A REMARK ON NON-INTEGRAL p-ADIC SLOPES FOR MODULAR FORMS

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ABSTRACT. We give a sufficient condition, namely "Buzzard irregularity", for there to exist a cuspidal eigenform which does not have integral p-adic slope.

RÉSUMÉ. ***Une remarque sur les pentes p-adiques non-entières des formes modulaires.*** On donne une condition suffisante, à savoir irrégularité au sens de Buzzard, pour qu'il existe une forme parabolique propre de pente p-adique non-entière.

1. Statement of result

Let p be a prime number. If k and M are integers then we write $S_k(\Gamma_0(M))$ for the space of weight k cusp forms of level $\Gamma_0(M)$. The p-th Hecke operator acting on $S_k(\Gamma_0(M))$ is written T_p if $p \nmid M$ and U_p otherwise.

For $T = T_p$ or U_p, we define the slopes of T to be the slopes of p-adic Newton polygon of the inverse characteristic polynomial $\det(1 - TX)$. This is the same as the list of the p-adic valuations of the non-zero eigenvalues of T, counted with algebraic multiplicity.

To state our theorem we need a definition due to Buzzard [4].

Definition 1.1. *Let $N \geq 1$ be an integer with $p \nmid N$.*

(a) An odd prime p is $\Gamma_0(N)$-regular if the slopes of T_p acting on $S_k(\Gamma_0(N))$ are all zero for $2 \leq k \leq \frac{p+3}{2}$.

(b) The prime $p = 2$ is $\Gamma_0(N)$-regular if the slopes of T_2 acting on $S_2(\Gamma_0(N))$ are all zero and the slopes of T_2 acting on $S_4(\Gamma_0(N))$ are all either zero or one.

This definition first appeared in [4] where Buzzard gives an elementary algorithm, depending on p and N, which on input k will output a list of integers. He conjectures that if p is $\Gamma_0(N)$-regular then this list is exactly the list of slopes of T_p acting on $S_k(\Gamma_0(N))$. The authors of the present work also have made a separate conjecture ([3]) which predicts the U_p-slopes of all p-adic modular forms of tame level $\Gamma_0(N)$ still assuming that p is $\Gamma_0(N)$-regular. The two conjectures are consistent with each other experimentally, but have not yet been shown to be consistent in general.

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Buzzard's conjecture clearly implies that every slope is an integer. (This implication is not at all clear from the conjectures in [3].) It is worth asking if the integrality of slopes is characteristic of $\Gamma_0(N)$-regularity. We show that it is. The proof occupies the second section.

Theorem 1.2. *If p is not $\Gamma_0(N)$-regular then there exists an even integer k such that U_p acting on $S_k(\Gamma_0(Np))$ has a slope strictly between zero and one.*

Coleman theory (which is used below) shows that no harm comes from assuming the witnessing weight in Theorem 1.2 is arbitrarily large. One could try to determine the minimum weight k which confirms Theorem 1.2. An effective bound should follow from [10], but it is likely suboptimal. Numerical data suggest that the optimal k, for p odd, is either $k = j$ or $k = j + (p - 1)$ where $2 \leq j \leq \frac{p+3}{2}$ is a low weight with a non-zero T_p-slope.

The theorem is also true for if we replace U_p and $S_k(\Gamma_0(Np))$ by T_p and $S_k(\Gamma_0(N))$. Indeed, if a_p is an eigenvalue for T_p acting on $S_k(\Gamma_0(N))$ then the polynomial $X^2 - a_pX + p^{k-1}$ divides the characteristic polynomial of U_p acting on $S_k(\Gamma_0(Np))$; the eigenvalues λ for U_p which are not roots of such polynomials are known to satisfy $\lambda^2 = p^{k-2}$. So, if $k > 2$ (which is sufficient by the previous paragraph) the slopes of U_p between zero and one are the same as the slopes of T_p between zero and one.

