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Locally connected Smale spaces, pinched spectrum, and infra-nilmanifolds

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ABSTRACT

We show that if $(\mathcal{X}, f)$ is a locally connected Smale space (e.g., a basic set of an Axiom-A diffeomorphism) such that the local product structure on $\mathcal{X}$ can be lifted by a covering with virtually nilpotent group of deck transformations to a global direct product, then $(\mathcal{X}, f)$ is topologically conjugate to a hyperbolic infra-nilmanifold automorphism. We use this result to give a generalization to Smale spaces of a theorem of M. Brin and A. Manning on Anosov diffeomorphisms with pinched spectrum, and to show that every locally connected codimension one Smale space is topologically conjugate to a hyperbolic automorphism of a torus.
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Contents

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1. Introduction

Smale spaces were introduced by D. Ruelle (see [33]) as generalizations of Anosov diffeomorphisms and basic sets of Axiom-A diffeomorphisms. They were also extensively studied before as "spaces with hyperbolic canonical coordinates" by R. Bowen [4,5].

A Smale space is a compact metric space $\mathcal{X}$ with a homeomorphism $f : \mathcal{X} \longrightarrow \mathcal{X}$ such that there exists a local direct product structure on $\mathcal{X}$ with respect to which f is expanding in one and contracting in the other direction. For example, every Anosov diffeomorphism of a compact manifold is a Smale space. Restrictions of Axiom-A homeomorphisms to the basic sets are also examples of Smale spaces. For more on Smale spaces, see [33,30,31].

Smale spaces are classical objects of the theory of dynamical systems, but many basic questions about them (and even about Anosov diffeomorphisms) remain to be open.

For example, it seems that the following question is open.

Question 1.1. Is it true that if $(\mathcal{X}, f)$ is a Smale space such that $\mathcal{X}$ is connected and locally connected, then $(\mathcal{X}, f)$ is topologically conjugate to an Anosov diffeomorphism?

Many well studied examples of Smale spaces are such that one or both of the factors of the local direct product structure are totally disconnected, e.g., the shifts of finite type, the Smale solenoid (see [10, Section 1.9]), Williams attractors, etc. See more examples in [38,2]. See also [37], where it is proved that all such Smale spaces are inverse limits of iterations of one self-map, i.e., are natural generalizations of solenoids.

Note that a question similar to Question 1.1 for expanding maps has a positive answer. Namely, the following theorem is proved in [24, Theorem 6.1.6] and [25, Theorem 5.9] using Gromov's theorem on groups of polynomial growth [17].

Theorem 1.2. *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a self-covering map of a locally connected and connected compact metric space. Suppose that there exists a covering map $\pi : \widetilde{\mathcal{X}} \longrightarrow \mathcal{X}$ such that f*

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can be lifted by π to an expanding homeomorphism of $\widetilde{\mathcal{X}}$. (This is true, for example, if $f : \mathcal{X} \longrightarrow \mathcal{X}$ is locally expanding and $\mathcal{X}$ is semi-locally simply connected.) Then $(\mathcal{X}, f)$ is topologically conjugate to an expanding infra-nilmanifold endomorphism.

Here an *infra-nilmanifold endomorphism* is a map $\phi : G\backslash L \longrightarrow G\backslash L$, where L is a simply connected nilpotent Lie group, G is a subgroup of the affine group $\mathrm{Aut}(L) \ltimes L$ acting on L freely, properly, and co-compactly, and ϕ is induced by an automorphism $\Phi : L \longrightarrow L$. If the automorphism Φ is expanding, i.e., if all eigenvalues of $D\Phi$ have absolute value greater than one, then we say that the corresponding endomorphism ϕ is *expanding*. If $D\Phi$ has no eigenvalues of absolute value one, then we say that ϕ is *hyperbolic*.

Note, that the case when $f : \mathcal{X} \longrightarrow \mathcal{X}$ in Theorem 1.2 is an expanding endomorphism of a Riemannian manifold, is a result of M. Gromov [17] (based on results of M. Shub [34]).

All known examples of Anosov diffeomorphisms, and hence apparently all known examples of locally connected Smale spaces are hyperbolic automorphisms of infra-nilmanifolds. See [35], and Problem 3 in the additional list of problems in [36]. It was proved by A. Manning in [22] that every Anosov diffeomorphism of an infra-nilmanifold is topologically conjugate to a hyperbolic automorphism of an infra-nilmanifold. Another result in this direction is a theorem of J. Franks [14] and S.E. Newhouse [28] stating that if $(\mathcal{X}, f)$ is an Anosov diffeomorphism such that stable or unstable manifolds of $\mathcal{X}$ are one-dimensional, then $(\mathcal{X}, f)$ is topologically conjugate to a hyperbolic linear automorphism of the torus $\mathbb{R}^n/\mathbb{Z}^n$.

One of the main obstacles for proving that every Anosov diffeomorphism is an automorphism of an infra-nilmanifold is showing that the foliations of $\mathcal{X}$ into stable and unstable manifolds when lifted to the universal covering $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ come from a direct product decomposition of $\widetilde{\mathcal{X}}$.

Definition 1.3. We say that a Smale space $(\mathcal{X}, f)$ is *splittable* if there exists a covering map $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ and a direct product decomposition $\mathcal{M} = A \times B$ of $\mathcal{M}$ such that π maps plaques $\{a\} \times B$ and $A \times \{b\}$ of the direct product decomposition of $\mathcal{M}$ bijectively to stable and unstable leaves of $\mathcal{X}$.

Here a *stable* (resp. *unstable*) *leaf* of $\mathcal{X}$ is the equivalence class with respect to the equivalence relation $\lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0$ (resp. $\lim_{n\to-\infty} d(f^n(x), f^n(y))$).

If $\mathcal{X}$ is locally connected and connected, then every splitting is a Galois covering with a finitely generated group of deck transformations, see Proposition 5.5.

One of the main results of our paper is the following theorem.

Theorem 1.4. *Let $(\mathcal{X}, f)$ be a Smale space such that $\mathcal{X}$ is connected and locally connected. Suppose that it has a splitting with a virtually nilpotent group of deck transformations. Then $(\mathcal{X}, f)$ is topologically conjugate to a hyperbolic infra-nilmanifold automorphism.*

Note that we do not assume in Theorem 1.4 that $\mathcal{X}$ is even locally simply connected. On the other hand, when restricted to the class of Anosov diffeomorphisms, it is a weaker statement than the result of [22]. It is not clear what should be the statement generalizing A. Manning's result in the class of locally connected Smale spaces.

M. Brin in [6,7] gave a "pinching" condition on the Mather spectrum of an Anosov diffeomorphism $(\mathcal{X}, f)$ (i.e., spectrum of the operator induced by f on the Banach space of vector fields on $\mathcal{X}$) ensuring that $(\mathcal{X}, f)$ has a splitting with a virtually nilpotent group of deck transformations. In the case of Anosov diffeomorphisms the splitting map $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ is necessarily the universal covering map, so that the group of deck transformations is the fundamental group of $\mathcal{X}$. M. Brin and A. Manning proved then in [9] that all Anosov diffeomorphisms satisfying the Brin's pinching condition are hyperbolic automorphisms of infra-nilmanifolds.

We generalize the results of M. Brin and A. Manning. Of course, we can not use the original pinching condition, since we do not have vector fields on Smale spaces. We find, however, a purely topological condition, which follows from Brin's condition in the case of Anosov diffeomorphism. In fact, we even improve the Brin's spectral pinching condition for Anosov diffeomorphisms.

Here is an informal description of our condition. Consider a finite covering $\mathcal{R}$ of $\mathcal{X}$ by sufficiently small open rectangles (i.e., such that their diameters are smaller than the expansivity constant). The covering will induce coverings of the stable and unstable leaves by the plaques of the elements of $\mathcal{R}$. Define, for an stable leaf V and $x, y \in V$, the combinatorial distance $d_{\mathcal{R}}(x, y)$ equal to the smallest length m of a chain $x \in R_0, R_1, \ldots, R_m \ni y$, $R_i \cap R_{i+1} \neq \emptyset$, of plaques of the elements of $\mathcal{R}$ (which can be infinite). Then $d_{\mathcal{R}}(f^{-n}(x), f^{-n}(y))$ grows exponentially for $x \neq y$, if it is finite. We say that $\alpha_0 > 0$ and $\alpha_1 > 0$ are *stable lower* and *upper exponents* if there exists $C > 1$ such that

$$C^{-1}e^{\alpha_0 n} \leq d_{\mathcal{R}}(f^{-n}(x), f^{-n}(y)) \leq Ce^{\alpha_1 n}$$

for all stably equivalent x, y such that the distance between x and y inside their stable leaf belongs to some fixed interval $[\epsilon_1, \epsilon_2]$ for $0 < \epsilon_1 < \epsilon_2$. *Stable upper and lower critical exponents* are the infimum and the supremum of all stable upper and lower exponents, respectively. We prove that the stable critical exponents are uniquely determined by the topological conjugacy class of the Smale space and are positive and finite (if the Smale space is locally connected and connected). The unstable upper and lower critical exponents are defined in the similar way (they are stable upper and lower critical exponents of $(\mathcal{X}, f^{-1})$). For more details, see Sections 3 and 4.

Note that if the stable or unstable leafs of $(\mathcal{X}, f)$ are not locally connected, then the corresponding upper exponents are infinite, since the metric $d_{\mathcal{R}}(x, y)$ will be infinite for some x, y (as we assume that the covering $\mathcal{R}$ consists of small rectangles).

Theorem 1.5. *Suppose that* $(\mathcal{X}, f)$ *is a Smale space such that* $\mathcal{X}$ *is connected and locally connected. Let* a_0, a_1, b_0, b_1 *be the stable lower and upper, and the unstable lower and upper critical exponents, respectively. If*

$$\frac{a_0}{a_1} + \frac{b_0}{b_1} > 1$$

then $(\mathcal{X}, f)$ *is topologically conjugate to a hyperbolic infra-nilmanifold automorphism.*

We show that the Brin's pinching condition on the Mather spectrum of an Anosov diffeomorphism implies our condition on the critical exponents.

As another application of Theorem 1.4, we show that the theorem of J. Franks and S.E. Newhouse on co-dimension one Anosov diffeomorphisms is true for all locally connected Smale spaces.

Theorem 1.6. *Let* $(\mathcal{X}, f)$ *be a Smale space such that* $\mathcal{X}$ *is connected and locally connected, and either stable or unstable leaves of* $(\mathcal{X}, f)$ *are homeomorphic (with respect to their intrinsic topology) to* $\mathbb{R}$*. Then* $(\mathcal{X}, f)$ *is topologically conjugate to a hyperbolic linear automorphism of a torus* $\mathbb{R}^n/\mathbb{Z}^n$*.*

Here intrinsic topology of a leaf is the direct limit topology coming from decomposition of a leaf into the union of plaques of rectangles of $\mathcal{X}$.

Theorem 1.6, for example, rules out basic sets of Axiom-A diffeomorphisms such that the stable leaves are homeomophic to $\mathbb{R}$, while the unstable leaves are locally connected but not homeomorphic to manifolds (e.g., are locally homeomorphic to the Sierpinski carpet).

Remark. A more general notion of an endomorphism of an infra-nilmanifold is discussed in [13,12]. It is also noted there that some of the results of [14] and [34] are based on a false result. The proof of Theorem 1.4 shows that it is enough to consider the narrower notion of an automorphism of an infra-nilmanifold in the classification of Smale spaces and Anosov diffeomorphisms up to topological conjugacy. We do not use the results of [14] (except for his proof of Theorem 2.2, which we repeat for our setting). The results of [34] are not used in the proof of Theorem 1.2, where also the narrower notion of an endomorphism of an infra-nilmanifold is used, see [24, Theorem 6.1.6].

Structure of the paper In Section 2, we collect basic facts and definitions related to Smale spaces, and fix the related notations.

We study lower exponents of a Smale space, and a family of metrics associated with lower exponents in Section 3. We also recall their properties of the SRB measures on leaves of Smale spaces.

Locally connected Smale spaces are studied in Section 4. We show that the following conditions for a Smale space $(\mathcal{X}, f)$ are equivalent (see Theorem 4.1):

(1) The space $\mathcal{X}$ is locally connected.
(2) All stable and unstable leaves of $\mathcal{X}$ are locally connected.
(3) All stable and unstable leaves of $\mathcal{X}$ are connected.
(4) $\mathcal{X}$ has finite stable and unstable upper exponents.

In Section 5 we study splittings of locally connected Smale spaces. We show that for any splitting $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ of a locally connected Smale space there exists a well defined group of deck transformations G, that G is finitely generated, and that there exists a lift $F : \mathcal{M} \longrightarrow \mathcal{M}$ of f to $\mathcal{M}$, which is unique up to compositions with elements of G.

The lift F defines then, for any point $x_0 \in \mathcal{M}$, an automorphism ϕ of G by the rule $F(g(x_0)) = \phi(g)(F(x_0))$.

Question 1.7. Does the pair (G, ϕ) uniquely determine the topological conjugacy class of $(\mathcal{X}, f)$?

We do not know the answer to this question, but we show that we can reconstruct $(\mathcal{X}, f)$ after adding an extra piece of information to (G, ϕ).

Definition 1.8. Let $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ be a splitting of a locally connected and connected Smale space $(\mathcal{X}, f)$. Let W_+ and W_- be stable and unstable plaques of a fixed point x_0 of a lift of f. We say that $\Sigma_+, \Sigma_- \subset G$ are *coarse stable and unstable plaques* if the Hausdorff distances between $\Sigma_+(x_0)$ and W_+ and between $\Sigma_-(x_0)$ and W_- are finite.

Here the distance in $\mathcal{M}$ is measured with respect to a G-invariant metric.

Theorem 1.9. *The quadruple $(G, \phi, \Sigma_+, \Sigma_-)$ uniquely determines the topological conjugacy class of $(\mathcal{X}, f)$.*

We prove Theorem 1.9 by representing W_+ and W_- as boundaries of Gromov hyperbolic graphs constructed using the quadruple. These graphs are quasi-isometric to Cayley graphs of the *Ruelle groupoids* associated with the Smale space. A general theory of Cayley graphs of *hyperbolic groupoids* is developed in [27]. We hope that these new techniques will be helpful in future studies of hyperbolic dynamics.

We get the following corollary of Theorem 1.9.

Theorem 1.10. *Let $(\mathcal{X}_i, f_i)$ for $i = 1, 2$ be connected and locally connected Smale spaces. Let $\pi_i : \mathcal{M}_i \longrightarrow \mathcal{X}_i$ be splittings, and let $F_i : \mathcal{M}_i \longrightarrow \mathcal{M}_i$ be lifts of f_i. Suppose that F_i have fixed points, and that the groups of deck transformations of π_i are both isomorphic to a group G. If there exists a continuous map $\Phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ such that $\Phi(g(x)) = g(\Phi(x))$ and $\Phi(F_1(x)) = F_2(\Phi(x))$ for all $x \in \mathcal{M}_1$ and $g \in G$, then the Smale spaces $(\mathcal{X}_1, f_1)$ and $(\mathcal{X}_2, f_2)$ are topologically conjugate.*

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Section 6 is devoted to the proof of Theorem 1.4. As the first step we prove the following.

Proposition 1.11. *Let $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ be a splitting of a locally connected and connected Smale space $(\mathcal{X}, f)$ such that the group G of deck transformations is torsion free nilpotent. Then f has a fixed point, and the associated automorphism $\phi : G \longrightarrow G$ is hyperbolic (i.e., its unique extension Φ to a simply connected nilpotent Lie group containing G as a lattice is hyperbolic).*

We prove both statements of Proposition 1.11 by induction on the nilpotency class of G. We show at first that the automorphism ϕ induces a hyperbolic automorphism of the center $Z(G) \cong \mathbb{Z}^n$ of G. Then we construct an action of $\mathbb{R}^n$ on $\mathcal{M}$ naturally extending the action of $\mathbb{Z}^n$, using the direct product structure on $\mathcal{M}$. The action induces an action of the torus $\mathbb{R}^n/\mathbb{Z}^n$ on $\mathcal{X}$ and agrees with the local product structure, metric on $\mathcal{X}$, and the dynamics, in such a way that the map induced by f on $(\mathbb{R}^n/\mathbb{Z}^n)\backslash\mathcal{X}$ is a Smale space with a splitting with the group of deck transformations isomorphic to $G/Z(G)$. This provides us the necessary inductive steps to prove Proposition 1.11. We use after that the arguments of [14, Theorem 2.2], Theorem 1.10, and some additional algebraic arguments to prove Theorem 1.4.

Theorem 1.5 generalizing the Brin's pinching condition to Smale spaces is proved in Section 7. We prove at first that every Smale space satisfying conditions of Theorem 1.5 has a splitting (Theorem 7.2). Then we prove that the group of deck transformations of the splitting is virtually nilpotent (Theorem 7.3) using Gromov's theorem on groups of polynomial growth. Both proofs are similar to the original proofs of M. Brin, except that in the proof of Theorem 7.2 we use results of Section 3 on lower exponents of a Smale space, which allows us to get a better pinching condition, and to prove the theorem for all locally connected Smale space, and not only for Anosov diffeomorphisms.

In Section 8, we show how our condition on critical exponents is related to M. Brin's pinching condition on the Mather spectrum of a diffeomorphism. We show that M. Brin's condition implies the condition of Theorem 1.5.

Section 9 is devoted to the proof Theorem 1.6 on co-dimension one Smale spaces. We prove it using Theorem 1.4 and the ideas of the proof of Theorem 1.5.

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2. Smale spaces

2.1. Local product structures

Definition 2.1. A *direct product structure* on a topological space R is defined by a continuous map $[\cdot,\cdot] : R \times R \longrightarrow R$ satisfying

(1) $[x, x] = x$ for all $x \in R$;
(2) $[[x, y], z] = [x, z]$ and $[x, [y, z]] = [x, z]$ for all $x, y, z \in R$.

We call a space with a direct product structure on it a *rectangle*.

If $R = A \times B$ is a decomposition of R into a direct product of two topological spaces, then the corresponding direct product structure is given by the operation

$$[(x_1, y_1), (x_2, y_2)] = (x_1, y_2). \tag{1}$$

Let R be a rectangle. For $x \in R$ the corresponding *plaques* are the sets

$$P_1(R, x) = \{y \in R \,:\, [x, y] = x\}, \qquad P_2(R, x) = \{y \in R \,:\, [x, y] = y\}. \tag{2}$$

See Fig. 1. If $R = A \times B$ with the corresponding direct product structure (1), then the plaques are given by

$$P_1(R, (a, b)) = A \times \{b\}, \qquad P_2(R, (a, b)) = \{a\} \times B.$$

The map $P_1(R, x) \times P_2(R, x) \longrightarrow R$ given by

$$(y_1, y_2) \mapsto [y_1, y_2]$$

is a homeomorphism.

For any pair $x, y \in R$ the natural maps $P_1(R, x) \longrightarrow P_1(R, y)$ and $P_2(R, x) \longrightarrow P_2(R, y)$ given by $z \mapsto [y, z]$ and $z \mapsto [z, y]$, respectively, are called *holonomy maps* inside R, and are homeomorphisms.

These homeomorphism agree with the homeomorphisms $P_1(R, x) \times P_2(R, x) \longrightarrow R$, so that we get a canonical decomposition of R into the direct product of two spaces A and B, which can be identified with $P_1(R, x)$ and $P_2(R, x)$, respectively.

Definition 2.2. Let $\mathcal{X}$ be a topological space. A *local product structure* on $\mathcal{X}$ is given by a covering $\mathcal{R}$ of $\mathcal{X}$ by open sets R with a direct product structure $[\cdot,\cdot]_R$ on each of them, such that for any $R_1, R_2 \in \mathcal{R}$, and for every $x \in \mathcal{X}$ there exists a neighborhood U of x such that $[y_1, y_2]_{R_1} = [y_1, y_2]_{R_2}$ for all $y_1, y_2 \in U \cap R_1 \cap R_2$.

Two coverings of $\mathcal{X}$ by open rectangles define the same local product structures if their union defines a local product structure, i.e., satisfied the above compatibility condition.

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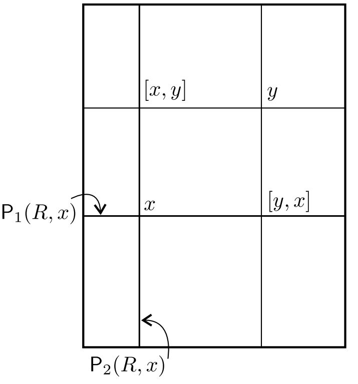

Fig. 1. Rectangle.

If $\mathcal{X}$ is a space with a local direct product structure, then an open subset $R \subset \mathcal{X}$ with a direct product structure $[\cdot, \cdot]$ is a *(sub-)rectangle* of $\mathcal{X}$ if the union of $\{R\}$ with a covering defining the local product structure satisfies the compatibility conditions of Definition 2.2.

Definition 2.3. We say that a continuous map $f : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ between spaces with local product structures *preserves the local product structures* if every point of $\mathcal{X}_1$ has a rectangular neighborhood U such that $f(U)$ is a rectangle of $\mathcal{X}_2$, and $f([x, y]_U) = [f(x), f(y)]_{f(U)}$ for all $x, y \in U$.

Definition 2.4. Let $\mathcal{X}$ be a space with a local product structure. We say that a metric d on $\mathcal{X}$ *agrees with the local product structure* if for every point $x \in \mathcal{X}$ there exists an open rectangular neighborhood $R = A \times B$ of x and metrics d_A and d_B on A and B, respectively, such that the restriction of d to R is bi-Lipschitz equivalent to the metric

$$d_R((x_1, y_1), (x_2, y_2)) = d_A(x_1, x_2) + d_B(y_1, y_2).$$

If a metric d agrees with the local product structure, then for every point $x \in \mathcal{X}$ there exists a rectangular neighborhood R of x such that all holonomy maps inside R are bi-Lipschitz with respect to the metric d with a fixed Lipschitz constant (depending only on R). Conversely, it is easy to see that a metric d agrees with the local product structure if for every $x \in \mathcal{X}$ there exists a rectangular neighborhood R of x such that the holonomies inside R are uniformly bi-Lipschitz, and $d(y, z)$ is bi-Lipschitz equivalent to $d([x, y], [x, z]) + d([y, x], [z, x])$.

2.2. Smale spaces

Definition 2.5. A *Smale space* is a compact metrizable space $\mathcal{X}$ together with a homeomorphism $f : \mathcal{X} \longrightarrow \mathcal{X}$ such that there exists a metric d on $\mathcal{X}$, constants $\lambda \in (0, 1)$

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and $C > 0$, and a local product structure on $\mathcal{X}$ such that f preserves the local product structure and for every $x \in \mathcal{X}$ there exists a rectangular neighborhood R of x such that for all $n \geq 0$ and $y, z \in P_1(R, x)$ we have

$$d(f^n(y), f^n(z)) \leq C\lambda^n d(y, z),$$

and for all $n \geq 0$ and $y, z \in P_2(R, x)$ we have

$$d(f^{-n}(y), f^{-n}(z) \leq C\lambda^n d(y, z).$$

We will denote $P_1(R, x) = P_+(R, x)$ and $P_2(R, x) = P_-(R, x)$.

Examples of Smale spaces are Anosov diffeomorphisms of *compact* manifolds, restrictions of Axiom-A diffeomorphisms to their basic sets, shifts of finite type, spaces of substitutional tilings, etc. See [31] for more examples. Note that pseudo-Anosov diffeomorphisms are not Smale spaces.

Definition 2.6. A homeomorphism $f : \mathcal{X} \longrightarrow \mathcal{X}$ of a compact space $\mathcal{X}$ is said to be *expansive* if there exists a neighborhood U of the diagonal in $\mathcal{X} \times \mathcal{X}$ such that $(f^n(x), f^n(y)) \in U$ for all $n \in \mathbb{Z}$ implies $x = y$.

Note that if U satisfies the conditions of the definition, then $\{(x, y) \in \mathcal{X}^2 : (x, y), (y, x) \in U\}$ also satisfies the conditions of the definition. Consequently, we may assume that U is symmetric.

Proposition 2.7. *Every Smale space is an expansive dynamical system.*

Proof. We can find a finite covering $\mathcal{R}$ of $\mathcal{X}$ by rectangles satisfying the conditions of Definition 2.5. Let $\epsilon > 0$ be a Lebesgue's number of the covering. There exists $\delta > 0$ such that for any points $x, y \in \mathcal{X}$ such that $d(x, y) < \delta$ and any rectangle $R \in \mathcal{R}$ such that $x, y \in R$ we have $d(x, [x, y]) < C^{-1}\epsilon$ and $d(y, [x, y]) < C^{-1}\epsilon$.

Let $x, y \in \mathcal{X}$ be such that $d(f^n(x), f^n(y)) < \delta$ for all $n \in \mathbb{Z}$. Then for every $n \in \mathbb{Z}$ there exists a rectangle $R_n \in \mathcal{R}$ such that $f^n(x), f^n(y) \in R_n$. Then $d(f^n(x), [f^n(x), f^n(y)]) < C^{-1}\epsilon$. Note that $f^n(x)$ and $[f^n(x), f^n(y)]$ belong to one plaque $P_-(R_n, f^n(x))$. It follows that $d(f^{n-k}(x), [f^{n-k}(x), f^{n-k}(y)]) \leq C\lambda^k d(f^n(x), [f^n(x), f^n(y)]) < C\lambda^k C^{-1}\epsilon = \lambda^k\epsilon$ for all $k \geq 0$ and all $n \in \mathbb{Z}$. In particular, $d(x, [x, y]) < \lambda^k\epsilon$ for all $k \geq 0$, i.e., $x = [x, y]$. It is shown in the same way that $y = [x, y]$, which implies that $x = y$. Therefore, the set $U \subset \mathcal{X} \times \mathcal{X}$ equal to the set of pairs (x, y) such that $d(x, y) < \delta$ satisfies the conditions of Definition 2.6. □

Definition 2.8. A *log-scale* on a set X is a function $\ell : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R} \cup \{\infty\}$ satisfying the following conditions:

(1) $\ell(x, y) = \ell(y, x)$ for all $x, y \in X$;

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(2) $\ell(x, y) = \infty$ if and only if $x = y$;
(3) there exists $\Delta > 0$ such that

$$\ell(x, z) \geq \min\{\ell(x, y), \ell(y, z)\} - \Delta$$

for all $x, y, z \in X$.

We say that two log-scales ℓ_1, ℓ_2 are *bi-Lipschitz equivalent* if the difference $|\ell_1(x, y) - \ell_2(x, y)|$ is uniformly bounded for all $x \neq y$.

Let us describe the natural class of metrics on expansive dynamical systems defined in [15], using log-scales.

Let $(\mathcal{X}, f)$ be an expansive dynamical system. Let U be a symmetric neighborhood of the diagonal, satisfying the conditions of Definition 2.6. Define $\ell(x, y)$ for $x, y \in \mathcal{X}$ to be maximal n such that $(f^k(x), f^k(y)) \in U$ for all $k \in [-n, n]$.

Lemma 2.9. *The defined function ℓ is a log-scale. It does not depend, up to bi-Lipschitz equivalence, on the choice of U.*

We call ℓ *the standard log-scale* of the expansive dynamical system.

Proof. We have $\ell(x, y) = \ell(y, x)$, since we assume that U is symmetric. We also have $\ell(x, y) = \infty$ if and only if $x = y$, by Definition 2.6.

It remains to show that there exists Δ such that $\ell(x, z) \geq \min\{\ell(x, y), \ell(y, z)\} - \Delta$ for all $x, y, z \in \mathcal{X}$.

Since a compact set has a unique uniform structure consisting of all neighborhoods of the diagonal (see [3]), there exists a neighborhood of the diagonal $V \subset \mathcal{X}^2$ such that $(x, y), (y, z) \in V$ implies $(x, z) \in U$.

