alpha_{m,n}$, we use the property that $x\mathsf{j}y = xy^*\mathsf{j}$ and $x\mathsf{j}^*y = -xy^*\mathsf{j}$ for $x \in \mathbb{C}$ and $y \in \mathbb{C}$ [Proof in the Appendix]. As the entries of $\mathbf{A}_{\mathrm{h}}$ and $\mathbf{A}_{\mathrm{v}}$ are elements in $\mathbb{C}$, this property can be used in (16) to show that $\alpha_{m,n}$ is the $(m,n)^{\mathrm{th}}$ entry of $[\mathbf{A}_{\mathrm{v}}\star\mathbf{A}_{\mathrm{h}} - \mathbf{A}_{\mathrm{h}}\star\mathbf{A}_{\mathrm{v}}]\mathsf{j}$. □

Theorem 2. *The complex components* $\mathbf{A}_{\mathrm{h}}$ *and* $\mathbf{A}_{\mathrm{v}}$ *of a perfect quaternion array* $\mathbf{A}$ *form a PCP.*

Proof. When $\mathbf{A}$ is a perfect quaternion array, i.e., $\mathbf{R_A} = MN\boldsymbol{\delta}$, the result in Lemma 1 leads to

$$\mathbf{R}_{\mathbf{A}_{\mathrm{h}}} + \mathbf{R}_{\mathbf{A}_{\mathrm{v}}} + [\mathbf{A}_{\mathrm{v}}\star\mathbf{A}_{\mathrm{h}} - \mathbf{A}_{\mathrm{h}}\star\mathbf{A}_{\mathrm{v}}]\mathsf{j} = MN\boldsymbol{\delta}. \quad (17)$$

We interpret the quaternion matrix on the left hand side of (17) as a sum of $\mathbf{R}_{\mathbf{A}_{\mathrm{h}}} + \mathbf{R}_{\mathbf{A}_{\mathrm{v}}}$ and $[\mathbf{A}_{\mathrm{v}}\star\mathbf{A}_{\mathrm{h}} - \mathbf{A}_{\mathrm{h}}\star\mathbf{A}_{\mathrm{v}}]\mathsf{j}$. As $\mathbf{A}_{\mathrm{h}}$ and $\mathbf{A}_{\mathrm{v}}$ are matrices in $\mathbb{C}^{M\times N}$, the first term $\mathbf{R}_{\mathbf{A}_{\mathrm{h}}} + \mathbf{R}_{\mathbf{A}_{\mathrm{v}}} \in \mathbb{C}^{M\times N}$ and does not have any j and k components. The second term, i.e., $[\mathbf{A}_{\mathrm{v}}\star\mathbf{A}_{\mathrm{h}} - \mathbf{A}_{\mathrm{h}}\star\mathbf{A}_{\mathrm{v}}]\mathsf{j}$, is a multiplication of a matrix in $\mathbb{C}^{M\times N}$ with j. Such a matrix has zero real and zero i components. The matrix on the right hand side, however, is purely real. Putting these observations together, it can be concluded that the equality in (17) holds only when

$$\mathbf{R}_{\mathbf{A}_\mathrm{h}} + \mathbf{R}_{\mathbf{A}_\mathrm{v}} = MN\boldsymbol{\delta}, \text{ and} \quad (18)$$
$$\mathbf{A}_\mathrm{v} \star \mathbf{A}_\mathrm{h} = \mathbf{A}_\mathrm{h} \star \mathbf{A}_\mathrm{v}. \quad (19)$$

The result in Theorem 2 follows from (18). □

An important observation from Theorem 2 is that the complex components of a PQA, i.e., $\mathbf{A}_\mathrm{h}$ and $\mathbf{A}_\mathrm{v}$, satisfy (19) in addition to the PCP property in (18). Equivalently, our method results in PCPs which are commutative with conjugate-free periodic cross-correlation. We now explain an example to generate a PCP in $\mathbb{C}^{2\times 2}$ from the PQA in (13). The matrix $\mathbf{D}$ in (13) can be expressed as $\mathbf{D}_\mathrm{h} + \mathbf{D}_\mathrm{v}\mathsf{j}$, where

$$\mathbf{D}_\mathrm{h} = \begin{pmatrix} 1 & \mathsf{i} \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{D}_\mathrm{v} = \begin{pmatrix} 0 & 0 \\ 1 & \mathsf{i} \end{pmatrix}. \quad (20)$$

It can be verified that $\mathbf{D}_\mathrm{h}$ and $\mathbf{D}_\mathrm{h}$ satisfy the definition of a PCP in (11), equivalently (18). These matrices, however, contain entries which are not in the quaternary alphabet $\mathcal{C}$. In Section IV, we show how Theorem 2 can still be used to construct quaternary PCPs from PQAs.