For p odd, the converse to Theorem 1.2 is also true. Namely, if there exists an even integer k such that $S_k(\Gamma_0(N))$ has a slope strictly between zero and one then p is not $\Gamma_0(N)$-regular. See [5, Theorem 1.6]. Its proof uses the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ and is thus significantly deeper than the present work. Combining the two results, the following two conditions are equivalent for an odd prime p:

(a) The prime p is not $\Gamma_0(N)$-regular.
(b) There exists an even integer k such that T_p acting on $S_k(\Gamma_0(N))$ has a slope strictly between zero and one.

There is a natural third condition, implied by (b):

(c) There exists an integer k such that T_p acting on $S_k(\Gamma_0(N))$ has a non-integral slope.

It is conjectured (see [6]) that all three conditions are equivalent, but this seems difficult.

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2. The proof

We fix algebraic closures $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p$ and write $v_p(-)$ for the induced p-adic valuation on $\overline{\mathbf{Q}}$ normalized so that $v_p(p) = 1$. We also fix an embedding $\overline{\mathbf{Q}} \subset$. We assume now that $N \geq 1$ is an integer co-prime to p.

If η is a Dirichlet character of modulus p we write $S_k(\Gamma_1(Np), \eta)$ for the subspace of forms in $S_k(\Gamma_1(Np))$ with character given by η (η promoted to a character of modulus Np). An eigenform f in particular means a normalized eigenform for the standard Hecke operators and the diamond operators. For such an f, its p-th Hecke eigenvalue is written $a_p(f)$.

Corresponding to the choice of embeddings, each eigenform has an associated two-dimensional p-adic Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$. Write $\overline{\rho}_f$ for its reduction modulo p and $\overline{\rho}_{f,p}$ (resp. $\rho_{f,p}$) for the restriction of $\overline{\rho}_f$ (resp. ρ_f) to the decomposition group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ induced from the embedding $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p$. Note that the construction of $\overline{\rho}_f$ requires the choice of a Galois-stable lattice, but that the semi-simplification of $\overline{\rho}_f$ is independent of this choice. In particular, whether or not $\overline{\rho}_{f,p}$ is irreducible is also independent of the choice of a stable lattice.

Lemma 2.1. *Let η be a Dirichlet character of conductor p and f an eigenform in $S_2(\Gamma_1(Np), \eta)$. If $v_p(a_p(f))$ equals 0 or 1, then $\rho_{f,p}$ is reducible.*

Proof. If $v_p(a_p(f)) = 0$ then it is well known that $\rho_{f,p}$ is reducible. For example, see [11, Lemma 2.1.5] and the references therein. (This is also commonly attributed to a letter from Deligne to Serre in the 1970s which has never been made public.)

Now suppose that $v_p(a_p(f)) = 1$. Then, there is an eigenform f' in $S_2(\Gamma_1(Np), \eta^{-1})$ with $v_p(a_p(f')) = 0$ and ρ_f isomorphic to $\rho_{f'}$ up to a twist. (The form f' is sometimes called the Atkin–Lehner involute of f; see [2, Proposition 3.8].) Since the first argument applies to f', we deduce that $\rho_{f',p}$ and its twist $\rho_{f,p}$ are both reducible. □

Proposition 2.2. *If p is odd and not $\Gamma_0(N)$-regular then there exists an even Dirichlet character η of modulus p such that U_p acting on $S_2(\Gamma_1(Np), \eta)$ has a slope strictly between zero and one.*

Proof. Choose an integer $2 \leq k \leq \frac{p+3}{2}$ and an eigenform $f \in S_k(\Gamma_0(N))$ with $v_p(a_p(f)) > 0$. By [9, Theorem 2.6], $\overline{\rho}_{f,p}$ is irreducible.

Suppose first that f has weight 2. Then, the polynomial $X^2 - a_p(f)X + p$ divides the characteristic polynomial of U_p acting on $S_k(\Gamma_0(Np))$ (as in the remarks after Theorem 1.2). The theory of the Newton polygon implies that the roots of this polynomial have valuation strictly between zero and one, so we can choose η to be the trivial character and we are done in this case.

