Optimality of Frequency Flat Precoding in Frequency Selective Millimeter Wave Channels

Kiran Venugopal, Nuria González-Prelcic, and Robert W. Heath, Jr.

Abstract

Millimeter wave (mmWave) MIMO communication is a key feature of next generation wireless systems. The selection of precoders and combiners for wideband mmWave channels has involved frequency selective designs based on channel state information. In this letter, we show that under some assumptions, the dominant subspaces of the frequency domain channel matrices are equivalent. This means that semi-unitary frequency flat precoding and combining are sufficient to achieve the maximum spectral efficiency when there is not too much scattering in the channel. It also motivates the use of techniques such as compressive subspace estimation as an alternative to estimating the full channel.

Index Terms

Millimeter wave communications, hybrid architecture, subspace estimation.

I. Introduction

The use of large antenna arrays and directional transmission and reception are key enabling technologies for wireless systems operating at mmWave frequencies [1]. Both analog-only and hybrid beamforming architectures have been proposed to reduce the cost and power consumption in mixed signal components of a fully-digital MIMO system operating at mmWave [2], [3]. While hybrid architectures are desirable for supporting multi-stream communication at mmWave, the analog processing stage is frequency flat, so it can not perfectly reproduce the optimum frequency selective precoders.

Some prior work on frequency selective mmWave systems [4]–[6] involved precoding with a frequency flat analog precoder followed optionally by a frequency selective baseband precoder. This was found to be optimum in terms of the achievable spectral efficiency for a given analog

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This work was supported by the National Science Foundation under Grant NSF-CCF-1514275, ECCS-1711702, and CNS-1731658.

codebook and a flexible baseband precoder. The potential optimality of frequency flat precoders and combiners was not studied.

In this paper, we show that frequency flat precoding and combining, assuming semi-unitary precoding, is optimum in frequency selective MIMO channels with few paths, as found in mmWave systems. This means that frequency selective precoding is not necessarily required in MIMO systems operating at mmWave frequencies to achieve the maximum spectral efficiency. Further, this result motivates the design of the precoder based on compressive covariance or subspace estimation instead of explicit channel estimation exploiting sparsity [7], [8].

Notation: In this paper, we use $\mathbf{A}$ to denote a matrix, $\mathbf{a}$ to denote a vector and a to denote a scalar. Conjugate transpose and the conjugate of $\mathbf{A}$ are $\mathbf{A}^*$ and $\bar{\mathbf{A}}$, and the (i,j)th entry of $\mathbf{A}$ is $[\mathbf{A}]_{i,j}$. We use $\|\cdot\|$ to denote the Euclidean norm, $\|\cdot\|_F$ to denote Frobenius norm, and $\mathrm{trace}(\cdot)$ to denote the trace. The determinant of a square matrix $\mathbf{A}$ is denoted as $|\mathbf{A}|$.

II. System model

Consider a wideband mmWave system with N_t transmit antennas and N_r receive antennas. A geometric channel model [8], [9] consisting of R paths is assumed for representing the frequency selective channel. The ℓth path has a complex gain $\alpha_\ell \in \mathbb{C}$, delay $\tau_\ell \in \mathbb{R}$, and angles of arrival and departure (AoA/AoD) $\phi_\ell \in [0, 2\pi)$ and $\theta_\ell \in [0, 2\pi)$. Assuming a pulse shaping filter denoted as $p(\tau)$, the discrete-time, frequency selective channel with N_c delay taps can be represented in terms of the frequency independent antenna array response vectors [1] of the receiver $\mathbf{a}_\mathrm{R}(\phi_\ell) \in \mathbb{C}^{N_\mathrm{r}\times 1}$ and the transmitter $\mathbf{a}_\mathrm{T}(\theta_\ell) \in \mathbb{C}^{N_\mathrm{t}\times 1}$ [3], [5], [9]. Sampling with period T_s, the discrete-time MIMO channel for $d = 0, 1, \ldots, N_\mathrm{c} - 1$ is

$$\mathbf{H}_d = \sum_{\ell=1}^{R} \alpha_\ell p(dT_\mathrm{s} - \tau_\ell)\mathbf{a}_\mathrm{R}(\phi_\ell)\mathbf{a}_\mathrm{T}^*(\theta_\ell). \tag{1}$$