Note that the sets $U_n = \{(x, y) : \ell(x, y) \geq n\} = \bigcap_{k=-n}^{n} f^k(U)$ are neighborhoods of the diagonal, $U_{n+1} \subseteq U_n$ for all n, and $\bigcap_{n \geq 1} U_n$ is equal to the diagonal. In particular, by compactness of $\mathcal{X}$, there exists $\Delta > 0$ such that $U_\Delta \subset V$.

Denote by $V_n = \bigcap_{k=-n}^{n} f^k(V)$ the set of pairs (x, y) such that $(f^k(x), f^k(y)) \in V$ for all $k = -n, \ldots, n$. Then $(x, y), (y, z) \in V_n$ implies $(x, z) \in U_n$.

Then for every $n > \Delta$ we have $U_n \subset V_{n-\Delta}$, since the conditions that $(f^k(x), f^k(y)) \in U$ for all $|k| \leq n$ implies $(f^k(x), f^k(y)) \in U_\Delta \subset V$ for all $|k| \leq n - \Delta$.

Let $\min\{\ell(x, y), \ell(y, z)\} = m$. Then $(x, y), (y, z) \in U_m \subset V_{m-\Delta}$, hence $(x, z) \in U_{m-\Delta}$, i.e., $\ell(x, z) \geq m - \Delta$.

Let us show that ℓ does not depend on the choice of U. Let U' and U'' be two neighborhoods of the diagonal, satisfying the conditions of Definition 2.6. Then, as above, there exists $C > 0$ such that $U'_C \subset U''$ and $U''_C \subset U'$. By the same arguments as above, we conclude that $U'_{n+C} \subset U''_n$ and $U''_{n+C} \subset U'_n$ for all $n \geq 0$. But this implies that the values of the log-scales defined by U' and U'' differ from each other not more than by C. $\square$

It is proved in [27, Lemma 5.4.2] that for every Smale space $(\mathcal{X}, f)$ the log-scale ℓ agrees with the local product structure on $\mathcal{X}$.

Definition 2.10. Let $(\mathcal{X}, f)$ be an expansive dynamical system, and let ℓ be the standard log-scale. We say that $x, y \in \mathcal{X}$ are *stably equivalent* (denoted $x \sim_+ y$) if $\ell(f^n(x), f^n(y)) \to +\infty$ as $n \to +\infty$. They are *unstably equivalent* (denoted $x \sim_- y$) if $\ell(f^{-n}(x), f^{-n}(y)) \to +\infty$ as $n \to +\infty$. We call stable and unstable equivalence classes *stable and unstable leaves*.

Note that $\ell(x_n, y_n) \to \infty$ is equivalent to $d(x_n, y_n) \to 0$ for any pair of sequences $x_n, y_n \in \mathcal{X}$ and for any metric d on $\mathcal{X}$.

Two points $x, y \in \mathcal{X}$ are stably equivalent if and only if $(f^n(x), f^n(y)) \in U$ for all n big enough. Denote, for $x \in \mathcal{X}$ and $n \in \mathbb{Z}$, by $W_{n,+}(x)$ the set of points $y \in \mathcal{X}$ such that $(f^k(x), f^k(y)) \in U$ for all $k \geq -n$. Similarly, we denote by $W_{n,-}(x)$ the set of points $y \in \mathcal{X}$ such that $(f^k(x), f^k(y)) \in U$ for all $k \leq n$.

Then $W_{n,+}(x)$ and $W_{n,-}(x)$ are decreasing sequences of sets, and $W_+(x) = \bigcup_{n\in\mathbb{N}} W_{-n,+}(x)$ and $W_-(x) = \bigcup_{n\in\mathbb{N}} W_{-n,-}(x)$ are equal to the stable and unstable leaves of x, respectively.

Note that for all $n \in \mathbb{N}$, $x \in \mathcal{X}$, $* \in \{+, -\}$, and $y_1, y_2 \in W_{n,*}(x)$ we have $\ell(y_1, y_2) \geq n$.

If W is a stable leaf, then we denote, for $y_1, y_2 \in W$, by $\ell_+(y_1, y_2)$ or $\ell_W(y_1, y_2)$ the biggest n_0 such that $(f^n(y_1), f^n(y_2)) \in U$ for all $n \geq -n_0$.

The following properties of ℓ_+ follow directly from the definitions.

- $\ell(y_1, y_2) \leq \ell_+(y_1, y_2)$ for all stably equivalent y_1, y_2;
- if y_1, y_2 are stably equivalent and $\ell_+(y_1, y_2) > 0$, then $\ell(y_1, y_2) = \ell_+(y_1, y_2)$;
- for all stably equivalent y_1, y_2 we have

$$\ell_+(f(y_1), f(y_2)) = \ell_+(y_1, y_2) + 1. \tag{3}$$

Similarly, if W is an unstable leaf, then $\ell_-(y_1, y_2) = \ell_W(y_1, y_2)$, for $y_1, y_2 \in W$, is the biggest n_0 such that $(f^n(y_1), f^n(y_2)) \in U$ for all $n \leq n_0$. We also have $\ell(y_1, y_1) \leq \ell_-(y_1, y_2)$, $\ell(y_1, y_2) = \ell_-(y_1, y_2)$ if $\ell_-(y_1, y_2) > 0$, and

$$\ell_-(f(y_1), f(y_2)) = \ell_-(y_1, y_2) - 1 \tag{4}$$

for all pairs y_1, y_2 of unstably equivalent points.

Lemma 2.11. *Let W be a stable or unstable leaf. Then the corresponding function ℓ_+ or ℓ_- is a log-scale on W.*

Proof. If $\ell_+(x, y), \ell_+(y, z), \ell_+(x, z)$ are all positive, then they are equal to the corresponding values of ℓ, hence, by Lemma 2.9, $\ell_+(x, z) \geq \min(\ell_+(x, y), \ell_+(y, z)) - \Delta$, for Δ not depending on x, y, z.

If they are not positive, then we can find $n \geq 0$ such that $\ell_+(f^n(x), f^n(y)) = \ell_+(x, y) + n$, $\ell_+(f^n(y), f^n(z)) = \ell_+(y, z) + n$, and $\ell_+(f^n(x), f^n(z)) = \ell_+(x, z) + n$ are positive, and applying the above argument for $f^n(x), f^n(y)$, and $f^n(z)$ conclude that $\ell_+(x, z) + n \geq \min(\ell_+(x, y) + n, \ell_+(y, z) + n) - \Delta$, which is equivalent to $\ell_+(x, z) \geq \min(\ell_+(x, y), \ell_+(y, z)) - \Delta$. $\square$

We call the log-scales ℓ_+ and ℓ_- the *internal* log-scales on the respective leaf.

The *internal topology* on a leaf is the topology defined by the corresponding log-scale ℓ_+ or ℓ_-. Here topology defined by a log-scale ℓ on a set X is given by the basis $B(n, x) = \{y \in X \ : \ \ell(x, y) \geq n\}$ of neighborhoods of points $x \in X$. Note that $B(n, x)$ is not necessarily open or closed.

Equivalently, the internal topology of a leaf W is equal to the direct limit topology of representation of W as the union of the sequence $W_{-n,*}(x)$ for $n \in \mathbb{N}$ and $x \in W$.

Note also that leaves of a Smale space are locally compact, since neighborhoods of points of a leaf are continuous images of neighborhoods of points of $\mathcal{X}$.

2.3. Irreducible Smale spaces

Let $(\mathcal{X}, f)$ be a Smale space. A point $x \in \mathcal{X}$ is said to be *non-wandering* if for every neighborhood U of x there exists a positive integer n such that $f^n(U) \cap U \neq \emptyset$. The set of non-wandering points is obviously f-invariant and closed.

We say that $(\mathcal{X}, f)$ is *irreducible* if for every pair of open sets $U, V \subset \mathcal{X}$ there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$. We say that it is *mixing* if for every pair of open sets $U, V \subset \mathcal{X}$ there exists N such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

Smale spaces were introduced by D. Ruelle [33] as purely topological generalizations of basic sets of Axiom-A diffeomorphisms. Note, however, that general Smale spaces may have non-empty wandering sets. For example, every shift of finite type is a Smale space, but shifts of finite type may have wandering points.

On the other hand, the Smale's Spectral Decomposition Theorem [35, Theorems 6.2, 6.6] holds for Smale spaces.

Theorem 2.12. *Let $(\mathcal{X}, f)$ be a Smale space. Then the dynamical system $(NW(\mathcal{X}), f)$, where $NW(\mathcal{X})$ is the set of non-wandering points, is a Smale space. The set $NW(\mathcal{X})$ can be decomposed into a finite disjoint union of closed f-invariant sets $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ such that $(\mathcal{X}_i, f)$ is an irreducible Smale space without wandering points.*

We write $\mathcal{X}_i \prec \mathcal{X}_j$ if there exists a wandering point $x \in \mathcal{X}$ such that the set of the accumulation points of $f^n(x)$, $n \geq 0$, is contained in $\mathcal{X}_j$, and the set of the accumulation points of $f^n(x)$, $n \leq 0$, is contained in $\mathcal{X}_i$. Then $\prec$ is a partial order on the set $\{\mathcal{X}_1, \ldots, \mathcal{X}_n\}$.

The sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ are called *irreducible components* of the Smale space $(\mathcal{X}, f)$.

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A proof of the above theorem is similar to the proof of the classical Smale's spectral decomposition theorem, and can be found in [31]. We also have the following relation between the notions of an irreducible and mixing Smale spaces, see [31].

Theorem 2.13. *Suppose that $(\mathcal{X}, f)$ is an irreducible Smale space. Then $\mathcal{X}$ can be decomposed into a finite union $\mathcal{X} = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k$ of disjoint clopen sets cyclically permuted by f and such that (A_i, f^k) are mixing Smale spaces.*

3. Lower exponents

3.1. Lower exponents of log-scales

We say that d is a metric *associated* with a log-scale ℓ, if there exist constants $\alpha > 0$ and $C > 1$ such that

$$C^{-1} e^{-\alpha \ell(x,y)} \le d(x,y) \le C e^{-\alpha \ell(x,y)}.$$

The number α is called the *exponent* of the metric. Topology defined by an associated metric obviously coincides with the topology defined by the log-scale.

Note that if d is a metric associated with ℓ of exponent α, then for any $0 < r < 1$ the function $(d(x,y))^r$ is a metric associated with ℓ of exponent $r\alpha$. It follows that the set of exponents α for which there exists a metric associated with a given log-scale is an interval of the form $(0, \alpha_0)$ or $(0, \alpha_0]$, where $\alpha_0 \in [0, \infty]$. We will see later that $\alpha_0 > 0$ (see also [27]). The number α_0 is the *metric critical exponent* of the log-scale.

Let X be a set with a log-scale ℓ. Let Γ_n, for $n \in \mathbb{R}$, be the graph with the set of vertices X in which two points x, y are connected by an edge if and only if $\ell(x,y) \ge n$. Denote then by d_n the combinatorial distance in Γ_n (we assume that $d_n(x,y) = \infty$ if x and y belong to different connected components of Γ_n).

Proposition 3.1. *Let Δ be such as in Definition 2.8. There exist $C > 0$ such that*

$$d_n(x,y) \ge C e^{\alpha(n - \ell(x,y))}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, where $\alpha = \frac{\log 2}{\Delta}$.

Proof. If (x_0, x_1, x_2) is a path in Γ_n, then $\ell(x_0, x_2) \ge n - \Delta$, hence (x_0, x_2) is a path in $\Gamma_{n-\Delta}$. It follows that $d_{n-\Delta}(x,y) \le \frac{1}{2}(d_n(x,y) + 1)$. In other terms:

$$d_{n+\Delta}(x,y) \ge 2 d_n(x,y) - 1.$$

If $\ell(x,y) = m$, then $d_{m+1}(x,y) \ge 2$, and hence

$$d_{m+1+k\Delta}(x,y) \ge 2^{k+1} - 2^{k-1} - 2^{k-2} - \cdots - 1 = 2^k + 1$$

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Note that $d_m(x,y) \geq d_n(x,y)$ whenever $m \geq n$. It follows that for $k = \left\lfloor \frac{n-\ell(x,y)-1}{\Delta} \right\rfloor \geq \frac{n-\ell(x,y)-1}{\Delta} - 1$ we have

$$d_n(x,y) \geq d_{\ell(x,y)+1+k\Delta}(x,y) > 2^k.$$

Consequently,

$$d_n(x,y) \geq 2^{(n-\ell(x,y)-1-\Delta)/\Delta} = Ce^{\alpha(n-\ell(x,y))}$$

for all $x, y \in X$ and $n \in \mathbb{R}$, where $C = 2^{(-1-\Delta)/\Delta}$ and $\alpha = \frac{\ln 2}{\Delta}$. □

Definition 3.2. We say that α is a *lower exponent* of a log-scale ℓ if there exists $C > 0$ such that

$$d_n(x,y) \geq Ce^{\alpha(n-\ell(x,y))}$$

for all $x, y \in X$ and $n \in \mathbb{Z}$. The supremum of all lower exponents is called the *lower critical exponent.*

The proof of the following proposition is straightforward.

Proposition 3.3. *Let ℓ_1 and ℓ_2 be bi-Lipschitz equivalent log-scales on X. A number $\alpha > 0$ is a lower exponent of ℓ_1 if and only if it is a lower exponent of ℓ_2.*

Theorem 3.4. *The metric critical exponent of a log-scale ℓ is equal to its lower critical exponent. In particular, the metric critical exponent is positive.*

Proof. Let d be a metric on X of exponent α associated with ℓ, and let $C_1 > 1$ be such that

$$C_1^{-1}e^{-\alpha\ell(x,y)} \leq d(x,y) \leq C_1e^{-\alpha\ell(x,y)}$$

for all $x, y \in X$.

Then for every n the inequality $\ell(x,y) \geq n$ implies $d(x,y) \leq C_1e^{-\alpha n}$, hence

$$d(x,y) \leq C_1 d_n(x,y)e^{-\alpha n}$$

for all x, y. It follows that

$$d_n(x,y) \geq C_1^{-1}d(x,y)e^{\alpha n} \geq C_1^{-2}e^{\alpha n-\alpha\ell(x,y)},$$

for all $x, y \in X$ and $n \in \mathbb{N}$, i.e., α is a lower exponent.

Let α be a lower exponent. Let β be an arbitrary number such that $\alpha > \beta > 0$. It is enough to show that there exists a metric on X of exponent β associated with ℓ.

Define, for $x, y \in X$, $d_\beta(x,y)$ as the infimum of $\sum_{i=1}^m e^{-\beta\ell(x_{i-1},x_i)}$ over all sequences $x_0 = x, x_1, x_2, \ldots, x_m = y$. The function $d_\beta(x,y)$ obviously satisfies the triangle inequality, is symmetric, and

$$d_\beta(x,y) \le e^{-\beta\ell(x,y)}$$

for all $x, y \in X$.

It remains to prove that there exists a constant C_2 such that for any sequence $x_0 = x, x_1, x_2, \ldots, x_m = y$ we have

$$\sum_{i=1}^m e^{-\beta\ell(x_{i-1},x_i)} \ge C_2 e^{-\beta\ell(x,y)}.$$

Let C be such that $0 < C < 1$ and $d_n(x,y) \ge Ce^{\alpha(n-\ell(x,y))}$ for all $x, y \in X$ and all n. Let us prove our statement by induction on m for $C_2 = \exp\left(\frac{\beta(\log C - 2\alpha\Delta)}{\alpha-\beta}\right)$.

The statement is true for $m = 1$, since $C_2 < 1$. Suppose that we have proved it for all $k < m$, let us prove it for m.

Lemma 3.5. *Let $x_0, x_1, \ldots, x_m$ be a sequence such that $\ell(x_i, x_{i+1}) \ge n$ for all $i = 0, 1, \ldots, m-1$. Let $n_0 \le n$. Then there exists a sub-sequence $y_0 = x_0, y_1, \ldots, y_{t-1}, y_t = x_m$ of the sequence x_i such that*

$$n_0 - 2\Delta \le \ell(y_i, y_{i+1}) < n_0$$

for all $i = 0, 1, \ldots, t-1$.

Proof. Let us construct the subsequence y_i by the following algorithm. Define $y_0 = x_0$. Suppose we have defined $y_i = x_r$ for $r < m$. Let s be the largest index such that $s > r$ and $\ell(x_r, x_s) \ge n_0$. Note that since $\ell(x_r, x_{r+1}) \ge n \ge n_0$, such s exists.

If $s < m$, then $\ell(x_r, x_{s+1}) < n_0$, and

$$\ell(x_r, x_{s+1}) \ge \min\{\ell(x_r, x_s), \ell(x_s, x_{s+1})\} - \Delta \ge \min\{n_0, \ell(x_s, x_{s+1})\} - \Delta = n_0 - \Delta.$$

Define then $y_{i+1} = x_{s+1}$. We have

$$n_0 - \Delta \le \ell(y_i, y_{i+1}) < n_0.$$

If $s + 1 = m$, we stop and get our sequence $y_0, \ldots, y_t$, for $t = i + 1$.

If $s = m$, then $\ell(x_r, x_m) = \ell(y_i, x_m) \ge n_0$, and

$$\ell(y_{i-1}, x_m) \ge \min\{\ell(y_{i-1}, y_i), \ell(y_i, x_m)\} - \Delta \ge \min\{n_0 - \Delta, n_0\} - \Delta = n_0 - 2\Delta$$

and

$$\ell(y_{i-1}, x_m) < n_0,$$

since y_i was defined and was not equal to x_m. Then we redefine $y_i = x_m$ and stop the algorithm.

In all the other cases we repeat the procedure. It is easy to see that at the end we get a sequence y_i satisfying the conditions of the lemma. □

Let $x_0 = x, x_1, \ldots, x_m = y$ be an arbitrary sequence of points of X. Let n_0 be the minimal value of $\ell(x_i, x_{i+1})$. Let $y_0 = x, y_1, \ldots, y_t = y$ be a sub-sequence of the sequence x_i satisfying conditions of Lemma 3.5.

Suppose at first that

$$n_0 < \ell(x, y) + \frac{2\alpha\Delta - \log C}{\alpha - \beta}.$$

Remember that $n_0 = \ell(x_i, x_{i+1})$ for some i, hence

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1},x_i)} \geq e^{-\beta n_0} > \exp\left(-\beta\ell(x, y) - \frac{\beta(2\alpha\Delta - \log C)}{\alpha - \beta}\right) = C_2 e^{-\beta\ell(x,y)},$$

and the statement is proved.

Suppose now that $n_0 \geq \ell(x, y) + \frac{2\alpha\Delta - \log C}{\alpha - \beta}$, which is equivalent to

$$(\alpha - \beta)n_0 - (\alpha - \beta)\ell(x, y) - 2\alpha\Delta + \log C \geq 0. \tag{5}$$

If $t = 1$, then $n_0 - 2\Delta \leq \ell(x, y) < n_0$, hence

$$n_0 \leq \ell(x, y) + 2\Delta = \ell(x, y) + \frac{2\alpha\Delta - 2\beta\Delta}{\alpha - \beta} < \ell(x, y) + \frac{2\alpha\Delta - \log C}{\alpha - \beta},$$

since $\log C < 0 < 2\beta\Delta$. But this contradicts our assumption.

Therefore $t > 1$, and the inductive assumption implies

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1},x_i)} \geq \sum_{i=0}^{t-1} C_2 e^{-\beta\ell(y_i,y_{i+1})} > tC_2 e^{-\beta n_0}.$$

We have $t \geq d_{n_0-2\Delta}(x, y) \geq Ce^{\alpha(n_0-2\Delta-\ell(x,y))}$, hence

$$\sum_{i=1}^{m} e^{-\beta\ell(x_{i-1},x_i)} \geq CC_2 e^{-\beta n_0+\alpha n_0-2\alpha\Delta-\alpha\ell(x,y)} =$$

$$C_2 \exp\left(\log C - \beta n_0 + \alpha n_0 - 2\alpha\Delta - \alpha\ell(x, y)\right) =$$

$$C_2 \exp\left(-\beta\ell(x, y) + (\alpha - \beta)n_0 - (\alpha - \beta)\ell(x, y) - 2\alpha\Delta + \log C\right) \geq C_2 e^{-\beta\ell(x,y)},$$

by (5). □

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3.2. Lower exponents of Smale spaces

Definition 3.6. Let $(\mathcal{X}, f)$ be a Smale space. A number $\alpha > 0$ is a *stable (resp. unstable) lower exponent* of the Smale space if there exists a constant $C > 0$ such that for any stable (resp. unstable) leaf W and any $x, y \in W$ we have

$$d_n(x, y) \geq Ce^{\alpha(n-\ell_W(x,y))}$$

for the internal log-scale on W. The supremum of the stable (resp. unstable) lower exponents is called the *stable* (resp. *unstable*) *lower critical exponent*.

Note that by Proposition 3.1 lower stable and unstable exponents exist and are positive for any Smale space. Proposition 3.3 implies that the lower critical exponents of a Smale space depend only on the topological conjugacy class of the Smale space.

Proposition 3.7. *Let $l \in \mathbb{R}$. A number $\alpha > 0$ is a lower stable (resp. unstable) exponent of $(\mathcal{X}, f)$ if and only if there exists $C_l > 0$ such that for every stable (resp. unstable) leaf W and for every two points $x, y \in W$ such that $\ell_W(x, y) \leq l$ we have*

$$d_n(x, y) \geq C_l e^{\alpha n} \tag{6}$$

for all n.

Proof. Let us assume that W is a stable leaf (the proof for an unstable leaf is the same). If α is a lower exponent and C is as in Definition 3.6, then for any $x, y \in W$ such that $\ell_W(x, y) \leq l$ we have

$$d_n(x, y) \geq Ce^{\alpha(n-\ell_W(x,y))} \geq Ce^{-\alpha l} \cdot e^{\alpha n},$$

and we can take $C_l = Ce^{-\alpha l}$.

Suppose now that $C_l > 0$ is such that $d_n(x, y) \geq C_l e^{\alpha n}$ for all x, y belonging to one stable leaf W and such that $\ell_W(x, y) \geq l$.

Let x and y be arbitrary stably equivalent points of $\mathcal{X}$. Let W_0 be their stable leaf. Denote $n_0 = \ell_{W_0}(x, y)$. Then $\ell_W(f^{l-n_0}(x), f^{l-n_0}(y)) = l$, where W is the stable leaf of $f^{l-n_0}(x) \sim_+ f^{l-n_0}(y)$. Consequently,

$$d_n(f^{l-n_0}(x), f^{l-n_0}(y)) \geq C_l e^{\alpha n}$$

for all n.

The map $z \mapsto f^{l-n_0}(z)$ transforms every path in $\Gamma_n(W_1)$ to a path in $\Gamma_{n+l-n_0}(W_2)$, where W_1 and W_2 are the stable leaves of z and $f^{l-n_0}(z)$, see (3).

It follows that

$$d_n(x, y) \geq d_{n+l-n_0}(f^{l-n_0}(x), f^{l-n_0}(y)) \geq C_l e^{\alpha(n+l-n_0)} = C_l e^l \cdot e^{\alpha(n-\ell_{W_0}(x,y))},$$

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which shows that α is a lower exponent. □

3.3. Metric properties of leaves

Let $(\mathcal{X}, f)$ be a Smale space, and let ℓ, ℓ_+, and ℓ_- be the standard log-scale on $\mathcal{X}$, and the internal log-scales on the stable and unstable leaves of $\mathcal{X}$. Let U be a neighborhood of the diagonal satisfying the conditions of Definition 2.6.

The following theorem describes the classical theory of Bowen-Margulis measure on Smale spaces, see [5]. See its exposition in [26], which is notationally close to our paper.

Denote by d, d_+, and d_- metrics associated with the log-scales ℓ, ℓ_+, and ℓ_-, respectively. Denote by $B_*(r, x)$ the ball of radius r, with respect to the metric d_*, with center in x, where $* \in \{+, -\}$.

Theorem 3.8. *Suppose that $(\mathcal{X}, f)$ is mixing. There exists a number $\eta > 0$ (called the* entropy *of $(\mathcal{X}, f)$), and a family of Radon measures μ_+ and μ_- on the stable and unstable leaves of $\mathcal{X}$ satisfying the following properties.*

(1) *There exists a number $C > 1$ such that*

$$C^{-1} r^{\eta/\alpha_*} \leq \mu_*(B_*(r, x)) \leq C r^{\eta/\alpha_*}$$

for all $r \geq 0$, $x \in \mathcal{X}$, and $* \in \{+, -\}$, where α_* is the exponent of the metric d_*.

(2) *The measures are preserved under holonomies.*

(3) *The measures are quasi-invariant with respect to f, and $\frac{df_*(\mu_+)}{d\mu_+} = e^{\eta}$, $\frac{df_*(\mu_-)}{d\mu_-} = e^{-\eta}$.*

It follows from condition (1) of the theorem, that μ_+ and μ_- are equivalent to the Hausdorff measures of the metrics d_+ and d_- of dimension $\frac{\eta}{\alpha_+}$ and $\frac{\eta}{\alpha_-}$, respectively.

4. Locally connected Smale spaces and upper exponents

4.1. Connectivity

The aim of this section is to prove the following description of locally connected Smale spaces.

Theorem 4.1. *Let $(\mathcal{X}, f)$ be a Smale space. The following conditions are equivalent.*

(1) *The space $\mathcal{X}$ is locally connected.*
(2) *All stable and unstable leaves are locally connected.*
(3) *All stable and unstable leaves are connected.*
(4) *All stable and unstable leaves are locally path connected.*
(5) *All stable and unstable leaves are path connected.*

(6) *The graphs $\Gamma_0(W)$ are connected for every (stable or unstable) leaf W.*
(7) *The graphs $\Gamma_n(W)$ are connected for every leaf W and every n.*
(8) *There exist $\alpha > 0$ and $C > 0$ such that for every leaf W we have*

$$d_n(x, y) \le Ce^{\alpha(n-\ell_W(x,y))}$$

for all $x, y \in W$ and all $n \ge \ell_W(x, y)$.

Recall that for a stable or unstable leaf W, we denote by $\Gamma_n(W)$ the graph with the set of vertices W in which two vertices x, y are connected by an edge if and only if $\ell_W(x, y) \ge n$, where ℓ_W is the corresponding (ℓ_+ or ℓ_-) internal log-scale on W.

Let us start by proving equivalence of conditions (1) and (2).

Proposition 4.2. *Let $(\mathcal{X}, f)$ be a Smale space. The space $\mathcal{X}$ is locally connected if and only if each leaf is locally connected.*

Proof. Each point $x \in \mathcal{X}$ has a neighborhood homeomorphic to the direct product of the neighborhoods of x in the corresponding stable and unstable leaves. It follows that if x has bases of connected neighborhoods in the leaves, then x has a basis of connected neighborhoods in $\mathcal{X}$.

In the other direction, if x has a basis of connected neighborhoods in $\mathcal{X}$, then for any rectangular neighborhood R of x there exists a connected neighborhood $U \subset R$ of x. Its projection onto the direct factors of R will be connected, hence the point x has bases of connected neighborhoods in its leaves. □

Proposition 4.3. *If every stable leaf of $(\mathcal{X}, f)$ is locally connected, then every stable leaf of $(\mathcal{X}, f)$ is connected.*

Proof. For every point $x \in \mathcal{X}$ there exists a connected neighborhood U of x in its stable leaf and a rectangular neighborhood R of x in $\mathcal{X}$ such that $P_+(R, x) = U$. Then each plaque of R will be homeomorphic to U, hence will be connected. It follows that every point of $\mathcal{X}$ has a rectangular neighborhood R such that all its stable plaques are connected. Since $\mathcal{X}$ is compact, there exists a finite covering $\mathcal{R} = \{R_i\}$ of $\mathcal{X}$ by open rectangles with connected stable plaques.