IV. Construction of quaternary PCPs from PQAs

We define the basic unit quaternion alphabet as $\mathbb{H} = \{1, -1, \mathsf{i}, -\mathsf{i}, \mathsf{j}, -\mathsf{j}, \mathsf{k}, -\mathsf{k}\}$. PQAs with entries in $\mathbb{H}$ were constructed in [14], [16]–[18]. In this section, we show how to construct PCPs with entries in $\mathcal{C}$ from such PQAs.

To generate quaternary PCPs, we first construct a matrix $\tilde{\mathbf{A}} = \mathbf{A}(1+\mathsf{j})$ where $\mathbf{A}$ is a PQA in $\mathbb{H}^{M\times N}$. The periodic autocorrelation of $\tilde{\mathbf{A}}$, i.e., $\mathbf{R}_{\tilde{\mathbf{A}}}$, is then $\mathbf{A}(1+\mathsf{j}) \star (\mathbf{A}(1+\mathsf{j}))^*$. The autocorrelation can be further simplified to $\mathbf{R}_{\tilde{\mathbf{A}}} = \mathbf{A}(1+\mathsf{j}) \star (1+\mathsf{j})^*\mathbf{A}^*$ using (6). As $(1+\mathsf{j})(1+\mathsf{j})^* = 2$, it can be shown that $\mathbf{R}_{\tilde{\mathbf{A}}} = 2\mathbf{R}_{\mathbf{A}}$. Now, it follows from (12) that $\tilde{\mathbf{A}}$ is a PQA whenever $\mathbf{A}$ is a PQA.

We observe that the complex components of $\tilde{\mathbf{A}}$, i.e., $\tilde{\mathbf{A}}_\mathrm{h}$ and $\tilde{\mathbf{A}}_\mathrm{v}$ form a PCP using Theorem 1. These components can be expressed in terms of $\mathbf{A}_\mathrm{h}$ and $\mathbf{A}_\mathrm{v}$ as

$$\begin{aligned} \tilde{\mathbf{A}}_\mathrm{h} &= \mathbf{A}_\mathrm{h} - \mathbf{A}_\mathrm{v} \text{ and} \\ \tilde{\mathbf{A}}_\mathrm{v} &= \mathbf{A}_\mathrm{h} + \mathbf{A}_\mathrm{v}. \end{aligned} \quad (21)$$

When $\mathbf{A}$ is a PQA in $\mathbb{H}^{M\times N}$, the entries of $\mathbf{A}_\mathrm{h}$ and $\mathbf{A}_\mathrm{v}$ are elements in $\mathcal{C} \cup \{0\}$. In addition, $\forall k, \ell,\ A_\mathrm{h}(k,\ell) = 0$ whenever $A_\mathrm{v}(k,\ell) \neq 0$ and vice versa. Putting these observations together, the entries of $\tilde{\mathbf{A}}_\mathrm{h}$ and $\tilde{\mathbf{A}}_\mathrm{v}$ are elements in $\{1, \mathsf{i}, -1, -\mathsf{i}\}$. Therefore, $\tilde{\mathbf{A}}_\mathrm{h}$ and $\tilde{\mathbf{A}}_\mathrm{v}$ form a quaternary PCP in $\mathcal{C}^{M\times N}$ whenever $\mathbf{A}$ is a PQA over $\mathbb{H}^{M\times N}$.

