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Optimal Communication Topology and Static Output Feedback of Networked Collocated Actuator/Sensor Pairs in Distributed Parameter Systems

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Abstract— **This paper is motivated by economic aspects of fixed initial and operating costs for control of spatially distributed systems. In particular, the paper investigates the possibility of a large number of inexpensive actuating and sensing devices, as an alternative to (a reduced number of) expensive high capacity devices. While such an alternative reduces the fixed initial costs associated with actuators and sensors, it may also lead to increased operating costs resulting from communication requirements between the now-networked actuator-sensor-control units. To simplify the controller architecture, a proportional controller is assumed that amounts to a static output feedback controller. In a network of n actuator-sensor pairs, an all-to-all communication topology results in a fully populated static output feedback matrix with as much as $n(n-1)$ communication links. In addition to a traditional performance index used to obtain the static output feedback gain matrix, this paper proposes a mixed index wherein both the traditional performance index and the number of communication links (representing operating costs associated with information exchange links), are taken into account. As an example, the proposed scheme is applied to a parabolic partial differential equation having four actuator-sensor pairs. The resulting optimization produces a sparse static gain matrix with a communication topology that has half the graph edges of the fully connected case and with essentially the same performance.**

I. Introduction

Traditional approaches to control design paid little or no attention to economic factors. In particular, the use of multiple inexpensive actuating and sensing devices over fewer more expensive actuating and sensing devices is still an open problem. In spatially distributed systems, the economic aspects of the hardware (fixed costs) and the implementation (operating costs) go beyond the problem of centralized vs distributed control design and implementation for large scale systems. A centralized design provides controller efficiency and performance, whereas a distributed one provides reduced controller complexity and robustness with respect to failures.

A centralized scheme with a reduced number of high capacity actuating and sensing devices will provide superb controller performance at the expense of a large initial fixed cost. On the other hand, a possibly distributed design utilizing a large number of inexpensive actuating and sensing devices can provide adequate controller performance with a significantly reduced initial fixed cost. The challenge may come at the operating costs for either controller approach. The distributed scheme employs a larger (and cheaper) number of actuating and sensing devices, but may have to rely on a rather large connectivity of the networked actuating and sensing devices to produce comparable controller performance. One strategy to defray the operating costs emanating from a large connectivity amongst the control units is to optimize the communication topology. This problem in the context of sparsity promoting control design has been extensively considered by Jovanović and co-workers in [1], [2] for the finite dimensional case. A somewhat similar optimization was considered in [3] in the context of communication optimization of multi agent systems. Thus, a distributed scheme with a large number of networked actuating and sensing devices utilizing a carefully chosen sparse interconnection topology can prove to have an economic advantage over a centralized scheme with a small number of expensive actuating and sensing devices, both in terms of fixed and operating costs. In addition, a distributed scheme will likely exhibit several advantages in terms of fault tolerance and reconfigurability (in case some of the actuating/sensing devices malfunction or some of the network links cease to exist); robustness is also a typical advantage of distributed/decentralized schemes.

This paper considers large systems, such as those representing spatially distributed processes and described by partial differential equations (PDEs). It assumes that a large number of inexpensive actuating and sensing devices are employed. To simplify the controller architecture, it assumes that the actuating devices are collocated with the sensing devices. This has the advantage of having the control unit (processor, actuator and sensor) reside at the same physical location. Then, the paper assumes that the control signals generated by each actuating device are scalar multiples of the measured outputs, which in essence implements a proportional controller. Since each actuating device can obtain the different scalar multiples of all networked sensor measurements, the resulting controller becomes a static output feedback controller. To minimize the operating costs caused by a large number of interconnections amongst the control units, *the paper proposes an optimization scheme to promote sparsity of the static feedback gain thus reducing the communication links between the networked control units and thereby minimizing the operating costs.* To achieve this,

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a traditional control performance index, represented by the total energy over a time interval, whose solution is given by the associated coupled Riccati-Lyapunov equations of the closed-loop system, is combined with a sparsity measure represented by the number of nonzero entries of the adjacency matrix associated with a given communication topology.

