ON TOPOLOGICAL FULL GROUPS OF $\mathbb{Z}^d$-ACTIONS

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ABSTRACT. We give new examples of simple finitely generated groups arising from actions of free abelian groups on the Cantor sets. As particular examples, we discuss groups of interval exchange transformations, and a group naturally associated with the Penrose tilings. Many groups in this class are amenable.

1. INTRODUCTION

The motivation of this paper is to present a new interesting source of finitely generated simple groups, with interesting properties like non-elementary amenability.

A group G is amenable if there exists a finitely additive translation invariant probability measure on all subsets of G. This definition was given by John von Neumann, [19], in a response to Banach-Tarski, and Hausdorff paradoxes. He singled out the property of a group which forbids paradoxical actions.

The class of *elementary amenable groups*, denoted by EG, was introduced by Mahlon Day in [6], as the smallest class of groups that contain finite and abelian groups and is closed under taking subgroups, quotients, extensions and directed unions. The fact that the class of amenable groups is closed under these operations was already known to von Neumann, [19], who noted at that at that time there was no known amenable group which did not belong to EG.

No substantial progress in understanding this class has been made until the 80s, when Chou, [5], showed that all elementary amenable groups have either polynomial or exponential growth, and Rostislav Grigorchuk, [7] gave an example of a group with intermediate growth. Grigorchuk's group served as a starting point in developing the theory of groups with intermediate growth, all of them being non-elementary amenable. In the same paper Chou showed that every simple finitely generated infinite group is not elementary amenable. In [10] it was shown that the topological full group of Cantor minimal system is amenable. By the results of Matui, [15], this group has a simple and finitely generated commutator subgroup, in particular, it is not elementary amenable. This was the first example of infinite simple finitely generated amenable group.

Currently there are only two sources of non-elementary amenable groups: groups acting on rooted trees and topological full groups of Cantor minimal systems. In [8], the author gives a unified approach to non-elementary amenability of groups acting on rooted trees. Here we give more examples of non-elementary amenable groups coming from topological full groups of Cantor minimal systems. In [24], Vorobets showed that the commutator group of the interval exchange transformation group

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is simple. However, this group is obviously uncountable, thus, can not be finitely generated.

Theorem 1. *Consider a minimal faithful action of* $\mathbb{Z}^d$ *on a Cantor set conjugate to the action on a closed* $\mathbb{Z}^d$*-invariant subset of* $\mathsf{X}^{\mathbb{Z}^d}$ *for some finite alphabet* X*. Then the commutator subgroup of the topological full group* $[[\mathbb{Z}^d]]$ *is finitely generated.*

We describe an explicit generating set, and give a simple direct proof, by extending the ideas of H. Matui's result [15] on $\mathbb{Z}$-actions. Subsequent paper [18] gives a less direct proof of the finite generation of special subgroups $\mathsf{A}(G)$ of the topological full groups in the case of expansive action. The groups $\mathsf{A}(G)$ are closely related to the derived subgroups of the full groups (in particular, they coincide in the case of the actions of abelian groups), but precise relation is still not well understood.

It was proved in [17], that the commutator subgroup of $[[\mathbb{Z}^d]]$ is simple. This also can be proved using [2] and [Proposition 3.18]LatremolierOrmes. The topological full groups that correspond to interval exchange transformation group were studied in [13]. The authors prove that subgroups of *rank* equals to 2 are amenable. These groups can be realized as topological full groups of minimal action of $\mathbb{Z}^2$ on the Cantor set. Therefore, Theorem 1 in the combination with [17], [13] gives more examples of simple finitely generated infinite amenable groups, and thus by the result of Chou non-elementary amenable groups. Matui, [17], showed that $[[T_1, \dots, T_m]] = [[T_1, \dots, T_n]]$ implies $m = n$. In particular, this implies that the groups from the corollary are different from the one previously obtained in [10].

In the last section we associate a group $\mathcal{P}$ to the Penrose tiling. It is the group of all permutations of the tiles *defined by local rules*. It has a natural minimal action on the Cantor set. We show that with respect to this action, $\mathcal{P}$ is the topological full group of an action of a finitely generated abelian group (isomorphic to $\mathbb{Z}^4 \oplus (\mathbb{Z}/5\mathbb{Z})$). Using our results about the topological full groups of free abelian groups we prove the following.

Theorem 2. *The derived subgroup of* $\mathcal{P}$ *is simple and finitely generated.*

It is an open question to decide if the group $\mathcal{P}$ is amenable. Another interesting open question is description of the defining relations in $\mathcal{P}$.

2. A finite generating set of the derived subgroup of $[[\mathbb{Z}^d]]$

Recall that an action of a group G on a set X is called *faithful* if only the identity element of G acts identically on X. It is called *free* if $g(x) = x$ for $g \in G$ and $x \in X$ implies $g = 1$. Obviously, every free action is faithful. An action of G on a topological space is *minimal* if all G-orbits are dense.

Lemma 3. *Let* G *be an abelian group. If the action of* G *on a Cantor set* $\mathbf{C}$ *is minimal and faithful, then it is free.*

Proof. Suppose that $g \in G$ is a non-zero element. By faithfulness of the action, there exists $x \in \mathbf{C}$ such that $g(x) \neq x$. Then there exists a neighborhood U of x such that $g(U) \cap U = \emptyset$. Let $y \in \mathbf{C}$ be an arbitrary point. By minimality, there exists $h \in G$ such that $h(y) \in U$. Since U and $g(U)$ are disjoint, the points $h(y)$ and $gh(y)$ are different. Then $g(y) = h^{-1}gh(y) \neq h^{-1}h(y) = y$. It follows that g has no fixed points in $\mathbf{C}$. □

Let us fix a minimal action of the free abelian group $\mathbb{Z}^d$ on a closed shift-invariant subset $\mathbf{C} \subset \mathsf{X}^{\mathbb{Z}^d}$ of the full shift over a finite alphabet X. Then $\mathbf{C}$ is homeomorphic to the Cantor set. We use the additive notation for the group $\mathbb{Z}^d$. If $w : \mathbb{Z}^d \longrightarrow \mathsf{X}$ is a point of $\mathsf{X}^{\mathbb{Z}^d}$, then its image under the action of $g \in \mathbb{Z}^d$ is defined by the rule

$$g(w)(h) = w(h - g).$$

In other words, elements of $\mathsf{X}^{\mathbb{Z}^d}$ are labelings of the points of $\mathbb{Z}^d$ by elements of X, and the elements $g \in \mathbb{Z}^d$ act by shifting all the labels by g. Alternatively, we may imagine the action of g as the shift of the "origin of coordinates" in a given sequence w by $-g$.