Next, we define the matrices $\mathbf{A}_\mathrm{R} \in \mathbb{C}^{N_\mathrm{r}\times R}$, $\mathbf{A}_\mathrm{T} \in \mathbb{C}^{N_\mathrm{t}\times R}$, and a diagonal matrix $\mathbf{P}_d \in \mathbb{C}^{R\times R}$. The columns of $\mathbf{A}_\mathrm{R}$ and $\mathbf{A}_\mathrm{T}$ are given by $\{\mathbf{a}_\mathrm{R}(\theta_\ell)\}_{\ell=1}^{R}$ and $\{\mathbf{a}_\mathrm{T}(\theta_\ell)\}_{\ell=1}^{R}$ respectively, and the ℓth diagonal entry of $\mathbf{P}_d$ is $\alpha_\ell p(dT_\mathrm{s} - \tau_\ell)$. Then,

$$\mathbf{H}_d = \mathbf{A}_\mathrm{R}\mathbf{P}_d\mathbf{A}_\mathrm{T}^* \tag{2}$$

gives a compact matrix representation of $\mathbf{H}_d$.

[1]While the spatial angles in the arguments of the steering vectors may include a frequency dependency called beam-squint [10], this behavior needs further study before including it into millimeter wave MIMO channel models.

Multicarrier (MC) transmission or a single carrier with frequency division multiplexing (SC-FDM) is assumed, with $K > N_\mathrm{c}$ denoting the number of subcarriers in the frequency domain. The channel matrix in the frequency domain is

$$\mathbf{H}[k] = \sum_{d=0}^{N_\mathrm{c}-1} \mathbf{H}_d e^{-\mathrm{j}\frac{2\pi k d}{K}} \tag{3}$$

for $k = 0, 1, \ldots, K-1$. To express (3) in matrix form, let $\mathbf{P}[k] = \sum_{d=0}^{N_\mathrm{c}-1} \mathbf{P}_d e^{-\mathrm{j}\frac{2\pi k d}{K}}$. Substituting (2) into (3), using the fact that the matrices $\mathbf{A}_\mathrm{R}$ and $\mathbf{A}_\mathrm{T}$ are independent of d then

$$\mathbf{H}[k] = \sum_{d=0}^{N_\mathrm{c}-1} \mathbf{A}_\mathrm{R} \mathbf{P}_d \mathbf{A}_\mathrm{T}^* e^{-\mathrm{j}\frac{2\pi k d}{K}} \tag{4}$$

$$= \mathbf{A}_\mathrm{R} \mathbf{P}\left[k\right] \mathbf{A}_\mathrm{T}^*. \tag{5}$$

The number of paths R is assumed to be smaller than the number of delay taps N_c in the frequency selective mmWave channel, which is reasonable due to the memory in the pulse shaping function.

Appropriate signal processing (for MC/SC-FDM) can be used at the transmitter and the receiver to obtain K parallel narrowband channels in the frequency domain. For N_s stream transmission, let $\mathbf{x}[k] \in \mathbb{C}^{N_\mathrm{s} \times 1}$ denote the complex symbol transmitted in the kth subcarrier of the data frame. Assuming frequency-selective precoding with a matrix $\mathbf{F}[k] \in \mathbb{C}^{N_\mathrm{t} \times N_\mathrm{s}}$ and combining with $\mathbf{W}[k] \in \mathbb{C}^{N_\mathrm{r} \times N_\mathrm{s}}$ during the transmission-reception of the data frame, the post combining received symbol in the kth subcarrier can be written as

$$\mathbf{y}\left[k\right] = \mathbf{W}^*[k]\mathbf{H}[k]\mathbf{F}[k]\mathbf{x}\left[k\right] + \mathbf{W}[k]^*\mathbf{n}\left[k\right], \tag{6}$$

where $\mathbf{n}[k] \sim \mathcal{N}\left(0, \sigma^2 \mathbf{I}\right)$ is the circularly symmetric complex Gaussian distributed additive noise vector of size N_r. Next we argue that in some cases $\mathbf{F}[k]$ and $\mathbf{W}[k]$ do not have to vary with frequency k, and a frequency flat precoder $\mathbf{F}_\mathrm{ff}$ and a frequency flat combiner $\mathbf{W}_\mathrm{ff}$ can be used without loss of performance.