We now discuss an example of a 2×2 quaternary PCP, and provide a list of quaternary PCPs that can be derived from PQAs using the proposed procedure. For the PQA in (13), the matrices $\tilde{\mathbf{D}}_\mathrm{h} = \mathbf{D}_\mathrm{h} + \mathbf{D}_\mathrm{v}$ and $\tilde{\mathbf{D}}_\mathrm{v} = \mathbf{D}_\mathrm{h} - \mathbf{D}_\mathrm{v}$ are

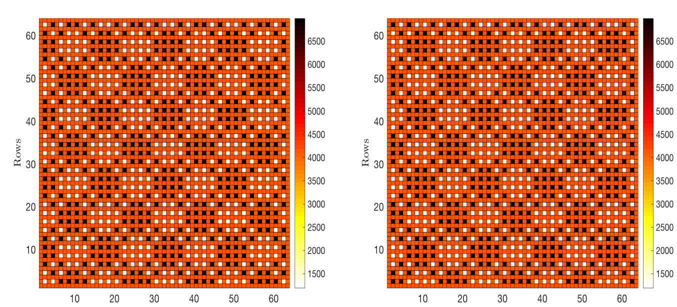
$$\tilde{\mathbf{D}}_\mathrm{h} = \begin{pmatrix} 1 & \mathsf{i} \\ 1 & \mathsf{i} \end{pmatrix} \text{ and } \tilde{\mathbf{D}}_\mathrm{v} = \begin{pmatrix} 1 & \mathsf{i} \\ -1 & -\mathsf{i} \end{pmatrix}. \quad (22)$$

The pair in (22) forms a PCP over $\mathcal{C}$. Our procedure can also be used to transform one-dimensional or multi-dimensional PQAs into PCPs of the same size. For instance, one-dimensional quaternary PCPs with lengths $4, 6, 8, 10, 14, 16, 18, 26, 30, 38, 42, 50, 54, 62, 74, 82, 90$ and 98 can be derived using the perfect quaterion sequences in [16]. Similarly, quaternary PCP matrices of size $2^n \times 2^n$, and quaternary PCP tensors of size $2^n \times 2^n \times 2^n \times 2^n$ can be constructed for $2 \leq n \leq 6$ from the PQAs in [16]. The PQA constructions in [16] are based on recursive algorithms or exhaustive search over a class of functions to generate such arrays. It is important to note that the periodic autocorrelation for the tensor case is multi-dimensional. For instance, the periodic autocorrelation of a 4D-array $\boldsymbol{\mathcal{A}} \in \mathbb{Q}^{M\times N\times S\times T}$ is defined as $\boldsymbol{\mathcal{R}}_{\mathcal{A}}$ where $\mathcal{R}_{\boldsymbol{\mathcal{A}}}(m,n,s,t) = \sum_{k,\ell,u,v} \mathcal{A}(k,\ell,u,v)\mathcal{A}^*(\langle k+m\rangle_M, \langle \ell + n\rangle_N, \langle u+s\rangle_S, \langle v+t\rangle_T)$. Our procedure can also be used to decompose the PQAs in [14], [17], and [18] into PCPs.

Now, we focus on quaternary PCP matrices and study their complementary property using the 2D-discrete Fourier transform (2D-DFT). For $\mathbf{X} \in \mathbb{C}^{M\times N}$, we define $\mathcal{F}_{\mathrm{2D}}(\mathbf{X})$ as the 2D-DFT of $\mathbf{X}$. For example, $\mathcal{F}_{\mathrm{2D}}(\boldsymbol{\delta}) = \mathbf{1}$. An interesting property of the 2D-DFT is that $\mathcal{F}_{\mathrm{2D}}(\mathbf{R}_\mathbf{X}) = |\mathcal{F}_{\mathrm{2D}}(\mathbf{X})|^2$ [20]. For a PCP $\tilde{\mathbf{A}}_\mathrm{h}, \tilde{\mathbf{A}}_\mathrm{v} \in \mathcal{C}^{M\times N}$, applying 2D-DFT on both sides of $\mathbf{R}_{\tilde{\mathbf{A}}_\mathrm{h}} + \mathbf{R}_{\tilde{\mathbf{A}}_\mathrm{v}} = 2MN\boldsymbol{\delta}$ results in [5]

$$|\mathcal{F}_{\mathrm{2D}}(\tilde{\mathbf{A}}_\mathrm{h})|^2 + |\mathcal{F}_{\mathrm{2D}}(\tilde{\mathbf{A}}_\mathrm{v})|^2 = 2MN\mathbf{1}. \quad (23)$$