A *patch* $\pi = (f, P)$ is a finite subset $P \subset \mathbb{Z}^d$ together with a map $f : P \longrightarrow \mathsf{X}$. The set P is called the *support* of the patch. We say that an element $w \in \mathsf{X}^{\mathbb{Z}^d}$ (a $\mathbb{Z}^d$*-sequence*) *contains* the patch (f, P) if $w|_P = f$. The set $\mathcal{W}_\pi$ of all sequences containing a given patch π is a clopen subset of $\mathsf{X}^{\mathbb{Z}^d}$ called the *cylindrical set* defined by the patch, and the set of all such clopen subsets forms a basis of topology on $\mathsf{X}^{\mathbb{Z}^d}$, by definition.

We say that two patches $\pi_1 = (f_1, P_1)$ and $\pi_2 = (f_2, P_2)$ are *compatible* if there exists sequence $w \in \mathbf{C}$ containing π_1 and π_2. In other words, the patches are compatible if the intersection of the associated cylindrical sets is non-empty. If the patches π_1 and π_2 are compatible then there *union* $\pi_1 \cup \pi_2$ is the patch $(f, P_1 \cup P_2)$ where $f|_{P_1} = f_1$ and $f|_{P_2} = f_2$. Note that in terms of the cylindrical sets, we have $\mathcal{W}_{\pi_1 \cup \pi_2} = \mathcal{W}_{\pi_1} \cap \mathcal{W}_{\pi_2}$.

If $\pi = (f, P)$ is a patch, and Q is a finite set containing P, then $\mathcal{W}_\pi$ is equal to the disjoint union of the sets $\mathcal{W}_{(\tilde{f}, Q)}$, where $\tilde{f}$ runs through the set of all maps $\tilde{f} : Q \longrightarrow \mathsf{X}$ such that $\tilde{f}|_P = f$. Note that some of these sets may be empty (if the corresponding patch is not allowed for the elements of $\mathbf{C}$).

If $\pi = (f, P)$ is a patch and $g \in \mathbb{Z}^d$, then we have $g(\mathcal{W}_\pi) = \mathcal{W}_{\pi+g}$, where $\pi + g = (f + g, P + g)$, where $(f + g) : (P + g) \longrightarrow \mathsf{X}$ is given by $(f + g)(h) = f(h - g)$, in accordance with the definition of the action of $\mathbb{Z}^d$ on $\mathsf{X}^{\mathbb{Z}^d}$.

Lemma 4. *Let $A \subset \mathbb{Z}^d$ be a finite set not containing zero. Then there exists $B \subset \mathbb{Z}^d$ such that for every $w \in \mathbf{C}$ and every $g \in A$ the patches $(w|_B, B)$ and $((w + g)|_B, B)$ are not compatible.*

Proof. Define the following metric on $\mathsf{X}^{\mathbb{Z}^d}$. The distance $|w_1 - w_2|$ is equal to 2^{-R}, where R is the biggest number such that restrictions of w_1 and w_2 to the ball of radius R in $\mathbb{Z}^d$ with center in 0 (for example in the ℓ_∞ norm) coincide. The it is enough to prove that there exists ϵ such that $|g(w) - w| > \epsilon$ for all $g \in A$ and $w \in \mathbf{C}$. Namely, for every ϵ there exists a finite set $B \subset \mathbb{Z}^d$ such that for every $w \in \mathsf{X}^{\mathbb{Z}^d}$ the set of all $u \in \mathsf{X}^{\mathbb{Z}^d}$ such that $(u|_B, B) = (w|_B, B)$ is contained in the ϵ-neighborhood of w.

Suppose that it is not true, i.e., that for every $\epsilon > 0$ there exist w and $g \in A$ such that $|g(w) - w| \leq \epsilon$. Since A is finite, this implies that there exists $g \in A$ and a sequence of points $w_n \in \mathsf{C}$ such that $|g(w_n) - w_n| \to 0$ as $n \to \infty$. Since $\mathbf{C}$ is homeomorphic to the Cantor set, this implies that g has a fixed point, which is a contradiction Lemma 3. $\square$

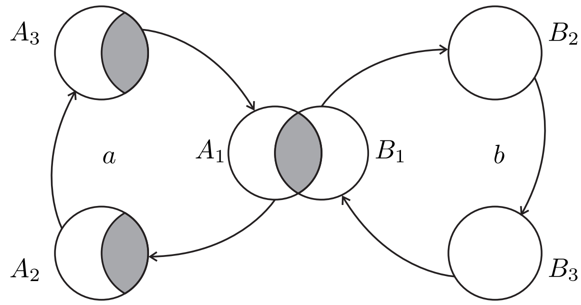

FIGURE 1. Lemma 5

Let $U \subset \mathbf{C}$ be a clopen set, and $g_1, g_2, g_3 \in \mathbb{Z}^d$ elements such that $g_1(U), g_2(U), g_3(U)$ are pairwise disjoint. Denote by $T_{U,(g_1,g_2,g_3)}$ the element of $[[\mathbb{Z}^d]]$ given by

$$T_{U,(g_1,g_2,g_3)}(w) = \begin{cases} (g_2 - g_1)(w) & \text{if } w \in g_1(U); \\ (g_3 - g_2)(w) & \text{if } w \in g_2(U); \\ (g_1 - g_3)(w) & \text{if } w \in g_3(U); \\ w & \text{if } w \notin g_1(U) \cup g_2(U) \cup g_3(U). \end{cases}$$

In other words, $T_{U,(g_1,g_2,g_3)}$ cyclically permutes $g_1(U)$, $g_2(U)$, and $g_3(U)$ in the natural way. We will denote $T_{\pi,(g_1,g_2,g_3)} = T_{\mathcal{W}_\pi,(g_1,g_2,g_3)}$, for a patch π.

Lemma 5. *Let $A_1, A_2, A_3, B_1, B_2, B_3$ be subsets of a set X such that only A_1 and B_1 have non-empty intersection, while all the other pairs of subsets are disjoint. Let a be a permutation of X acting trivially on $X \setminus (A_1 \cup A_2 \cup A_3)$, and satisfying $a(A_1) = A_2$, $a(A_2) = A_3$, $a(A_3) = A_1$, and $a^3 = 1$. Similarly, let b be a permutation acting trivially on $X \setminus (B_1 \cup B_2 \cup B_3)$ and satisfying $b(B_1) = B_2$, $b(B_2) = B_3$, $b(B_3) = A_3$, and $b^3 = 1$. Then $[[b^{-1}, a^{-1}], [b, a]]$ acts as a on the set $(A_1 \cap B_1) \cup a(A_1 \cap B_1) \cup a^2(A_1 \cap B_1)$ and identically outside of it.*

See Figure 1 illustrating Lemma 5. Note that we use the left action here, but the usual commutator $[g, h] = g^{-1}h^{-1}gh$.