III. Optimality of frequency-flat precoding

Under some assumptions, we now show that the K MIMO channel matrices defined in (3) have the same row and column subspaces. Let $\mathcal{C}\left(\mathbf{A}\right)$ denote the column space of $\mathbf{A}$ and $\mathcal{R}\left(\mathbf{A}\right)$ the row space of $\mathbf{A}$. Using this notation, we define the subspaces $\mathcal{H}_c = \mathcal{C}\left(\mathbf{A}_\mathrm{R}\right)$ and $\mathcal{H}_r = \mathcal{R}\left(\mathbf{A}_\mathrm{T}^*\right)$.

Proposition 1: Assuming that $\mathbf{A}_{\mathrm{R}}$ and $\mathbf{A}_{\mathrm{T}}$ have full column rank and $\mathsf{P}[k]$ is full rank, the frequency domain MIMO channel matrices $\mathsf{H}[k]$, $k = 0, 1, \cdots, K-1$, have the same column space $\mathcal{H}_c$ and the same row space $\mathcal{H}_r$.

Proof: From (5), $\mathcal{R}(\mathsf{H}[k]) \subseteq \mathcal{H}_r$ with equality when $\mathbf{A}_{\mathrm{R}}\mathsf{P}[k]$ has full column rank, which is true when $\mathbf{A}_{\mathrm{R}}$ has full column rank and $\mathsf{P}[k]$ is full rank. Similarly, $\mathcal{C}(\mathsf{H}[k]) \subseteq \mathcal{H}_c$ with equality when $\mathsf{P}[k]\mathbf{A}_{\mathrm{T}}^*$ has full row rank, which is true when $\mathbf{A}_{\mathrm{T}}$ has full column rank and $\mathsf{P}[k]$ is full rank. ∎

Assuming that $\mathsf{P}[k]$ is full rank is reasonable since the diagonal elements are the Fourier transform of shifted sampled Nyquist pulse shapes and the path gains are non-zero. The assumption that $\mathbf{A}_{\mathrm{R}}$ and $\mathbf{A}_{\mathrm{T}}$ have full column rank holds for typical array geometries, like the uniform linear array or uniform patch array, with small enough element spacing and distinct angles of arrival and departure.

Corollary 2: Frequency flat precoding (and combining) is optimum in frequency selective mmWave channels under the assumptions of semi-unitary precoding (and combining) and a small number of paths ($R < N_{\mathrm{t}}$ and $R < N_{\mathrm{r}}$), if $\mathbf{A}_{\mathrm{R}}$ and $\mathbf{A}_{\mathrm{T}}$ have full column rank.

Proof: The conventional semi-unitary frequency-selective precoder design is based on the singular value decomposition (SVD) of $\mathsf{H}[k] = \mathbf{U}[k]\mathbf{\Lambda}[k]\mathbf{V}^*[k]$, where $\mathsf{F}[k] = [\mathbf{V}[k]]_{:,1:N_{\mathrm{s}}}$ with $N_{\mathrm{s}} \leq \min(N_{\mathrm{t}}, N_{\mathrm{r}})$. If the number of paths is small, and $\mathbf{A}_{\mathrm{T}}$ has full column rank then $\mathrm{rank}(\mathbf{A}_{\mathrm{T}}) = R$ and it suffices to take $N_{\mathrm{s}} = R$. In this case, the columns of $\mathsf{F}[k]$ are a basis for $\mathcal{H}_r$. Based on Proposition 1, though, $\mathcal{H}_r$ is the same for all k thus a common basis for $\mathcal{H}_r$ given by $\mathbf{F}_{\mathrm{ff}}$ can be used for all subcarriers. A similar argument applies to using a single $\mathbf{W}_{\mathrm{ff}}$ for combining across all subcarriers. ∎