Let W be a stable leaf, and let $x, y \in W$. By Lebesgue's covering lemma, there exists n such that $f^n(x)$ and $f^n(y)$ belong to one plaque V of a rectangle $R_i \in \mathcal{R}$. Then $f^{-n}(V)$ is a connected subset of W containing x and y. We have shown that any two points of W belong to one connected component of W, i.e., that W is connected. □

Let $\mathcal{R} = \{R_i\}_{i \in I}$ be a finite covering of $\mathcal{X}$ by open rectangles. Let $\mathcal{R}_+$ be the set of all stable plaques of elements of $\mathcal{R}$.

Every stable leaf W is a union $\bigcup_{T\in\mathcal{R}_+, T\subset W} T$ of stable plaques contained in W. Each plaque is an open subset of W and the internal topology on W coincides with the direct limit topology of the union of the plaques.

Denote by $\Gamma'_n(W)$ the graph with the set of vertices W in which two vertices are connected by an edge if they belong to one set of the form $f^n(T)$, $T\in\mathcal{R}_+$.

The map $f : W \longrightarrow f(W)$ induces isomorphisms $\Gamma_n(W) \longrightarrow \Gamma_{n+1}(f(W))$ and $\Gamma'_n(W) \longrightarrow \Gamma'_{n+1}(f(W))$, see (3) and (4) for the first isomorphism.

Lemma 4.4. *There exists a number k_0 such that if $x, y \in W$ are adjacent in $\Gamma_n(W)$, then they are adjacent in $\Gamma'_{n-k_0}(W)$, and if $x, y \in W$ are adjacent in $\Gamma'_n(W)$, then they are adjacent in $\Gamma_{n-k_0}(W)$.*

Proof. There exists k_1 such that for every plaque $V \in \mathcal{R}_+$ and every pair $x, y \in V$ we have $(f^k(x), f^k(y)) \in U$ for all $k \geq k_1$ (where U is a neighborhood of the diagonal defining ℓ, ℓ_+, and ℓ_-). If $x, y \in W$ are connected by an edge in $\Gamma'_n(W)$, then $x, y \in f^n(T)$ for $T \in \mathcal{R}_+$, hence $(f^{k-n}(x), f^{k-n}(y)) \in U$ for all $k \geq k_1$, hence $\ell_+(x, y) \geq n - k_1$, i.e., x and y are connected by an edge in $\Gamma_{n-k_1}(W)$.

By Lebesgue's covering lemma, there exists k_2 such that if $x, y \in W$ are such that $\ell_W(x, y) \geq k_2$, then x and y belong to one plaque $V \in \mathcal{R}_+$. Then every edge of $\Gamma_{k_2}(W)$ is an edge in $\Gamma'_0(W)$. Consequently, every edge of $\Gamma_n(W)$ is an edge in $\Gamma_{n-k_2}(W)$. □

Proposition 4.5. *The following conditions are equivalent.*

(1) *The graph $\Gamma_0(W)$ is connected for every stable leaf W.*
(2) *The graph $\Gamma_n(W)$ is connected for every stable leaf W and for every $n \in \mathbb{Z}$.*
(3) *The graph $\Gamma'_0(W)$ is connected for every stable leaf W.*
(4) *The graph $\Gamma'_n(W)$ is connected for every stable leaf W and for every $n \in \mathbb{Z}$.*

Proof. Since the map $f^k : W \longrightarrow f^k(W)$ induces isomorphisms $\Gamma_n(W) \longrightarrow \Gamma_{n+k}(f^k(W))$ and $\Gamma'_n(W) \longrightarrow \Gamma'_{n+k}(f^k(W))$, (1) is equivalent to (2), and (3) is equivalent to (4).

Let k_0 be as in Lemma 4.4. If all graphs $\Gamma_n(W)$ are connected, then all graphs $\Gamma'_{n-k_0}(W) \supseteq \Gamma_n(W)$ are connected. If all graphs $\Gamma'_n(W)$ are connected, then all graphs $\Gamma_{n-k_0}(W) \supseteq \Gamma'_n(W)$ are connected. This shows that all conditions (1)–(4) are equivalent to each other. □

Proposition 4.6. *If a stable leaf W is connected, then the graph $\Gamma'_n(W)$ is connected for every n.*

Proof. Suppose that W is a connected stable leaf. Let A be a connected component of $\Gamma'_0(W)$. Let W_A be the union of the plaques $V \in \mathcal{R}_+$ containing vertices of A. It follows from the definition of the graph $\Gamma'_0(W)$ that every plaque $V \in \mathcal{R}_+$ is either contained in W_A, or is disjoint with it. Consequently, W_A is clopen, which implies that $W = W_A$, hence $A = W$, and $\Gamma'_0(W)$ is connected. □

Proposition 4.7. *Suppose that the graphs $\Gamma'_n(W)$ are connected for all stable leaves W and all n. Then there exists $A \geq 1$ such that any two adjacent vertices in $\Gamma'_0(W)$ are on distance at most A in $\Gamma'_1(W)$.*

Proof. Let E be the closure in $\mathcal{X} \times \mathcal{X}$ of the set pairs of points (x, y) such that there exists a plaque $V \in \mathcal{R}_+$ such that $x, y \in V$. It is easy to see that E is compact. It contains the set of edges of every graph $\Gamma'_0(W)$, and is contained in the stable equivalence relation.

Any pair of points x, y such that $(x, y) \in E$ is connected by a path in $\Gamma'_1(W)$, as all graphs $\Gamma'_n(W)$ are connected. It means that there exists a sequence of rectangles $R_1, \ldots, R_n \in \mathcal{R}$, and a sequence of points $x_i \in R_i$ such that x and x_1 belong to the same stable plaque of R_1, y and x_n belong to the same stable plaque of R_n, and the stable plaque of x_i in R_i intersects with the stable plaque of x_{i+1} in R_{i+1}. This sequence $R_1, \ldots, R_n$ will define a path in $\Gamma'_1(W)$ connecting any two points (x', y') belonging to a neighborhood of (x, y) in E. It follows then from compactness of E that we can find a finite upper bound on the length of a path connecting any two points of E, which finishes the proof. □

Recall that $d_n(x, y)$ is the distance in the graph $\Gamma_n(W)$.

Proposition 4.8. *Suppose that all graphs $\Gamma'_n(W)$ are connected. Then there exist positive constants α and C such that for any two points $x, y \in W$ we have*

$$d_n(x, y) \leq Ce^{\alpha(n - \ell_W(x,y))}$$

for all $n \geq \ell_W(x, y)$.

Proof. Let A and k_0 be as in Propositions 4.7 and 4.4, and let $x, y \in W$ be arbitrary. Denote $n_0 = \ell_W(x, Y)$. The points x and y are connected by an edge in $\Gamma_{n_0}(W)$, hence they are connected by an edge in $\Gamma'_{n_0 - k_0}(W)$. It follows from Proposition 4.7 that for every $k \geq 0$, distance between x and y in $\Gamma'_{n_0 - k_0 + k}(W)$ is not greater than A^k. As the set of edges of $\Gamma'_{n_0 - k_0 + k}(W)$ is contained in the set of edges of $\Gamma_{n_0 - 2k_0 + k}(W)$, we have

$$d_{n_0 - 2k_0 + k}(x, y) \leq A^k,$$

for all $k \geq 0$. Consequently,

$$d_n(x, y) \leq A^{n - n_0 + 2k_0} = A^{2k_0} \cdot A^{n - \ell_W(x,y)}$$

for all $n \geq \ell_W(x, y) - 2k_0$. □

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Proposition 4.9. *Suppose that all graphs $\Gamma'_n(W)$ are connected. Let d be a metric on W associated with ℓ_W. There exists a constant C such that for any two points $x, y \in W$ there exists a curve $\gamma : [0,1] \longrightarrow W$ connecting x to y and such that the diameter the range of γ is not larger than $Cd(x,y)$.*

Proof. Let A and k_0 be as in Propositions 4.7 and 4.4. Let $C_1 > 1$ and $\alpha > 0$ be such that $C_1^{-1}e^{-\alpha\ell_W(x,y)} \le d(x,y) \le C_1 e^{-\alpha\ell_W(x,y)}$ for all $x, y \in W$.

Take arbitrary $x, y \in W$. Let $n_0 = \ell_W(x,y)$. Then x and y are adjacent in $\Gamma'_{n_0-k_0}(W)$, hence they are connected by a path $\gamma_1 : \{x = x_{1,1}, x_{1,2}, \ldots, x_{1,m_1} = y\}$ of length at most A in $\Gamma'_{n_0-k_0+1}(W)$. Each pair of points $x_{1,i}, x_{1,i+1}$ is connected by a path of length at most A in $\Gamma'_{n_0-k_0+2}(W)$. We get then a path $\gamma_2 = \{x = x_{2,1}, x_{2,2}, \ldots, x_{1,m_2}\}$ of length at most A^2 in $\Gamma'_{n_0-k_0+2}(W)$, containing γ_1. We get then inductively defined sequence of paths $\gamma_n = \{x = x_{n,1}, x_{n,2}, \ldots, x_{n,m_n} = y\}$ in $\Gamma'_{n_0-k_0+n}(W)$ such that each next path γ_n is obtained from γ_{n-1} by inserting at most $A-1$ points between each pair of neighbors of γ_n.

Every pair of points $x_{n,j}, x_{n,j+1}$ is adjacent in $\Gamma'_{n_0-k_0+n}(W)$, hence $\ell_W(x_{n,j}, x_{n,j+1}) \ge n_0 - 2k_0 + n$, hence $d(x_{n,j}, x_{n,j+1}) \le C_1 e^{-\alpha(n+n_0-2k_0)}$. In particular, for every point t_1 of γ_{n+1} there exists a point t_2 of γ_n such that $d(t_1, t_2) < AC_1 e^{-\alpha(n+1+n_0-2k_0)}$.

It follows that the diameter of the set γ_n is not greater than

$$C_1 e^{-\alpha(n_0-2k_0)} + 2\sum_{i=1}^{n} AC_1 e^{-\alpha(i+n_0-2k_0)} <$$

$$2C_1 A e^{-\alpha(n_0-2k_0)} \cdot \frac{1}{1-e^{-\alpha}} =$$

$$\frac{2C_1 A e^{2\alpha k_0}}{1-e^{-\alpha}} \cdot e^{-\alpha\ell_W(x,y)} \le \frac{2C_1^2 A e^{2\alpha k_0}}{1-e^{-\alpha}} \cdot d(x,y).$$

Since $d(x_{n,j}, x_{n,j+1}) \le C_1 e^{-\alpha(n_0-2k_0+n)}$, the closure of $\bigcup_{n=1}^{\infty} \gamma_n$ is the image of a continuous curve connecting x to y. Diameter of the image of the curve is not greater than $\frac{2C_1^2 A e^{2\alpha k_0}}{1-e^{-\alpha}} \cdot d(x,y)$. □

Let us summarize now the proof of Theorem 4.1. The equivalence of (1) and (2) is shown in Proposition 4.2. The implication (2)⇒(3) is given in Proposition 4.3. The equivalence of (6) and (7) is contained in Proposition 4.5. Proposition 4.6 shows then that (3) implies (6). Proposition 4.8 proves that (7) implies (8). Condition (8) obviously implies (7). Proposition 4.9 shows that (7) implies path connectivity and local path connectivity of the leaves, i.e., that (7) implies (4) and (5). The implications (3)⇒(1), (5)⇒(3), and (4)⇒(2) are obvious. This finishes the proof of Theorem 4.1.

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4.2. Local product structure on locally connected Smale spaces

Proposition 4.10. *Let R be a sub-rectangle of a Smale space $(\mathcal{X}, f)$. If R is connected and locally connected, then the direct product structure on R compatible with the local product structure on $\mathcal{X}$ is unique.*

Proof. Suppose that, on the contrary, there exist two different direct product structures $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$, both compatible with the local product structure on $\mathcal{X}$.

By Definition 2.2, there exists a covering $\mathcal{R}$ of $\mathcal{X}$ by open rectangles such that for any $U \in \mathcal{R}$ and $x, y \in R \cap U$ we have $[x, y]_U = [x, y]_1 = [x, y]_2$. Then for every $U \in \mathcal{R}$ the intersection $U \cap R$ is a (possibly empty) sub-rectangle of R with respect to both direct product structures; and restrictions of the direct products structures $[\cdot,\cdot]_i$, $i = 1, 2$, onto $U \cap R$ coincide.

Since R is connected, all plaques of R (with respect to both direct products structures) are connected. Let P_+ be a stable plaque of $(R, [\cdot,\cdot]_1)$. Let $x, y \in P_+$. Since P_+ is connected, there exists a sequence of points $x_0 = x, x_1, \ldots, x_n = y$ and a sequence of rectangles $U_0, U_1, \ldots, U_n \in \mathcal{R}$ such that $x_i \in U_i$, and $U_i \cap U_{i+1} \cap P_+ \neq \emptyset$. The set $U_i \cap P_+$ is a plaque of the rectangle $U_i \cap R$, hence it is a subset of the stable plaque of $(R, [\cdot,\cdot]_2)$. We get a sequence $U_i \cap P_+$ of subsets of plaques of $(R, [\cdot,\cdot]_2)$ such that $(U_i \cap P_+) \cap (U_{i+1} \cap P_+) \neq \emptyset$. But it means that $U_i \cap P_+$ are subsets of one plaque of $(R, [\cdot,\cdot]_2)$. We have shown that if two points belong to one stable plaque of $(R, [\cdot,\cdot]_1)$, then they belong to one stable plaque of $(R, [\cdot,\cdot]_2)$. The converse is proved in the same way. Consequently, the stable plaques of R with respect to $[\cdot,\cdot]_1$ are the same as the stable plaques of R with respect to $[\cdot,\cdot]_2$. The same statement is obviously true for the unstable plaques, which implies that the direct product structures $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ on R coincide. □

4.3. Upper exponents

Definition 4.11. A positive number $\alpha > 0$ is a *stable (resp. unstable) upper exponent* of the Smale space if there exists $C > 0$ such that for any stable (resp. unstable) leaf W and every pair of points $x, y \in W$ we have

$$d_n(x, y) \leq Ce^{\alpha(n - \ell_W(x,y))}$$

for all $n \geq \ell_W(x, y)$.

Note that changing ℓ_W to a bi-Lipschitz equivalent log-scale, one does not change the set of upper exponents, i.e., this notion is well defined and depends only on the topological conjugacy class of the Smale space (see Lemma 2.9).

By Theorem 4.1 a finite upper exponent exists if $\mathcal{X}$ is locally connected.

The proof of the next proposition is analogous to the proof of Proposition 3.7.

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Proposition 4.12. *Fix $l \in \mathbb{R}$. A number $\alpha > 0$ is a stable (resp. unstable) upper exponent if and only if there exists a constant $C_l > 0$ such that for any stable (resp. unstable) leaf W and any $x, y \in W$ such that $\ell_W(x, y) \geq l$ we have*

$$d_n(x, y) \leq C_l e^{\alpha n}$$

for all $n \geq 0$.

5. Splittings of Smale spaces

5.1. Groups of deck transformations

Definition 5.1. Let $(\mathcal{X}, f)$ be a Smale space. A *splitting* of $(\mathcal{X}, f)$ is a covering map $\pi : \mathcal{M} \longrightarrow \mathcal{X}$, where $\mathcal{M}$ is a space with a (global) direct product structure, such that

(1) π preserves the local product structures on $\mathcal{M}$ and $\mathcal{X}$, see Definition 2.3;
(2) restriction of π onto every plaque $P_1(\mathcal{M}, x)$ of $\mathcal{M}$ is a homeomorphism with the stable leaf $W_+(\pi(x))$, and restriction of π onto every plaque $P_2(\mathcal{M}, x)$ of $\mathcal{M}$ is a homeomorphism with the unstable leaf $W_-(\pi(x))$, with respect to their intrinsic topology.

Proposition 5.2. *If there exists a splitting of a Smale space $(\mathcal{X}, f)$, then the Smale space is irreducible and the set of non-wandering points of $(\mathcal{X}, f)$ is equal to $\mathcal{X}$. If, in addition, $\mathcal{X}$ is connected, then $(\mathcal{X}, f)$ is mixing.*

Proof. The proof is the same as, for example, the proof of Theorem 5 in [11]. Suppose that $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ is a splitting. Let $\mathcal{X}_1, \mathcal{X}_2 \subset \mathcal{X}$ be irreducible components of $(\mathcal{X}, f)$, see Theorem 2.12. Take $x_1, x_2 \in \mathcal{M}$ such that $\pi(x_1) \in \mathcal{X}_1$ and $\pi(x_2) \in \mathcal{X}_2$. Consider the point $[x_1, x_2] \in \mathcal{M}$. Then $\pi([x_1, x_2]) \in W_+(\pi(x_1)) \cap W_-(\pi(x_2))$. It follows that the set of the accumulation points of $f^n(\pi([x_1, x_2]))$ for $n \geq 0$ belongs to $\mathcal{X}_1$, and the set of the accumulation points of $f^n(\pi([x_1, x_2]))$ for $n \leq 0$ belongs to $\mathcal{X}_2$, hence $\mathcal{X}_2 \prec \mathcal{X}_1$. But we will also have $\mathcal{X}_1 \prec \mathcal{X}_2$ by considering $[x_2, x_1]$, which implies $\mathcal{X}_1 = \mathcal{X}_2$, by Theorem 2.12. Consequently, $(\mathcal{X}, f)$ has no wandering points and is irreducible. The rest of the proposition follows from Theorem 2.13. □

Proposition 5.3. *Suppose that $\mathcal{X}$ is connected and locally connected. Let $\pi_1 : \mathcal{M}_1 \longrightarrow \mathcal{X}$ and $\pi_2 : \mathcal{M}_2 \longrightarrow \mathcal{X}$ be splittings of $(\mathcal{X}, f)$. If $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$ are such that $\pi_1(x_1) = \pi_2(x_2)$, then there exists a unique homeomorphism $F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ preserving the local product structures such that $\pi_1 = \pi_2 \circ F$ and $F(x_1) = x_2$.*

Proof. Since the leaves of locally connected Smale spaces are connected and locally connected, the spaces $\mathcal{M}_i$ are connected and locally connected.

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Denote $x = \pi_1(x_1) = \pi_2(x_2)$. Restriction of π_i onto the plaques $P_1(\mathcal{M}_i, x_i)$ and $P_2(\mathcal{M}_i, x_i)$ are homeomorphisms with the leaves $W_+(x)$ and $W_-(x)$, respectively. Therefore, the only possible way to define F is by the equality

$$F([y_1, y_2]_{\mathcal{M}_1}) = [z_1, z_2]_{\mathcal{M}_2},$$

where $y_1 \in P_1(\mathcal{M}_1, x_1)$, $y_2 \in P_2(\mathcal{M}_1, x_1)$ are arbitrary, while $z_1 \in P_1(\mathcal{M}_2, x_2)$, $z_2 \in P_2(\mathcal{M}_2, x_2)$ are uniquely determined by the condition $\pi_1(y_1) = \pi_2(z_1)$ and $\pi_1(y_2) = \pi_2(z_2)$.

The defined map F is a homeomorphism, since it is a direct product of two homeomorphisms. Consequently, $\pi_2 \circ F : \mathcal{M}_1 \longrightarrow \mathcal{X}$ is a covering map. Since F and π_2 preserve the local product structures of $\mathcal{M}_i$ and $\mathcal{X}$, their composition $\pi_2 \circ F$ preserves the local product structures, i.e., the image of the direct product structure on $\mathcal{M}_1$ by F defines the same local product structure on $\mathcal{M}_2$ as the direct product structure $[\cdot, \cdot]_{\mathcal{M}_2}$. By the same arguments as in the proof of Proposition 4.10, the direct product structure on $\mathcal{M}_2$ is uniquely determined by the corresponding local product structure. Consequently, F preserves the direct product structures, i.e., $F([y_1, y_2]_{\mathcal{M}_1}) = [F(y_1), F(y_2)]_{\mathcal{M}_2}$ for all $y_1, y_2 \in \mathcal{M}_1$. It follows that if $\pi_1(y) = \pi_2(F(y))$ for $y \in \mathcal{M}_1$, then $\pi_1 = \pi_2 \circ F$ on a rectangular neighborhood of y. Consequently, the set of points $y \in \mathcal{M}_1$ such that $\pi_1(y) = \pi_2(F(y))$ is open and closed, it contains x, hence it is equal to $\mathcal{M}_1$. □

We assume now that $(\mathcal{X}, f)$ is a locally connected and connected Smale space.

Let $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ be a splitting. Let G be the set of all homeomorphisms $g : \mathcal{M} \longrightarrow \mathcal{M}$ preserving the direct product structure on $\mathcal{M}$ and such that $\pi = \pi \circ F$. Then G is obviously a group. By Proposition 5.3 (for the case $\pi_1 = \pi_2$), the action of the group on $\mathcal{M}$ is free and transitive on $\pi^{-1}(x)$ for every $x \in \mathcal{X}$. We call G the *group of deck transformations* of the splitting. It acts properly on $\mathcal{M}$, since every point of $\mathcal{M}$ has a neighborhood U such that $\pi(U)$ is evenly covered, i.e., such that $g(U) \cap U = \emptyset$ for all not-trivial $g \in G$.

Note that if $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ is a splitting, then $\pi \circ f$ is also a splitting. Choose $x_1, x_2 \in \mathcal{M}$ such that $f \circ \pi(x_1) = \pi(x_2)$, and apply Proposition 5.3 to $\pi_1 = f \circ \pi$ and $\pi_2 = \pi$. We get that there exists a unique homeomorphism $F : \mathcal{M} \longrightarrow \mathcal{M}$ preserving the direct product structure such that $F(x_1) = x_2$ and $\pi \circ F = f \circ \pi$.

Definition 5.4. Let $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ be a splitting. We say that a homeomorphism $F : \mathcal{M} \longrightarrow \mathcal{M}$ preserving the direct product structure is a *lift of* f if $\pi \circ F = f \circ \pi$.

It follows from the above arguments that lifts of f exist, and if F_1 and F_2 are two lifts, then $F_1^{-1}F_2$ and $F_1F_2^{-1}$ belong to G.

It also follows that for every $g \in G$, and every lift F of f, we have $F^{-1}gF \in G$. The map $g \mapsto F^{-1}gF$ is an automorphism of G. We say that it is *induced by* f. Any two automorphisms induced by f on G differ from each other by an inner automorphism.

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Proposition 5.5. *The group of deck transformations of a splitting of a connected and locally connected Smale space is finitely generated.*

Proof. Let $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ be a splitting. Consider a finite covering $\mathcal{U}$ of $\mathcal{X}$ by open connected evenly covered by π subsets. Let $\mathcal{U}_{\mathcal{M}}$ be the union of the sets of connected components of $\pi^{-1}(U)$ for all $U \in \mathcal{U}$. Then $\mathcal{U}_{\mathcal{M}}$ is a covering of $\mathcal{M}$. Since $\mathcal{M}$ is connected, every two elements $U, V \in \mathcal{U}_{\mathcal{M}}$ are connected with each other by a chain of elements $U_0 = U, U_1, \dots, U_n = V$ such that $U_i \cap U_{i+1} \neq \emptyset$. It follows that there exists a connected finite union V of elements of $\mathcal{U}_{\mathcal{M}}$ such that $\pi(V) = \mathcal{X}$. Note that $\overline{V}$ is compact.

Let S be the set of elements $g \in G$ such that $g(V) \cap V \neq \emptyset$. It is finite, since the action of G on $\mathcal{M}$ is proper.

Let $g \in G$ and $x_0 \in V$. Since $\mathcal{M}$ is connected and $\mathcal{M} = \bigcup_{g \in G} g(V)$, there exists a sequence $g_i \in G$, $i = 0, 1, \dots, k$ such that $g_0 = 1$, $g_k = g$, and $g_i(V) \cap g_{i+1}(V) \neq \emptyset$ for all $i = 0, 1, \dots, k-1$. Note that $g_i(V) \cap g_{i+1}(V) \neq \emptyset$ implies $V \cap g_i^{-1} g_{i+1}(V) \neq \emptyset$, hence $g_i^{-1} g_{i+1} \in S$. We see that $g = g_0^{-1} g_1 \cdot g_1^{-1} g_2 \cdots g_{k-1}^{-1} g_k$ is a product of k elements of S. □

5.2. Splittable Smale spaces and hyperbolic graphs

A connection between Smale spaces and Gromov hyperbolic graphs described in this subsection is a particular case of the theory of Cayley graphs of hyperbolic groupoids, described in [27]. Since the theory for Smale spaces is simpler than the general case, and in order to make our paper more self-contained, we describe them directly.

Let $(\mathcal{X}, f)$ be a Smale space with locally connected and connected space $\mathcal{X}$, and let $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ be a splitting. We denote by $[\cdot, \cdot]$ the local product structures on $\mathcal{M}$ and $\mathcal{X}$. Let G be the group of deck transformations of the splitting. Let $F : \mathcal{M} \longrightarrow \mathcal{M}$ be a lift of f.

Let d be a metric on $\mathcal{X}$ associated with the standard log-scale ℓ. We will denote by d_+ and d_- metrics on the stable and unstable leaves of $\mathcal{X}$ associated with the respective standard log-scales ℓ_+ and ℓ_-. We assume that exponents of the metrics d, d_+, and d_- are equal (by taking them sufficiently small). Then we have the following corollary of [27, Lemma 7.8], see also [15].

Proposition 5.6. *There exist constants $\epsilon, L > 1$ such that for every point $x \in \mathcal{X}$ and every rectangle R contained in the ϵ-neighborhood of x the restriction of d to R is L-bi-Lipschitz equivalent to the metric given by*

$$d_x(y_1, y_2) = d_+([y_1, x], [y_2, x]) + d_-([x, y_1], [x, y_2]).$$

Let $\epsilon > 0$ be such that it satisfies the conditions of Proposition 5.6 and for every $x \in \mathcal{X}$ the ϵ-neighborhood of x is evenly covered by π.

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Define then $d_{\mathcal{M}}(x, y)$ as the infimum of the sum $\sum_{i=0}^{m-1} d(\pi(x_i), \pi(x_{i+1}))$ over all sequences $x_0, x_1, \ldots, x_m$ such that $x_0 = x$, $x_m = y$, and $d(\pi(x_i), \pi(x_{i+1})) < \epsilon$ for all $i = 0, 1, \ldots, m-1$.

Then $d_{\mathcal{M}}$ is a G-invariant metric on $\mathcal{M}$ such that $d_{\mathcal{M}}(x, y) = d(\pi(x), \pi(y))$ for all x, y such that $d_{\mathcal{M}}(x, y) < \epsilon$.

The map π bijectively identifies the plaques $W_+(x)$ and $W_-(x)$ of $\mathcal{M}$ with their images $W_+(\pi(x))$ and $W_-(\pi(x))$, respectively. We obtain the metrics d_+ and d_- on $W_+(x)$ and $W_-(x)$, respectively. Then Proposition 5.6 holds when we replace $\mathcal{X}$ by $\mathcal{M}$ and d by $d_{\mathcal{M}}$.

Let $R \subset \mathcal{M}$ be an open relatively compact rectangle such that $\bigcup_{g \in G} g(R) = \mathcal{M}$. Let W_+ be a stable plaque of $\mathcal{M}$. Denote by $\Omega_n(W_+, R)$, for $n \in \mathbb{Z}$, the set of elements $g \in G$ such that of $F^n(g(R))$ intersects W_+.