Proof. Let $C = A_1 \cap B_1$. Then the sets $C, a(C), a^2(C), b(C), b^2(C)$ are pairwise disjoint. The permutations a and b act as cycles of length three permuting $C, a(C), a^2(C)$ and $C, b(C), b^2(C)$, respectively. The element $[[b^{-1}, a^{-1}], [b, a]]$ acts then on these fives sets in the same way as the similar expression involving commutators of the permutations $a = (1, 2, 3)$ and $b = (1, 4, 5)$ act on $\{1, 2, 3, 4, 5\}$. We have $[b, a] = b^{-1}a^{-1}ba = (1, 5, 3)$ and $[b^{-1}, a^{-1}] = bab^{-1}a^{-1} = (1, 4, 2)$, and $[[b^{-1}, a^{-1}, [b, a]] = (1, 4, 2)^{-1}(1, 5, 3)^{-1}(1, 4, 2)(1, 5, 3) = (1, 2, 3) = a$.

The set $A' = (A_1 \setminus C) \cup a(A_1 \setminus C) \cup a^2(A_1 \setminus C)$ is a-invariant, and b acts trivially on it. It follows that the restriction of $[b^{-1}, a^{-1}]$ and $[b, a]]$ to A' is equal to the restriction of $[1, a^{-1}]$ and $[1, a]$, which are trivial. It follows that $[[b^{-1}, a^{-1}], [b, a]]$ acts trivially on A'. The same argument shows that $[[b^{-1}, a^{-1}], [b, a]]$ acts trivially on $(B_1 \setminus C) \cup b(B_1 \setminus C) \cup b^2(B_1 \setminus C)$. □

As a corollary, we get the following relation between the elements of the form $T_{\pi,(g_1,g_2,g_3)}$. (Compare with [15, Lemma 5.3].)

Corollary 6. *Let π_1, π_2 be patches, g_1, g_2, h_1, h_2 be elements of $\mathbb{Z}^d$ such that $\pi_1, \pi_1 + g_1, \pi_1 + g_2, \pi_2, \pi_2 + h_1, \pi_2 + h_2$ are pairwise incompatible except for the pair π_1 and π_2. Then*

$$[[T^{-1}_{\pi_2,(0,h_1,h_2)}, T^{-1}_{\pi_1,(0,g_1,g_2)}], [T_{\pi_2,(0,h_1,h_2)}, T_{\pi_1,(0,g_1,g_2)}]] = T_{\pi_1\cup\pi_2,(0,g_1,g_2)}.$$

We will use the usual ℓ_1 metric on $\mathbb{Z}^d$, i.e., the word metric associated with the standard generating set of $\mathbb{Z}^d$. Denote by $B(R)$ the ball of radius R with the center in $0 \in \mathbb{Z}^d$ for this metric.

By Lemma 4, there exists R_1 such that for every $w \in \mathbf{C}$ the patches $\pi = (B(R_1), w|_{B(R_1)})$ and $\pi + g$ are incompatible for every $g \in \mathbb{Z}^d$, $g \neq 0$, of length ≤ 3.

Let $\{e_1, e_2, \ldots, e_d\}$ be the standard generating set of $\mathbb{Z}^d$. Denote by $\mathcal{T}_R$ the set of elements of $[[\mathbb{Z}^d]]'$ of the form $T_{\pi,(0,e_i,-e_i)}$, where π runs through the set of all patches of the form $(B(R), w|_{B(R)})$ for $w \in \mathbf{C}$.

Proposition 7. *If $R \geq R_1 + 2$, then the group generated by $\mathcal{T}_R$ contains $\mathcal{T}_{R+1}$.*

Proof. Denote $S = \{\pm e_1, \pm e_2, \ldots, \pm e_d\}$. Let $A \subset \mathbb{Z}^d$ be a finite subset containing $B(R_1 + 2)$, and let $w \in \mathbf{C}$. Define the patches $\rho_0 = (w|_A, A)$, $\rho_h = (w|_{A+h}, A + h)$, for $h \in S$. Note that the patch ρ_h contains the patch $(w|_{B(R_1)}, B(R_1))$ and that $\rho_h - h = ((w - h)|_A, A)$.

Let us apply Corollary 6 for $\pi_1 = \rho_0$, $\pi_2 = \rho_h$, $g_1 = g$, $g_2 = -g$, $h_1 = h$, $h_2 = 2h$, where g, h are different elements of S. Since ρ_0 and ρ_h both contain the patch $(w|_{B(R_1)}, B(R_1))$, the patches $\pi_1, \pi_1 + g_1$, $\pi_1 + g_2$, π_2, $\pi_2 + h_1$, and $\pi_2 + h_2$ are pairwise incompatible, except for π_1 and π_2, which are both patches of w. It follows that we can apply Corollary 6, hence

$$[[T^{-1}_{\rho_h,(0,h,2h)}, T^{-1}_{\rho_0,(0,g,-g)}], [T_{\rho_h,(0,h,2h)}, T_{\rho_0,(0,g,-g)}]] = T_{\rho_0\cup\rho_h,(0,g,-g)}.$$

Note that $T_{\rho_h,(0,h,2h)} = T_{\rho_h-h,(-h,0,h)} = T_{\rho_h-h,(0,h,-h)}$.

Let us apply now Corollary 6 to $\pi_1 = \rho_0 - g$, $\pi_2 = \rho_g - g$, $g_1 = -2g$, $g_2 = -g$, $h_1 = h$, $h_2 = -h$. The patches π_1 and π_2 are patches of $w - g$, and contain $((w - g)|_{B(R_1)}, B(R_1))$. It follows that, in the same way as above, we can apply Corollary 6, and get

$$[[T^{-1}_{\rho_g-g,(0,h,-h)}, T^{-1}_{\rho_0-g,(0,-2g,-g)}], [T_{\rho_g-g,(0,h,-h)}, T_{\rho_0-g,(0,-2g,-g)}]] = T_{(\rho_0-g)\cup(\rho_h-g),(0,-2g,-g)}.$$

Recall that $\rho_g - g = ((w-g)|_A, A)$ and that we have $T_{\rho_0-g,(0,-2g,-g)} = T_{\rho_0,(g,-g,0)} = T_{\rho_0,(0,g,-g)}$. We also have $T_{(\rho_0-g)\cup(\rho_h-g),(0,-2g,-g)} = T_{\rho_0\cup\rho_h,(0,g,-g)}$.

We have shown that the group generated by the set

$$\{T_{\pi,(0,g,-g)} \ : \ g \in S, \pi = (w|_A, A), w \in \mathbf{C}\}$$

contains the set

$$\{T_{\pi,(0,g,-g)} \ : \ g \in S, \pi = (w|_{A\cup A+h}, A \cup A + h), w \in \mathbf{C}, h \in S\}.$$

Since $B(R + 1) = \bigcup_{h\in S} B(R) + h$, this finishes the proof of the proposition. □

It follows that the group generated by $\mathcal{T}_{R_1+2}$ contains $\mathcal{T}_R$ for every $R \geq R_1 + 2$. For every cylindrical set $U \subset \mathbf{C}$ there exists R such that U is equal to the disjoint union of cylindrical sets $\mathcal{W}_\pi$ such that π is a patch with support $B(R)$. It follows that every element of the form $T_{\pi,(0,e_i,-e_i)}$ can be written as a product of elements $T_{\pi',(0,e_j,-e_j)}$ such that π' is a patch with support $B(R)$ for some R big enough. Consequently, the group generated by $\mathcal{T}_{R_1+2}$ contains all elements of the form $T_{\pi,(0,e_i,-e_i)}$.