IV. Achievable rate with frequency-flat precoders

We choose the number of streams N_{s} to be equal to the rank of the MIMO channel matrices for each subcarrier k, given by $\min(R, N_{\mathrm{r}}, N_{\mathrm{t}})$. Based on the SVD of $\mathsf{H}[k]$ in the proof of Corollary 2, the conventional frequency selective precoder $\mathsf{F}[k] = [\mathbf{V}[k]]_{:,1:N_{\mathrm{s}}}$ and combiner $\mathsf{W}[k] = [\mathbf{U}[k]]_{:,1:N_{\mathrm{s}}}$ results in achievable spectral efficiency

$$\mathsf{R}_{\mathrm{fs}} = \frac{1}{K}\sum_{k=1}^{K}\log_2\left|\mathbf{I} + \frac{\mathrm{SNR}}{N_{\mathrm{s}}}\hat{\mathbf{\Lambda}}^2[k]\right|, \tag{7}$$

where $\mathrm{SNR} = \frac{P}{K\sigma^2}$, P is the average transmitted power, and $\hat{\mathbf{\Lambda}}[k] = [\mathbf{\Lambda}[k]]_{1:N_{\mathrm{s}},1:N_{\mathrm{s}}}$. The maximal achievable rate can be derived by performing an additional spatial water-filling step.

Let the frequency flat precoder $\mathbf{F}_{\text{ff}}$ be a semi-unitary matrix whose columns span $\mathcal{H}_r$. Similarly, let $\mathbf{W}_{\text{ff}}$ be a semi-unitary matrix whose columns span $\mathcal{H}_c$. Since all the column spaces of the matrices are the same, we can write $\mathbf{W}_{\text{ff}} = \mathbf{W}[k]\mathbf{Q}_{\mathsf{W}}[k]$, where $\mathbf{Q}_{\mathsf{W}}[k] \in \mathbb{C}^{N_{\text{s}} \times N_{\text{s}}}$ is a unitary rotation matrix that accounts for the unitary invariance in the representation of a point on the Grassmann manifold. Similarly, we can write $\mathbf{F}_{\text{ff}} = \mathbf{F}[k]\mathbf{Q}_{\mathsf{F}}[k]$, where $\mathbf{Q}_{\mathsf{F}}[k] \in \mathbb{C}^{N_{\text{s}} \times N_{\text{s}}}$ is another unitary rotation matrix. Further, since the frequency flat combiner is semi-unitary, the noise covariance of the post combining received signal in (6) is $\sigma^2\mathbf{I}$. Therefore, the achievable spectral efficiency using $\mathbf{F}_{\text{ff}}$ and $\mathbf{W}_{\text{ff}}$ is

$$\mathsf{R}_{\text{ff}} = \frac{1}{K}\sum_{k=1}^{K} \log_2 \left| \mathbf{I} + \frac{\text{SNR}}{N_{\text{s}}} \mathbf{W}_{\text{ff}}^* \mathbf{H}[k] \mathbf{F}_{\text{ff}} \mathbf{F}_{\text{ff}}^* \mathbf{H}^*[k] \mathbf{W}_{\text{ff}} \right|. \tag{8}$$

Note that the effective channel matrix with the optimal frequency flat precoder and combiner is $\mathbf{H}_{\text{eff}}[k] = \mathbf{W}_{\text{ff}}^*\mathbf{H}[k]\mathbf{F}_{\text{ff}}$, which can be written as

$$\mathbf{H}_{\text{eff}}[k] = \mathbf{Q}_{\mathsf{W}}^*[k]\mathbf{W}^*[k]\mathbf{U}[k]\mathbf{\Lambda}[k]\mathbf{V}^*[k]\mathbf{F}[k]\mathbf{Q}_{\mathsf{F}}[k] \tag{9}$$

$$= \mathbf{Q}_{\mathsf{W}}^*[k]\hat{\mathbf{\Lambda}}[k]\mathbf{Q}_{\mathsf{F}}[k]. \tag{10}$$