The problem at hand is motivated in Section II and the formulation of the mixed optimization and its solution are summarized in Section III. Numerical results for a diffusion PDE employing four actuator-sensor pairs is presented in Section IV. Conclusions follow in Section V.

II. Problem motivation

The class of systems under consideration is described by the following evolution equation

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \qquad x(0) = x_0 \in \mathcal{D}(A), \tag{1}$$

defined over the Hilbert space $\{H, \langle\cdot,\cdot\rangle_H, |\cdot|_H\}$. The state operator $\mathcal{A}$ and the input operator $\mathcal{B}$ must be defined in appropriate spaces. Towards that, define the reflexive Banach space (interpolating space) $\{V, \|\cdot\|_V\}$ that is densely and continuously embedded in H with $V \hookrightarrow H \hookrightarrow V^*$ where the embeddings are dense and continuous, [4]. The space V^* denotes the continuous dual of V. Then, the operator $\mathcal{A} \in \mathcal{L}(V, V^*)$. Assuming, for simplicity, a single input, then the input operator $\mathcal{B} \in \mathcal{L}(\mathbb{R}^1, V^*)$. The input operator represents an actuating device with high capacity and bandwidth capable of delivering large control signals to the process. It also represents a prohibitively expensive option for controlling the infinite dimensional process (1). Assuming access to full state, a feedback controller of the form

$$u(t) = -\mathcal{K}x(t) \tag{2}$$

where the feedback operator $\mathcal{K} \in \mathcal{L}(V, \mathbb{R}^1)$ can be designed to optimize a performance index, such as LQR or H_2/H_∞, [5]. However, the control law (2) requires:

(A.1) access to the full infinite dimensional state $x(t)$,
(A.2) an expensive actuator represented by the input operator $\mathcal{B}$, required to deliver the control signal to the process.

Relaxing requirement (A.1) typically involves the use of process measurements employed by a state estimator to provide the estimate $\widehat{x}(t)$ of $x(t)$. The control law becomes

$$u(t) = -\mathcal{K}\widehat{x}(t). \tag{3}$$

The control law (3) requires:

(B.1) an expensive sensing device to obtain measurements,
(B.2) an expensive actuator represented by the input operator $\mathcal{B}$, required to deliver the control signal to the process,
(B.3) a state estimator to reconstruct the state $x(t)$.

Requirement (B.1) is realized via the measurement

$$y(t) = \mathcal{C}x(t) \tag{4}$$

where the output operator $\mathcal{C} \in \mathcal{L}(V, \mathbb{R}^1)$ represents a single expensive sensing device. Requirement (B.3) is satisfied by

$$\dot{\widehat{x}}(t) = \mathcal{A}\widehat{x}(t) + \mathcal{B}u(t) + \mathcal{F}\left(y(t) - \mathcal{C}\widehat{x}(t)\right), \tag{5}$$

with $\widehat{x}(0) = \widehat{x}_0 \in \mathcal{D}(A)$, where $\mathcal{F} \in \mathcal{L}(\mathbb{R}^1, V^*)$ denotes the filter operator gain. The filter gain design can be based on a Kalman filter of a Luenberger observer, [5].

A possible solution that enables one to reduce the computational and design complexity of implementing the state estimator (5) in real-time, is to use a static output feedback controller, whenever applicable. In this case, the controller (3) is replaced by the static controller

$$u(t) = -\Gamma y(t), \qquad \Gamma : \mathbb{R}^1 \to \mathbb{R}^1, \tag{6}$$

where Γ is the static output feedback gain, which requires:

(C.1) an expensive sensing device, represented by the output operator $\mathcal{C}$, to obtain process measurements,
(C.2) an expensive actuator represented by the input operator $\mathcal{B}$, required to deliver the control signal to the process,
(C.3) the additional condition of static stabilizability for the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ which guarantees the existence of the gain Γ such that the closed-loop operator $\mathcal{A} - \mathcal{B}\Gamma\mathcal{C}$ generates an exponentially stable C_0 semigroup, [6].