The proof of Theorem 1 is finished by the following, since the set $\mathcal{T}_{R_1+2}$ is finite.

Proposition 8. *The derived subgroup of the full group of the action of $\mathbb{Z}^d$ on* C *is generated by the set of all elements of the form $T_{\pi,(0,e_i,-e_i)}$, where $i = 1, 2, \ldots, d$, and π is a finite patch such that $T_{\pi,(0,e_i,-e_i)}$ is defined.*

Proof. It is known, see [16], that the derived subgroup of $[[\mathbb{Z}^d]]$ is simple and is contained in every non-trivial normal subgroup of $[[\mathbb{Z}^d]]$.

Consider the set $\mathcal{T} \subset [[\mathbb{Z}^d]]$ of all elements elements of order three permuting cyclically three disjoint clopen subsets U_1, U_2, U_3 of $\mathbf{C}$ and acting identically outside their union. The set $\mathcal{T}$ is obviously invariant under conjugation by elements of $[[\mathbb{Z}^d]]$, hence the group generated by $\mathcal{T}$ is normal. On the other hand, we have $\mathcal{T} \subset [[\mathbb{Z}^d]]'$, as every element of $\mathcal{T}$ is equal to the commutator of two transformations: one permuting U_1 with U_2, and the other permuting U_2 with U_3. Consequently, $\mathcal{T}$ generates $[[\mathbb{Z}^d]]'$.

For every element $T \in \mathcal{T}$ permuting cyclically clopen sets U_1, U_2, U_3, there exists partitions of U_i into cylindrical sets such that T maps a piece of the partition to a piece of the partition, and restriction of T to every piece of the partitions is equal to the restriction of an element of $\mathbb{Z}^d$. It follows that T is a product of a finite set of elements of the form $T_{\pi,(g_1,g_2,g_3)}$. It remains to show that we can generate all elements of the form $T_{\pi,(g_1,g_2,g_3)}$ by elements of the form $T_{\pi,(0,e_i,-e_i)}$. It is well known that the alternating group A_n is generated by cycles $(k, k+1, k+2)$. It follows that the group generated by $T_{\pi,(0,e_i,-e_i)}$ contains the set of elements of the form $T_{\pi,(g_1,g_2,g_3)}$, where g_i belong to one direct factor of $\mathbb{Z}^d$.

Let us prove the following technical lemma.

Lemma 9. *Let $X_d = \{x_1 \ldots x_d | x_i \in \{a, b, c\},\ 1 \le i \le d\}$ be the 3^d-element set of d-letter words over the alphabet $\{a, b, c\}$, and let S_{X_d} be the symmetric group of permutations of X_d. Denote the alternating subgroup of even permutations of X_d by A_{X_d}. Consider the set B_d of all elements of the type $(XaY\ XbY\ XcY) \in S_{X_d}$, where X and Y are arbitrary (possibly, empty) words such that $|X| + |Y| = d - 1$. Then A_{X_d} is generated by the set B_d.*

Proof. The lemma can be proved by induction on d.

For $d = 2$, we use the well-known fact that A_9 is generated by 3-cycles $\{(123), (234), \ldots, (789)\}$. To apply this fact, we need to show that all 7 elements $(aa\ ab\ ac)$, $(ab\ ac\ ba)$, $(ac\ ba\ bb)$, $(ba\ bb\ bc)$, $(bb\ bc\ ca)$, $(bc\ ca\ cb)$, $(ca\ cb\ cc)$ are generated by B_2. This can be checked by hand:

- $(aa\ ab\ ac) \in B_d$;
- $(ab\ ac\ ba) = (aa\ ba\ ca)(aa\ ab\ ac)(aa\ ca\ ba)$;
- $(ac\ ba\ bb) = (ac\ cc\ bc)(ba\ bb\ bc)(ac\ bc\ cc)$;
- $(ba\ bb\ bc) \in B_d$;
- $(bb\ bc\ ca) = (aa\ ba\ ca)(ba\ bb\ bc)(aa\ ca\ ba)$;
- $(bc\ ca\ cb) = (ac\ cc\ bc)(ca\ cb\ cc)(ac\ bc\ cc)$;
- $(ca\ cb\ cc) \in B_d$.

Suppose the statement holds for $d = k$ and consider the case $d = k+1$. Since the alternating group is generated by 3-cycles, it's sufficient to show that every 3-cycle is generated by B_{k+1}. Assume we have a cycle $(Ax\ By\ Cz)$, where $A, B, C \in X_k$ are pairwise distinct, and $x, y, z \in \{a, b, c\}$, not necessarily distinct. We know the following:

- $(Ax\ Bx\ Cx)$, $(Ay\ By\ Cy)$, $(Az\ Bz\ Cz)$ are generated by B_{k+1}. Indeed, we can take the elements of B_k generating $(A\ B\ C)$ and append the needed letter to each of them.
- $(Ax\ Ay\ Az)$, $(Bx\ By\ Bz)$, $(Cx\ Cy\ Cz)$ are in B_{k+1} by definition.

Then, applying the induction base for the set $\{A, B, C\} \times \{a, b, c\}$, we conclude that $(Ax\ By\ Cz)$ is also generated by B_{k+1}.

In case A, B, C are not distinct, we can use a slightly modified version of the proof above. If, for example, $A = B$ (which automatically implies $x \neq y$), we can take an arbitrary word $D \in X_k$ distinct from A and C in order to apply the induction base to $\{A, C, D\} \times \{a, b, c\}$. Clearly, the 3-cycle $(Ax\ Ay\ Cz)$ will belong to X_{k+1}

The induction step is complete. □

Lemma 9 implies that the group generated by all elements of the form $T_{\pi,(g_1,g_2,g_3)}$, where g_i belong to one factor of $\mathbb{Z}^d$, contains all elements of the form $T_{\pi,(g_1,g_2,g_3)}$, where $g_i \in \mathbb{Z}^d$ are now arbitrary. This finishes the proof of the proposition and Theorem 1. □

3. Topological full group and interval exchange group

One of classical situations where topological full groups of abelian group appear naturally are finitely generated groups of interval exchange transformations. An *interval exchange transformation* is a bijection $f : [0, 1) \longrightarrow [0, 1)$ such that $[0, 1)$ can be partitioned into a finitely many half-intervals $[a, b)$ on which f is equal to a translation $x \mapsto x + t$. It is easy to see that the set of all interval exchange transformations is a group. Let us describe subgroups of this group isomorphic to topological full groups of minimal actions of $\mathbb{Z}^d$.