Then (8) simplifies to

$$\mathsf{R}_{\text{ff}} = \frac{1}{K}\sum_{k=1}^{K} \log_2 \left| \mathbf{I} + \frac{\text{SNR}}{N_{\text{s}}} \mathbf{Q}_{\mathsf{W}}^*[k] \hat{\mathbf{\Lambda}}^2[k] \mathbf{Q}_{\mathsf{W}}[k] \right| \tag{11}$$

$$\stackrel{(a)}{=} \frac{1}{K}\sum_{k=1}^{K} \log_2 \left| \mathbf{I} + \frac{\text{SNR}}{N_{\text{s}}} \hat{\mathbf{\Lambda}}^2[k] \right| = \mathsf{R}_{\text{fs}}. \tag{12}$$

In (12), $\stackrel{(a)}{=}$ follows from the matrix identity $|\mathbf{I}+\mathbf{ABC}| = |\mathbf{I}+\mathbf{BCA}|$ and since $\mathbf{Q}_{\mathsf{W}}[k]$ is unitary.

It is important to note that the proposed precoder-combiner does not necessarily diagonalize the MIMO channel per-subcarrier $\mathbf{H}_{\text{eff}}[k]$, like the usual SVD-based frequency selective precoding and combining. Therefore, the transmitted symbol may be detected using optimal digital receiver strategies with the knowledge of the $N_{\text{s}} \times N_{\text{s}}$ matrix $\mathbf{H}_{\text{eff}}[k]$. This low dimensional effective channel may be estimated using convention digital MIMO channel estimation techniques and feedback of this channel is not required, since there is no additional per-subcarrier digital precoding layer. As shown in the simulations, small gains with frequency selective precoding can be achieved using water-filling across space and/or frequency. We also show in the simulations that the optimality of the frequency flat beamformers does not depend on the number of subcarriers K or the number of delay taps N_{c}.

V. Compressive subspace estimation

Given that all the channel matrices have the same row and column spaces, $\mathcal{H}_c$ and $\mathcal{H}_r$, the knowledge of these subspaces is sufficient for designing the optimal frequency flat precoders and combiners that achieve the maximum spectral efficiency. The purpose of this section is to illustrate how this insight from Section III can be used to replace the channel estimation stage, central for a conventional precoder design, by a compressive subspace estimation. Accordingly, the system design problem boils down to estimating the orthogonal bases of the row and column spaces of the $N_{\rm r} \times N_{\rm t}$ channel matrices.

While iterative approaches to estimate the right and left singular subspaces of mmWave MIMO channels have been considered previously [11], we use a variation of the nuclear norm minimization approach in [12] to perform the compressive subspace estimation. Compressive sensing based approaches are interesting because they only require a small number of random, linear measurements [12], [13] when estimating the low rank subspace of large dimensional matrices.

Let $\mathbf{F}_{(m)} \in \mathbb{C}^{N_{\rm t} \times N_{\rm s}}$ denote the precoder used for the mth training frame, and $\mathbf{W}_{(m)} \in \mathbb{C}^{N_{\rm r} \times N_{\rm s}}$ denote the corresponding combiner. Then, the compressed subspace estimation can be formulated as a low rank matrix estimation problem by nuclear norm relaxation [12]. Denoting the nuclear norm of matrix $\mathbf{A}$ as $\|\mathbf{A}\|_* \triangleq \operatorname{trace}\left(\sqrt{\mathbf{A}^*\mathbf{A}}\right)$, and with H denoting the variable of optimization, we solve

$$\min_{\substack{\mathsf{H} \in \mathbb{C}^{N_{\rm r} \times N_{\rm t}} \\ \text{s.t. } \|\mathbf{W}_{(m)}^* \mathsf{H} \mathbf{F}_{(m)} \mathbf{x}[k] - \mathbf{y}[k]\| \leq \epsilon \quad \text{for } m=1,2,\cdots,M.}} \|\mathsf{H}\|_* \quad , \tag{13}$$

for that subcarrier k which gives the best received SNR. Note that M is the number of compressed measurements in (13). The $N_{\rm s}$ orthogonal vectors of the row and column spaces $\mathcal{H}_c$ and $\mathcal{H}_r$, respectively of the estimated channel (i.e., the first $N_{\rm s}$ left and right singular vectors) are the columns of the optimal frequency flat combiner $\mathbf{W}_{\rm ff}$ and precoder $\mathbf{F}_{\rm ff}$. Most of the prior work on mmWave frequency selective systems proposes, however, a subcarrier specific baseband precoder-combiner and a frequency flat analog processing stage [5].