All three controllers, (2), (3) and (6), presented in decreasing controller complexity, require expensive actuating and sensing devices. To avoid the use of expensive hardware, one considers the use of *inexpensive* sensing and actuating devices. In its general form, the system in (1) is re-written

$$\begin{aligned}\dot{x}(t) &= \mathcal{A}x(t) + \sum_{i=1}^{n_a} \mathcal{B}_i u_i(t), \quad x(0) = x_0 \in \mathcal{D}(A),\\ y(t) &= \begin{bmatrix} y_1(t) \\ \vdots \\ y_{n_s}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_1(xt) \\ \vdots \\ \mathcal{C}_{n_s}x(t) \end{bmatrix},\end{aligned} \tag{7}$$

where n_a denotes the number of actuating devices and n_s the number of sensing devices. Both the actuating devices, represented by the input operators $\mathcal{B}_i \in \mathcal{L}(\mathbb{R}^1, V^*)$, $i = 1, \ldots, n_a$, and the sensing devices, represented by the output operators $\mathcal{C}_j \in \mathcal{L}(V, \mathbb{R}^1)$, $j = 1, \ldots, n_s$, represent inexpensive devices with possibly reduced capacities and bandwidths.

Many open problems arise from the formulation in (7) regarding the architecture of each controller signal $u_i(t)$, $i = 1, \ldots, n_a$. However, we will limit ourselves to the special case of a simple proportional controller (or static output feedback). In this case, the control signals are given by

$$u_i(t) = -\sum_{j=1}^{n_s} \gamma_{ij} y_j(t), \quad i = 1, \ldots, n_a, \ j = 1, \ldots, n_s, \tag{8}$$

where γ_{ij} are the proportional gains. Careful examination of (8) reveals the underlined complexity vis-à-vis the information exchange between each actuator $\mathcal{B}_i$, $i = 1, \ldots, n_a$ and each sensor $\mathcal{C}_j$, $j = 1, \ldots, n_s$. For the specific choice of equal number of actuating and sensing devices with $n_a = n_s$ that are collocated $\mathcal{C}_i = \mathcal{B}_i^*$, $i = 1, \ldots, n_a$, then

$$u_i(t) = -\gamma_{ii} y_i(t), \quad i = 1, \ldots, n_a, \tag{9}$$

represent *completely disconnected* actuators-sensors (referred to as a decentralized control architecture), whereas

$$u_i(t) = -\sum_{j=1}^{n_a} \gamma_{ij} y_j(t), \quad i = 1, \ldots, n_a, \tag{10}$$

represents an actuator-sensor network with *full connectivity* and is depicted in Figure 1. In other words, each control unit has access to *all* sensor outputs and uses them to generate the control signals (10). This paper considers the case between (9) and (10), namely between noninteracting and fully connected control units.

Before continuing with the proposed controller that will result in reduced network complexity, we provide the framework for the information exchange between the actuators and sensors. A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to describe the information exchange between each actuator-sensor pair. The nodes $\mathcal{V} = \{1, 2, \ldots, n_a\}$ represent the control units (actuator-sensor pairs) and the edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represent the communication links between these networked control units. In particular, edge (i, j) indicates that node j can send information to node i. The set of in-neighbors that a given i^{th} control unit receives information from, is defined as $\mathcal{N}_i^- = \{j : (i, j) \in \mathcal{E}\}$, whereas the set of out-neighbors that a given i^{th} control unit sends information to, is defined as $\mathcal{N}_i^+ = \{l : (j, i) \in \mathcal{E}\}$.

A digraph forms an undirected graph if whenever $(i, j) \in \mathcal{E}$, we also have $(j, i) \in \mathcal{E}$. In such case, the set of in-neighbors and out-neighbors of node i are the same with $\mathcal{N}_i^- = \mathcal{N}_i^+ = \mathcal{N}_i$. The graph Laplacian matrix associated with an undirected graph $\mathcal{G}$ is denoted by $\mathbf{L}$ and is given by $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the degree matrix and $\mathbf{A}$ is the adjacency matrix, [7].