Let $\alpha_1, \alpha_2, \dots, \alpha_d$ be irrational numbers such that $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are linearly independent over $\mathbb{Q}$. Then the additive groups $\langle \alpha_1, \alpha_2, \dots, \alpha_d \rangle$ and $H = \langle \alpha_1, \alpha_2, \dots, \alpha_d \rangle / \mathbb{Z}$ are isomorphic to $\mathbb{Z}^d$. The group H is a subgroup of the circle $\mathbb{R}/\mathbb{Z}$, and hence acts on it in the natural way. By the classical Kronecker's theorem, the action of each subgroup $\langle \alpha_i \rangle$ on $\mathbb{R}/\mathbb{Z}$ is minimal, hence the action of H on $\mathbb{R}/\mathbb{Z}$ is also minimal.

Let us lift H as a set to $[0, 1]$ by the natural quotient map $[0, 1] \to \mathbb{R}/\mathbb{Z}$, and let $W \subset [0, 1]$ be the obtained set.

Let us replace each number $q \in W \subset [0, 1]$ by two copies: q_{-0} and q_{+0}. Here we identify 0_{-0} with 1 and 0_{+0} with 0, 1_{-0} with 1 and 1_{+0} with 0, according to the natural cyclic order on $\mathbb{R}/\mathbb{Z}$ (seen also as the quotient of the interval $[0, 1]$). Denote by R_H the obtained set (equal to the disjoint union of $[0, 1] \setminus W$ and the set of doubled points W). The set R_H is ordered in the natural way (we assume that $q_{-0} < q_{+0}$), and the order is linear (total).

Let us introduce the order topology on R_H. Recall, that it is the topology generated by the open intervals $(a, b) = \{x \in R_H\ :\ a < x < b\}$.

Lemma 10. *The space R_W is homeomorphic to the Cantor set.*

Proof. We use the following formulation of Brouwer's theorem: A topological space is a Cantor space if and only if it is non-empty, compact, totally disconnected, metrizable and has no isolated points. Note that by classical metrization theorems, we can replace metrizability by Hausdorffness and second countability.

The space R_H is obviously non-empty, has no isolated points. For any $a, b \in W \cap [0, 1]$ such that $a < b$, we have $[a_{+0}, b_{-0}] = (a_{-0}, b_{+0})$, hence the intervals

(a_{-0}, b_{+0}) are clopen. The set of such intervals is a basis of topology, since the set W is dense. We also see that the space R_H is second countable and Hausdorff.

Let $A \subset R_H$ be an arbitrary subset. Let us show that $\sup A$ and $\inf A$ exist, which will imply compactness. Let $\hat{A}$ be the image of A in $[0,1]$. We know that $\sup \hat{A}, \inf \hat{A} \in [0,1]$ exist. If $\sup \hat{A} \notin W$, then the corresponding element of R_H is also a supremum of A. If $\sup \hat{A} \in W$, then $\sup A = \sup \hat{A}_{-0}$, unless $\sup \hat{A}_{+0} \in A$, in which case $\sup A = \sup \hat{A}_{+0}$. Infima are treated in the same way. □

The action of H on $\mathbb{R}/\mathbb{Z}$ naturally lifts to an action on R_H: we just set $h(q_{+0}) = h(q)_{+0}$ and $h(q_{-0}) = h(q)_{-0}$.

Denote by IET_H the topological full group of the action (H, R_H). For every element $g \in IET_H$ there exists a finite partition of R_H into clopen subsets such that the action of g on each of the subsets coincides with a translation by an element of H. Clopen subsets of R_H are finite unions of intervals of the form (a_{+0}, b_{-0}) for $a, b \in H$. It follows that g is an *interval exchange transformation*: it splits the interval $[0,1]$ into a finite number of intervals and then rearranges them. The endpoints of the intervals belong to W. Conversely, every interval exchange transformation such that the endpoints of the subintervals belong to W is lifted to an element of IET_H.

We have proved the following.

Lemma 11. *The group IET_H is naturally isomorphic to the group of all interval exchange transformations of $[0,1]$ such that the endpoints of the intervals into which $[0,1]$ is split belong to H.*

Theorem 1 now implies the following.

Theorem 12. *The derived subgroup of IET_H is simple and finitely generated.*

A two-dimensional version of an interval exchange transformation group is considered in the next section.

4. Penrose tiling group

Let us describe another classical situation, where minimal actions of free abelian groups appear: Penrose tilings. A relation between Penrose tilings and minimal actions of free abelian groups is well known, see the works [22, 23, 21, 1] for this and other dynamical properties of Penrose tilings and their generalizations. We describe here this relation and give a natural interpretation of the associated topological full group (as a group of permutations of the set of tiles, and as a group of rearrangements of a polygon).

There are several versions of the Penrose tiling [20], let us describe one of them. The tiles are two types of rhombi of equal side length 1. The angles of one rhombus are 72° and 108°. The angles of the other are 36° and 144°. We call these rhombuses "thick" and "thin", respectively. Mark a vertex of angle 72° in the thick rhombus, and a vertex of angle 144° of the thin rhombus. Mark the sides adjacent to the marked vertex by single arrows pointing towards the marked vertex. Mark the other edges by double arrows, so that in the thick rhombus they point away from the unmarked vertex of angle 72° and in the thin rhombus they point towards the unmarked vertex of angle 144°, see Figure 2. A *Penrose tiling* is a tiling of the whole plane by such rhombi, where markings of the edges match (adjacent tiles