From a beamforming perspective, $|\mathcal{F}_{\mathrm{2D}}(\tilde{\mathbf{A}}_\mathrm{h})|^2$ is the power of the discrete beam pattern generated when $\tilde{\mathbf{A}}_\mathrm{h}$ is applied to a planar antenna array [21]. When $\tilde{\mathbf{A}}_\mathrm{h}$ and $\tilde{\mathbf{A}}_\mathrm{v}$ are applied along the orthogonal polarizations of a dual polarized beamforming (DPBF) system, it can be observed from (23) that the sum of the beam powers taken across both polarizations is constant at all the discrete beam pattern locations. As a result, PCPs result in quasi-omnidirectional beams when applied to DPBF systems. The quaternary nature of the PCPs derived in this paper allows their application to DPBF systems with just two-bit phase shifters.



(a) $|\mathcal{F}_{\mathrm{2D}}(\tilde{\mathbf{A}}_\mathrm{h})|^2$ (b) $|\mathcal{F}_{\mathrm{2D}}(\tilde{\mathbf{A}}_\mathrm{v})|^2$

Fig. 1: The squared 2D-DFT magnitudes of the quaternary PCPs constructed from a 64×64 PQA in [16]. The sum of these matrices is 2×64^2 at all the locations as per (23).

For 4×4, 8×8, 16×16, and 32×32 PCPs derived from the PQAs in [16], we observed that $|\mathcal{F}_{\mathrm{2D}}(\tilde{\mathbf{A}}_\mathrm{h})|^2 = MN\mathbf{1}$. Equivalently, $\mathbf{R}_{\tilde{\mathbf{A}}_\mathrm{h}} = MN\boldsymbol{\delta}$. In such a case, the PCP is a collection of two perfect quaternary arrays [22] as $\tilde{\mathbf{A}}_{\mathrm{h}}$ and $\tilde{\mathbf{A}}_{\mathrm{v}}$ have a perfect periodic autocorrelation. These arrays are different from the perfect quaternary arrays used in [18] to construct PQAs. The construction in [18] is based on an inflation technique which transforms a $M \times N$ perfect quaternary array into an $Md \times Nd$ PQA, where $d = MN - 1$ is prime. Our method generates PCPs which have the same size of the underlying PQA and is not the same as the reverse construction in [18].

The proposed construction does not always result in PCPs which contain two perfect quaternary arrays. For example, it can be observed that $|\mathcal{F}_{\mathrm{2D}}(\tilde{\mathbf{D}}_{\mathrm{h}})|^2/4 \neq \mathbf{1}$ and $|\mathcal{F}_{\mathrm{2D}}(\tilde{\mathbf{D}}_{\mathrm{v}})|^2/4 \neq \mathbf{1}$ for the PCP in (22). In this case, $\tilde{\mathbf{D}}_{\mathrm{h}}$ and $\tilde{\mathbf{D}}_{\mathrm{v}}$ are not perfect quaternary arrays although they form a PCP. Another example of a non-trivial PCP is one that is generated from a 64×64 PQA. The squared 2D-DFT magnitudes of the 64×64 matrices in this PCP are shown in Fig. 1. As the 2D-DFT magnitudes in Fig. 1 vary across the entries, the matrices in this PCP are not perfect quaternary arrays.

We would like to mention that the main focus of this paper is on constructing PCPs from PQAs. An interesting question that arises is if PQAs can be derived from PCPs using the reverse of the proposed construction. To answer this question, we consider a PCP $\mathbf{B}_{\mathrm{h}}, \mathbf{B}_{\mathrm{v}} \in \mathbb{C}^{M \times N}$ such that $\|\mathbf{B}_{\mathrm{h}}\|_{\mathrm{F}} = \sqrt{MN}$ and $\|\mathbf{B}_{\mathrm{v}}\|_{\mathrm{F}} = \sqrt{MN}$. By definition, $\mathbf{R}_{\mathbf{B}_{\mathrm{h}}} + \mathbf{R}_{\mathbf{B}_{\mathrm{v}}} = 2MN\boldsymbol{\delta}$. From Lemma 1, it can be concluded that the quaternion matrix $\mathbf{B} = \mathbf{B}_{\mathrm{h}} + \mathbf{B}_{\mathrm{v}}\mathsf{j}$ is perfect when $\mathbf{B}_{\mathrm{h}} \star \mathbf{B}_{\mathrm{v}} = \mathbf{B}_{\mathrm{v}} \star \mathbf{B}_{\mathrm{h}}$. Therefore, PCPs which are commutative with conjugate-free cross correlation can be used to construct PQAs.