Denote by $\Xi(W_+, R)$ the graph with the set of vertices $\bigcup_{n \in \mathbb{Z}} \Omega_n(W_+, R) \times \{n\}$ in which two vertices are connected by an edge if and only if they are either of the form (g_1, n), (g_2, n), where $g_1(R) \cap g_2(R) \neq \emptyset$, or of the form (g_1, n), $(g_2, n+1)$, where $g_1(R) \cap F(g_2(R)) \neq \emptyset$. In other words, we connect two vertices of $\Xi(W_+, R)$ if and only if they belong to the same or neighboring levels $\Omega_n(W_+, R)$ and the corresponding rectangles intersect.

Theorem 5.7. *The graph $\Xi = \Xi(W_+, R)$ is Gromov hyperbolic. There exists a point $\omega \in \partial\Xi$ such that all paths of the form $(g_n, -n) \in \Xi$, $n \geq 0$, converge to ω. The correspondence mapping the limit in $\partial\Xi$ of a path (g_n, n), $n \geq 0$ to the limit of the intersections of $F^n(g_n(R))$ with W_+ induces a homeomorphism between $\partial\Xi \setminus \{\omega\}$ and W_+.*

Proof. Consider two vertices $(g, 0)$ and $(h, 0)$. Let m be the distance between them in $\Omega_0(W_+, R)$.

Let $(g_k, -k)$ and $(h_k, -k)$ for $k \geq 0$ be arbitrary paths in Ξ such that $g_0 = g$ and $h_0 = h$.

There exists $\epsilon > 0$ such that for every $x \in \mathcal{M}$ there exist $g \in G$ such that the ball of radius ϵ with center x is contained in $g(R)$. It follows that for every $m \in \mathbb{N}$ there exists $N_m > 0$ such that for any path $(a_0, 0), (a_1, 0), \ldots, (a_m, 0)$ of length m in $\Omega_0(W_+, R)$ there exists $g \in G$ such that $F^{N_m}(a_0(R)) \cup \cdots \cup F^{N_m}(a_m(R)) \cap W_+ \subset g(R) \cap W_+$. It follows that $(g_{N_m}, -N_m)$ and $(h_{N_m}, -N_m)$ are on distance at most 2 in $\Omega_{-N_m}(W_+, R)$.

It follows now from [27, Theorem 2.10] that the graph Ξ is Gromov hyperbolic, that the level function $(g(R), n) \mapsto n$ is a Busemann function of a point $\omega \in \partial\Xi$, and that every path (g_n, n) for $n \geq 0$ converges to a point of $\partial\Xi \setminus \omega$, whereas every path of the form $(g_n, -n)$, $n \geq 0$, converges to ω.

Lemma 5.8. *Every point of $\partial\Xi \setminus \omega$ is the limit of a sequence of the form (g_n, n).*

Proof. Note that since $(g, n) \mapsto n$ is a Busemann function associated with $\omega \in \partial\Xi$, every point $\xi \in \partial\Xi \setminus \omega$ is the limit of a sequence of the form (g_n, n) (which is not necessarily a path). For every $n \geq 0$ there exists a path $(h_{n,k}, k)$, $k \leq n$, such that $h_{n,n} = g_n$. All these paths converge to ω, and by the above arguments, for any n_1, n_2, distance from $(h_{n_1,k}, k)$ to $(h_{n_2,k}, k)$ is not more than 2 for all k smaller than some $k(n_1, n_2)$. Moreover, $k(n_1, n_2) \to \infty$ as $n_1, n_2 \to \infty$. It follows then by compactness arguments (since Ξ has bounded valency) that there exists a path (h_n, n), $n \geq 0$, converging to ξ. □

Let (g_n, n), for $n \geq 0$, be a path in Ξ. The sets $V_n = F^n(g_n(R)) \cap W_+$ are compact, their diameters decrease exponentially, and we have $V_n \cap V_{n+1} \neq \emptyset$ for every n. It follows that the sequence V_n converges (in the Hausdorff metric) to a point $x \in W_+$. Let us show that the map $\Lambda : \lim_{n\to\infty}(g_n, n) \mapsto \lim_{n\to\infty} F^n(g_n(R))$ is a homeomorphism between $\partial\Xi \setminus \omega$ and W_+.

The arguments basically repeat the proof of [27, Theorem 6.9]. Let us show that the map is well defined. If (g_n, n) and (h_n, n) converge to the same limit in $\partial\Xi$, then the distance between g_n and h_n in $\Omega_n(W_+, R)$ is uniformly bounded. But this implies that the Hausdorff distance between $F^n(g_n(R))$ and $F^n(h_n(R))$ is exponentially decreasing, hence $\lim_{n\to\infty} F^n(g_n(R)) = \lim_{n\to\infty} F^n(h_n(R))$. The same argument shows that the map Λ is continuous, since if ξ_1 and ξ_2 are close to each other, then the sequences (g_n, n) and (h_n, n) are close to each other for an initial interval $n = 0, \ldots, L$, where L is big. But then the limits $\lim_{n\to\infty} F^n(g_n(R))$ and $\lim_{n\to\infty} F^n(h_n(R))$ are close to each other.

The map Λ is onto, since for every point $x \in W_+$ there exists a path (g_n, n) defined by the condition $F^n(g_n(R)) \ni x$.

Using Lebesgue's covering lemma, we show that if x and y are close to each other, then there exists a sequence (g_n, n) such that $F^n(g_n(R)) \supset \{x, y\}$ for all $n = 0, \ldots, L$, where L is big. This shows that Λ^{-1} exists and is continuous. □

Suppose now that the map $F : \mathcal{M} \longrightarrow \mathcal{M}$ has a fixed point x_0. Let $\phi : G \longrightarrow G$ be the automorphism defined by the condition $F(g(x_0)) = \phi(g)(x_0)$. Let W_+ and W_- be the stable and the unstable plaques of $\mathcal{M}$ containing x_0. Let R, $\Omega_n(W_+, R)$, and $\Xi(W_+, R)$ be as above. We assume that R is connected and $x_0 \in R$.

Note that $F^n(g(R)) \cap W_+ \neq \emptyset$ is equivalent to $g(R) \cap F^{-n}(W_+) = g(R) \cap W_+ \neq \emptyset$. It follows that the set $\Omega_n(W_+, R)$ does not depend on n.

The graph $\Xi(W_+, R)$ is isomorphic then to the graph with the set of vertices $\Omega_0(W_+, R) \times \mathbb{Z}$ in which two vertices are connected by an edge if and only if they are either of the form (g_1, n) and (g_2, n), where $g_1, g_2 \in \Omega_0(W_+, R)$ and $g_1(R) \cap g_2(R) \neq \emptyset$, or of the form (g_1, n) and $(g_2, n+1)$, where $g_1, g_2 \in \Omega_0(W_+, R)$ and $F^n(g_1(R)) \cap F^{n+1}(g_2(R)) \neq \emptyset$, which is equivalent to $g_1(R) \cap \phi(g_2)(F(R)) \neq \emptyset$. Note that $(g, n) \mapsto (g, n+1)$ is an automorphism of $\Xi(W_+, R)$.

Let $A \subset G$ be a finite set containing the identity, and let S be a finite generating set of G. We assume that S contains all elements $g \in G$ such that $R \cap g(R) \neq \emptyset$ or $R \cap g(F(R)) \neq \emptyset$ and that $A \subset S$.

Let $\Omega_0' \subset G$ be any set such that $\Omega_0(W_+, R) \subset \Omega_0' \subset \Omega_0(W_+, R)A$.

Denote then by Ξ' the graph with the set of vertices $\Omega_0' \times \mathbb{Z}$ with edges of two kinds: vertical and horizontal. The horizontal edges connect two vertices $(g_1, n), (g_2, n)$ if and

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only if $g_1^{-1}g_2 \in S$. The vertical edges connect a vertex (g_1, n) to a vertex $(g_2, n+1)$ if and only if $g_1^{-1}\phi(g_2) \in S$.

Note that if $g_1(R) \cap g_2(R) \neq \emptyset$, then $R \cap g_1^{-1}g_2(R) \neq \emptyset$. If $g_1(R) \cap F(g_2(R)) \neq \emptyset$, then $g_1(R) \cap \phi(g_2)(F(R)) \neq \emptyset$, hence $R \cap g_1^{-1}\phi(g_2)(F(R)) \neq \emptyset$. It follows that $\Xi(W_+, R)$ is a sub-graph of Ξ'.

Proposition 5.9. *The inclusion $\Xi(W_+, R) \hookrightarrow \Xi'$ is a quasi-isometry.*

Proof. Let us prove at first the following lemmas.

Lemma 5.10. *There exists $n_1 > 0$ such that $\phi^{-n_1}(\Omega_0(W_+, R)) \subset \Omega_0(W_+, R)$.*

For every such n_1 there exists D_{n_1} such that distance from (g, n) to $(\phi^{-n_1}(g), n+n_1)$ in $\Xi(W_+, R)$ is less than D_{n_1}.

Proof. By Lebesgue's covering lemma, there exists $\epsilon > 0$ such that for every $x \in \mathcal{M}$ there exists $g \in G$ such that the ϵ-neighborhood of x is contained in $g(R)$. It follows that if distance from x to W_+ is less than ϵ, then there exists $g \in \Omega_0(W_+, R)$ such that $x \in g(R)$.

There exists an upper bound (equal to the diameter of R) on the distance from $g(x_0)$ to W_+ for all $g \in \Omega_0(W_+, R)$. Consequently, there exists $n_1 > 0$ such that $\phi^{-n_1}(\Omega_0(W_+, R)) \subset \Omega_0(W_+, R)$.

Let us prove the second part of the lemma. Let $(g, n) \in \Xi(W_+, R)$, and let $x \in g(R) \cap W_+$. Choose for $k = 1, \ldots, n_1$, $h_k \in \Omega_0(W_+, R)$ such that $F^{-k}(x) \in h_k(R)$. Then $F^{-1}(x) \in h_1(R)$ and $x \in g(R)$, hence $F(h_1(R)) \cap g(R) \neq \emptyset$, which implies that (g, n) is connected to $(h_1, n+1)$. Similarly, $F^{-k}(x) \in h_k(R)$ and $F^{-(k+1)}(x) \in h_{k+1}(R)$, hence $F(h_{k+1}(R)) \cap h_k(R) \neq \emptyset$, so that $(h_k, n+k)$ is connected to $(h_{k+1}, n+k+1)$. We have $g(x_0) \in g(R)$, hence $F^{-n_1}(x) \in h_{n_1}(R)$ and $F^{-n_1}(x), \phi^{-n_1}(g)(x_0) \in F^{-n_1}(g(R)) = \phi^{-n_1}(g)(F^{-n_1}(R))$. The set $F^{-n_1}(g(R))$ is connected, hence there exists a path $f_1, f_2, \ldots, f_m$ in $\Omega_0(W_+, R) \times \{n+n_1\}$ connecting $(h_{n_1}, n+n_1)$ to $(\phi^{-n_1}(g), n+n_1)$. Since $F^{-n_1}(R)$ is relatively compact, there exists a uniform bound M such that we may assume that $m \leq M$. It follows that the distance in $\Xi(W_+, R)$ from (g, n) to $(\phi^{-n_1}(g), n+n_1)$ is not more than $n_1 + m - 1$. □

Lemma 5.11. *For every finite set $B \subset G$ there exists $D > 0$ such that if $g_1^{-1}g_2 \in B$ for $g_1, g_2 \in \Omega_0(W_+, R)$, then the distance between (g_1, n) and (g_2, n) in $\Xi(W_+, R)$ is not greater than D.*

Proof. Let R_B be a compact connected rectangle of $\mathcal{M}$ containing $B(x_0)$. Note that it follows from Proposition 5.6 that there exists a uniform upper bound on the d_--diameter of the unstable plaques of R_B. Then it is also a uniform upper bound on the d_--distance from W_+ to a point of $g(R_B)$ for $g \in \Omega_0(W_+, R)$.

It follows that for every $\epsilon > 0$ there exists $n_2 > 0$ such that for every $g \in \Omega_0(W_+, R)$ the set $F^{-n_2}(g(R_B))$ belongs to the ϵ-neighborhood of W_+, hence (if ϵ is small enough)

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it is covered by the sets $h(R)$ for $h \in \Omega_0(W_+, R)$. Since $F^{-n_2}(g(R_B))$ is connected, for every two points $x, y \in g(R_B)$ there exists a sequence $h_1, h_2, \ldots, h_m \in \Omega_0(W_+, R)$ such that $F^{-n_2}(x) \in h_1(R)$, $F^{-n_2}(y) \in h_m(R)$, $h_i(R) \cap h_{i+1}(R) \neq \emptyset$ for all $i = 1, \ldots, m-1$, and h_i are pairwise different. Since $F^{-n_2}(g(R_B)) = \phi^{-n_2}(g)(F^{-n_2}(R_B))$ belong to the G-orbit of $F^{-n_2}(R_B)$, there exists a uniform upper bound $M(n_2)$ on the length $m-1$ of the corresponding path $h_1, \ldots, h_m$ in $\Omega_0(W_+, R) \times \{0\}$. We can choose n_2 bigger than the number n_1 from Lemma 5.10.

Let $g_1, g_2 \in \Omega_0(W_+, R)$ be such that $g_1^{-1}g_2 \in B$. Since $g_1^{-1}g_2 \in B$, $g_1(x_0), g_2(x_0) \in g_1(R_B)$. It follows that there exists a path $h_1, \ldots, h_m \in \Omega_0(W_+, R)$ such that $m < M$, and $F^{-n_2}(g_1(x_0)) \in h_1(R)$, $F^{-n_2}(g_2(x_0)) \in h_m(R)$. The last two conditions are equivalent to $\phi^{-n_2}(g_1)(x_0) \in h_1(R)$ and $\phi^{-n_2}(g_2)(x_0) \in h_m(R)$, which imply that $(\phi^{-n_2}(g_1), 0)$ and $(\phi^{-n_2}(g_2), 0)$ are connected to $(h_1, 0)$ and $(h_m, 0)$, respectively, by horizontal edges. By Lemma 5.10, we have a uniform bound on the distances from (g_1, n) to $(\phi^{-n_2}(g_1), n+n_1)$ and from (g_2, n) to $(\phi^{-n_2}(g_2), n+n_1)$, which finishes the proof. □

The proof of the following lemma is analogous.

Lemma 5.12. *For every finite set $B \subset G$ there exists $D > 0$ such that if $g_1^{-1}\phi(g_2) \in B$ for $g_1, g_2 \in \Omega_0(W_+, R)$, then the distance between (g_1, n) and $(g_2, n+1)$ in $\Xi(W_+, R)$ is not greater than D.*

Let us go back to proving Proposition 5.9. The image of $\Xi(W_+, R)$ under the inclusion map is a 1-net in Ξ'. Distance between vertices in $\Xi(W_+, R)$ is not less than the distance between them in Ξ'.

Let us show that there exists a constant $D > 1$ such that distance between $(g_1, n_1), (g_2, n_2)$ in $\Xi(W_+, R)$ is not more than D times the distance from (g_1, n_1) to (g_2, n_2) in Ξ'. Let $(g_1, n_1) = v_0, v_1, \ldots, v_n = (g_2, n_2)$ be a geodesic path in Ξ'. Since $\Xi(W_+, R)$ is a net in Ξ', there exists a constant $C > 1$ such that every such geodesic path can be replaced by a path $(g_1, n_1), v_1', \ldots, v_{n-1}', (g_2, n_2)$, where $v_i' \in \Xi(W_+, R)$, and distance from v_i' to v_{n-1}' in Ξ' is bounded from above by C. Moreover, we may assume that each v_i' belongs to the same level Ω_n' as v_i. Then v_i' and v_{i+1}' either belong to one level, or to two neighboring levels. Then Lemmas 5.11 and 5.12 finish the proof. □

Definition 5.13. Let G be the group of deck transformations of the splitting $\pi : \mathcal{M} \longrightarrow \mathcal{X}$, and let $F : \mathcal{M} \longrightarrow \mathcal{M}$ be a lift of f with a fixed point x_0. A set $\Sigma \subset G$ is a *coarse stable (resp. unstable) plaque* if the stable plaque $W_+(x_0)$ (resp. unstable plaque $W_-(x_0)$) and the set Σ are of finite Hausdorff distance from each other.

Recall that two subsets A_1, A_2 of a metric space (X, d) are of a finite Hausdorff distance from each other if there exists $D > 0$ such that for every $x \in A_1$ there exists $y \in A_2$, and for every $y \in A_2$ there exists $x \in A_1$ such that $d(x, y) < D$.

Theorem 5.14. *Let $(\mathcal{X}, f)$ be a connected and locally connected Smale space. Suppose that there exists a splitting $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ and a lift $F : \mathcal{M} \longrightarrow \mathcal{M}$ of f with a fixed point x_0. Let ϕ be the associated automorphism of G, and let Σ_+ and Σ_- be coarse stable and unstable plaques of x_0. Then $(\mathcal{X}, f)$ is uniquely determined, up to topological conjugacy by the quadruple $(G, \phi, \Sigma_+, \Sigma_-)$.*

Proof. The dynamical system $(\mathcal{X}, f)$ is uniquely determined by the G-space $\mathcal{M}$ and the map $F : \mathcal{M} \longrightarrow \mathcal{M}$. The group G acts on the plaques W_+ and W_- by the actions

$$g : x \mapsto [g(x), x], \qquad g : x \mapsto [x, g(x)],$$

respectively. The action of G on $\mathcal{M} \cong W_+ \times W_-$ is reconstructed from these actions by the formula

$$g(x) = [[g(x), x], [x, g(x)]].$$

Similarly, the map $F : \mathcal{M} \longrightarrow \mathcal{M}$ is determined by the action of F on W_+ and W_-, since

$$F([x, y]) = [F(x), F(y)].$$

Consequently, it is enough to show that the quadruple $(G, \phi, \Sigma_+, \Sigma_-)$ uniquely determines the dynamical systems (W_+, G), (W_+, F), (W_-, G), and (W_-, F), up to topological conjugacy. Let us prove that the triple (G, ϕ, Σ_+) uniquely determines the dynamical systems (W_+, G) and (W_+, F). The same proof will show that (G, ϕ, Σ_-) uniquely determines (W_-, G) and (W_-, F).

Let $R \subset \mathcal{M}$ be a relatively compact open rectangle such that $x_0 \in R$ and $\bigcup_{g \in G} g(R) = \mathcal{M}$. Let $\Omega_0(W_+, R)$ be, as before, the set of elements $g \in G$ such that $g(R) \cap W_+ \neq \emptyset$.

For a set $\Sigma \subset G$ and a finite generating set S of G, denote by $\Xi(\Sigma, S)$ the graph with the set of vertices $\Sigma \times \mathbb{Z}$ in which two vertices are adjacent either if they are of the form (g, n) and (gs, n) for $g, gs \in \Sigma$ and $s \in S$, or of the form (g, n) and $(\phi^{-1}(gs), n + 1)$ for $g, \phi^{-1}(gs) \in \Sigma$ and $s \in S$. Note that the map $(g, n) \mapsto (g, n + 1)$ is an automorphism of $\Xi(\Sigma, S)$.

If A is big enough, then $\Sigma_+ A$ contains $\Omega_0(W_+, R)$. Then, by Proposition 5.9, the identical embedding $\Xi(W_+, R) \hookrightarrow \Xi(\Sigma_+ A, S)$ is a quasi-isometry, provided S is big enough. It follows then from Theorem 5.7 that $\Xi(\Sigma_+ A, S)$ is Gromov hyperbolic, and that the boundary of $\Xi(\Sigma_+ A, S)$ minus the common limit ω of quasi-geodesic paths of the form $(g_n, -n)$, $n \geq 1$, is homeomorphic to W_+. Moreover, it follows directly from Theorem 5.7 that the natural homeomorphism $\Phi : \partial\Xi(\Sigma_+ A, S) \setminus \omega \longrightarrow W_+$ maps the limit of a sequence $(g_n, n) \in \Xi(\Sigma_+ A, S)$ to the limit of the sequence $[F^n(g_n(x_0)), x_0] = [\phi^n(g_n)(x_0), x_0] \in W_+$.

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Consequently, the homeomorphism Φ conjugates $F : W_+ \longrightarrow W_+$ with the map on the boundary of $\Xi(\Sigma_+ A, S)$ induced by the automorphism $(g, n) \mapsto (g, n+1)$. This shows that the dynamical system (F, W_+) is uniquely determined by (G, ϕ, Σ_+).

It remains to show that for every $g \in G$ the homeomorphism $g : x \mapsto [g(x), x_0]$ of W_+ is uniquely determined by (G, ϕ, Σ_+) and g.

Let ξ be the limit of a sequence $(g_n, n) \in \Xi(\Sigma_+ A, S)$, where $s_n = g_n^{-1}\phi(g_{n+1}) \in S$ for all $n \geq 0$. Note that every point of $\partial\Xi(\Sigma_+ A, S)$ can be represented this way, provided S is big enough (see Theorem 5.7).

There exists $\epsilon > 0$ such that for every $x \in \mathcal{M}$ there exists $g \in G$ such that the ϵ-neighborhood of x is contained in $g(R)$. Let $x \in W_+$. Choose for every $n \geq 0$ an element $g_n \in G$ such that the ϵ-neighborhood of $F^{-n}(x)$ is contained in $g_n(R)$. Then x is contained in $F^n(g_n(R))$. In particular, $F^n(g_n(R)) \cap F^{n+1}(g_{n+1}(R)) \neq \emptyset$, i.e., the sequence (g_n, n) is a path in $\Xi(W_+, R)$, and its limit in $\partial\Xi(W_+, R)$ is mapped by the natural homeomorphism to x.

The rectangles $gF^n(g_n(R))$ contain $g(x)$ for all n. Since F is expanding in the unstable direction, the sets $gF^n(g_n(R))$ intersects W_+, i.e., $\phi^{-n}(g)g_n \in \Omega_0(W_+, R)$, for all n big enough.

The limit of the intersections of $gF^n(g_n(R))$ with W_+ is equal to $[x_0, g(x)]$. It follows that $(\phi^{-n}(g)g_n, n)$, where n is big enough, is a path in $\Xi(W_+, R)$ converging to the point of $\partial\Xi(W_+, R)$ corresponding to $[x_0, g(x)] \in W_+$, i.e., to the image of x under the action of g on W_+.

Note that the left multiplication by g preserves the distances between the vertices of the graph $\Xi(\Sigma_+ A, S)$ (when the images of the vertices belong to the graph). It also follows from the classical properties of Gromov hyperbolic graphs that there exists a constant Δ_1 such that if two paths (g_n, n) and (h_n, n) of $\Xi(\Sigma_+ A, S)$ converge to the same point of the boundary, then the distance between (g_n, n) and (h_n, n) is less than Δ_1 for all n big enough.

It follows that the action of g on W_+ can be modeled on the boundary of $\Xi(\Sigma_+ A, S)$ by the following rule. Take a path $(g_n, n) \in \Xi(\Sigma_+ A, S)$ converging to a point $\xi \in \partial\Xi(\Sigma_+ A, S)$. If (gg_n, n) for n big enough belong to $\Xi(\Sigma_+ A, S)$, then its limit is $g(\xi)$. Since $\Xi(W_+, R) \subset \Xi(\Sigma_+ A, S)$, this rule will determine the action of g on $\partial\Xi(\Sigma_+ A, S) \setminus \{\omega\}$. □

Theorem 5.15. *Let $(\mathcal{X}_1, f_1)$ and $(\mathcal{X}_2, f_2)$ be connected and locally connected Smale spaces. Suppose that there exist fixed points of f_i and splittings $\pi_i : \mathcal{M}_i \longrightarrow \mathcal{X}_i$. Let G_i be the groups of deck transformations of the splittings. Let F_i be lifts of f_i, with fixed points $x_i \in \mathcal{M}_i$. If there exists a continuous map $\Phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ and an isomorphism $\Psi : G_1 \longrightarrow G_2$ such that $\Phi(x_1) = x_2$, and*

$$\Phi(F_1(x)) = F_2(\Phi(x)), \qquad \Phi(F_1(g(x))) = F_2(\psi(g)(\Phi(x)))$$

for all $x \in \mathcal{M}_1$ and $g \in G_1$, then $(\mathcal{X}_1, f_1)$ and $(\mathcal{X}_2, f_2)$ are topologically conjugate.

Note that we do not require Φ to be a homeomorphism.

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Proof. The map Φ is proper as an equivariant map between two proper actions (see, for instance [25, Lemma 5.2]).

Let U be a compact neighborhood of $x_2 \in \mathcal{M}_2$. Then $\bigcup_{n\ge 1}\bigcap_{k\ge n} F_2^k(U)$ is equal to the unstable plaque $W_-(x_2)$ in $\mathcal{M}_2$. Similarly,

$$\bigcup_{n\ge 1}\bigcap_{k\ge n} F_1^k(\Phi^{-1}(U)) = \Phi^{-1}\left(\bigcup_{n\ge 1}\bigcap_{k\ge n} F_2^k(U)\right)$$

is equal to the unstable plaque $W_-(x_1)$ in $\mathcal{M}_1$. It follows that $\Phi^{-1}(W_-(x_2)) = W_-(x_1)$. Similarly, $\Phi^{-1}(W_+(x_2)) = W_+(x_1)$.

Let K_2 be a compact subset of $\mathcal{M}_2$ such that K_2 and $K_1 = \Phi^{-1}(K)$ are G_i-transversals, i.e., intersect every G_i-orbit. They exist, since the actions of G_i are co-compact, proper, and the map Φ is continuous and proper.

Then $g_2(K_2) \cap W_-(x_2) \neq \emptyset$ for $g_2 \in G_2$ is equivalent to $\Psi^{-1}(g_2)(K_1) \cap W_-(x_1) \neq \emptyset$. The same is true for the stable plaques $W_+(x_1)$ and $W_+(x_2)$. It follows that the sets $\Sigma_{i,*} = \{g_i(K_i) \cap W_*(x_i) \neq \emptyset\}$ for $* \in \{+,-\}$ and $i \in \{1,2\}$ are coarse stable and unstable plaques for which we can use Theorem 5.14 to show that $(\mathcal{X}_1, f_1)$ and $(\mathcal{X}_2, f_2)$ are topologically conjugate. □

In fact, it follows from Theorem 5.7 and the proof of Theorem 5.15 that any continuous map Φ satisfying the conditions of Theorem 5.15 is a homeomorphism.

6. Smale spaces with virtually nilpotent splitting

Let L be a simply connected nilpotent Lie group. Let G be a finitely generated subgroup of $\mathrm{Aut}\,L \ltimes L$ such that the action of G on L is free, proper, and co-compact. Here we identify the elements of A with the transformations $g \mapsto \alpha(g)\cdot h$ of L, where $\alpha \in \mathrm{Aut}\,L$ and $h \in L$.

Let $F \in \mathrm{Aut}\,L$ be a *hyperbolic automorphism* of L (i.e., such that its differential DF at the identity of L has no eigenvalues of absolute value one). Then F induces an automorphism ϕ of $\mathrm{Aut} \ltimes L$ by conjugation. Suppose that G is invariant under this automorphism. Then F induces an Anosov homeomorphism $f : G\backslash L \longrightarrow G\backslash L$. Such homeomorphisms are called *hyperbolic infra-nilmanifold automorphisms.*

The aim of this section is to prove the following description of locally connected Smale spaces that have a splitting with a virtually nilpotent group of deck transformation.

Theorem 6.1. *Let $(\mathcal{X}, f)$ be a Smale space such that $\mathcal{X}$ is connected and locally connected, and there exists a splitting $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ with a virtually nilpotent group of deck transformations. Then $(\mathcal{X}, f)$ is topologically conjugate to a hyperbolic infra-nilmanifold automorphism.*

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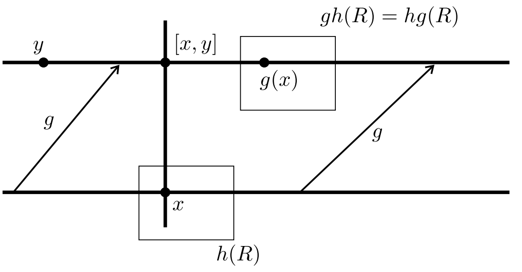

Fig. 2. Central elements.