We can now define the proposed control laws

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} \gamma_{ij} y_j(t), \quad i = 1, \ldots, n_a. \tag{11}$$

The realization and implementation of the control laws in (11) lead to many optimization problems. To demonstrate aspects of the optimization involving the optimal placement of the actuating and sensing devices (actuator and sensor locations), and to provide an appreciation of the spatial effects hidden in the abstract representation (7), we consider the diffusion PDE in one spatial dimension

$$\begin{aligned}
&\frac{\partial x}{\partial t}(t, \xi) = \alpha \frac{\partial^2 x}{\partial \xi^2}(t, \xi) + \sum_{i=1}^{n_a} b_i(\xi) u_i(t) \\
&x(t, 0) = x(t, \ell) = 0, \;\; x(0, \xi) = x_0(\xi) \in L_2(\Omega), \\
&y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_{n_a}(t) \end{bmatrix} = \begin{bmatrix} \int_0^\ell c_1(\xi) x(t, \xi) \, d\xi \\ \vdots \\ \int_0^\ell c_{n_a}(\xi) x(t, \xi) \, d\xi \end{bmatrix},
\end{aligned} \tag{12}$$

where $[0, \ell] = \Omega$ denotes the spatial domain, $b_i(\xi)$, $i = 1 \ldots, n_a$ denote the spatial distributions associated with the input operators $\mathcal{B}_i$, $i = 1 \ldots, n_a$, in (7), and $c_i(\xi)$, $i = 1 \ldots, n_a$ denote the spatial distributions associated with the output operators $\mathcal{C}_i$, $i = 1 \ldots, n_a$ in (7). The state space $H = L_2(\Omega)$ with $V = H_0^1(\Omega)$ and $V^* = H^{-1}(\Omega)$, [5]. In particular, one defines the input and output operators in weak form via

$$\langle \mathcal{B}_i u_i(t), \phi \rangle = \int_0^\ell b_i(\xi) u_i(t) \phi(\xi) \, d\xi, \quad i = 1, \ldots, n_a,$$

$$\mathcal{C}_j \phi = \int_0^\ell c_j(\xi) \phi(\xi) \, d\xi, \quad j = 1, \ldots, n_a,$$

for all test functions $\phi \in H_0^1(\Omega)$. The state operator is

$$\langle \mathcal{A}\phi, \psi \rangle = \int_0^\ell \alpha \frac{d^2 \phi(\xi)}{d\xi^2} \psi(\xi) \, d\xi = -\int_0^\ell \alpha \frac{d\phi(\xi)}{d\xi} \frac{d\psi(\xi)}{d\xi} \, d\xi.$$

Finally, the infinite dimensional state $x(t)$ in (7) is identified as the solution to the PDE in (12) via $x(t) = x(t, \cdot)$.

To further reveal some aspects of the optimization not found in the finite dimensional case, we consider the placement of the actuating and sensing devices as another level of optimization. Towards that we assume that each of these devices is modelled by a Dirac delta spatial function with the interpretation of a pointwise-in-space actuation for $b_i(\xi)$ and pointwise-in-space measurement for $c_i(\xi)$; thus

$$\begin{aligned}
\int_0^\ell b_i(\xi_i) u_i(t) \phi(\xi) \, d\xi &= \int_0^\ell \delta(\xi - \xi_i) u_i(t) \phi(\xi) \, d\xi \\
&= \phi(\xi_i) u_i(t)
\end{aligned} \tag{13}$$

for $i = 1 \ldots, n_a$, and

$$\int_0^\ell c_j(\xi_j) \phi(\xi) \, d\xi = \int_0^\ell \delta(\xi - \xi_j) \phi(\xi) \, d\xi = \phi(\xi_j) \tag{14}$$

for $j = 1 \ldots, n_a$. The actuator locations ξ_i, $i = 1, \ldots, n_a$ in (13) and the sensor locations ξ_j, $j = 1, \ldots, n_a$ in (14) are termed the *actuator and sensor centroids*. They can be arbitrary and their location can be selected via the optimization of an appropriate performance metric, or a priori selected.

III. Problem Formulation

To formulate the various optimization problems arising from the implementation of the distributed controllers (11), we formally make the following assumptions.

Assumption 1 (Actuator and sensor devices): The number n_a of actuators and n_s of sensors in (7) are equal.

Assumption 2 (Identical devices): The spatial distributions of the actuating devices are identical, in the sense $b_i(\xi_k) = b_j(\xi_k)$, $i, j = 1 \ldots, n_a$, and only differ in their location. Similarly the sensing devices are identical with $c_i(\xi_k) = c_j(\xi_k)$, $i, j = 1 \ldots, n_a$, and only differ in their location.