Now, we identify sufficient conditions to construct PQAs in $\mathbb{H}^{M \times N}$ from quaternary PCPs. For a quaternary PCP $\mathbf{B}_{\mathrm{h}}, \mathbf{B}_{\mathrm{v}} \in \mathcal{C}^{M \times N}$, we define $\mathbf{A}_{\mathrm{h}} = (\mathbf{B}_{\mathrm{h}} + \mathbf{B}_{\mathrm{v}})/2$ and $\mathbf{A}_{\mathrm{v}} = (\mathbf{B}_{\mathrm{h}} - \mathbf{B}_{\mathrm{v}})/2$. Now, it can be shown that $\mathbf{A} = \mathbf{A}_{\mathrm{h}} + \mathbf{A}_{\mathrm{v}}\mathsf{j}$ is a PQA when $\mathbf{B}_{\mathrm{h}} \star \mathbf{B}_{\mathrm{v}} = \mathbf{B}_{\mathrm{v}} \star \mathbf{B}_{\mathrm{h}}$. Furthermore, all the entries of $\mathbf{A}$ are elements in $\mathbb{H}$ only when $A_{\mathrm{h}}(k,\ell)A_{\mathrm{v}}(k,\ell) = 0$ for every k, ℓ. This condition translates to $B_{\mathrm{h}}(k,\ell) = \pm B_{\mathrm{v}}(k,\ell)$ for every k, ℓ. In conclusion, the reverse of our construction allows mapping a quaternary PCP $\mathbf{B}_{\mathrm{h}}, \mathbf{B}_{\mathrm{v}}$ to a PQA in $\mathbb{H}^{M \times N}$ if the PCP satisfies the following properties:

$$
\begin{aligned}
&(a)\ \mathbf{R}_{\mathbf{B}_{\mathrm{h}}} + \mathbf{R}_{\mathbf{B}_{\mathrm{v}}} = 2MN\boldsymbol{\delta},\\
&(b)\ \mathbf{B}_{\mathrm{h}} \star \mathbf{B}_{\mathrm{v}} = \mathbf{B}_{\mathrm{v}} \star \mathbf{B}_{\mathrm{h}}, \text{ and}\\
&(c)\ B_{\mathrm{h}}(k,\ell) = \pm B_{\mathrm{v}}(k,\ell)\ \forall k, \ell.
\end{aligned}
$$

To the best of our knowledge, the conditions $(a)-(c)$ have not been presented in prior work. We believe that these conditions can provide new insights into constructing PQAs over $\mathbb{H}$.

V. Conclusions and future work

In this paper, we established a connection between perfect arrays over quaternions and periodic complementary arrays over complex numbers. We also demonstrated how perfect quaternion arrays can be transformed to quaternary periodic complementary pairs. Finally, we identified sufficient conditions to construct perfect quaternion arrays over the basic unit quaternions from quaternary periodic complementary pairs. In future work, we will study the use of perfect quaternion arrays for beamforming in low resolution phased arrays.

Appendix

We first prove that $x\mathsf{j}y = xy^*\mathsf{j}$ when $x, y \in \mathbb{C}$. Using the representation in (3), $x\mathsf{j}y$ can be written as

$$
\begin{aligned}
x\mathsf{j}y &= (x_1 + x_2\mathsf{i})\mathsf{j}(y_1 + y_2\mathsf{i})\\
&= (x_1 + x_2\mathsf{i})(y_1\mathsf{j} - y_2\mathsf{k})\\
&= (x_1y_1 + x_2y_2)\mathsf{j} + (x_2y_1 - x_1y_2)\mathsf{k}\\
&= [(x_1y_1 + x_2y_2) + (x_2y_1 - x_1y_2)\mathsf{i}]\mathsf{j}. \qquad (24)
\end{aligned}
$$

It can be observed that the right hand side of (24) is $xy^*\mathsf{j}$. Using $\mathsf{j}^* = -\mathsf{j}$ and the result in (24), it can be shown that $x\mathsf{j}^*y = -xy^*\mathsf{j}$ for any $x, y \in \mathbb{C}$.