Proof. Let $(\mathcal{X}, f)$ satisfy conditions of the theorem. Let us assume at first that the group G of deck transformations is nilpotent and torsion free.

Let F be a lift of f, and let $F(x_0) = x_1$. Then the map $g \mapsto \phi(g)$ defined by $F(g(x_0)) = \phi(g)(x_1)$ is an automorphism of G.

Denote by $Z(G)$ the *center* of G, i.e., the set of elements of G that commute with every element of G.

The group $Z(G)$ is obviously abelian and torsion free. It is finitely generated, since all subgroups of a finitely generated nilpotent group are finitely generated (see [32, 5.2.17]). Consequently, $Z(G)$ is isomorphic to $\mathbb{Z}^d$ for some d.

Lemma 6.2. *Let $g \in Z(G)$. There exist positive constants D_- and D_+ such that for every stable (resp. unstable) plaque V of $\mathcal{M}$ and any $x \in V$, $y \in g(V)$ we have $d_-(x, [x, y]) \le D_-$ (resp. $d_+(x, [y, x]) \le D_+$).*

Proof. Let us prove the lemma for stable plaques. Note that $[x, y]$ is equal to the intersection of $g(V)$ with $W_-(x)$, and so does not depend on the choice of $y \in g(V)$. Therefore, it is enough to show that $d_-(x, [x, g(x)])$ is bounded for all $x \in \mathcal{M}$, see Fig. 2. Let $R \subset \mathcal{M}$ be a compact rectangle such that $\pi(R) = \mathcal{X}$. Let D be an upper bound on the value of $d_-(x, [x, y])$ for $x \in R$ and $y \in g(R)$. It is finite, since there exists a compact rectangle P such that $P \supset R \cup g(R)$ (see also Proposition 5.6).

For every $x \in \mathcal{M}$ there exists $h \in G$ such that $h(x) \in R$. Then $d_-(x, [x, g(x)]) = d_-(h(x), [h(x), hg(x)]) = d_-(h(x), [h(x), gh(x)]) < D$. □

Denote for $g \in Z(G)$

$$D_-(g) = \sup_{x \in \mathcal{M}} d_-(x, [x, g(x)]), \qquad D_+(g) = \sup_{x \in \mathcal{M}} d_+(x, [g(x), x]),$$

which are finite by Lemma 6.2. Note that we obviously have

$$D_+(g_1 g_2) \le D_+(g_1) + D_+(g_2), \qquad D_-(g_1 g_2) \le D_-(g_1) + D_-(g_2) \tag{7}$$

for all $g_1, g_2 \in Z(G)$.

Let $\lambda \in (0,1)$ and $C > 1$ be such that for any two stably (resp. unstably) equivalent points $x, y \in \mathcal{X}$ we have

$$d_+(f^n(x), f^n(y)) \le C\lambda^n d_+(x,y)$$

(resp. $d_-(f^{-n}(x), f^{-n}(y)) \le C\lambda^n d_-(x,y)$) for all $n \ge 0$. Then the same estimates will hold for F and $x, y \in \mathcal{M}$ belonging to one stable (resp. unstable) plaque.

Note that the center $Z(G)$ is characteristic (i.e., invariant under automorphisms of G), hence $\phi(Z(G)) = Z(G)$.

Proposition 6.3. *For every $g \in Z(G)$ and $n \ge 0$ we have*

$$D_+(\phi^n(g)) \le C\lambda^n D_+(g), \qquad D_-(\phi^{-n}(g)) \le C\lambda^n D_-(g).$$

Proof. Let us prove the first inequality. The second is proved the same way.

Let V be an unstable plaque of $\mathcal{M}$, and let $x \in V$ and $y \in \phi^n(g)(V)$ be such that x and y belong to the same stable plaque. It is enough to prove that $d_+(x,y) \le C\lambda^n D_+(g)$.

The points $F^{-n}(x)$ and $F^{-n}(y)$ belong to one stable plaque, and $F^{-n}(x) \in F^{-n}(V)$, $F^{-n}(y) \in F^{-n}(\phi^n(g)(V)) = g(F^{-n}(V))$, hence $d_+(F^{-n}(x), F^{-n}(y)) \le D_+(g)$. But this implies $d_+(x,y) \le C\lambda^n d_+(x,y)$. □

Proposition 6.4. *For every finite set $S \subset Z(G)$ there exists a constant $D_S > 0$ satisfying the following condition. For every finite set $A \subset Z(G)$ there exists n_0 such that for all $n \ge n_0$, $g_1, g_2 \in \phi^n(A)\phi^{n-1}(S)\phi^{n-2}(S)\cdots\phi(S)S$, and every unstable plaque V we have*

$$d_+(x, [y,x]) < D_S$$

for all $x \in g_1(V)$ and $y \in g_2(V)$.

Proof. Let Δ_S and Δ_A be upper bounds on $D_+(g)$ for $g \in S$ and $g \in A$, respectively.

Then, by (7) and Proposition 6.3, we have, for all $h \in A$, $g_i \in S$, and all n big enough,

$$D_+(\phi^n(h)\phi^{n-1}(g_1)\cdots\phi(g_{n-1})g_n) \le C\lambda^n\Delta_A + C(\lambda^{n-1} + \cdots + \lambda + 1)\Delta_S < 1 + \frac{C\Delta_S}{1-\lambda}.$$

It follows that we can take $D_S = 2 + \frac{2C\Delta_S}{1-\lambda}$. □

Proposition 6.5. *The restriction of the automorphism ϕ to $Z(G) \cong \mathbb{Z}^d$ is hyperbolic, i.e., has no eigenvalues of absolute value 1.*

Proof. Suppose that on the contrary, there exists an eigenvalue $\cos\alpha + i\sin\alpha$ of ϕ of absolute value 1. Suppose at first that $\alpha \notin \pi \cdot \mathbb{Z}$. Then there exists a two-dimensional subspace $L \leq \mathbb{R}^d$ and a Euclidean structure on it such that ϕ acts on L as a rotation by the angle α. Denote $K = \{(x_i)_{i=1}^d \in \mathbb{R}^d \ : \ |x_i| < 1\}$, and let S be the set of elements $g \in Z(G) = \mathbb{Z}^d$ such that $\phi(K) \cap (K + g) \neq \emptyset$ or $\phi^{-1}(K) \cap (K + g) \neq \emptyset$. The set S is obviously finite.

Let $R > 0$ be arbitrary, and consider the circle γ of radius R in L with center in the origin. Then $\phi(\gamma) = \gamma$. Let A_R be the set of elements $g \in Z(G)$ such that $K + g \cap \gamma \neq \emptyset$. It is finite and non-empty. Note that union of the sets A_R for all $R > 0$ is infinite.

Let h be an arbitrary element of A_R, and let $x \in K + h \cap \gamma$. Then $\phi^{-1}(x) \in \gamma$, and there exists $g \in \mathbb{Z}^d$ such that $\phi^{-1}(x) \in K + g$. Then $g \in A_R$, and $x \in \phi(K) + \phi(g) \cap K + h$. It follows that $K + h - \phi(g) \cap \phi(K) \neq \emptyset$, so that $h - \phi(g) \in S$. We see that $h = \phi(g) + (h - \phi(g)) \in \phi(A_R) + S$. We have proved that $A_R \subset \phi(A_R) + S$. It is proved the same way that $A_R \subset \phi^{-1}(A_R) + S$. By induction we conclude that

$$A_R \subset \phi^n(A_R) + \phi^{n-1}(S) + \cdots + \phi(S) + S$$

and

$$A_R \subset \phi^{-n}(A_R) + \phi^{-(n-1)}(S) + \cdots + \phi^{-1}(S) + S$$

for all $n \geq 1$.

Fix an arbitrary point $x_0 \in \mathcal{M}$. Since $\phi^n(A_R) + \phi^{n-1}(S) + \cdots \phi(S) + S \supset A_R$ for all n, it follows from Proposition 6.4 that there exists $D_S > 0$, not depending on R, such that $d_+(x_0, [g(x_0), x_0]) < D_S$ and $d_-(x_0, [x_0, g(x_0)]) < D_S$ for all $g \in A_R$. It follows that $g(x_0)$ belongs to the rectangle $[B_+, B_-]$, where $B_\pm$ are the balls of radius D_S with center in x_0 in the corresponding plaque containing x_0. Note that the set $\{g \in G \ : \ g(x_0) \in [B_+, B_-]\}$ is finite, does not depend on R, and contains A_R. But this is a contradiction.

The case when the eigenvalue is equal to ± 1 is similar (with one-dimensional space L). $\square$

Let E_+ (resp. E_-) be the sum of the root subspaces of $\mathbb{R}^d$ of the eigenvalues λ of ϕ such that $|\lambda| < 1$ (resp. $|\lambda| > 1$). We have $\mathbb{R}^d = E_+ \oplus E_-$. Denote by P_+ and $P_- = 1 - P_+$ the projections onto E_+ and E_-, respectively.

Denote $K = \{(x_i)_{i=1}^d \in \mathbb{R}^d \ : \ |x_i| < 1\}$. Let $S \subset Z(G)$ be a finite set containing all elements $g \in Z(G)$ such that $K + g \cap (\phi(K) \cup \phi^{-1}(K)) \neq \emptyset$.

Proposition 6.6. *For every point $x \in E_+$ there exists a sequence $g_i \in S$, $i = 1, 2, \ldots$, and an element $g_0 \in Z(G)$ such that*

$$x = \lim_{n\to\infty} P_+(\phi^n(g_n) + \phi^{n-1}(g_{n-1}) + \cdots + \phi(g_1) + g_0).$$

There exists a finite set $N \subset Z(G)$ such that an equality

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$$\lim_{n\to\infty} P_+(\phi^n(g_n)+\phi^{n-1}(g_{n-1})+\cdots+\phi(g_1)+g_0) = \lim_{n\to\infty} P_+(\phi^n(g'_n)+\phi^{n-1}(g'_{n-1})+\cdots+\phi(g'_1)+g'_0)$$

holds for $g_i, g'_i \in S$, $i \ge 1$, and $g_0, g'_0 \in Z(G)$ if and only if there exists a sequence $h_n \in N$ such that

$$\phi^n(g'_n)+\phi^{n-1}(g'_{n-1})+\cdots+\phi(g'_1)+g'_0 = \phi^n(h_n+g_n)+\phi^{n-1}(g_{n-1})+\cdots+\phi(g_1)+g_0$$

for all n big enough.

Proof. The sets $K+g$ cover $\mathbb{R}^d$ for $g \in Z(G) = \mathbb{Z}^d$, and the group $Z(G)$ is ϕ-invariant, hence for every $x \in E_+$ and $n \ge 0$ there exists $h_n \in Z(G)$ such that $x \in \phi^n(K)+h_n$. We have then $\phi^{n-1}(K)+h_{n-1} \cap \phi^n(K)+h_n \ne \emptyset$, hence $\phi^{-1}(K) \cap K + \phi^{-n}(h_n) - \phi^{-n}(h_{n-1}) \ne \emptyset$ which implies that $\phi^{-n}(h_n) - \phi^{-n}(h_{n-1}) = g_n \in S$, i.e., $h_n = \phi^n(g_n) + h_{n-1}$. It follows that there exists a sequence $g_i \in S$ such that $h_n = \phi^n(g_n)+\phi^{n-1}(g_{n-1})+\cdots+\phi(g_1)+h_0$.

Note that since $x \in \phi^n(K)+h_n$, we have $\|P_+(h_n) - x\| < C\lambda^n$ for some constant C. It follows that $x = \lim_{n\to\infty} P_+(h_n)$.

Note that the set of all limits $\lim_{n\to\infty} P_+(\phi^n(g_n)+\phi^{n-1}(g_{n-1})+\cdots+\phi(g_1))$ for all choices of $g_i \in S$ is a bounded subset T_+ of E_+.

Suppose that

$$\lim_{n\to\infty} P_+(\phi^n(g_n)+\phi^{n-1}(g_{n-1})+\cdots+\phi(g_1)+h) = \lim_{n\to\infty} P_+(\phi^n(g'_n)+\phi^{n-1}(g'_{n-1})+\cdots+\phi(g'_1)+h')$$

for $g_i, g'_i \in S$ and $h, h' \in Z(G)$.

Then for every $n \ge 0$ we have

$$P_+(\phi^n(g_n)+\phi^{n-1}(g_{n-1})+\cdots+\phi(g_1)+h) - P_+(\phi^n(g'_n)+\phi^{n-1}(g'_{n-1})+\cdots+\phi(g'_1)+h') = \big(P_+(\phi^{n+1}(g'_{n+1}))+P_+(\phi^{n+2}(g'_{n+2}))+\cdots\big) - \big(P_+(\phi^{n+1}(g_{n+1}))+P_+(\phi^{n+2}(g_{n+2}))+\cdots\big) \in \phi^n(T_+ - T_+).$$

It follows that

$$P_+((g_n+\phi^{-1}(g_{n-1})+\cdots+\phi^{-(n-1)}(g_1)+\phi^{-n}(h)) - (g'_n+\phi^{-1}(g'_{n-1})+\cdots+\phi^{-(n-1)}(g'_1)+\phi^{-n}(h'))) \in T_+ - T_+.$$

Since ϕ^{-1} is contracting on E_-, there exists a compact set $T_- \subset E_-$ such that for any h and any sequence $g_i \in S$ we have

$$P_-(g_n+\phi^{-1}(g_{n-1})+\cdots+\phi^{-(n-1)}(g_1)+\phi^{-n}(h)) \in T_-$$

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for all n big enough.

It follows that for all n big enough the difference

$$(g_n + \phi^{-1}(g_{n-1}) + \cdots + \phi^{-(n-1)}(g_1) + \phi^{-n}(h)) - (g'_n + \phi^{-1}(g'_{n-1}) + \cdots + \phi^{-(n-1)}(g'_1) + \phi^{-n}(h'))$$

belongs to a bounded set $T = (T_+ - T_+) \oplus (T_- - T_-)$, hence we can take $N = T \cap \mathbb{Z}^d$. □

Fix a stable plaque $W_+ = W_+(x_0)$ of $\mathcal{M}$. The group G acts on W_+ by $x \mapsto [g(x), x]$, since G preserves the direct product structure of $\mathcal{M}$.

Let $v \in \mathbb{R}^d$, and denote $v_+ = P_+(v)$ and $v_- = P_-(v)$. Using Proposition 6.6, find a sequence $g_i \in S$, $i \geq 1$, and $g_0 \in Z(G)$ such that

$$v_+ = \lim_{n\to\infty} P_+(\phi^n(g_n) + \cdots + \phi(g_1) + g_0),$$

and define for $x \in W_+$

$$v_+(x) = \lim_{n\to\infty} [(\phi^n(g_n) + \cdots + \phi(g_1) + g_0)(x), x]. \tag{8}$$

We also define for $x \in W_-$, where W_- is an unstable plaque:

$$v_-(x) = \lim_{n\to\infty} [x, (\phi^{-n}(g_n) + \cdots + \phi^{-1}(g_1) + g_0)(x)], \tag{9}$$

where $g_i \in S$, for $i \geq 1$, and $g_0 \in Z(G)$ are such that

$$v_- = \lim_{n\to\infty} P_-(\phi^{-n}(g_n) + \cdots + \phi^{-1}(g_1) + g_0).$$

(Replacing in Proposition 6.6 ϕ by ϕ^{-1} and E_+, P_+ by E_-, P_-, we see that such a sequence g_n exists.)

Proposition 6.7. *The limit (8) exists and depends only on v_+ and x. The limit (9) exists and depends only on v_- and x.*

Proof. It follows directly from (7), Propositions 6.3 and 6.6. □

Theorem 6.8. *The limits (8) and (9) define continuous actions of E_+ and E_- on $W_+ = W_+(x_0)$ and $W_- = W_-(x_0)$, respectively. Their direct sum is a continuous action of $\mathbb{R}^d$ on $\mathcal{M}$. This action satisfies the following conditions:*

(1) *it is free and proper;*
(2) *its restriction onto $Z(G) = \mathbb{Z}^d < \mathbb{R}^d$ coincides with the original action of $Z(G)$ on $\mathcal{M}$;*

(3) *it preserves the direct product structure, i.e.,* $v([x, y]) = [v(x), v(y)]$ *for all* $v \in \mathbb{R}^d$ *and* $x, y \in \mathcal{M}$*;*
(4) $F(v(x)) = \phi(v)(F(x))$ *for all* $x \in \mathcal{M}$ *and* $v \in \mathbb{R}^d$*;*
(5) *the action commutes with* G*, i.e.,* $v(g(x)) = g(v(x))$ *for all* $x \in \mathcal{M}$*,* $g \in G$*, and* $v \in \mathbb{R}^d$*;*
(6) *if* $g(x) = v(x)$ *for* $g \in G$ *and* $v \in \mathbb{R}^d$*, then* $g = v \in Z(G)$*.*

Proof. The fact that conditions (8) and (9) define actions follows directly from the fact that the limits do not depend on S and the choice of the sequences g_i.

Let us prove that the action is continuous. It is enough to prove that the action of E_+ on W_+ is continuous. We have to show that for every $v_1 \in E_+$, $x \in W_+$, and $\epsilon > 0$ there exists $\delta > 0$ such that if $v_2 \in E_+$ and $y \in W_+$ are such that $\|v_1 - v_2\| < \delta$ and $d_+(x, y) < \delta$, then $d_+(v_1(x), v_2(y)) < \epsilon$.

Take an arbitrary $\epsilon > 0$. For every n there exists $\delta_1(n)$ and a sequence $g_0 \in H$, $g_i \in S$, $i \geq 1$, such that $v_1 = \lim_{m\to\infty} P_+(g_0 + \phi(g_1) + \cdots + \phi^m(g_m))$ and all points v_2 in the $\delta_1(n)$-neighborhood of v_1 can be represented as limits $v_2 = \lim_{m\to\infty} P_+(h_0 + \phi(h_1) + \cdots + \phi^m(h_m))$ for $h_0 \in Z(G)$, and $h_i \in S$, $i \geq 1$, such that $h_i = g_i$ for $i = 0, 1, \ldots, n$ (see the proof of Proposition 6.6 and use Lebesgue's covering lemma). There exists $\delta_2(n)$ such that if $y \in W_+$ is such that $d_+(x, y) < \delta_2(n)$, then

$$d_+([(g_0 + \phi(g_1) + \cdots + \phi^n(g_n))(x), x], [(g_0 + \phi(g_1) + \cdots + \phi^n(g_n))(y), y]) < \epsilon/2,$$

since the function $y \mapsto [(g_0 + \phi(g_1) + \cdots + \phi^n(g_n))(y), y]$ is continuous. There exist constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$d_+(u(z), [(g_0 + \phi(g_1) + \cdots + \phi^n(g_n))(z), z]) < C\lambda^n$$

for all $z \in W_+$ and $u \in E_+$ such that $u = \lim_{m\to\infty} P_+(g_0 + \phi(g_1) + \cdots + \phi^m(g_m))$ for $g_0 \in Z(G)$ and $g_i \in S$ for $i \geq 1$.

Take $n \geq -\frac{\log(\epsilon/4C)}{\log\lambda}$. Then for all $v_2 \in E_+$ and $y \in W_+$ such that $\|v_1 - v_2\| < \delta_1(n)$ and $d_+(x, y) < \delta_2(n)$ we have

$$\begin{aligned} d_+(v_1(x), v_2(y)) \leq \\ & d_+(v_1(x), [(g_0 + \phi(g_1) + \cdots + \phi^n(g_n)(x), x]) + \\ & d_+([(g_0 + \phi(g_1) + \cdots + \phi^n(g_n)(x), x], [(g_0 + \phi(g_1) + \cdots + \phi^n(g_n)(y), y]) + \\ & d_+(v_2(y), [(g_0 + \phi(g_1) + \cdots + \phi^n(g_n)(y), y]) \leq \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon. \end{aligned}$$

Which shows that the action of E_+ on W_+ is continuous.

The same arguments (using Proposition 6.3 and inequalities (7)) as in the proof of the criterion of equality of two limits in Proposition 6.6 show that an equality

$$\lim_{n\to\infty}[(\phi^n(g_n) + \cdots \phi(g_1) + g_0)(x), x] = \lim_{n\to\infty}[(\phi^n(g'_n) + \cdots \phi(g'_1) + g'_0)(x), x]$$

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for $g_i, g_i' \in S$, $i \geq 1$, and $g_0, g_0' \in Z(G)$ is equivalent to the equality

$$\lim_{n\to\infty} P_+(\phi^n(g_n) + \cdots \phi(g_1) + g_0) = \lim_{n\to\infty} P_+(\phi^n(g_n') + \cdots \phi(g_1') + g_0').$$

This (and a similar statement for P_- and the action on W_-) shows that the action of $\mathbb{R}^d$ is free.

Let us show that the action is proper. Let $B \subset \mathcal{M}$ be a compact set. We have to show that the set $\{v \in \mathbb{R}^d : v(B) \cap B \neq \emptyset\}$ is compact. It is closed, since the action is continuous.

Denote $K = \{(x_i)_{i=1}^d \in \mathbb{R}^d : |x_i| \leq 1\}$. Then for every $v \in \mathbb{R}^d$ there exists $h \in Z(G) = \mathbb{Z}^d$ such that $v - h \in K$. The set $K(B) = \{v(x) : x \in B, v \in K\}$ is compact, since the action is continuous and the sets B and K are compact. The action of G on $\mathcal{M}$ is proper, hence the set A of elements $h \in Z(G)$ such that $h(K(B)) \cap B \neq \emptyset$ is finite.

Suppose that $x \in B$ and $v \in \mathbb{R}^d$ are such that $v(x) \in B$. There exists $h \in Z(G)$ such that $v - h \in K$. Then $v(x) = (h + v - h)(x) \in h(K(B)) \cap B$, hence $h \in A$, so that $v \in K + A$. But the set $K + A$ is compact, which proves that the action of $\mathbb{R}^d$ on $\mathcal{M}$ is proper.

The proof of statements (2)–(5) is straightforward, using the fact that the action does not depend on the choice of S.

Let us prove the last statement. Suppose that $g(x) = v(x)$ for $g \in G$ and $v \in \mathbb{R}^d$. Then g leaves invariant the orbit $\mathbb{R}^d(x)$ of x. Let G_1 be the group of all elements leaving the $\mathbb{R}^d(x)$ invariant. The action of $Z(G)$ on $\mathbb{R}^d(x)$ is co-compact, the action of G on $\mathcal{M}$ is proper, hence the index of $Z(G)$ in G_1 is finite, i.e., the image of G_1 in $G/Z(G)$ is finite. But $G/Z(G)$ is torsion free (see [32, 5.2.19]). Consequently, $G_1 = Z(G)$. Since the action of $\mathbb{R}^d$ on $\mathbb{R}^d(x)$ is free, this implies that $g = v \in Z(G)$. □

Proposition 6.9. *If G is abelian, then the action of $\mathbb{R}^d$ on $\mathcal{M}$ is transitive (i.e., has exactly one orbit).*

Proof. It is enough to show that for every point $x \in W_+$ there exists a sequence $g_i \in S$, $i \geq 1$, and an element $g_0 \in G$ such that

$$x = \lim_{n\to\infty} [(\phi^n(g_n) + \cdots + \phi(g_1) + g_0)(x_0), x_0]. \tag{10}$$

The action of G on $\mathcal{M}$ is co-compact, hence there exists a relatively compact open rectangle $R \subset \mathcal{M}$ containing x_0 and such that $\bigcup_{h\in G} h(R) = \mathcal{M}$.

Then for every $x \in W_+$ and every $n \geq 0$ there exists $h_n \in G$ such that $x \in h_n(F^n(R))$. Assume that S is big enough so that it contains all elements $h \in G$ such that $h(R) \cap (F(C) \cup F^{-1}(C)) \neq \emptyset$. Then the same arguments as in the proof of Proposition 6.6 show that there exists a sequence $g_i \in S$, $i \geq 1$, and an element $g_0 \in G$ for which (6.6) holds. □

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Theorem 6.10. *Let $(\mathcal{X}, f)$ be a locally connected and connected Smale space which has a splitting with a free abelian group of deck transformations $G \cong \mathbb{Z}^d$. Let ϕ be the automorphism of G induced by a lift of f. Then $(\mathcal{X}, f)$ is topologically conjugate to the hyperbolic automorphism of the torus $\mathbb{R}^d/\mathbb{Z}^d$ induced by ϕ. In particular, $(\mathcal{X}, f)$ has a fixed point.*

Proof. By Proposition 6.9, the action of $\mathbb{R}^d$ on $\mathcal{M}$ defined in Theorem 6.8 is transitive.

Fix a basepoint $x_0 \in \mathcal{M}$, define $\rho_0 : \mathbb{R}^d \longrightarrow \mathcal{M}$ by $v \mapsto v(x_0)$. The map ρ_0 is a homeomorphism, since it is continuous, bijective, and proper. Denote $v_0 = \rho_0^{-1}(F(x_0))$, i.e., $v_0 \in \mathbb{R}^d$ is such that $v_0(x_0) = F(x_0)$.

Then the map $\phi_0 = \rho_0^{-1} F \rho_0 : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies

$$\phi_0(v) = \rho_0^{-1}(F(v(x_0)) = \rho_0^{-1}(\phi(v)(F(x_0))) = \rho_0^{-1}((\phi(v) + v_0)(x_0)) = \phi(v) + v_0.$$

The linear operator $1 - \phi$ is invertible, since ϕ is hyperbolic. Therefore, there exists $w_0 \in \mathbb{R}^d$ such that $w_0 - \phi(w_0) = v_0$, i.e., $\phi_0(w_0) = \phi(w_0) + v_0 = w_0$. Define then

$$\rho_1(v) = \rho_0(v + w_0).$$

We have then

$$F(\rho_1(v)) = F(\rho_0(v + w_0)) = \rho_0(\phi_0(v + w_0)) = \\ \rho_0(\phi(v + w_0) + v_0) = \rho_0(\phi(v) + \phi(w_0) + v_0) = \rho_0(\phi(v) + w_0) = \rho_1(\phi(v)).$$

The statement of the theorem follows now directly from Theorem 6.8. □

Let us go back to the case when G is torsion free nilpotent.

Proposition 6.11. *The action of $\mathbb{R}^d$ on $\mathcal{M}$ is uniformly locally Lipschitz, i.e., there exist $\epsilon > 0$ and $C > 1$ such that for every $v \in \mathbb{R}^d$ and all $x, y \in \mathcal{M}$ such that $d_{\mathcal{M}}(x, y) < \epsilon$ we have*

$$d_{\mathcal{M}}(v(x), v(y)) \le C d_{\mathcal{M}}(x, y).$$

Note that Proposition 6.11 implies that $C^{-1} d_{\mathcal{M}}(x, y) \le d_{\mathcal{M}}(v(x), v(y))$ for all $x, y \in \mathcal{M}$ such that $d_{\mathcal{M}}(x, y) \le C^{-1}\epsilon$.

Proof. By Theorem 6.8, G maps $\mathbb{R}^d$-orbits to $\mathbb{R}^d$-orbits.

Let $K = \{(x_i)_{i=1}^d \in \mathbb{R}^d \ : \ |x_i| \le 1\}$, and let $R \subset \mathcal{M}$ be a relatively compact rectangle such that $\pi(R) = \mathcal{X}$. Let $\epsilon > 0$ and $C > 1$ be such that

$$C^{-1} d_{\mathcal{M}}(x, y) \le d_+(x, [y, x]) + d_-(x, [x, y]) \le C d_{\mathcal{M}}(x, y) \tag{11}$$

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for all $x, y \in \mathcal{M}$ such that $d_{\mathcal{M}}(x, y) < \epsilon$, see Subsection 5.2, where the metric $d_{\mathcal{M}}$ is defined. We also assume that ϵ is sufficiently small so that for all $x, y \in \mathcal{M}$ such that $d_{\mathcal{M}}(x, y) < \epsilon$ there exists $g \in G$ such that $g(x), g(y) \in R$.