Assumption 3 (Collocated actuators and sensors): The actuating devices $b_i(\xi)$ in (12) are collocated to the sensing devices $c_j(\xi)$ with $b_i(\xi) = c_i(\xi)$, $i = 1, \ldots, n_a$.

Remark 1: Assumption 2 implies that the input and output operators in (7) are related via $\mathcal{B}_i = \mathcal{C}_i^*$, $i = 1 \ldots, n_a$.

In view of Assumptions 1, 2, 3 and Remark 1, the system (7) with the proposed controllers (11) can be written as

$$\dot{x}(t) = \mathcal{A}x(t) + \begin{bmatrix} \mathcal{B}_1 & \ldots & \mathcal{B}_{n_a} \end{bmatrix} \Gamma \begin{bmatrix} \mathcal{B}_1^* x(t) \\ \vdots \\ \mathcal{B}_{n_a}^* x(t) \end{bmatrix}, \tag{15}$$

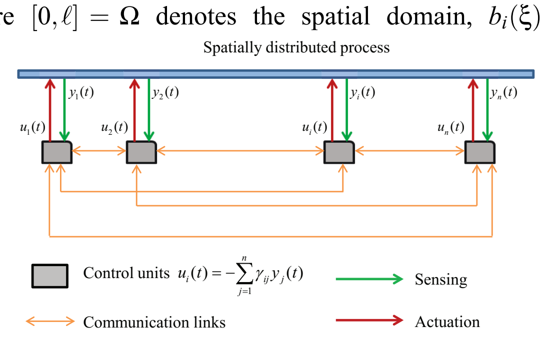

Fig. 1. Spatially distributed process with interconnected actuator-sensors.

where the $n_a \times n_a$ matrix Γ has $\{\Gamma\}_{ij} = \gamma_{ij}$. In terms of the specific PDE (12), it is given by

$$\begin{aligned}\frac{\partial x}{\partial t}(t,\xi) &= \alpha\frac{\partial^2 x}{\partial \xi^2}(t,\xi) - \sum_{i=1}^{n_a} b_i(\xi) \sum_{j\in\mathcal{N}_i} \gamma_{ij} y_j(t)\\ &= \alpha\frac{\partial^2 x}{\partial \xi^2}(t,\xi)\\ &\quad - \left[\begin{array}{ccc} b_1(\xi) & \ldots & b_{n_a}(\xi)\end{array}\right]\Gamma \left[\begin{array}{c}\int_0^\ell b_1(\xi)x(t,\xi)\,\mathrm{d}\xi\\ \vdots\\ \int_0^\ell b_{n_a}(\xi)x(t,\xi)\,\mathrm{d}\xi\end{array}\right].\end{aligned} \tag{16}$$

The various optimization problems associated with (16), or its abstract representation (15), can now be presented in increasing complexity. These involve the physical location of the collocated actuating/sensing devices via the centroids ξ_i, the connectivity of the control units as given by the nonzero entries of Γ, and the numerical value of the gains γ_{ij} in (11). These optimizations echo the ones presented in [8].

(O1) Given a number n_a of actuating-sensing devices in fixed locations and *a priori* selected information exchange channels described by the graph topology $\mathcal{G}$, which immediately defines the nonzero entries of Γ, find
  1) the numerical values of the nonzero entries of Γ,

  by minimizing a suitable performance metric.

(O2) Given an *a priori* selected information exchange described by the graph topology $\mathcal{G}$, which immediately defines the nonzero entries of the gain matrix Γ, find
  1) the location of the n_a actuating-sensing devices,
  2) the numerical values of the nonzero entries of Γ,

  by minimizing a suitable performance metric.

(O3) Given a fixed location of the n_a actuating-sensing devices, find
  1) the optimal connectivity as described by the nonzero entries of Γ (i.e. which entries of Γ are to be nonzero),
  2) the numerical values of the nonzero entries of Γ,

  by minimizing a suitable performance metric.

(O4) Given a number n_a of the actuating-sensing devices, find
  1) the location of the n_a actuating-sensing devices,
  2) the optimal connectivity as described by the nonzero entries of Γ (i.e. which entries of Γ are to be nonzero),
  3) the numerical values of the nonzero entries of Γ,

  by minimizing a suitable performance metric.