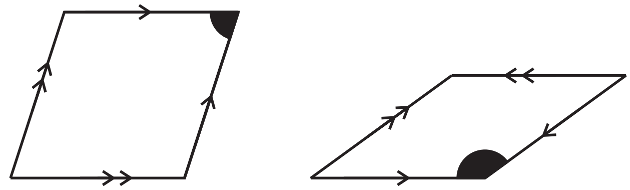

FIGURE 2. Tiles of the Penrose tilings

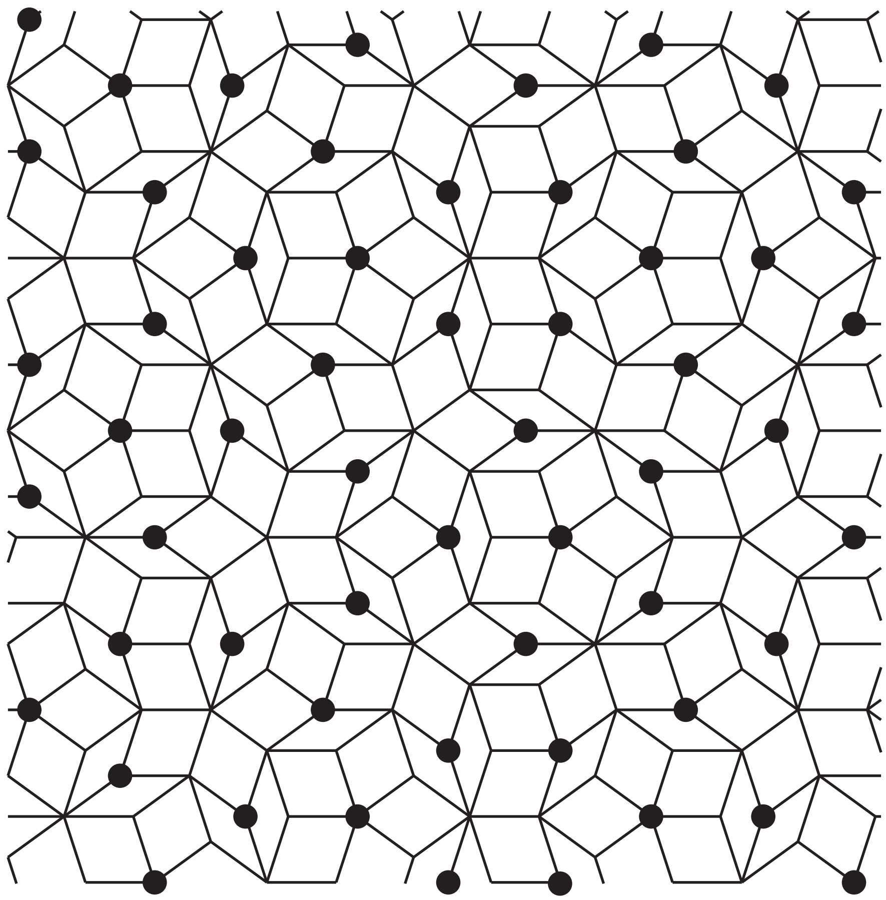

FIGURE 3. Penrose tiling

must have same number of arrows pointing in the same direction). See Figure 3 for an example of a patch of a Penrose tiling.

There are uncountably many different (up to translation and rotation) Penrose tilings. Each of them is *aperiodic*, i.e., does not admit a translational symmetry.

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$, and consider all Penrose tilings by rhombi such that their sides are parallel to the lines $e^{k\pi i/5}\mathbb{R}$, $k \in \mathbb{Z}$. A *pointed* Penrose tiling is a Penrose tiling with a marked vertex of a tile. Let $\mathcal{T}$ be the set of all such pointed Penrose tilings, up to translations (two pointed tilings correspond to the same element of $\mathcal{T}$ if and only if there exists a translation mapping one tiling to the other and the marked vertex of one tiling to the marked vertex of the other). We sometimes identify a tiling with the set of vertices of its tiles. Note that the tiling is uniquely determined by the set of the vertices of the tiles.

Let us introduce a topology on $\mathcal{T}$ in the following way. Let $A \subset T$ be a finite set of vertices of a Penrose tiling T (a *patch*), and let $v \in A$. The corresponding open set $\mathcal{U}_{A,v}$ is the set of all pointed tilings (T, u) such that $A + u - v \subset T$. In other words, a pointed tiling (T, u) belongs to $\mathcal{U}_{A,v}$ if we can see the pointed patch (A, v) around u as a part of T. Then the natural topology on $\mathcal{T}$ is given by the basis of open sets of the form $\mathcal{U}_{A,v}$ for all finite pointed patches (A, v) of Penrose tilings. It follows from the properties of Penrose tilings that the space $\mathcal{T}$ is homeomorphic to the Cantor set, and that for every Penrose tiling T the set of pointed tilings (T, v) is dense in $\mathcal{T}$. The space $\mathcal{T}$ is called sometimes *transversal*.

Consider a patch A with two marked vertices $v_1, v_2 \in A$. Then we have a natural homeomorphism $F_{A,v_1,v_2} : \mathcal{U}_{A,v_1} \longrightarrow \mathcal{U}_{A,v_2}$ mapping $(T, u) \in \mathcal{U}_{A,v_1}$ to $(T, u + v_2 - v_1) \in \mathcal{U}_{A,v_2}$. The homeomorphism F_{A,v_1,v_2} moves in every patch A the marking from the vertex v_1 to the vertex v_2. It is easy to see that F_{A,v_1,v_2} is a homeomorphism between clopen subsets of $\mathcal{T}$.

Definition 13. *The* topological full group of Penrose tilings *is the group $\mathcal{P}$ of homeomorphisms of $\mathcal{T}$ that are locally equal to the homeomorphisms of the form F_{A,v_1,v_2}.*

The set of all pointed tilings (T, v) obtained from a given tiling T is dense in $\mathcal{T}$ (see the remark after Proposition 14) and invariant under the action of the topological full group. It follows that every element of the full group is uniquely determined by the permutation it induces on the set of vertices of the tiling. In terms of permutations of T the full group can be defined in the following way.

We say that a map $\alpha : T \longrightarrow T$, where T is a tiling, *is defined by local rules* if there exists R such that for every $x \in T$ the value of $x - \alpha(x)$ depends only on the set $B_R \cap (T - x)$, where B_R is the disc of radius R around the origin $(0, 0) \in \mathbb{R}^2$.

The following is straightforward.

Proposition 14. *A permutation $\alpha : T \longrightarrow T$ is induced by the element of the full group if and only if α is defined by a local rule. Consequently, the full group is isomorphic to the group of all permutations of T defined by local rules.*

It is well known (starting from the original results of R. Penrose) that any two Penrose tilings are locally isomorphic, i.e., that an isomorphic copy of any finite patch of one tiling is contained in any other Penrose tiling. It follows that the topological full group acts minimally. We will also reprove this fact below (it follows directly from Corollary 17).

Let us describe a more explicit model of the space $\mathcal{T}$ and the full group $\mathcal{P}$ using a description of the Penrose tilings given in the papers [3, 4].

Denote $\zeta = e^{\frac{2\pi i}{5}}$, and let

$$P = \left\{ \sum_{j=0}^{4} n_j \zeta^j \ : \ n_j \in \mathbb{Z}, \sum_{j=0}^{4} n_j = 0 \right\} = (1 - \zeta)\mathbb{Z}[\zeta]$$

be the group generated by the vectors on the sides of the regular pentagon $S = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$. Note that $5 = 4 - \zeta - \zeta^2 - \zeta^3 - \zeta^4 \in P$. As an abelian group, P is isomorphic to $\mathbb{Z}^4$.

Denote by $\mathcal{L}$ the set of lines of the form $i\zeta^j\mathbb{R} + w$, for $j = 0, 1, \ldots, 4$ and $w \in P$. It is easy to see that for any two intersecting lines $l_1, l_2 \in \mathcal{L}$ and any generator $z \in \{1 - \zeta, \zeta - \zeta^2, \zeta^2 - \zeta^3, \zeta^3 - \zeta^4\}$ there exists $z' \in P$ such that z' is parallel to l_2 and $l_1 + z = l_1 + z'$. It follows that for any pair of intersecting lines $l_1, l_2 \in \mathcal{L}$ the intersection point $l_1 \cap l_2$ belongs to P. Consequently, a point $\xi \in \mathbb{C}$ belongs either to 0, 1, or to 5 lines from $\mathcal{L}$. If $\xi \in \mathbb{C}$ does not belong to any line $l \in \mathcal{L}$, then we call ξ *regular*.