Let δ is such that $d_{\mathcal{M}}(v(x), v(y)) < \epsilon$ for all $v \in K$ and all $x, y \in R$ such that $d_{\mathcal{M}}(x, y) < \delta$. It exists, since the action of $\mathbb{R}^d$ is continuous, and the set K and the closure of R are compact. For all $x, y \in \mathcal{M}$ such that $d_{\mathcal{M}}(x, y) < \delta$ and all $v \in \mathbb{R}^d$ there exists $g \in G$ and $h \in Z(G)$ such that $g(x), g(y) \in R$, and $v + h \in K$. Then $d_{\mathcal{M}}(g(x), g(y)) < \delta$, hence

$$d_{\mathcal{M}}(v(x), v(y)) = d_{\mathcal{M}}(g^{-1} \cdot v \cdot g(x), g^{-1} \cdot v \cdot g(y)) = \\ d_{\mathcal{M}}((h + v)(g(x)), (h + v)(g(y))) < \epsilon.$$

We have shown that for all $x, y \in \mathcal{M}$ such that $d_{\mathcal{M}}(x, y) < \delta$ and all $v \in \mathbb{R}^d$ we have $d_{\mathcal{M}}(v(x), v(y)) < \epsilon$.

Let $x, y \in \mathcal{M}$ be such that $d_{\mathcal{M}}(x, y) < \epsilon$. Let n be the biggest positive integer such that $d_+(F^{-n}(x), F^{-n}([y, x])) < C^{-1}\delta$. Then n is equal, up to an additive constant, to $-\log d_+(x, [y, x])/\alpha_+$, where α_+ is the exponent of d_+.

We have then $d_+(v(F^{-n}(x)), v(F^{-n}([y, x])) < \epsilon$ for all $v \in \mathbb{R}^d$. Applying F^n, and using the fact that ϕ is an automorphism of $\mathbb{R}^d$, we get that $d_+(u(x), u([y, x])) \le C_1 e^{-n\alpha_+} \le C_2 d_+(x, [y, x])$ for all $u \in \mathbb{R}^d$, where C_1, C_2 are constant (not depending on x, y).

In the same way we prove that $d_-(u(x), u([x, y])) < C_3 d_-(x, [x, y])$ for all $u \in \mathbb{R}^d$, if $d_{\mathcal{M}}(x, y) < \epsilon$. It follows then from (11) that there exist $\epsilon_1 > 0$ and $C_4 > 0$ such that if $x, y \in \mathcal{M}$ are such that $d_{\mathcal{M}}(x, y) < \epsilon_1$, then $d_{\mathcal{M}}(u(x), u(y)) < C_4 d_{\mathcal{M}}(x, y)$ for all $u \in \mathbb{R}^d$. □

Let $\overline{\mathcal{M}}$ be the quotient of $\mathcal{M}$ by the $\mathbb{R}^d$-action defined in Theorem 6.8. We denote for $x \in \mathcal{M}$ by $\overline{x}$ the $\mathbb{R}^d$-orbit of x. Since G maps $\mathbb{R}^d$-orbits to $\mathbb{R}^d$-orbits, the action of G on $\mathcal{M}$ induces a well-defined action of G on $\overline{\mathcal{M}}$. Denote $\overline{G} = G/Z(G)$, it is a torsion-free finitely generated nilpotent group of nilpotency class one less than the class of G. By Theorem 6.8, $Z(G)$ is equal to the kernel of the action of G on $\overline{\mathcal{M}}$, and the action of $\overline{G}$ on $\overline{\mathcal{M}}$ is free.

The action of $\mathbb{R}^d$ on $\mathcal{M}$ descends to a free action of $\mathbb{R}^d/\mathbb{Z}^d$ on $\mathcal{X}$, whose orbits are the images of the $\mathbb{R}^d$-orbits under the map $\pi : \mathcal{M} \longrightarrow \mathcal{X}$. Let $\overline{\pi} : \overline{\mathcal{M}} \longrightarrow \overline{\mathcal{X}}$ be the corresponding map induced by $\pi : \mathcal{M} \longrightarrow \mathcal{X}$.

Denote, for $\overline{x}, \overline{y} \in \overline{\mathcal{M}}$

$$\delta(\overline{x}, \overline{y}) = \inf\{x_1 \in \overline{x}, y_1 \in \overline{y} \ : \ d_{\mathcal{M}}(x_1, y_1)\}.$$

Lemma 6.12. *There exist $\epsilon > 0, C > 1$ and a $\overline{G}$-invariant metric $\overline{d}$ on $\overline{\mathcal{M}}$ such that*

$$C^{-1}\delta(\overline{x}, \overline{y}) \le \overline{d}(\overline{x}, \overline{y}) \le C\delta(\overline{x}, \overline{y})$$

for all x, y such that $\delta(\overline{x}, \overline{y}) < \epsilon$.

Proof. The function δ is $\overline{G}$-invariant, since the metric $d_{\mathcal{M}}$ is G-invariant. Note that $\delta(\overline{x}, \overline{y}) > 0$ for all $\overline{x} \neq \overline{y}$, since the quotient map $\pi : (\mathcal{M}, d_{\mathcal{M}}) \longrightarrow (\mathcal{X}, d)$ is a local isometry, the images of $\overline{x}$ and $\overline{y}$ in $\mathcal{X}$ are compact, hence distance between any two points of $\pi(\overline{x})$ and $\pi(\overline{y})$ are bounded from below.

Let ϵ be as in Proposition 6.11. Define $\overline{d}(\overline{x}, \overline{y})$ as infimum of $\sum \delta(\overline{x}_i, \overline{x}_{i+1})$ over all sequences $\overline{x} = \overline{x}_0, \ldots, \overline{x}_n = \overline{y}$ such that $\delta(\overline{x}_i, \overline{x}_{i+1}) < \epsilon$. Note that by Proposition 6.11 there exists $C > 1$ such that $\delta(\overline{x}, \overline{y}) \leq C \sum \delta(\overline{x}_i, \overline{x}_{i+1})$, hence

$$\delta(\overline{x}, \overline{y}) \leq C\overline{d}(\overline{x}, \overline{y})$$

for all $\overline{x}, \overline{y} \in \overline{\mathcal{M}}$. We also have

$$\overline{d}(\overline{x}, \overline{y}) \leq \delta(\overline{x}, \overline{y})$$

for all $\overline{x}, \overline{y}$ such that $\delta(\overline{x}, \overline{y}) < \epsilon$. □

Proposition 6.13. *The topology defined by the metric $\overline{d}$ coincides with the quotient topology on $\overline{\mathcal{M}}$. The map $\overline{\pi} : \overline{\mathcal{M}} \longrightarrow \overline{\mathcal{X}}$ is uniformly locally bi-Lipschitz with respect to $\overline{d}$ and the metric on $\overline{\mathcal{X}}$ coming from the Hausdorff distance between compact subsets of $\mathcal{X}$.*

Proof. By Proposition 6.11 and the definition of $\overline{d}$, there exist $C > 1$ and $\epsilon > 0$ such that if $\overline{d}(\overline{x}, \overline{y}) < \epsilon$, then for every $y \in \overline{y}$ there exists $x \in \overline{x}$ such that $d_{\mathcal{M}}(x, y) < C\epsilon$. Suppose that $\overline{U} \subset \overline{\mathcal{M}}$ is open with respect to $\overline{d}$. Let $U \subset \mathcal{M}$ be the preimage of $\overline{U}$. Then for every $\overline{x} \in \overline{U}$ there exists $\epsilon > 0$ such that the ϵ-neighborhood of $\overline{x}$ (with respect to $\overline{d}$) is contained in $\overline{U}$. Let $\overline{y}$ be such that $\overline{d}(\overline{x}, \overline{y}) < C^{-1}\epsilon$. Then for every $y \in \overline{y}$ there exists $x \in \overline{x}$ such that $d_{\mathcal{M}}(x, y) < \epsilon$. It follows that the ϵ-neighborhood of the set $\overline{x}$ contains the set $\overline{y}$. It follows that U is open in $\mathcal{M}$.

Suppose that $U \subset \mathcal{M}$ is an $\mathbb{R}^d$-invariant open subset of $\mathcal{M}$. Then for every $x \in U$ there exists $\epsilon > 0$ such that the ϵ-neighborhood of x is contained in U. Suppose that $\overline{y} \subset U$ is such that $\overline{d}(\overline{x}, \overline{y}) < C^{-1}\epsilon/2$. Then there exists $y \in \overline{y}$ such that $d_{\mathcal{M}}(x, y) \leq \epsilon$. Then $y \in U$, hence $\overline{y} \subset U$, since U is $\mathbb{R}^d$-invariant. We have shown that every set that is open in the quotient topology is also open with respect to $\overline{d}$.

The statement about the Hausdorff distance also follows directly from the definition of $\overline{d}$ and Proposition 6.11 (and the fact that $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ is a local isometry). □

Note that $v(x) = [v_+(x), v_-(x)]$, where $v_+ = P_+(v)$ and $v_- = P_-(v)$. It follows that for any $v, u \in \mathbb{R}^d$ and $x, y \in \mathcal{M}$ we have

$$[v(x), u(y)] = [v_+(x), u_-(y)] = [(v_+ + u_-)(x), (v_+ + u_-)(y)] = (v_+ + u_-)([x, y]),$$

i.e., the value of $\overline{[x, y]}$ depends only on $\overline{x}$ and $\overline{y}$. It follows that the function $[\overline{x}, \overline{y}] = \overline{[x, y]}$ is well defined and satisfies the equalities (1) and (2) of Definition 2.1. We will prove that it is continuous in the next proposition.

We also have a well defined homeomorphism $\overline{F}(\overline{x}) = \overline{F(x)}$, by condition (4) of Theorem 6.8.

Lemma 6.14. *The metric $\overline{d}$ agrees with the local product structure on $\overline{\mathcal{M}}$. In particular, the map $[\cdot,\cdot]$ is continuous.*

Proof. We know that the metric $d_{\mathcal{M}}$ on $\mathcal{M}$ agrees with the local product structure on $\mathcal{M}$, since it is locally isometric to the standard metric on $\mathcal{X}$. Let $\overline{x}$, $\overline{y}$ be points of $\overline{\mathcal{M}}$ such that $\delta(\overline{x},\overline{y})$ is small. Let us prove that

$$\delta(\overline{x},\overline{y}) \asymp \delta(\overline{x},[\overline{x},\overline{y}]) + \delta(\overline{y},[\overline{x},\overline{y}]) \tag{12}$$

for all $\overline{x}$ and $\overline{y}$ that are close enough to each other. (Here $F_1 \asymp F_2$ means that there exists a constant $C > 1$ such that $C^{-1}F_1 \le F_2 \le CF_1$.)

There exist points $x \in \overline{x}$ and $y \in \overline{y}$ such that $d_{\mathcal{M}}(x,y) \le 2\delta(\overline{x},\overline{y})$. Since d agrees with the local product structure on $\mathcal{M}$, we have

$$d_{\mathcal{M}}(x,y) \asymp d_{\mathcal{M}}(x,[x,y]) + d_{\mathcal{M}}(y,[x,y]).$$

We have $\delta(\overline{x},[\overline{x},\overline{y}]) \le d_{\mathcal{M}}(x,[x,y])$ and $\delta(\overline{y},[\overline{x},\overline{y}]) \le d_{\mathcal{M}}(y,[x,y])$, hence there exists a constant $C_1 > 1$ such that

$$\delta(\overline{x},\overline{y}) \ge C_1^{-1}(\delta(\overline{x},[\overline{x},\overline{y}]) + \delta(\overline{y},[\overline{x},\overline{y}])).$$

On the other hand, since δ is equivalent to a metric (see Lemma 6.12), there exists $C_2 > 1$ such that

$$\delta(\overline{x},\overline{y}) \le C_2(\delta(\overline{x},[\overline{x},\overline{y}]) + \delta(\overline{y},[\overline{x},\overline{y}])),$$

by the triangle inequality. This proves (12).

Lemma 6.15. *There exist $C > 1$ and $\epsilon > 0$ such that if $\overline{x},\overline{y} \in \overline{\mathcal{M}}$ are such that $[\overline{x},\overline{y}] = \overline{x}$ (i.e., $\overline{x}$ and $\overline{y}$ belong to the same stable plaque of $\overline{\mathcal{M}}$) and $\delta(\overline{x},\overline{y}) < \epsilon$, then there exist $x \in \overline{x}$ and $y \in \overline{y}$ such that $[x,y] = x$ and $d_{\mathcal{M}}(x,y) \le C\delta(\overline{x},\overline{y})$.*

Proof. Let x, y belong to one stable plaque of $\mathcal{M}$. There exist $\epsilon > 0$ and $C > 1$ (not depending on x, y) such that if $d_{\mathcal{M}}([x,g(x)],[y,h(y)]) < \epsilon$ for $g, h \in G$, then

$$\begin{aligned} d_{\mathcal{M}}([x,g(x)],[y,h(y)]) \ge \\ C^{-1}(d_{\mathcal{M}}([x,g(x)],[y,g(y)]) + d_{\mathcal{M}}([y,g(y)],[y,h(y)])) \ge \\ C^{-1}d_{\mathcal{M}}([x,g(x)],[y,g(y)]), \end{aligned}$$

see Fig. 3.

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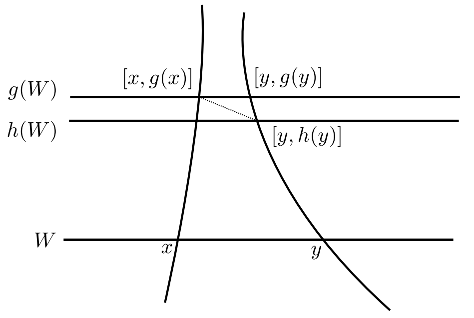

Fig. 3. Distance between plaques.

It follows that

$$\inf\{d_{\mathcal{M}}(v(x), u(y)) \;:\; v, u \in E_-\} \geq C^{-1} \inf\{d_{\mathcal{M}}(v(x), v(y)) \;:\; v \in E_-\}, \qquad (13)$$

if the left hand side of the inequality is less than ϵ.

We always can find $x_1 \in \overline{x}$ and $y_1 \in \overline{y}$ such that $d_{\mathcal{M}}(x_1, y_1) \leq 2\delta(\overline{x}, \overline{y})$. Then $[x_1, y_1] = v(x_1)$ for some $v \in \mathbb{R}^d$.

Note that $[x_1, y_1]$ and x_1 belong to the same unstable plaque of $\mathcal{M}$, hence $v \in E_-$. Then $[v(x_1), y_1] = [x_1, y_1] = v(x_1)$, i.e., the points $v(x_1)$ and y_1 belong to the same stable plaque.

It follows from (13) that

$$\inf\{d_{\mathcal{M}}(uv(x_1), u(y_1)) \;:\; u \in E_-\} \leq C \inf\{d_{\mathcal{M}}(uv(x_1), w(y_1)) \;:\; u, w \in E_-\} \leq C d_{\mathcal{M}}(x_1, y_1) \leq 2C\delta(\overline{x}, \overline{y}).$$

It follows that there exists a pair of points $x' \in \overline{x}, y' \in \overline{y}$ such that x' and y' belong to the same stable plaque, and $d_{\mathcal{M}}(x', y') \leq 2C\delta(\overline{x}, \overline{y})$. □

Let $\overline{x}, \overline{y} \in \overline{\mathcal{M}}$ be such that $\delta(\overline{x}, \overline{y})$ is small, and $[\overline{x}, \overline{y}] = \overline{x}$. Let $\overline{x}_2 \in \overline{\mathcal{M}}$ be a point close to $\overline{x}$. Using Lemma 6.15, find $x \in \overline{x}$ and $y \in \overline{y}$ such that $[x, y] = x$ and $d_{\mathcal{M}}(x, y) \leq C_1\delta(\overline{x}, \overline{y})$. By Proposition 6.11, there exists $x_2 \in \overline{x}_2$ such that $d_{\mathcal{M}}(x, x_2) \leq C_2\delta(\overline{x}, \overline{x}_2)$. We conclude from this, and from the fact that $d_{\mathcal{M}}$ agrees with the local product structure on $\mathcal{M}$, that there exists a constant $C_3 > 1$ such that if $d_{\mathcal{M}}(x, y)$ and $\delta(\overline{x}, \overline{x}_2)$ are small enough, we have

$$d_{\mathcal{M}}([x, x_2], [y, x_2]) \leq C_3 d_{\mathcal{M}}(x, y).$$

Consequently,

$$\delta([\overline{x}, \overline{x}_2], [\overline{y}, \overline{x}_2]) \leq d_{\mathcal{M}}([x, x_2], [y, x_2]) \leq C_3 d_{\mathcal{M}}(x, y) \leq C_3 C\delta(\overline{x}, \overline{y}).$$

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It follows that the maps $W_+(\overline{x}_1) \longrightarrow W_+(\overline{x}_2) : \overline{x} \mapsto [\overline{x}, \overline{x}_2]$ are locally Lipschitz with respect to the metric $\overline{d}$. Since the inverses of these maps are also maps of the same form, they are in fact locally bi-Lipschitz. This shows that the local product structure on $\overline{\mathcal{M}}$ agrees with $\overline{d}$. □

Proposition 6.16. *The dynamical system $(\overline{\mathcal{X}}, \overline{f})$, where $\overline{f}$ is the map induced by $\overline{F}$ is a connected and locally connected Smale space. The quotient map $\overline{\mathcal{M}} \longrightarrow \overline{\mathcal{X}}$ is a splitting with the group of deck transformations $\overline{G}$.*

Proof. It follows from Theorem 6.8 that $\overline{f} : \overline{\mathcal{X}} \longrightarrow \overline{\mathcal{X}}$ is a well defined homeomorphism. The plaques of the direct product structure of $\overline{\mathcal{M}}$ are continuous images of plaques of $\mathcal{M}$, hence they are connected. The map $\overline{F}$ is a lift of $\overline{f}$.

The map $\overline{\pi} : \overline{\mathcal{M}} \longrightarrow \overline{\mathcal{X}}$ is uniformly locally bi-Lipschitz with respect to $\overline{d}$ and the Hausdorff distance on $\overline{\mathcal{X}}$, by Proposition 6.13. It follows that the Hausdorff distance agrees with the quotient topology on $\overline{\mathcal{X}}$ and that $\overline{\pi}$ is a covering.

It follows from Proposition 6.13, Lemma 6.14, and the fact that $\overline{G}$ preserves the direct product structure on $\overline{\mathcal{M}}$, that the image under $\overline{\pi}$ of the local product structure on $\overline{\mathcal{M}}$ is a well defined local product structure on $\overline{\mathcal{X}}$.

Suppose that $\overline{x}, \overline{y} \in \overline{\mathcal{M}}$ are such that $[\overline{x}, \overline{y}] = \overline{x}$ (i.e., $\overline{x}$ and $\overline{y}$ belong to the same stable plaque of $\overline{\mathcal{M}}$). By Lemma 6.15, there exist $x \in \overline{x}$ and $y \in \overline{y}$ belonging to the same stable plaque of $\mathcal{M}$. Note also that it follows from Lemmas 6.12 and 6.14 that there exists a constant $C > 1$, not depending on $\overline{x}$ and $\overline{y}$, such that we can find x, y satisfying

$$C^{-1}\overline{d}(\overline{x}, \overline{y}) \leq d_{\mathcal{M}}(x, y) \leq C\overline{d}(\overline{x}, \overline{y}),$$

provided $\overline{d}(\overline{x}, \overline{y})$ is small enough.

Then $d_{\mathcal{M}}(F^n(x), F^n(y)) \leq C\lambda^n d_{\mathcal{M}}(x, y)$ for some fixed $C > 1$ and $\lambda \in (0, 1)$. It follows that there exists a constant $C_2 > 1$ such that for any two points $\overline{x}, \overline{y} \in \overline{\mathcal{M}}$ such that $[\overline{x}, \overline{y}] = \overline{x}$, and $\overline{d}(\overline{x}, \overline{y})$ is small enough we have $\overline{d}(\overline{F}^n(\overline{x}), \overline{F}^n(\overline{y})) \leq C_2\lambda^n\overline{d}(\overline{x}, \overline{y})$ for all n. Analogous statement about the unstable plaques of $\overline{\mathcal{M}}$ is proved in the same way.

The map $\overline{\pi} : \overline{\mathcal{M}} \longrightarrow \overline{\mathcal{X}}$ is locally bi-Lipschitz with respect to $\overline{d}$ and the Hausdorff distance on $\overline{\mathcal{X}}$. It follows that the images of the stable and unstable plaques of $\overline{\mathcal{M}}$ are stable and unstable leaves of $(\overline{\mathcal{X}}, \overline{f})$.

Suppose that $\overline{x}, \overline{y} \in \overline{\mathcal{M}}$ are such that $\overline{\pi}(\overline{x}) = \overline{\pi}(\overline{y})$, i.e., the $\mathbb{R}^d$-orbits $\overline{x}$ and $\overline{y}$ are mapped to the same set in $\mathcal{X}$. Then there exist $x \in \overline{x}$ and $y \in \overline{y}$ such that $\pi(x) = \pi(y)$, i.e., there exists $g \in G$ such that $g(x) = y$. Then $\overline{g}(\overline{x}) = \overline{y}$. Since the action of $\overline{G}$ on $\overline{\mathcal{M}}$ is free, we conclude that $\overline{G}$ is the group of deck transformations of the splitting $\overline{\pi} : \overline{\mathcal{M}} \longrightarrow \overline{\mathcal{X}}$. □

Proposition 6.17. *If a connected and locally connected Smale space $(\mathcal{X}, f)$ has a splitting with a nilpotent torsion free group of deck transformations, then f has a fixed point.*

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Proof. We argue by induction on the nilpotency class. We know that the statement is true for abelian groups of deck transformations, see Theorem 6.10. The Smale space $(\overline{\mathcal{X}}, \overline{f})$ is a locally connected Smale space with the group of deck transformations $\overline{G}$ of lower class. Therefore, by the inductive hypothesis, $\overline{f}$ has a fixed point. Its preimage in $\mathcal{X}$ is an f-invariant torus $T \subset \mathcal{X}$ equal to an orbit of the $\mathbb{R}^d/\mathbb{Z}^d$-action. It follows from the definition of the action of $\mathbb{R}^d$ on $\mathcal{M}$ that T is locally closed with respect to the local product operation $[\cdot, \cdot]$, hence (T, f) is a Smale space, and f restricted to this torus is a hyperbolic automorphism, hence it has a fixed point (see Theorem 6.10). □

If G is a torsion-free finitely generated nilpotent group, then there exists a unique simply connected nilpotent Lie group L such that G is isomorphic to a co-compact lattice in L, see [21]. Moreover, every automorphism of G is uniquely extended to L. The Lie group L is called the *Malcev completion* of G.

Let G be a finitely generated torsion free nilpotent group, and let ϕ be its automorphism. We say that $\phi : G \longrightarrow G$ is *hyperbolic* if its unique extension $\phi : L \longrightarrow L$ to the Malcev completion is hyperbolic, i.e., if the automorphism $D\phi$ of the Lie algebra of L has no eigenvalues on the unit circle.

Proposition 6.18. *Let $(\mathcal{X}, f)$ be a connected and locally connected Smale space with a splitting $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ with a torsion free nilpotent group of deck transformations G. Let ϕ be an automorphism of G induced by a lift F of f which has a fixed point in $\mathcal{M}$. Then ϕ is hyperbolic.*

Proof. Let us prove our proposition by induction on the nilpotency class of G. It is true for abelian groups, by Proposition 6.5.

Suppose that we have proved the proposition for all nilpotent groups of class n. Suppose that nilpotency class of G is $n + 1$. By Proposition 6.16 and the inductive hypothesis, the automorphism of $\overline{G}$ induced by ϕ is hyperbolic.

Let $x_0 \in \mathcal{M}$ be the fixed point of F. Then $\overline{x}_0 \in \overline{\mathcal{M}}$ is a fixed point of $\overline{F}$. The image of $\overline{x}_0$ in $\mathcal{X}$ is an f-invariant torus T, such that (T, f) is topologically conjugate to a hyperbolic automorphism of the torus.

The map $\pi : \overline{x}_0 \longrightarrow T$ is its splitting with the group of deck transformations equal to $Z(G) = \mathbb{Z}^d$. It follows then from Proposition 6.5 that the restriction of ϕ onto $Z(G)$ is hyperbolic.

We see that restriction of $\phi : L \longrightarrow L$ onto $Z(L)$ and the automorphism induced by ϕ on $L/Z(L)$ both are hyperbolic, hence ϕ itself is hyperbolic. □

Theorem 6.19. *Let $(\mathcal{X}, f)$ be a connected and locally connected Smale space with a splitting $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ with a torsion free nilpotent group of deck transformations G. Let ϕ be an automorphism of G induced by a lift of f. Let $f_L : G\backslash L \longrightarrow G\backslash L$ be the diffeomorphism induced by ϕ. Then $(\mathcal{X}, f)$ and $(G\backslash L, f_L)$ are topologically conjugate.*

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Proof. Let F be a lift of f to $\mathcal{M}$ with a fixed point x_0. Extend the automorphism $\phi : G \longrightarrow G$ to an automorphism $\phi : L \longrightarrow L$ of the Lie group.

Proposition 6.20. *There exists a G-equivariant map $h : \mathcal{M} \longrightarrow L$ such that $\phi \circ h = h \circ F$ and $h(x_0) = 1$.*

Proof. Let us show at first that there exists a G-equivariant map $h_0 : \mathcal{M} \longrightarrow L$. The space $\mathcal{X}$ is a quotient of the Cantor set under a finite-to-one map (see [4,16]) with an upper bound on the cardinality of its fibers. It follows then from Hurewicz formula [20] that $\mathcal{X}$ has finite topological dimension.

By a theorem of Alexandroff [1], $\mathcal{X}$ is homeomorphic to an inverse limit of simplicial complexes, which are nerves of finite open coverings of $\mathcal{X}$. We can make the elements of the coverings sufficiently small, so that they can be lifted to a G-invariant covering of $\mathcal{M}$. It follows that $\mathcal{M}$ is an inverse limit of a sequence of simplicial complexes with G-actions and G-equivariant maps between them. In particular, there exists a G-equivariant map A from $\mathcal{M}$ to a simplicial complex Δ with a G action on it. Since L is homeomorphic to $\mathbb{R}^n$, there exists a G-equivariant map $B : \Delta \longrightarrow L$. Composition $h_0 = B \circ A$ is then a G-equivariant map from $\mathcal{M}$ to L.

Let us show now that there exists a G-equivariant map $h : \mathcal{M} \longrightarrow L$ such that $\phi \circ h = h \circ F$. We will use the arguments of [14, Theorem 2.2], which we repeat here for the sake of completeness, and since our setting is slightly different.

Consider the space Q of all continuous maps $\gamma : \mathcal{X} \longrightarrow L$ such that $\gamma(\pi(x_0)) = 1$ with the topology of uniform convergence on $\mathcal{X}$. It is a nilpotent group (of the same class as L) with respect to pointwise multiplication. Define $\Phi_0(\gamma) = \phi^{-1} \circ \gamma \circ f$. Then Φ_0 is a continuous automorphism of the group Q.