Now assume that f has weight at least 4 and thus also $p \geq 5$. By [1, Theorem 3.5(a)], which assumes $p \geq 5$, there exists an even Dirichlet character η necessarily of conductor p (because f has weight at most $\frac{p+3}{2} < p+1$) and an eigenform $g \in S_2(\Gamma_1(Np), \eta)$ such that $\overline{\rho}_g$ and $\overline{\rho}_f$ have isomorphic semi-simplifications. Since $\overline{\rho}_{f,p}$ is irreducible, $\overline{\rho}_{g,p}$ is as well. Thus, $\rho_{g,p}$ is irreducible, and Lemma 2.1 implies that $v_p(a_p(g))$ is strictly between zero and one. □

Proposition 2.2 is an analog of Theorem 1.2 for weight two forms with character, and its proof confirms our theorem when there is a weight 2 form of level $\Gamma_0(N)$ with positive T_p-slope. To prove Theorem 1.2 in general, we use the theory of p-adic modular forms. We refer to [8] for the facts in the next two paragraphs.

If $\kappa : \mathbf{Z}_p^\times \to \overline{\mathbf{Q}}_p^\times$ is a continuous character (a "p-adic weight") then we write $S_\kappa^\dagger(N)$ for the space of overconvergent p-adic cusp forms of weight κ and tame level $\Gamma_0(N)$ equipped with its U_p-operator. If k is an integer and $\kappa(z) = z^k$ then we write this space as $S_k^\dagger(N)$; it contains $S_k(\Gamma_0(Np))$ as a U_p-compatible subspace. Likewise, if $\kappa(z) = z^k\eta(z)$ where η is a non-trivial finite order character of $\mathbf{Z}_p^\times$ then $S_{z^k\eta}^\dagger(N)$ contains $S_k(\Gamma_1(Np^{f_\eta}), \eta)$ as a U_p-compatible subspace (where p^{f_η} is the conductor of η).

By Coleman theory we mean the following: suppose that κ is a p-adic weight and h is the p-adic valuation of a non-zero eigenvalue for U_p appearing in $S_\kappa^\dagger(N)$. Then, for any sequence of p-adic weights $(\kappa_n)_{n\geq 0}$ such that κ_n and κ agree on the torsion subgroup of $\mathbf{Z}_p^\times$, and $\kappa_n(1+2p) \to \kappa(1+2p)$ as $n \to \infty$, we have that h is also a U_p-slope in $S_{\kappa_n}^\dagger(N)$ for $n \gg 0$.

We can now give the proof of the theorem.

Proof of Theorem 1.2. Assume first that p is odd. By Proposition 2.2 there exists an even Dirichlet character η of modulus p and rational number $0 < h < 1$ which appears as a U_p-slope in $S_2(\Gamma_1(Np), \eta)$. Thus, the slope h appears as a U_p-slope in $S_{z^2\eta}^\dagger(N)$. Choose $j \geq 0$ even so that $\eta|_{\mathbf{F}_p^\times}$ is of the form $z \mapsto z^j$. Then, for $n \gg 0$ and $k_n = 2 + j + (p-1)p^n$, the slope h is a U_p-slope in $S_{k_n}^\dagger(N)$ by Coleman theory described above. For such k we have $h < 1 < k-1$ and so h is U_p-slope in $S_k(\Gamma_0(Np))$ by [7, Theorem 6.1].

The proof for $p = 2$ is similar to the argument in Proposition 2.2 when $k = 2$. If either $S_2(\Gamma_0(N))$ or $S_4(\Gamma_0(N))$ has a non-integral slope we are done. If not, then either $S_2(\Gamma_0(N))$ contains a slope one form, or $S_4(\Gamma_0(N))$ contains a form of slope two or three. In either case, the corresponding 2-adic refinements will have fractional slope. □

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