As mentioned in [8], one can also consider another level of optimization by finding the minimum number of the actuating-sensing devices n_a necessary to render the triple $(\mathcal{A},\mathcal{B},\mathcal{B}^*)$ a statically stabilizable triple and the minimum number of communication links, also necessary for ensuring that the closed-loop operator $\mathcal{A}-\mathcal{B}\Gamma\mathcal{B}^*$ generates an exponentially stable C_0 semigroup, [6]. These two additional levels of optimization will not be considered here; however, their influence trickles down to the definition of an admissible parameter space. The set of admissible gain matrices Γ that define the parameter space, couples the requirement of static stabilizability and graph connectivity.

The parameter space Θ consists of all $n_a \times n_a$ constant matrices with nonzero entries at the same locations as the graph adjacency matrix $\mathbf{A}$ (along with nonzero entries on the main diagonal). However, for a prescribed topology, one must ensure that the resulting closed-loop operator $\mathcal{A}-\mathcal{B}\Gamma\mathcal{B}^*$ generates an exponentially stable C_0 semigroup. As the number of links between the control units decreases, then the set of admissible gain matrices Γ that render the triple $(\mathcal{A},\mathcal{B},\mathcal{B}^*)$ statically stabilizable decreases. The graph-dependent parameter space $\Theta(\mathcal{G})$ is defined as the space of $n_a \times n_a$ constant matrices that have nonzero entries at the same locations as the matrix $\mathbf{I}_{n_a}+\mathbf{A}$ *and* ensure that the resulting operator $\mathcal{A}-\mathcal{B}\Gamma\mathcal{B}^*$ generates an exponentially stable C_0 semigroup. The requirement of having nonzero entries at the same location as matrix $\mathbf{I}_{n_a}+\mathbf{A}$ can be expressed in terms of setting the gain matrix admit the expansion

$$\Gamma = (\mathbf{I}_{n_a}+\mathbf{A})\circ M,$$

where $\circ$ denotes the Hadamard (entrywise) product [9] and M is any $n_a \times n_a$ fully populated matrix. The Boolean matrix $(\mathbf{I}_{n_a}+\mathbf{A})$ has 1's at the same locations as the nonzero entries of the graph Laplacian $\mathbf{L}$ and provides a "memory" of the connectivity amongst the networked control units; if the ij^{th} entry of $(\mathbf{I}_{n_a}+\mathbf{A})$ is equal to 1, it means that the i^{th} actuator receives information from the j^{th} sensor. Equivalently, it means that γ_{ij} is nonzero. Similarly, if the ij^{th} entry of $(\mathbf{I}_{n_a}+\mathbf{A})$ is equal to 0, it means that the j^{th} sensor does not transmit its output $y_j(t)$ to the i^{th} actuator. Formally, the parameter space is defined as

$$\Theta(\mathcal{G}) = \left\{\begin{array}{l}\Gamma\in\mathbb{R}^{n_a\times n_a} : \Gamma = (\mathbf{I}_{n_a}+\mathbf{A})\circ M, M\in\mathbb{R}^{n_a\times n_a}\\ \mathcal{A}-\mathcal{B}\Gamma\mathcal{B}^* \text{ generates an e.s. } C_0 \text{ semigroup}\end{array}\right\} \tag{17}$$

In this paper, we will be concerned with the optimization problem (O3), namely finding the optimal connectivity (i.e. which entries of Γ should be nonzero) and subsequently finding their numerical values. Note that there are two (possibly conflicting) objectives associated this problem. We discuss these two objectives in more detail below.