Let $\mathfrak{L}$ be the Lie algebra of L, and let $\exp : \mathfrak{L} \longrightarrow L$ be the exponential map. It is a diffeomorphism, since L is simply connected and nilpotent. Let $\mathfrak{Q}$ be the Banach space of continuous maps $\mathcal{X} \longrightarrow \mathfrak{L}$ mapping $\pi(x_0)$ to zero. Then $\mathrm{Log} : \gamma \mapsto \exp^{-1} \circ \gamma$ is a homeomorphism of Q with $\mathfrak{Q}$.

Let $T_0 : Q \longrightarrow Q$ be defined by $T_0(\gamma) = \Phi_0(\gamma)\gamma^{-1}$. Let us show that T_0 is a homeomorphism. We show at first that it is a local homeomorphism at the identity (i.e., the constant map $x \mapsto 1$), using the homeomorphism $\mathrm{Log} : Q \longrightarrow \mathfrak{Q}$ and computing the derivative of $T = \mathrm{Log} \circ T_0 \circ \mathrm{Log}^{-1}$. Denote $\Phi = \mathrm{Log} \circ \Phi_0 \circ \mathrm{Log}^{-1}$.

We have $\exp \circ d\phi = \phi \circ \exp$, where $d\phi$ is the derivative of $\phi : L \longrightarrow L$ at the identity. It follows that $d\phi^{-1} \circ \exp^{-1} = \exp^{-1} \circ \phi^{-1}$, and for every $\gamma \in \mathfrak{Q}$ we have

$$\Phi(\gamma) = \mathrm{Log} \circ \Phi_0 \circ \mathrm{Log}^{-1}(\gamma) = \exp^{-1} \circ \phi^{-1} \circ \exp \circ \gamma \circ f = d\phi^{-1} \circ \exp^{-1} \circ \exp \circ \gamma \circ f = d\phi^{-1} \circ \gamma \circ f.$$

It follows that $\Phi : \mathfrak{Q} \longrightarrow \mathfrak{Q}$ is linear.

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For every $\gamma \in \mathfrak{Q}$, we have

$$T(\gamma) = \operatorname{Log} \circ T_0 \circ \operatorname{Log}^{-1}(\gamma) = \exp^{-1}(T_0(\exp \circ \gamma)) = \exp^{-1}(\Phi_0(\exp \circ \gamma) \cdot (\exp \circ \gamma)^{-1}) = \exp^{-1}(\exp \circ \Phi(\gamma) \cdot (\exp \circ \gamma)^{-1}) = \operatorname{Log}(\operatorname{Log}^{-1}(\Phi(\gamma)) \cdot \operatorname{Log}^{-1}(\gamma)^{-1}) = \operatorname{Log}(\operatorname{Log}^{-1}(\Phi(\gamma)) \cdot \operatorname{Log}^{-1}(-\gamma)),$$

since $(\operatorname{Log}^{-1}(\gamma(x)))^{-1} = (\exp(\gamma(x)))^{-1} = \exp(-\gamma(x))$ for all $x \in \mathcal{X}$.

Let us compute the derivative of T at zero. If $\gamma \in \mathfrak{Q}$, then

$$\lim_{t\to 0} \frac{1}{t} T(t\gamma) = \lim_{t\to 0} \frac{1}{t} \operatorname{Log}(\operatorname{Log}^{-1}(\Phi(t\gamma)) \operatorname{Log}^{-1}(-t\gamma)) = \Phi(\gamma) - \gamma,$$

by the Campbell-Hausdorf formula.

Since $d\phi$ is hyperbolic, there exist a direct sum decomposition $\mathfrak{L} = \mathfrak{L}_+ \oplus \mathfrak{L}_-$ and constants $C > 0$ and $0 < \lambda < 1$ such that $\|d\phi^n(v)\| \le C\lambda^n \|v\|$ for all $n \ge 0$ and $v \in \mathfrak{L}_+$, and $\|d\phi^{-n}(v)\| \le C\lambda^n\|v\|$ for all $n \ge 0$ and $v \in \mathfrak{L}_-$. Define

$$\mathfrak{Q}_* = \{\gamma \in \mathfrak{Q} \,:\, \gamma(\mathcal{X}) \subset \mathfrak{L}_*\},$$

for $* \in \{+, -\}$. Since $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are $d\phi$-invariant, the spaces $\mathfrak{Q}_+$ and $\mathfrak{Q}_-$ are Φ-invariant. We obviously have $\mathfrak{Q} = \mathfrak{Q}_+ \oplus \mathfrak{Q}_-$, $\|\Phi^n(\gamma)\| \le C\lambda^n\|\gamma\|$ for $n \ge 0$, $\gamma \in \mathfrak{Q}_+$, and $\|\Phi^{-n}(\gamma)\| \le C\lambda^n\|\gamma\|$ for $n \ge 0$, $\gamma \in \mathfrak{Q}_-$. It follows that $\Phi - I$ is invertible.

This shows that T is a local homeomorphism at zero. Consequently, T_0 is a local homeomorphism at the identity of Q. Let us show that T_0 is surjective. Let $Z_1(Q) = Z(Q) \subset Z_2(Q) \subset \cdots \subset Z_n(Q) = Q$ be the upper central series of Q. Let us prove by induction on i that $T_0(Q) \supset Z_i(Q)$. It is easy to see that $T_0(\gamma_1\gamma_2) = T_0(\gamma_1)T_0(\gamma_2)$ for all $\gamma_1, \gamma_2 \in Z(Q)$. Since T_0 is a local homeomorphism at the identity and $Z(Q)$ is generated by any neighborhood of the identity (as any connected topological group, see [29, Theorem 15, p. 76], this implies that $Z(Q) \subset T_0(Q)$.

Suppose that we have proved that $Z_i(Q) \subset T_0(Q)$. Let $T_0(\gamma_1), T_0(\gamma_2) \in Z_{i+1}(Q)$. Then

$$T_0(\gamma_1)T_0(\gamma_2) = \Phi_0(\gamma_1)\gamma_1^{-1}\Phi_0(\gamma_2)\gamma_2^{-1} = \Phi_0(\gamma_1)\Phi_0(\gamma_2) \cdot \left(\Phi_0(\gamma_2)^{-1}\gamma_1^{-1}\Phi_0(\gamma_2)\gamma_2^{-1}\gamma_1\gamma_2\right) \cdot \gamma_2^{-1}\gamma_1^{-1}.$$

We have

$$\gamma' = \Phi_0(\gamma_2)^{-1}\gamma_1^{-1}\Phi_0(\gamma_2)\gamma_2^{-1}\gamma_1\gamma_2 = \gamma_2^{-1}(\Phi_0(\gamma_2)\gamma_2^{-1})^{-1}\gamma_1^{-1}(\Phi_0(\gamma_2)\gamma_2^{-1})\gamma_1\gamma_2 = \gamma_2^{-1}T_0(\gamma_2)^{-1}\gamma_1^{-1}T_0(\gamma_2)\gamma_1\gamma_2 = \gamma_2^{-1}[T_0(\gamma_2), \gamma_1]\gamma_2 \in Z_i(Q)$$

(here $[T_0(\gamma_2), \gamma_1]$ is the commutator, and has nothing to do with the direct product decomposition). By the inductive hypothesis, there exists $\gamma_3 \in Q$ such that $\gamma' = T_0(\gamma_3)$. Then

$$T_0(\gamma_1)T_0(\gamma_2) = \Phi_0(\gamma_1)\Phi_0(\gamma_2)\Phi_0(\gamma_3)\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1} = T_0(\gamma_1\gamma_2\gamma_3) \in T_0(Q).$$

Since $Z_{i+1}(Q)$ is connected, hence generated by any neighborhood of the identity, it follows that $Z_{i+1}(Q) \subset T(Q)$.

Since $\Phi - I$ is invertible, the only fixed point of Φ is 0. Consequently, the only fixed point of Φ_0 is the unit of Q. If $T(\gamma_1) = T(\gamma_2)$, then $\Phi_0(\gamma_1)\gamma_1^{-1} = \Phi_0(\gamma_2)\gamma_2^{-1}$, hence $\Phi_0(\gamma_2^{-1}\gamma_1) = \gamma_2^{-1}\gamma_1$. But then $\gamma_1 = \gamma_2$, as the identity is the only fixed point of Φ_0.

We have proved that $T_0 : Q \longrightarrow Q$ is a homeomorphism. Let $h_0 : \mathcal{M} \longrightarrow L$ be any G-equivariant map. Consider the map

$$\overline{\gamma}(x) \mapsto (\phi^{-1} \circ h_0 \circ F(x))^{-1} \cdot h_0(x).$$

It is easy to see that for every $g \in G$ we have

$$\overline{\gamma}(g(x)) = \overline{\gamma}(x),$$

i.e., $\overline{\gamma}$ is constant on G-orbits, hence it descends to a continuous map $\gamma : \mathcal{X} \longrightarrow L$, which is an element of Q. There exists $\gamma' \in Q$ such that $T(\gamma') = \gamma$. Then $F_0(\gamma')(\gamma')^{-1} = \gamma$, which means that $\gamma'(f(x)) = \phi(\gamma(x)) \cdot \phi(\gamma'(x))$.

Then the map $h(x) = h_0(x) \cdot \gamma'(\pi(x))$ is G-equivariant, and

$$\begin{aligned}h(F(x)) = h_0(F(x)) \cdot \gamma'(f(\pi(x))) = \\ h_0(F(x)) \cdot \phi(\overline{\gamma}(x)) \cdot \phi(\gamma'(\pi(x))) = \\ h_0(F(x))h_0(F(x))^{-1}\phi(h_0(x))\phi(\gamma'(\pi(x))) = \\ \phi(h_0(x)\gamma'(\pi(x))) = \phi(h(x)),\end{aligned}$$

which finishes the proof of the proposition. □

Theorem 5.15 shows now that $(\mathcal{X}, f)$ and $(G\backslash L, f_L)$ are topologically conjugate. □

Let us finish the proof of Theorem 6.1. Let $(\mathcal{X}, f)$ be a connected and locally connected Smale space, and let $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ be its splitting with a virtually nilpotent group of deck transformations G. Let F be a lift of f to $\mathcal{M}$. Let ϕ be the automorphism induced by F on G.

Every finitely generated virtually nilpotent group G contains a torsion free nilpotent subgroup G_0 of finite index (see [19, 17.2.2]). For every $g \in G$ and $n \in \mathbb{Z}$ the subgroup $g^{-1}\phi^n(G_0)g$ has the same index in G as G_0. There exists only a finite number of subgroups of given index in a finitely generated group. Taking then intersection of all

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subgroups of the form $g^{-1}\phi^n(G_0)g$ for $n \in \mathbb{Z}$ and $g \in G$, we get a normal ϕ-invariant torsion free nilpotent subgroup G_1 of finite index in G. It will be finitely generated as a finite index subgroup of a finitely generated group.

Then $G_1\backslash\mathcal{M}$ together with the map f_1 induced by F is a Smale space. It is a finite covering of $\mathcal{X}$, and its group of deck transformations is G_1. Then, by Proposition 6.17, f_1 has a fixed point, hence we may assume that F has a fixed point x_0. We assume then that $\phi : G \longrightarrow G$ is given by $\phi(g)(x_0) = F(g(x_0))$.

Let L be the Malcev completion of G_1. Extend $\phi : G_1 \longrightarrow G_1$ to an automorphism $\phi : L \longrightarrow L$. Then, by Theorem 6.19, there exists a homeomorphism $\Phi : L \longrightarrow \mathcal{M}$ conjugating the actions of G_1 on $\mathcal{M}$ and L, and such that $\Phi \circ F = \phi \circ L$. Note that $\Phi(1) = x_0$, $\Phi(L_+) = W_+$, and $\Phi(L_-) = W_-$, where $L_+ = W_+(1), L_- = W_-(1)$ are the stable and unstable plaques of the identity element of L, and $W_+ = W_+(x_0), W_- = W_-(x_0)$ are the stable and unstable plaques of x_0.

Note that $L_+ = \{g \in L : \lim_{n\to+\infty} \phi^n(g) = 1\}$ and $L_- = \{g \in L : \lim_{n\to-\infty} \phi^n(g) = 1\}$ are closed subgroups of L (they are closed since they are plaques of a splitting). The stable plaques of L are the left cosets of L_+; the unstable plaques of L are the left cosets of L_-.

Consider the action of G on L obtained by conjugating by Φ the action of G on $\mathcal{M}$. The action of its subgroup $G_1 \le G$ will coincide with the natural action of $G_1 \le L$ on L by left multiplication.

The Smale space $(\mathcal{X}, f)$ is then topologically conjugate to the homeomorphism induced by ϕ on $G\backslash L$.

Proposition 6.21. *The group G acts on L by affine transformations.*

Proof. The action of $G_1 \le G$ on L coincides with the natural left action of G_1 on L as a subgroup of L. The action of G on G_1 by conjugation can be uniquely extended to an action of G on L by automorphism. Denote by $\alpha_g(h)$ for $g \in G$ and $h \in L$ the image of h under the automorphism of L equal to the extension of the automorphism $h \mapsto ghg^{-1}$ of G_1.

Let $g \in G$. Then $a_g = g(1) = \Phi^{-1}(g(x_0))$ is an element of L. Let us prove that the action of g on L is given by the formula

$$g(x) = \alpha_g(x) \cdot a_g.$$

Consider the map $A_g(x) = a_g^{-1}\alpha_g(x)a_g : L \longrightarrow L$. Note that if $h \in G_1$, then $A_{hg} = A_g$, since $a_{hg} = hg(1) = ha_g$ and $\alpha_{hg}(x) = h\alpha_g(x)h^{-1}$. It follows that there is a finite number of possibilities for A_g, since G_1 has finite index in G. Note also that $\phi(a_g) = \phi(g(1)) = \phi(g)(1)$ and $\phi(\alpha_g(x)) = \alpha_{\phi(g)}(\phi(x))$, so that $\phi(A_g(x)) = A_{\phi(g)}(\phi(x))$.

Suppose that $x \in L_+$. Then $\phi^n(x) \to 1$ as $n \to \infty$. Since the set of possible maps of the form $A_{\phi^n(g)}$ is finite and they are continuous, we have

$$\phi^n(A_g(x)) = A_{\phi^n(g)}(\phi^n(x)) \to 1,$$

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i.e., $A_g(x) \in L_+$. Consequently, the maps A_g preserve L_+.

If x, y belong to one stable plaque, then $x^{-1}y \in L_+$, hence

$$A_g(x^{-1}y) = (\alpha_g(x)a_g)^{-1}(\alpha_g(y)a_g) \in L_+.$$

Consequently, the affine map $x \mapsto \alpha_g(x)a_g$ preserves the stable plaques of L. It is proved in the same way that it preserves the unstable plaques, hence it preserves the local product structure.

Let $h \in G_1$. Then $g(hL_+) = ghg^{-1}(g(L_+)) = \alpha_g(h)a_gL_+$, since $g(L_+)$ is the stable plaque a_gL_+ of the point $a_g = g(1)$. By the same argument, $g(hL_-) = \alpha_g(h)a_gL_-$ for all $h \in G_1$.

Let $R \subset L$ be a relatively compact open rectangle such that $G_1\backslash R = G_1\backslash L$ and $1 \in R$. Then for every $x \in L_-$ and every $n \in \mathbb{N}$ there exists $g_n \in G_1$ such that $x \in \phi^n(g_nR)$. Note that then distance from x to $\phi^n(g_nL_+)$ is exponentially decreasing with n. It follows that the union of the stable plaques of the form hL_+ for $h \in G_1$ is dense in L. Similarly, the union of the unstable plaques of the form hL_- for $h \in G_1$ is also dense in L.

The actions of the maps $x \mapsto g(x)$ and $x \mapsto \alpha_g(x)a_g$ on the stable and unstable plaques of the form $hL_\pm$ for $h \in G_1$ coincide. Both maps are continuous on L and preserve the direct product structure, hence they are equal. □

This finishes the proof of Theorem 6.1. □

7. Smale spaces with pinched spectrum

7.1. Splitting

Definition 7.1. Let $(\mathcal{X}, f)$ be a Smale space such that $\mathcal{X}$ is connected and locally connected. Let a_0, a_1 be the stable lower and upper critical exponents, and let b_0, b_1 be the unstable lower and upper critical exponents.

We say that the Smale space has *pinched spectrum* if

$$\frac{a_0}{a_1} + \frac{b_0}{b_1} > 1,$$

Theorem 7.2. *A Smale space with pinched spectrum is splittable.*

Proof. Choose numbers $\alpha_0, \alpha_1, \beta_0, \beta_1$ such that $0 < \alpha_0 < a_0 \le a_1 < \alpha_1$, $0 < \beta_0 < b_0 \le b_1 < \beta_1$, and

$$\frac{\alpha_0}{\alpha_1} + \frac{\beta_0}{\beta_1} > 1.$$

Let d_+ and d_- be metrics associated with the internal log-scales ℓ_+ and ℓ_- on the stable and unstable leaves of the exponents α_0 and β_0, respectively. All distances inside the leaves will be measured using these metrics.

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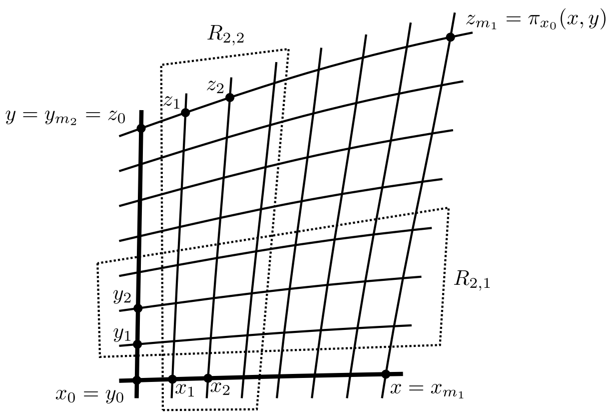

Fig. 4. Splitting.

Let $\mathcal{R}$ be a finite covering of $\mathcal{X}$ by small open rectangles. We assume (making the rectangles small enough) that the holonomies inside the rectangles $R \in \mathcal{R}$ are bi-Lipschitz. Then it follows from equalities (3) and (4) in Section 2 that the holonomies inside $f^k(R)$ for all $R \in \mathcal{R}$ and $k \in \mathbb{Z}$ are bi-Lipschitz with a common Lipschitz constant $L > 1$.

There exists $\epsilon > 0$ such that for any $x \in \mathcal{X}$ there exists a rectangle $R \in \mathcal{R}$ such that the ϵ-neighborhood of x in $W_+(x)$ (with respect to d_+) and the ϵ-neighborhood of x in $W_-(x)$ (with respect to d_-) are contained in R. Then for some constant $c > 0$ and for every $k \in \mathbb{Z}$, $x \in \mathcal{X}$ there exists $R \in \mathcal{R}$ such that the $ce^{-\alpha_0 k}$-neighborhood of x in $W_+(x)$ is contained in $f^k(R)$, and the $ce^{\alpha_0 k}$-neighborhood of x in $W_-(x)$ is contained in $f^k(R)$.

Choose $x_0 \in \mathcal{X}$. Let us construct a splitting $\pi_{x_0} : W_+(x_0) \times W_-(x_0) \longrightarrow \mathcal{X}$. Let $x \in W_+(x_0)$, and let n be a positive integer. Since α_1 is a stable upper exponent, there exists a sequence $x_0, x_1, \ldots, x_{m_1} = x$ of points of $W_+(x_0)$ such that $m_1 \le C_1 e^{\alpha_1(n - \ell_-(x_0, x))}$, and $\ell_+(x_i, x_{i+1}) \ge n$, for some constant C_1.

Passing to d_+, we get that

$$m_1 \le C_2 d_+(x_0, x)^{\alpha_1/\alpha_0} e^{\alpha_1 n}, \qquad d_+(x_i, x_{i+1}) \le C_3 e^{-\alpha_0 n}.$$

For every $k \in \mathbb{Z}$ there exist rectangles $R_{i,1} \in \mathcal{R}$ such that the $ce^{-\alpha_0 k}$-neighborhood of x_i in $W_+(x_i) = W_+(x_0)$ and the $ce^{\beta_0 k}$-neighborhood of x_i in $W_-(x_i)$ belong to $f^k(R_{i,1})$. If $ce^{-\alpha_0 k} > C_3 e^{-\alpha_0 n}$, then $R_{i,1}$ contains x_{i-1} and x_{i+1}. The last inequality is equivalent to $k \le n - r_1$ for some constant $r_1 \in \mathbb{Z}$. Choose $k_1 = n - r_1$, and find a sequence of rectangles $R_{i,1}$ satisfying the above conditions for $k = k_1$. (See Fig. 4.)

Let $y \in W_-(x_0)$ be such that $d_-(x_0, y) \le ce^{\alpha_0 k_1} = C_4 e^{\alpha_0 n}$ (where $C_4 = ce^{-\alpha_0 r_1}$). Denote $z_0 = y$, $z_1 = [x_1, z_0]_{f^{k_1}(R_{0,1})}$, $z_2 = [x_2, z_1]_{f^{k_1}(R_{1,1})}$, etc. If all points $z_0, \ldots, z_{m_1}$ are defined, then we say that *y can be continued to x*, and denote $\pi_{x_0}(x, y) = z_{m_1}$. Note that x_0 can be continued to x and $\pi_{x_0}(x_0, x) = x$.

If y can be continued to x, then, in the above notation,

$$d_+(z_i, z_{i+1}) \le L d_+(x_i, x_{i+1}) \le C_5 e^{-\alpha_0 n},$$

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hence

$$d_+(y, \pi_{x_0}(x, y)) \le C_5 m_1 e^{-\alpha_0 n} \le C_6 d_+(x_0, x)^{\alpha_1/\alpha_0} e^{(\alpha_1-\alpha_0)n}.$$

Let k_2 be such that

$$ce^{\alpha_0 k_2} \ge C_6 d_+(x_0, x)^{\alpha_1/\alpha_0} e^{(\alpha_1-\alpha_0)n}, \tag{14}$$

so that for every point $z \in \mathcal{X}$ there exists a rectangle $R \in \mathcal{R}$ such that the $C_6 d_+(x_0, x)^{\alpha_1/\alpha_0} e^{(\alpha_1-\alpha_0)n}$-neighborhood of z in $W_+(z)$ is contained in $f^{-k_2}(R)$, and the $ce^{-\beta_0 k_2}$-neighborhood of z in $W_-(z)$ is also contained in $f^{-k_2}(R)$.

Inequality (14) follows from an inequality

$$k_2 \ge \frac{\alpha_1}{\alpha_0^2} \log d_+(x_0, x) + \frac{\alpha_1 - \alpha_0}{\alpha_0} n + s$$

for some constant s. Consequently, we can take

$$k_2 = \frac{\alpha_1}{\alpha_0^2} \log d_+(x_0, x) + \frac{\alpha_1 - \alpha_0}{\alpha_0} n + s_1, \tag{15}$$

where $s_1 > 0$ is bounded from above.

Let n_2 be such that $\ell_-(z_1, z_2) \ge n_2$ implies $d_-(z_1, z_2) \le ce^{-\beta_0 k_2}$. There exists a constant r_2 (not depending on k_2) such that we can take $n_2 = k_2 + r_2$. Let $y \in W_-(x_0)$. There exists a sequence $y_0 = x_0, y_1, y_2, \ldots, y_{m_2} = y$ such that

$$m_2 \le C_7 d_-(y, x_0)^{\beta_1/\beta_0} e^{\beta_1 n_2}, \qquad d_-(y_i, y_{i+1}) \le ce^{-\beta_0 n_2} \le ce^{-\beta_0 k_2}.$$

Choose a sequence of rectangles $R_{i,2} \in \mathcal{R}$ such that the $ce^{-\beta_0 n_2}$-neighborhood of y_i in $W_-(y_i) = W_-(x_0)$, and the $ce^{\alpha_0 n_2}$-neighborhood of y_i in $W_+(y_i)$ are contained in $f^{-n_2}(R)$.

Suppose that y_i can be continued to x. Then $d_+(y_i, \pi_{x_0}(x, y_i)) \le ce^{\alpha_0 k_2} \le ce^{\alpha_0 n_2}$, hence $\pi_{x_0}(x, y_i) \in f^{-n_2}(R_{i,2})$. Define then a sequence $z_{0,i} = y_{i+1}$, $z_{1,i} = [x_1, z_{0,i}]_{f^{k_1}(R_0)}$, $z_{2,i} = [x_2, z_{1,i}]_{f^{k_1}(R_1)}$, etc. Each of the points $z_{j,i}$ will be defined, provided $d_-(z_{j-1,i}, x_{j-1}) \le ce^{\beta_0 k_1}$. We have an estimate

$$\begin{aligned} d_-(z_{j-1,i}, x_{j-1}) \le C_8(d_-(x_0, y_1) + d_-(y_1, y_2) + \cdots + d_-(y_{j-2}, y_{j-1})) \le \\ C_8 m_2 ce^{-\beta_0 k_2} \le C_9 d_-(y, x_0)^{\beta_1/\beta_0} e^{(\beta_1-\beta_0)k_2}. \end{aligned}$$

(We used that $n_2 = k_2 + r_1$ for some constant r_1.)

It follows that y can be continued to x if

$$C_9 d_-(y, x_0)^{\beta_1/\beta_0} e^{(\beta_1-\beta_0)k_2} \le ce^{\beta_0 k_1},$$

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i.e., if

$$(\beta_1 - \beta_0)k_2 + \frac{\beta_1}{\beta_0}\log d_-(y, x_0) + s_2 \leq \beta_0 n$$

for some constant s_2.

Replacing k_2 by the value given in (15), we get that y can be continued to x if

$$(\beta_1 - \beta_0)\left(\frac{\alpha_1 - \alpha_0}{\alpha_0}n + \frac{\alpha_1}{\alpha_0^2}\log d_+(x_0, x)\right) + \frac{\beta_1}{\beta_0}\log d_-(y, x_0) + s_3 \leq \beta_0 n \tag{16}$$

for some constant s_3. If

$$\frac{(\beta_1 - \beta_0)(\alpha_1 - \alpha_0)}{\alpha_0} < \beta_0, \tag{17}$$

then taking n big enough, we can guarantee that inequality (16) is satisfied. Inequality (17) is equivalent to

$$(\beta_1 - \beta_0)(\alpha_1 - \alpha_0) < \alpha_0\beta_0,$$

i.e., to

$$\beta_1\alpha_1 < \beta_0\alpha_1 + \beta_1\alpha_0 \Longleftrightarrow \frac{\alpha_0}{\alpha_1} + \frac{\beta_0}{\beta_1} > 1.$$

It follows that if the Smale space has pinched spectrum, then every point $y \in W_-(x_0)$ can be continued to every $x \in W_+(x_0)$, and we can define $\pi_{x_0} : W_+(x_0) \times W_-(x_0)$ using the rules described above.

Let us show that the map $\pi_{x_0} : W_+(x_0) \times W_-(x_0) \longrightarrow \mathcal{X}$ is well defined, i.e., does not depend on the choice of the rectangles $R_{i,1}$ (we did not use the rectangles $R_{i,2}$ in the definition of π_{x_0}).

It follows from the construction that the map $y \mapsto \pi_{x_0}(x, y)$ is equal to composition of holonomy maps of a sequence of rectangles $R_{i,1} \in f^{k_1}(\mathcal{R})$ for some positive k_1.

We also showed that for every y, the germ of the map $y \mapsto \pi_{x_0}(x, y)$ is equal to a germ of a holonomy in a rectangle $R_{j,1} \in f^{-n_2}(\mathcal{R})$ for some positive n_2.

Note also that given such a sequence $R_{i,1} \in f^{k_1}(\mathcal{R})$ we can find a sequence $R'_i \in f^m(\mathcal{R})$ such that m is arbitrarily big and the map $y \mapsto \pi_{x_0}(x, y)$ defined by the original sequence $R_{1,i}$ is a restriction of the maps defined by the new sequence R'_i.