VI. Simulation results

In this section, we present simulation results to illustrate the optimality of the proposed frequency flat precoder and combiner proposed in Section III. We assume a uniform linear array with half wavelength antenna element separation at the transceivers for the simulations.

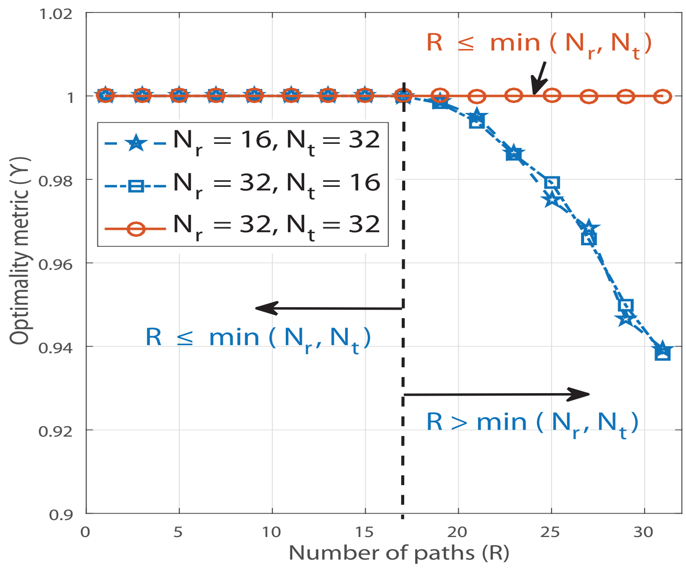

Fig. 1: Optimality metric Υ, as a function of the number of paths R. The frequency flat precoder-combiner is optimal when the number of paths R is small in comparison to $\min(N_{\mathrm{r}}, N_{\mathrm{t}})$.

The metric used to evaluate the optimality of the proposed procoders and combiners is the normalized energy Υ forced into the dominant subspaces when using the optimal frequency flat precoder and combiner. This energy can be computed as $\Upsilon = \frac{1}{K}\sum_{k=1}^{K}\gamma_k$, with

$$\gamma_k = \frac{\|\mathbf{W}_{\mathrm{ff}}^{*}\mathbf{H}[k]\mathbf{F}_{\mathrm{ff}}\|_F^2}{\|\mathbf{H}[k]\|_F^2}. \tag{14}$$

Fig. 1 shows the optimality metric versus the number of paths R in the frequency selective channel with parameters $N_{\mathrm{c}} = 8$ and $K = 16$, for various values of N_{r} and N_{t}. The rank of the channel matrices is upper bounded by $\min(R, N_{\mathrm{r}}, N_{\mathrm{t}})$. When R is small compared to the number of transmit and receive antennas, from (5), the rank of each of the MIMO channel matrices is at most R. In this case, the proposed frequency flat precoder and combiner are optimal. When $R > \min(N_{\mathrm{r}}, N_{\mathrm{t}})$, the semi-unitary precoding-combining solution is sub-optimum, as seen from Fig. 1.

In Fig. 2 and Fig. 3, we assume $N_{\mathrm{c}} = 16$ and $K = 64$ for the channel. The achievable rate using the proposed frequency flat precoder and combiner as a function of the number of paths R for $\mathrm{SNR} = 0\mathrm{dB}$, $N_{\mathrm{t}} = 64$ and $N_{\mathrm{r}} = 16$ is shown in Fig. 2. The achievable rates with the conventional frequency selective precoder and combiner, with and without water-filling are also plotted. The plots in Fig. 2 show that when the MIMO channel rank equals the number of