References

[1] E. E. Fenimore and T. M. Cannon, "Coded aperture imaging with uniformly redundant arrays," *OSA Appl. optics*, vol. 17, no. 3, pp. 337–347, 1978.

[2] P. Fan and M. Darnell, *Sequence design for communications applications*. Wiley, 1996, vol. 1.

[3] N. Levanon, I. Cohen, and P. Itkin, "Complementary pair radar waveforms–evaluating and mitigating some drawbacks," *IEEE Aero. and Electronic Sys. Mag.*, vol. 32, no. 3, pp. 40–50, 2017.

[4] M. Golay, "Complementary series," *IRE Trans. on Inform. Theory*, vol. 7, no. 2, pp. 82–87, 1961.

[5] L. Bomer and M. Antweiler, "Periodic complementary binary sequences," *IEEE Trans. on Inform. Theory*, vol. 36, no. 6, pp. 1487–1494, 1990.

[6] D. Ž. Đoković and I. S. Kotsireas, "Some new periodic Golay pairs," *Springer Numerical Algorithms*, vol. 69, no. 3, pp. 523–530, 2015.

[7] X. Li, Z. Liu, Y. L. Guan, and P. Fan, "Two-valued periodic complementary sequences," *IEEE Signal Process. Lett.*, vol. 24, no. 9, pp. 1270–1274, 2017.

[8] F. Zeng and Z. Zhang, "Two dimensional periodic complementary array sets," in *IEEE Intl. Conf. on Wireless Commun. Networking and Mobile Comput. (WiCOM)*, 2010, pp. 1–4.

[9] C. Yang, "Maximal binary matrices and sum of two squares," *Math. of Comput.*, vol. 30, no. 133, pp. 148–153, 1976.

[10] Z. Zhou, J. Li, Y. Yang, and S. Hu, "Two constructions of quaternary periodic complementary pairs," *IEEE Commun. Lett.*, vol. 22, no. 12, pp. 2507–2510, 2018.

[11] F. Zeng, X. Zeng, Z. Zhang, and G. Xuan, "Quaternary periodic complementary/z-complementary sequence sets based on interleaving technique and gray mapping." *Adv. in Math. of Comm.*, vol. 6, no. 2, pp. 237–247, 2012.

[12] J.-W. Jang, Y.-S. Kim, S.-H. Kim, and D.-W. Lim, "New construction methods of quaternary periodic complementary sequence sets," *Adv. in Math. of Comm.*, vol. 4, no. 1, p. 61, 2010.

[13] S. O. Petersson, "Power-efficient beam pattern synthesis via dual polarization beamforming," *arXiv preprint arXiv:1910.10015*, 2019.

[14] O. Kuznetsov, "Perfect sequences over the real quaternions," in *IEEE Intl. Work. on Signal Des. and its Appl. in Commun.*, 2009, pp. 8–11.

[15] W. R. Hamilton, *Elements of quaternions*. Longmans, Green, and Company, 1899, vol. 1.

[16] S. Blake, "Perfect sequences and arrays over the unit quaternions," *arXiv preprint arXiv:1701.01154*, 2016.

[17] C. Bright, I. Kotsireas, and V. Ganesh, "New infinite families of perfect quaternion sequences and Williamson sequences," *arXiv preprint arXiv:1905.00267*, 2019.

[18] S. B. Acevedo and N. Jolly, "Perfect arrays of unbounded sizes over the basic quaternions," *Springer Crypt. and Commun.*, vol. 6, no. 1, pp. 47–57, 2014.

[19] N. J. Myers, "Quaternary periodic complementary pairs," https://github.com/nitinjmyers, 2020.

[20] H. D. Luke, "Sequences and arrays with perfect periodic correlation," *IEEE Trans. on Aero. and Elec. Sys.*, vol. 24, no. 3, pp. 287–294, 1988.

[21] X. Meng, X. Gao, and X.-G. Xia, "Omnidirectional precoding based transmission in massive MIMO systems," *IEEE Trans. on Commun.*, vol. 64, no. 1, pp. 174–186, 2015.

[22] K. Arasu and W. de Launey, "Two-dimensional perfect quaternary arrays," *IEEE Trans. on Inform. Theory*, vol. 47, no. 4, pp. 1482–1493, 2001.