Suppose that $h_i : W_-(x_0) \longrightarrow W_-(x)$ for $i = 1, 2$ are compositions of holonomies defined using two sequences $R_{1,i}$, and $R'_{1,i}$. We may assume that both sequences belong to $f^{k_1}(\mathcal{R})$ for some fixed k_1. Let y belong to the domain of both maps h_i. We may assume (taking k_1 big enough) that x and y belong to connected components of the domains of h_i. Then there exists a connected chain of rectangles $R_{2,i} \in f^{-n_2}(\mathcal{R})$, for some $n_2 > 0$, such that the restrictions of h_i to the corresponding plaques of $R_{2,i}$ are equal to holonomies in $R_{2,i}$. It follows then from $h_1(x_0) = h_2(x_0) = x$ that $h_1(y) = h_2(y)$. Consequently, π_{x_0} is well defined. The map π_{x_0} is obviously a local homeomorphism.

Note that if $R \in \mathcal{R}$ is such that $x_0 \in R$, then $\pi_{x_0} : P_+(R, x_0) \times P_-(R, x_0) \longrightarrow R$ coincides with $[\cdot, \cdot]_R$.

Let $(a, b) \in W_+(x_0) \times W_-(x_0)$ be an arbitrary point, and let $x_1 = \pi_{x_0}(a, b)$. Then it follows from the definition of the maps π_{x_i} and their uniqueness that

$$\pi_{x_0}(x, y) = \pi_{x_1}(\pi_{x_0}(x, b), \pi_{x_0}(a, y)). \tag{18}$$

It follows that π_{x_0} is onto, since its range contains every rectangle $R \in \mathcal{R}$ intersecting it. It also follows that the map π_{x_0} is a covering, since every point of $W_+(x_0) \times W_-(x_0)$ has a neighborhood mapped homeomorphically by π_{x_0} to an element of $\mathcal{R}$.

Another corollary of (18) is that π_{x_0} homeomorphically maps the plaques of $W_+(x_0) \times W_-(x_0)$ to the leaves of $\mathcal{X}$, since π_{x_1} maps the direct factors of $W_+(x_1) \times W_-(x_1)$ identically onto the leaves $W_+(x_1)$ and $W_-(x_1)$. This finishes the proof of the theorem. □

7.2. Polynomial growth

Theorem 7.3. *Let $(\mathcal{X}, f)$ be a Smale space with pinched spectrum. Then the group of deck transformations of the splitting of $(\mathcal{X}, f)$ has polynomial growth.*

Proof. Our proof essentially repeats the proof of the main theorem of [7]. Let $\mathcal{R}$ be a finite covering of $\mathcal{X}$ by open connected rectangles. Let $\pi : \mathcal{M} \longrightarrow \mathcal{X}$ be the splitting constructed in Theorem 7.2, where $\mathcal{M} = W_+(x_0) \times W_-(x_0)$. Denote by $\widetilde{\mathcal{R}}$ the union of the sets of connected components of $\pi^{-1}(R)$ for $R \in \mathcal{R}$.

Consider the graph Γ with the set of vertices identified with $\widetilde{\mathcal{R}}$ in which two vertices are connected if the corresponding sets have non-empty intersection. It is easy to show (see the proof of Proposition 5.5) that the graph Ξ is quasi-isometric to the Cayley graph of the group G of deck transformations of π and has the same growth rate as G.

Let $B(r)$ be the set of elements of $\widetilde{\mathcal{R}}$ that are on distance at most r in Γ from a vertex $R \in \widetilde{\mathcal{R}}$ such that $(x_0, x_0) \in R$.

Let $0 < \alpha_0 < a_0 \le a_1 < \alpha_1$ and $0 < \beta_0 < b_0 \le b_1 < \beta_1$ such that $\alpha_0/\alpha_1 + \beta_0/\beta_1 > 1$. Denote by d_+ and d_- the metrics of exponents α_0 and β_0 on the corresponding leaves of $\mathcal{X}$ and plaques of $W_+(x_0) \times W_-(x_0)$. (Every plaque of $\mathcal{M}$ is identified with a leaf of $\mathcal{X}$ by π.)

Take $R \in B(r)$. Choose a sequence $R_0 \ni (x_0, x_0), R_1, \ldots, R_m = R$ of elements of $\widetilde{\mathcal{R}}$ forming a chain in Γ of length $m \le r$.

We will denote by $[\cdot, \cdot]$ the direct product structure on $W_+(x_0) \times W_-(x_0)$. Let D_- and D_+ be the suprema of the d_-- and d_+-diameters of the sets $[y, \cup R_i]$ and $[\cup R_i, y]$ for all $y \in \cup R_i$ (see Fig. 5).

There exist constants C_1, C_2 not depending on r, and a number $n = n(r)$ such that $|n - \log D_-/\beta_0| < C_1$, and for every point $y \in \mathcal{M}$ there exists a rectangle $V \in f^n(\mathcal{R})$

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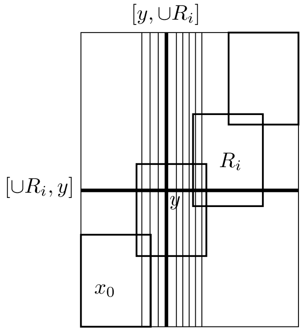

Fig. 5. Growth estimation

such that the $C_2 e^{-\alpha_0 n}$-neighborhood of y in the stable plaque of y (with respect to d_+) and the set $[y, \cup R_i]$ are both contained in V.

It follows that we can find a sequence of rectangles $V_i \in f^n(\mathcal{R})$ of length at most $C_3 e^{\alpha_1 n}$ such that $V_i \cap V_{i+1} \neq \emptyset$, the first rectangle in the sequence contains a given point of $R_{i-1} \cap R_i$, while the last one contains a given point of $R_i \cap R_{i+1}$.

Consequently, we can find a sequence of rectangles $V_0, \ldots, V_l \in f^n(\mathcal{R})$ of length at most

$$C_3 r e^{\alpha_1 n} \le C_4 r D_-^{\alpha_1/\beta_0}$$

such that $V_i \cap V_{i+1} \neq \emptyset$, $x_0 \in V_0$, V_m contains a point $x \in R = R_m$, and for every V_i there exists a point $z_i \in V_i$ such that $[z_i, \cup R_i] \subset V_i$. Moreover, we may assume that the chain V_i covers any given in advance three point $y_1, y_2, y_3 \in \cup R_i$.

It follows that the d_+-distance from $[y_1, y_2]$ to $[y_3, y_2]$ is bounded from above by

$$C_5 r D_-^{\alpha_1/\beta_0} \cdot e^{-\alpha_0 n} \le C_6 r D_-^{\alpha_1/\beta_0} D_-^{-\alpha_0/\beta_0} = C_6 r D_-^{\frac{\alpha_1 - \alpha_0}{\beta_0}}.$$

It follows that

$$D_+ \le C_6 r D_-^{\frac{\alpha_1 - \alpha_0}{\beta_0}},$$

and, by the same argument,

$$D_- \le C_7 r D_+^{\frac{\beta_1 - \beta_0}{\alpha_0}}.$$

Combining the inequalities, we get

$$D_+ \le C_6 r \left(C_7 r D_+^{\frac{\beta_1 - \beta_0}{\alpha_0}} \right)^{\frac{\alpha_1 - \alpha_0}{\beta_0}} = C_8 r^{1 + \frac{\alpha_1 - \alpha_0}{\beta_0}} D_+^{\frac{(\alpha_1 - \alpha_0)(\beta_1 - \beta_0)}{\alpha_0 \beta_0}},$$

hence

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$$D_+^{\frac{\alpha_1}{\alpha_0}+\frac{\beta_1}{\beta_0}-\frac{\alpha_1\beta_1}{\alpha_0\beta_0}} \leq C_8 r^{1+\frac{\alpha_1-\alpha_0}{\beta_0}}.$$

Note that $\frac{\alpha_1}{\alpha_0}+\frac{\beta_1}{\beta_0}-\frac{\alpha_1\beta_1}{\alpha_0\beta_0}=\frac{\alpha_1\beta_1}{\alpha_0\beta_0}\left(\frac{\alpha_0}{\alpha_1}+\frac{\beta_0}{\beta_1}-1\right)>0$, hence

$$D_+ \leq C_8 r^{p_+}$$

for $p_+=\left(1+\frac{\alpha_1-\alpha_0}{\beta_0}\right)\left(\frac{\alpha_1}{\alpha_0}+\frac{\beta_1}{\beta_0}-\frac{\alpha_1\beta_1}{\alpha_0\beta_0}\right)^{-1}$.

Similarly,

$$D_- \leq C_9 r^{p_-}$$

for $p_-=\left(1+\frac{\beta_1-\beta_0}{\alpha_0}\right)\left(\frac{\alpha_1}{\alpha_0}+\frac{\beta_1}{\beta_0}-\frac{\alpha_1\beta_1}{\alpha_0\beta_0}\right)^{-1}$.

In particular (taking $C_{10}=\max(C_8, C_9)$ and $p=\max(p_+, p_-)$) we have that $d_-([x_0, x], x_0)$ and $d_+([x, x_0], x_0)$ are less than $C_{10}r^p$ for any $x \in \bigcup_{R\in B(r)} R$.

Let μ_+ and μ_- be the measures satisfying the conditions of Theorem 3.8 (note that $(\mathcal{X}, f)$ is mixing by Proposition 5.2). Let μ be their direct product on $\mathcal{M} = W_+(x_0)\times W_-(x_0)$. Since the measures μ_+ and μ_- on the leaves of $\mathcal{X}$ are invariant under holonomies, G acts by measure preserving transformations on $\mathcal{M}$. It follows that there exist positive constants A_1 and A_2 such that

$$A_1|B(r)| \leq \mu\left(\bigcup_{R\in B(r)} R\right) \leq A_2|B(r)|.$$

By the proven above, the set $\bigcup_{R\in B(r)} R$ is contained in the direct product of the balls of radius $C_{10}r^p$ with center in x_0 in $W_+(x_0)$ and $W_-(x_0)$. By condition (1) of Theorem 3.8 volumes of these balls are bounded from above by $C(C_{10}r^p)^{\eta/\alpha_0}$ and $C(C_{10}r^p)^{\eta/\beta_0}$ for some constant C. It follows that $|B(r)|$ is bounded above by a polynomial in r. □

By the Gromov's theorem on groups of polynomial growth [17], G is virtually nilpotent. Theorem 6.1 now implies the following description of Smale spaces with pinched spectrum.

Theorem 7.4. *Every connected and locally connected Smale space with pinched spectrum is topologically conjugate to an infra-nilmanifold automorphism.*

8. Mather spectrum of Anosov diffeomorphisms

Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a diffeomorphism of a compact Riemann manifold $\mathcal{X}$. It induces a linear operator f_* on the Banach space of continuous vector fields by

$$f_*(\vec{X})(x) = Df \circ \vec{X}(f^{-1}(x)).$$

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By a theorem of J. Mather [23], f is an Anosov diffeomorphism if and only if the spectrum of f_+ does not intersect the unit circle. It belongs then to the set

$$\{z \in \mathbb{C} \ : \ \lambda_1 < |z| < \lambda_2\} \cup \{z \in \mathbb{C} \ : \ \mu_2 < |z| < \mu_1\},$$

where $0 < \lambda_1 < \lambda_2 < 1 < \mu_2 < \mu_1$. The tangent bundle $T\mathcal{X}$ is decomposed into a direct sum $W^s \oplus W^u$ such that there exists a constant $C > 1$ such that for all vectors $\vec{v}_+ \in W^s$, $\vec{v}_- \in W^u$, and for every positive integer n we have

$$C^{-1}\lambda_1^n\|\vec{v}_+\| \leq \|Df^n\vec{v}_+\| \leq C\lambda_2^n\|\vec{v}_+\|$$

and

$$C^{-1}\mu_2^n\|\vec{v}_-\| \leq \|Df^n\vec{v}_-\| \leq C\mu_1^n\|\vec{v}_-\|.$$

For more detail, see [23,6,7]. Every Anosov diffeomorphism is a Smale space, see [10, Proposition 5.10.1]. Stable and unstable leaves of $(\mathcal{X}, f)$ are manifolds, and the vectors of W^s and W^u are tangent to the stable and unstable leaves, respectively.

Proposition 8.1. *Let $\lambda_1, \lambda_2, \mu_1, \mu_2$ be as above. Then $\log\mu_2$ and $\log\mu_1$ are unstable lower and upper exponents of $(\mathcal{X}, f)$, and $-\log\lambda_2$ and $-\log\lambda_1$ are stable lower and upper exponents of $(\mathcal{X}, f)$.*

Proof. It is easy to show that for every fixed k_0 the metrics d_{k_0} on leaves $W_+(x_0)$ and $W_-(x_0)$ are quasi-isometric to the restrictions of the Riemannian metric of $\mathcal{X}$ onto $W_+(x_0)$ and $W_-(x_0)$, with the quasi-isometry constants depending only on k_0 (and $(\mathcal{X}, f)$). For every stable leaf W_+ of $(\mathcal{X}, f)$ and all $x, y \in W_+$, $n \in \mathbb{Z}$,

$$d_0(f^{-n}(x), f^{-n}(y)) = d_n(x, y).$$

It follows that there exist constants $C_1 > 1, \Delta > 0$ such that

$$C_1^{-1}\widetilde{d}_+(f^{-n}(x), f^{-n}(y)) - \Delta \leq d_n(x, y) \leq C_1\widetilde{d}_+(f^{-n}(x), f^{-n}(y)) + \Delta,$$

where $\widetilde{d}_+$ is the Riemannian metrics on the stable leaves.

If γ is a curve in the stable leaf connecting $f^{-n}(x)$ to $f^{-n}(y)$, then $f^n(\gamma)$ is a curve connecting x to y, and

$$\text{length}(f^n(\gamma)) = \int \left\|\frac{d}{dt}f^n \circ \gamma(t)\right\| \, dt = \int \left\|Df^n \circ \frac{d}{dt}\gamma(t)\right\| \, dt,$$

hence

$$C^{-1}\lambda_1^n \,\text{length}(\gamma) \leq \text{length}(f^n(\gamma)) \leq C\lambda_2^n \,\text{length}(\gamma),$$

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and

$$C^{-1}\lambda_1^n \cdot \widetilde{d}_+(f^{-n}(x), f^{-n}(y)) \le \widetilde{d}_+(x,y) \le C\lambda_2^n \cdot \widetilde{d}_+(f^{-n}(x), f^{-n}(y)).$$

It follows that

$$C_1^{-1}C^{-1}d_+(x,y)\cdot\lambda_2^{-n} - \Delta \le d_n(x,y) \le C_1 C\widetilde{d}_+(x,y)\cdot\lambda_1^{-n} + \Delta$$

for all stably equivalent x, y and all positive n. Then, by Propositions 3.7 and 4.12, $-\log\lambda_1$ and $-\log\lambda_2$ are upper and lower exponents.

The case of unstable leaves is proved in the same way. □

M. Brin considers in [6–9] Anosov diffeomorphisms such that either

$$1 + \frac{\log\mu_2}{\log\mu_1} > \frac{\log\lambda_1}{\log\lambda_2} \tag{19}$$

or

$$1 + \frac{\log\lambda_2}{\log\lambda_1} > \frac{\log\mu_1}{\log\mu_2}. \tag{20}$$

Note that since $\frac{\log\mu_1}{\log\mu_2}$ and $\frac{\log\lambda_1}{\log\lambda_2}$ are both greater than one, each of the inequalities (19) and (20) implies

$$\frac{\log\lambda_2}{\log\lambda_1} + \frac{\log\mu_2}{\log\mu_1} > 1. \tag{21}$$

For instance, in the case of (19):

$$\frac{\log\lambda_1}{\log\lambda_2} + \frac{\log\mu_1}{\log\mu_2} > 1 + \frac{\log\mu_1}{\log\mu_2} = \frac{\log\mu_1}{\log\mu_2}\left(\frac{\log\mu_2}{\log\mu_1} + 1\right) > \frac{\log\mu_1}{\log\mu_2}\cdot\frac{\log\lambda_1}{\log\lambda_2}.$$

Multiplying by $\frac{\log\mu_2}{\log\mu_1}\cdot\frac{\log\lambda_2}{\log\lambda_1}$, we get (21).

Note that if a_0 and a_1 are stable lower and upper critical exponents, and b_0 and b_1 are unstable lower and upper critical exponents of the Smale space, then $-\log\lambda_2 \le a_0$, and $-\log\lambda_1 \ge a_1$, $\log\mu_2 \le b_0$, and $\log\mu_1 \ge b_1$, so that

$$\frac{\log\mu_2}{\log\mu_1} \le \frac{a_0}{a_1}, \qquad \frac{\log\lambda_2}{\log\lambda_1} \le \frac{b_0}{b_1},$$

and therefore

$$\frac{a_0}{a_1} + \frac{b_0}{b_1} \ge \frac{\log\mu_2}{\log\mu_1} + \frac{\log\lambda_2}{\log\lambda_1}.$$

Consequently, each of the conditions (19), (20) implies the inequality

$$\frac{a_0}{a_1} + \frac{b_0}{b_1} > 1,$$

which is the condition of Theorems 7.2 and 7.3. In particular, we conclude that Theorem 7.4 is a generalization of the results of [9].

9. Co-dimension one Smale spaces

We say that a Smale space is of *co-dimension one* if its stable or unstable leaves are homeomorphic to $\mathbb{R}$ with respect to their intrinsic topology.

It was proved by Franks [14, Theorem 6.3] and Newhouse [28] that every co-dimension one *Anosov diffeomorphism* is topologically conjugate to a linear Anosov diffeomorphism of a torus. Here we prove this statement for all locally connected and connected Smale spaces.

Theorem 9.1. *A locally connected and connected co-dimension one Smale space is topologically conjugate to a co-dimension one hyperbolic automorphism of a torus* $\mathbb{R}^d/\mathbb{Z}^d$.

Proof. Let us prove at first that every co-dimension one locally connected and connected Smale space $f : \mathcal{X} \longrightarrow \mathcal{X}$ is splittable and irreducible. We assume that the stable leaves of $\mathcal{X}$ are homeomorphic to $\mathbb{R}$.

Let $d_+(x, y)$ and $d_-(x, y)$ be the metrics on the stable and unstable leaves of $\mathcal{X}$ associated with the natural log-scales and some exponents, as in the proof of Theorem 7.2. Let $d_n(x, y)$ be the distance inside the stable leaves of $\mathcal{X}$ defined in 3. Changing the log-scale ℓ used in the definition of $d_n(x, y)$ to a bi-Lipschitz equivalent one (e.g., changing the entourage U in the definition of ℓ) will change d_n to a metric d'_n satisfying $L^{-1}d_n(x, y) - C \le d'_n(x, y) \le Ld_n(x, y) + C$. In other words, the identity map will be a quasi-isometry.

In particular, up to quasi-isometry, the metric d_n can be defined in the following way. Fix a finite cover $\mathcal{R}$ of $\mathcal{X}$ by small open rectangles, such that the stable direction of every rectangle $R_i \in \mathcal{R}$ is homeomorphic to $\mathbb{R}$ (equivalently, to an open interval in $\mathbb{R}$). We will denote $f^n(\mathcal{R}) = \{f^n(R) \ : \ R \in \mathcal{R}\}$. By the Lebesgue covering lemma there exists $\delta_0 > 0$ such that for every $x \in \mathcal{X}$ there exists $R \in \mathcal{R}$ such that the δ_0-neighborhoods of x in its stable and unstable leaves belong to R.

Then $d_n(x, y)$ is equal (up to linear lower and upper bounds) to the smallest number k such that there exists a sequence of stable plaques $P_0, P_1, \ldots, P_k$ of rectangles R_i from $f^n(\mathcal{R})$ such that $P_i \cap P_{i+1} \neq \emptyset$, $x \in P_0$, $y \in P_k$.

Fix $\epsilon > 0$. For every $n \ge 1$, let D_n be the smallest value of $d_n(x, y)$ for stably equivalent x, y and such that $d(x, y) > \epsilon$. Let x and y be some points realizing the minimum D_n. Let $R_0, R_1, \ldots, R_{D'} \in f^n(\mathcal{R})$ be a sequence of rectangles as in the previous paragraph, were $L^{-1}D_n - C \le D' \le LD_n + C$ for some fixed L and C.

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Note that it follows from the inequality $d_{n-\Delta}(x,y) \le \frac{1}{2}(d_n(x,y)+1)$ for some $\Delta > 0$ (see the beginning of the proof of Proposition 3.3) and the fact that f^k induces an isomorphism $\Gamma_n(W) \longrightarrow \Gamma_{n+k}(f^k(W))$ that there exist positive constants k and l such that $d_n(f^k(x), f^k(y)) < d_n(x,y)$. Consequently, $d(f^k(x_n), f^k(y_n)) < \epsilon$. It follows (as f is bi-Lipschitz) that there exists a constant $C_1 > 0$ such that $d(x_n, y_n) < C_1\epsilon$ for all n. We may assume that $C_1\epsilon < \delta_0$ by choosing ϵ small enough.

Consider the composition of the holonomies between the unstable plaques of the rectangles R_i defined by the intersections $R_i \cap R_{i+1}$ and seen as a homeomorphism from a neighborhood $U_x \subset W_-(x)$ of x to a neighborhood $U_y \subset W_-(y)$ of y. Suppose that $x' \in U_x$, and let $y' \in U_y$ be its image under the composition. Then we get a chain of intersecting stable plaques of the rectangles R_i starting from the plaque of x' and ending in the plaque of y'. It follows that $d_n(x', y') \le L' d_n(x,y) + \Delta'$ for some fixed L', Δ'. Applying an appropriate iteration f^s (where s depends only on L' and Δ') we get $d_n(f^s(x'), f^s(y')) < d_n(x,y)$, which implies that $d(f^s(x'), f^s(y')) < \epsilon$. Consequently, there exists a constant $r > 0$ (not depending on n) such that $d(x', y') < r$.

We are using now arguments similar to the arguments of the proof of Theorem 7.2. By the Lebesgue covering condition for the rectangles R_i and the fact that f^{-1} uniformly expands the unstable leaves, we get that the rectangles R_i contain $C_2 \exp(\alpha n)$-neighborhoods in the unstable direction of their intersections with the stable leaf of x and y. If $x' \in U_x, y' \in U_y$ are as in the previous paragraph, then there exists a rectangle $R \in f^{-m}(\mathcal{R})$ containing x' and y' in one stable plaque and such that the δ_1-neighborhood of x' in the unstable direction belongs to R (where m and hence δ_1 do not depend on n, x, and y). Then the holonomy of the unstable leaf from x' to y' extends to this neighborhood (but may be not contained in U_x). The smallest number of steps of size at most δ from x to a point $z \in W_-(x)$ is bounded from above by a function of the form $C_3 d(x,z)^p + C_4$ (where p depends on the ratio of the exponent of the metric and the upper exponent of the unstable direction, see the proof of Theorem 7.2). The holonomy inside the rectangles $R \in f^{-s}(\mathcal{R})$ are uniformly Lipschitz. It follows that when we apply the holonomies between the unstable directions coming from the intersections $R_i \cap R_{i+1}$, one by one, then the distances between the images of x and z are bounded from above by a function of the form $C_5 d(x,z)^p + C_6$. It follows that $z \in U_x$ if $C_5 d(x,z)^p + C_6 < C_2 \exp(\alpha n)$. Consequently, the holonomy from x to y can be extended to a ball with center in x of exponentially big in n radius.

Let x_n, y_n be the points realizing the minimum D_n. Choose a strictly increasing sequence n_k such that each of the sequence x_n and y_n converge to points x, y. Since $d(x_n, y_n) < C_1\epsilon$ for all n, such a sequence exists and the points x, y are stably equivalent with $\epsilon \le d(x,y) \le C_1\epsilon$. The holonomy from the unstable plaque of x_{n_k} to the unstable plaque of x is a bi-Lipschitz map with the same bi-Lipschitz constant L defined on an exponentially big in n_k ball inside $W_-(x_{n_k})$, the same is true for y and y_{n_k}. By the proven above, the holonomy from $W_-(x_{n_k})$ to $W_-(y_{n_k})$ is defined on an exponentially big ball with center in x_{n_k}. It follows that the holonomy between the unstable leaves of x and y is everywhere defined.

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Note that we have not used the fact that the Smale space is of co-dimension one so far. We have shown that for every locally connected Smale space there exist two different stably equivalent points x and y such that local holonomies between their unstable plaques can be extended to a global holonomy of their unstable leaves $W_-(x)$ and $W_-(y)$. In the case when the stable leaves are one-dimensional, this is enough to show splittability of the Smale space. It is unclear, however, what we can deduce from this fact in the general case.

Namely, let x and y be as above. Then for every point $x' \in W_-(x)$ there exists N and a chain of rectangles $R_0, R_1, \ldots, R_D \in f^N(\mathcal{R})$ such that $x', x \in R_0$, $y \in R_D$, $R_i \cap R_{i+1} \neq \emptyset$, and the composition of all the holonomies between the unstable plaques of R_i defined by the intersections $R_i \cap R_{i+1}$ is a holonomy from $U_-(x) \subset W_-(x)$ to $U_-(y)$ such that $x, x' \in U_-(x)$ and $y \in U_-(y)$.

The stable directions of R_i are open sub-intervals of the stable leaf of x and y such that x and y are contained in the initial and the final intervals, and every two neighboring intervals intersect. It follows that these intervals cover the interval $I \subset W_+(x)$ with the endpoints x and y. Consequently, there exists a local product preserving map $\pi : I \times W_-(x) \longrightarrow \mathcal{X}$ identical on I and $W_-(x)$. Denote by π_n the map $f^{-n} \circ \pi \circ f^n : f^{-n}(I) \times W_-(f^{-n}(x)) \longrightarrow \mathcal{X}$. It is also local product preserving and identical on $f^{-n}(I) \times W_-(f^{-n}(x))$. Note that the diameter of $f^{-n}(I)$ inside $W_+(f^{-n}(x))$ grows exponentially with $n > 0$.

Let t be an interior point of the interval I. Let m_k be a sequence of positive integers converging to infinity such that the limit $t' = \lim_{k\to\infty} f^{-m_k}(t)$ exists. The maps $\pi_{m_k} : f^{-m_k}(I) \times W_-(f^{-m_k}(t)) \longrightarrow \mathcal{X}$ as $k \to \infty$ will converge to a splitting $W_+(t') \times W_-(t') \longrightarrow \mathcal{X}$.

Consequently, $f : \mathcal{X} \longrightarrow \mathcal{X}$ is irreducible (see Proposition 5.2). Let μ_+ be the measure on stable leaves described in Theorem 3.8. The group G of deck transformations of a splitting of $\mathcal{X}$ acts on a stable leaf $W_+(x_0) \cong \mathbb{R}$ by the transformations $x \mapsto [g(x), x_0]$. This action preserves the measure μ_+. Let us identify $(W_+(x_0), \mu_+)$ and $\mathbb{R}$ with the Lebesgue measure by a measure-preserving homeomorphism. Since G acts by measure-preserving transformations, the corresponding action of G on $\mathbb{R}$ is by transformations of the form $x \mapsto \pm x + a$ for $a \in \mathbb{R}$.

The action of G on $W_+(x_0)$ is free, since otherwise an unstable leaf is not mapped homeomorphically onto its image in $\mathcal{X}$. But this implies that G acts on $\mathbb{R}$ by translations, hence it is torsion-free abelian. Therefore, by Proposition 5.5 and Theorem 6.10, $(\mathcal{X}, f)$ is topologically conjugate to a hyperbolic automorphism of the torus $\mathbb{R}^d/\mathbb{Z}^d$ for some d. □

Note that the proof of Franks-Newhouse theorem due to K. Hiraide [18] also uses the measure μ_+ on the stable leave.

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