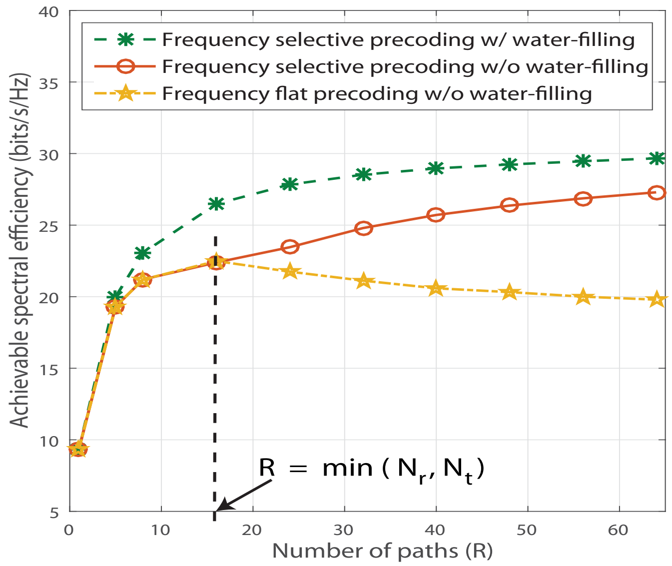

Fig. 2: Plot showing the achievable spectral efficiency versus the number of paths R with frequency selective, and with the proposed frequency flat precoder-combiner. The proposed frequency flat beamforming is optimal when $R \leq \min(N_\mathrm{r}, N_\mathrm{t})$.

paths $R \leq \min(N_\mathrm{r}, N_\mathrm{t})$, then the frequency flat precoder-combiner achieves the same rate as the SVD-based frequency selective precoder-combiner. This implies that the system design and implementation is simplified in low-rank, large dimensional frequency selective MIMO channels, which are common in wideband mmWave systems. Also, when the number of paths is small, the gains due to power allocation via spatial water-filling are small.

In Fig. 3, we compare the achievable spectral efficiency provided by the frequency flat precoders and combiners designed from compressive subspace estimation outlined in Section V, as a function of the number of paths R, for various training length M. We assume $N_\mathrm{r} = N_\mathrm{t} = 32$ here. When the number of paths is small, subspace recovery tools requiring small number of compressive measurements can be used to design the frequency flat precoders and combiners. In this case, the loss in performance relative to the optimal frequency flat precoders and combiners (assuming perfect channel knowledge) is also small. When the rank of the channel (which depends on the number of paths) is higher, a larger number of measurements is needed to obtain a good estimation of the subspaces and design the proposed frequency-flat precoders and combiners.

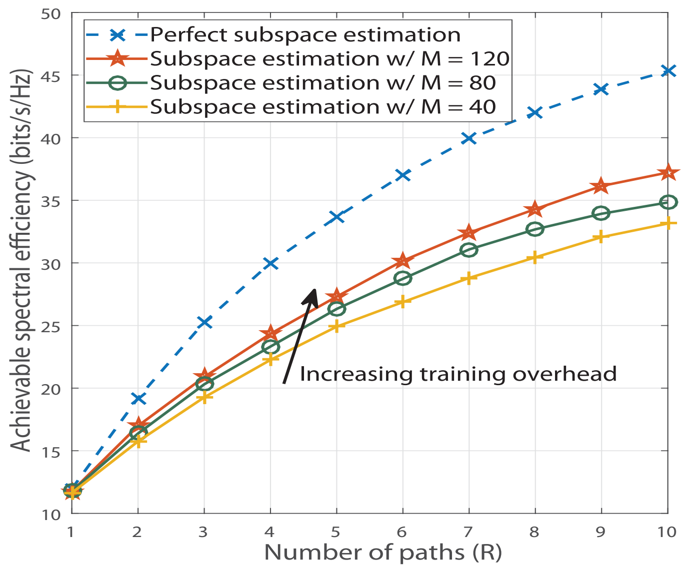

Fig. 3: Plot showing the achievable spectral efficiency for $\mathrm{SNR} = 10$dB with the compressive subspace estimation algorithm as a function of the number of paths R for various training lengths M.

The hardware limitations in mmWave systems make the use of fully digital precoders and combiners impractical [3]. These place constraints on the frequency selective hybrid precoder-combiner design as well. Nevertheless, efficient designs of frequency selective hybrid precoder-combiner guarantee achievable rates similar to all-digital systems [5]. The proposed frequency flat optimal precoder-combiner in this paper, can however, be implemented in the RF part of the transceiver architecture with additional constraints incorporating the limited resolution of phase shifters [3]. This not only makes the system design easier, but also makes the hardware implementation cost effective.