Similarly to the case of interval exchange transformations, let us double each line $l \in \mathcal{L}$. Let $\mathcal{C}$ be the obtained space and let $Q : \mathcal{C} \longrightarrow \mathbb{C}$ be the corresponding quotient map. If $\xi \in \mathbb{C}$ is regular, then $Q^{-1}(\xi)$ consists of a single point. If $\xi \in \mathcal{C} \setminus P$ belongs to a line $l \in \mathcal{L}$, then $Q^{-1}(\xi)$ consists of two points associated with each of the two half-planes into which l separates $\mathbb{C}$. Every point $\xi \in P$ has 10 preimages in $\mathcal{C}$ associated with each of the ten sectors into which the lines from $\mathcal{L}$ passing through ξ separate the plane. A sequence ξ_n of points of $\mathcal{C}$ converges to a point $\xi \in \mathcal{C}$ if and only if the sequence $Q(\xi_n)$ converges to $Q(\xi)$ and the sequence ξ_n eventually belongs (if $Q(\xi)$ is not regular) to the associated closed half-plane or sector. The space $\mathcal{C}$ is locally compact and totally disconnected. Polygons with sides belonging to lines from $\mathcal{L}$ form a basis of topology of $\mathcal{C}$.

The group P acts on $\mathcal{C}$ in the natural way, so that the action is projected by Q to the action of P on $\mathbb{C}$ by translations. Therefore, sums of the form $\tilde{\xi} + a$, for $\tilde{\xi} \in \mathcal{C}$ and $a \in P$, are well defined.

Let us describe, following [3, 4], how a Penrose tiling is associated with a point $\tilde{\xi} \in \mathcal{C}$. We will usually denote $\xi = Q(\tilde{\xi})$. Suppose that ξ is regular. The vertices of the corresponding tiling $T_{\tilde{\xi}}$ will be the points of the form $\sum_{j=0}^4 k_j\zeta^j$, where $k_j \in \mathbb{Z}$ are such that

$$\left(\sum_{j=0}^4 k_j, \quad \sum_{j=0}^4 k_j\zeta^{2j} + \xi\right) \in \bigcup_{s=1}^4 (s, V_s),$$

where V_1 is the pentagon with vertices ζ^j, V_2 is the pentagon with vertices $\zeta^j + \zeta^{j+1}$, $V_3 = -V_2$, and $V_4 = -V_1$. (Note that we have changed ξ to $-\xi$ comparing with [3, 4].)

If ξ is singular, then we can find a sequence $\tilde{\xi}_n$ of regular points converging in $\mathcal{C}$ to $\tilde{\xi}$, and then the tiling $T_{\tilde{\xi}}$ is the limit of the tilings $T_{\tilde{\xi}_n}$.

Let $v = \sum_{j=0}^4 n_j\zeta^{2j} \in P$ and $v' = \sum_{j=0}^4 n_j\zeta^j$. Then $x \in T_{\tilde{\xi}}$ if and only if $x - v' \in T_{\tilde{\xi}+v}$. It follows that action of P on $\mathcal{C}$ preserves the associated tilings up to translations. In fact, it is not hard to show that two tilings $T_{\tilde{\xi}_1}$ and $T_{\tilde{\xi}_2}$ are translations of each other if and only if $\tilde{\xi}_1$ and $\tilde{\xi}_2$ belong to one P-orbit, see [3, 4].

Note that sides of the pentagons $V_s' = V_s - s$ are contained in lines from the collection $\mathcal{L}$, hence they are naturally identified with compact open subsets of $\mathcal{C}$. See Figure 4 for the pentagons V_s'. Denote by $V' = \bigcup_{s=1}^4 (s, V_s')$.

For every $(s, \tilde{\xi}) \in V'$ the point $s = s + 0 \cdot \zeta + 0 \cdot \zeta^2 + 0 \cdot \zeta^3 + 0 \cdot \zeta^4$ belongs to the tiling $T_{\tilde{\xi}}$. We say that the pointed tiling $(T_{\tilde{\xi}}, s)$ *corresponds* to the point $(s, \tilde{\xi}) \in V'$.

Let $x = \sum_{j=0}^4 k_j\zeta^j \in T_{\tilde{\xi}}$, and let $s = \sum_{j=0}^4 k_j$. Then the numbers $v' = x - s$ and $v = \sum_{j=0}^4 k_j\zeta^{2j} - s$ belong to P, and the map $y \mapsto y - v'$ is a bijection $T_{\tilde{\xi}} \longrightarrow T_{\tilde{\xi}+v}$.

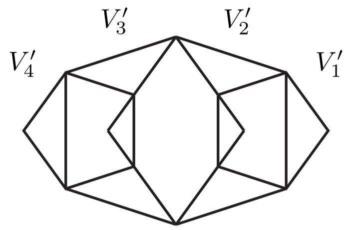

FIGURE 4. Pentagons V_s'

This maps moves x to the marked vertex $s = x - (x - s)$ of the tiling corresponding to $(s, \tilde{\xi} + v)$. Consequently, every pointed tiling, up to translation, corresponds to a point of V'. It is easy to see that every pointed Penrose tiling is represented by a unique point of V', so that we get a bijection between V' and the space $\mathcal{T}$. It follows from the results of [3, 4] that this bijection is a homeomorphism, and we get the following description of the action of the group $\mathcal{P}$ on $\mathcal{T}$.

Proposition 15. *The group $\mathcal{P}$ acts on $V' \cong \mathcal{T}$ locally by translations by elements of P. In other words, for every $\alpha \in \mathcal{P}$ there exists a partition of V' into disjoint clopen subsets (s_i, U_i) such that α acts on each of them by a translation $\alpha(s_i, x) = (s_i', x + \xi_i)$ for some $s_i' \in \{1, 2, 3, 4\}$ and $\xi_i \in P$.*

Let us find some elements $t_i \in P$ such that $V_s' + t_i$ are pairwise disjoint, and denote by $V'' \subset \mathcal{C}$ the union of the sets $V_s'' = V_s' + t_i$. Then it follows from Proposition 15 that $\mathcal{P}$ is the group of all transformations $V'' \longrightarrow V''$ that are locally equal to translations by elements of P.

Let us say that two clopen sets $U_1, U_2 \subset \mathcal{C}$ are *equidecomposable* if there exists a homeomorphism $\phi : U_1 \longrightarrow U_2$ locally equal to translations by elements of P. If U is any clopen subset which is equidecomposable with V'', then $\mathcal{P}$ is equal to the group of all transformations of U that are locally translations by elements of P.

Proposition 16. *The set V'' is equidecomposable with the parallelogram F with vertices $0, w_1 = \zeta^2 - \zeta^3, w_2 = 5(1 - \zeta^2 - \zeta^3 + \zeta^4), w_1 + w_2$.*

Proof. Let us cut the pentagons V_s'' into triangles as it is shown on Figure 5.

The obtained triangles can be grouped into pairs of triangles T, T' such that T' is obtained from T by a rotation by 2π (and translation). Such pairs can be put together to form parallelograms, as it is shown on Figure 6.