VII. Conclusion

In this letter, we established the optimality of the frequency flat precoder-combiner for frequency selective wideband mmWave channels with small number of paths. We proved that all the frequency domain MIMO channel matrices corresponding to all the subcarriers, share the same row and column subspaces. The combiners and precoders derived from these subspaces were shown to form maximum-spectral-efficiency-achieving optimal frequency selective combiners and precoders when the MIMO channel dimensions are large in comparison with the rank of the channel, which is a function of the number of paths.

References

[1] W. Roh, J. Y. Seol, J. Park, B. Lee, J. Lee, Y. Kim, J. Cho, K. Cheun, and F. Aryanfar, "Millimeter-wave beamforming as an enabling technology for 5G cellular communications: theoretical feasibility and prototype results," *IEEE Commun. Mag.*, vol. 52, pp. 106–113, Feb. 2014.

[2] T. S. Rappaport, R. W. Heath Jr, R. Daniels, and J. N. Murdock, *Millimeter Wave Wireless Communications*. Pearson Education, 2014.

[3] R. W. Heath Jr., N. González-Prelcic, S. Rangan, W. Roh, and A. M. Sayeed, "An overview of signal processing techniques for millimeter wave MIMO systems," *IEEE J. Sel. Topics Signal Process.*, vol. 10, pp. 436–453, April 2016.

[4] P. Sudarshan, N. B. Mehta, A. F. Molisch, and J. Zhang, "Channel statistics-based RF pre-processing with antenna selection," *IEEE Trans. Wireless Commun.*, vol. 5, pp. 3501–3511, Dec. 2006.

[5] A. Alkhateeb and R. W. Heath Jr., "Frequency selective hybrid precoding for limited feedback millimeter wave systems," *IEEE Trans. Commun.*, vol. 64, pp. 1801–1818, May 2016.

[6] S. Park, A. Alkhateeb, and R. W. Heath Jr., "Dynamic subarrays for hybrid precoding in wideband mmwave MIMO systems," *CoRR*, vol. abs/1606.08405, 2016. http://arxiv.org/abs/1606.08405.

[7] J. Lee, G. T. Gil, and Y. H. Lee, "Exploiting spatial sparsity for estimating channels of hybrid MIMO systems in millimeter wave communications," in *Proc. IEEE Global Telecommun. Conf. (GLOBECOM)*, pp. 3326–3331, Dec. 2014.

[8] A. Alkhateeb, O. E. Ayach, G. Leus, and R. W. Heath Jr., "Channel estimation and hybrid precoding for millimeter wave cellular systems," *IEEE J. Sel. Topics Signal Process.*, vol. 8, pp. 831–846, Oct. 2014.

[9] P. Schniter and A. Sayeed, "Channel estimation and precoder design for millimeter-wave communications: The sparse way," in *Proc. Asilomar Conf. Signals, Syst., Comput.*, pp. 273–277, Nov. 2014.

[10] J. H. Brady and A. M. Sayeed, "Wideband communication with high-dimensional arrays: New results and transceiver architectures," in *Proc. IEEE International Conference on Communication (ICC) Workshop (ICCW)*, 2015.

[11] H. Ghauch, T. Kim, M. Bengtsson, and M. Skoglund, "Subspace estimation and decomposition for large millimeter-wave MIMO systems," *IEEE J. Sel. Topics Signal Process.*, vol. 10, pp. 528–542, April 2016.

[12] K. Zhong, P. Jain, and I. S. Dhillon, "Efficient matrix sensing using rank-1 Gaussian measurements," in *Proc. Int. Conf. Algorithmic Learning Theory (ALT)*, pp. 3–18, Oct. 2015.

[13] S. Haghighatshoar and G. Caire, "Low-complexity massive MIMO subspace estimation and tracking from low-dimensional projections," *CoRR*, vol. abs/1608.02477, 2016. https://arxiv.org/abs/1608.02477.