Figure 6 also shows that each such parallelogram is equidecomposable with its rotation by $\pi/5$. It follows that each parallelogram is equidecomposable with its rotation by any angle of the form $k\pi/5$. Consequently, every parallelogram formed by the acute-angled triangles is equidecomposable with the parallelogram with the set of vertices $\{0, \zeta^2 - \zeta^3, 1 - \zeta^2, 1 - \zeta^3\}$, and each parallelogram formed by the obtuse-angled triangles is equidecomposable with the parallelogram with the set of vertices $\{0, \zeta^2 - \zeta^3, \zeta^4 - \zeta^3, \zeta^2 - 2\zeta^3 + \zeta^4\}$. We get 5 parallelograms of each kind. We can put all the obtained parallelograms together to form the parallelogram F. $\square$

The parallelogram F, seen as a subset of $\mathcal{C}$, is the fundamental domain of the group $\langle w_1, w_2 \rangle < P$. It is easy to check that $P/\langle w_1, w_2 \rangle$ is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z}$. The space of orbits $\mathcal{C}/\langle w_1, w_2 \rangle$ is naturally homeomorphic to the parallelogram F

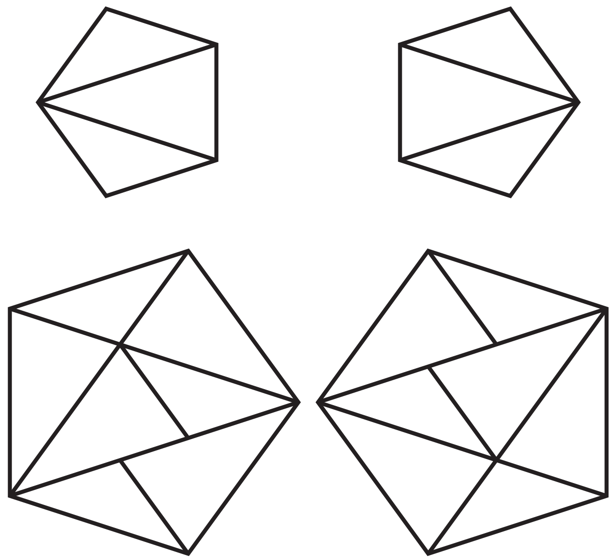

FIGURE 5. Cutting pentagons V_s into triangles

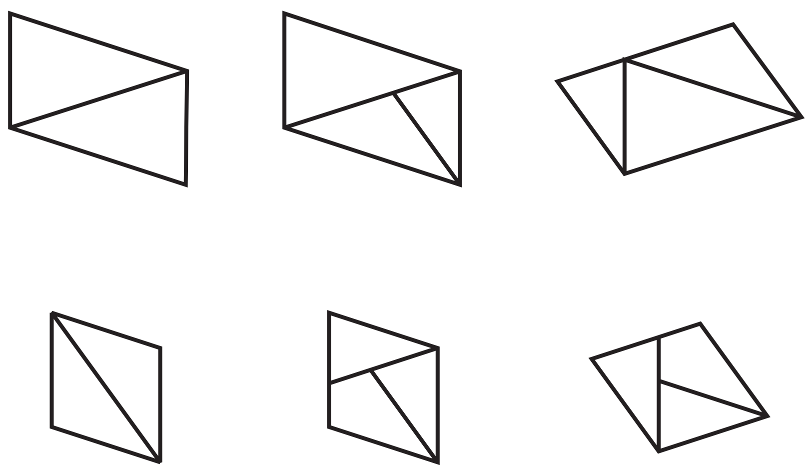

FIGURE 6. Equidecomposability of parallelograms

(and hence to the spaces V' and $\mathcal{T}$). We have then the following corollary of Propositions 15 and 16.

Corollary 17. *The group $\mathcal{P}$ is isomorphic to the full topological group of the action of $P/\langle w_1, w_2\rangle$ on the Cantor set $\mathcal{C}/\langle w_1, w_2\rangle$.*

Proposition 18. *The derived subgroup of $\mathcal{P}$ is simple and finitely generated.*

Proof. We can not apply Theorem 1 directly, since $P/\langle w_1, w_2\rangle$ is not a free abelian group. But it is easy to extend it to non-free finitely generated abelian groups. Instead of giving a general argument, let us show how this can be done in this particular example (the general case is analogous).

Let us cut the parallelogram $F \subset \mathcal{C}$ into five clopen parallelograms $F_k = F_0 + kw_2/5$, $k = 0, 1, \ldots, 4$, where F_0 is the parallelogram spanned by $w_1 = \zeta^2 - \zeta^3$ and $w_2/5 = 1 - \zeta^2 - \zeta^3 + \zeta^4$. Denote by $\mathcal{P}_l$ the subgroup of $\mathcal{P}$ consisting of the elements acting identically on all parallelograms F_k except for $k = l$. Then $\mathcal{P}_0$ is naturally identified with the full topological group of the action of $\mathcal{P}/\langle w_1, w_2/5\rangle \cong \mathbb{Z}^2$ acting on the Cantor set $\mathcal{C}/\langle w_1, w_2/5\rangle$. The action is minimal and is conjugate to the action of a $\mathbb{Z}^2$-shift, hence the derived subgroup of $\mathcal{P}_0$ is finitely generated, by Theorem 1. The groups $\mathcal{P}_l$ are just obtained by conjugating $\mathcal{P}_0$ by the parallel translations $z \mapsto z + lw_2/5$, hence the derived subgroups of $\mathcal{P}_l$ are also finitely generated.

Let us partition F_0 into two non-empty disjoint clopen sets A_0 and B_0. Let $A_k = A_0 + kw_2/5$ and $B_k = B_0 + kw_2/5$ be the corresponding partitions of F_k. Let S be the set of the elements of $\mathcal{P}$ equal to the length three cycles of the form $(A_{k_1}, A_{k_2}, A_{k_3})$ and $(B_{k_1}, B_{k_2}, B_{k_3})$, where the clopen sets are mapped to each other using the corresponding parallel translations by multiples of $w_2/5$. It follows from Lemma 5 that the group generated by $\bigcup_{k=0}^4 [\mathcal{P}_k, \mathcal{P}_k] \cup S$ contains restrictions of the elements of S to arbitrary clopen subsets of the parallelogram F. Then it is easy to see (for example, using Lemma 9) that the group generated by $\bigcup_{k=0}^4 [\mathcal{P}_k, \mathcal{P}_k] \cup S$ contains all elements of the form $T_{U,(g_1,g_2,g_3)}$. This implies, in the same way as in the proof of Theorem 1, that the derived subgroup of $\mathcal{P}$ is generated by the union of S with a finite generating set of the derived subgroups of $\mathcal{P}_0$. (Note that $\mathcal{P}_k$ is conjugate to $\mathcal{P}_0$ by the element $(A_0, A_k, A_l)(B_0, B_k, B_l)$, where $l \in \{1, 2, 3, 4\}$ is any index different from k.) □

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