Objective 1: Finding an appropriate performance index for the computation of Γ: Assume (for now) that the given n_a actuator/sensor units are associated with a fixed (pre-determined) interconnection topology. In general, this interconnection topology can be captured by a digraph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,2,...,n_a\}$ is the set of vertices (each corresponding to an actuator/sensor unit) and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of (directed, pairwise) communication links between them. Under these assumptions, the set of possible Γ is given by (17) (where $\mathbf{A}$ is the adjacency matrix of $\mathcal{G}$); by defining an appropriate performance index we can obtain optimal numerical values for the nonzero entries of the gain matrix Γ. For instance, related work in [8] proposed as performance index the energy of the closed-loop system (15) over the infinite horizon, which is given by the $L_1(0,\infty;E(t))$ norm of the closed-loop state with $E(t) = \langle x(t), Qx(t)\rangle_H$, namely

$$J = \int_0^\infty \langle x(\tau), Qx(\tau)\rangle_H \, d\tau \, . \tag{18}$$

The operator Q is a coercive operator chosen to reflect certain performance criteria. In its simplest choice, $Q = I$ and the above index simply becomes the $L_1(0,\infty; |x(t)|_H^2)$ norm.

Clearly, for a fixed interconnection topology $\mathcal{G}$, the optimization problem can be stated as

(O3 (*fixed topology*)):

$$\left\{ \text{ Find } \Gamma \in \Theta(\mathcal{G}) \text{ to minimize } J \text{ in (18). } \right\}$$

It should be clear from the above optimization that if the topology changes from $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ such that $\mathcal{E}' \subseteq \mathcal{E}$ then $J_{\mathcal{G}'} \geq J_{\mathcal{G}}$ (since an entry of Γ that is allowed to be nonzero can also assume the zero value).

Remark 2: It is worth pointing out that if $\mathcal{G}$ is the fully connected topology (i.e., $(\mathbf{I}_{n_a} + \mathbf{A}) = \mathbf{1}\mathbf{1}^T$, where $\mathbf{1}$ is the vector of 1's), the optimal Γ can be found using techniques from [10]; for the case when $\mathcal{G}$ is not the connected topology (i.e., certain entries of Γ are restricted to be zero), we propose (see example in Section IV) a heuristic approach (inspired from [10]) to obtain a Γ that satisfies the constraints (i.e., $\Gamma \in \Theta(\mathcal{G})$) and minimizes J.

Objective 2: Finding an appropriate connectivity (interconnection topology) among the n_a actuator/control units: A straightforward way to quantify optimality for the interconnection topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is via the total number of edges involved, which we denote by $q_{\mathcal{G}} = |\mathcal{E}|$. Note that $q_{\mathcal{G}}$ can also be obtained as

$$q_{\mathcal{G}} = \mathbf{1}^T \mathbf{A}_{\mathcal{G}} \mathbf{1}, \tag{19}$$

where $\mathbf{A}_{\mathcal{G}}$ is the adjacency matrix associated with the given digraph $\mathcal{G}$. More generally, if different communication links have different costs, we can talk about the weighted cost of interconnection topology $\mathcal{G}$, which we can define as

$$q_{w,\mathcal{G}} = \sum_{e \in \mathcal{E}} w_e \, ,$$

where $w_e \geq 0$ is the cost associated with edge $e \in \mathcal{E}$.

Clearly, the number $q_{\mathcal{G}}$ (or $q_{w,\mathcal{G}}$) can be used as a performance index for the optimization (O3). For instance, if all we are interested in is to ensure that $\mathcal{G}$ admits a Γ that generates an exponentially stable C_0 semigroup, then our goal can be stated as follows.

(O3 (*variable topology*)):

$$\left\{ \begin{array}{l} \text{Find } \mathcal{G} \in \Psi \text{ so that (i) } q_{\mathcal{G}} \text{ (or } q_{w,\mathcal{G}}) \text{ is minimized and} \\ \quad \text{(ii) } \Theta(\mathcal{G}) \text{ is non-empty} \end{array} \right\}$$

where $\Psi = \{\mathcal{G} = (\mathcal{V}, \mathcal{E}) \mid \mathcal{V} = \{1, 2, ..., n_a\}\}$ is the set of all possible directed topologies among n_a nodes. Note that there are $2^{n_a(n_a-1)}$ different digraphs in Ψ, since we can potentially have $n_a(n_a-1)$ directed edges between the n_a nodes (self-edges excluded).

Mixing Objective 1 and Objective 2: More generally, we may be interested in mixing the two objectives described above. There are many interesting ways in which we may attempt to optimize these two (conflicting) objectives. For example, we may want to optimize J over all topologies $\mathcal{G}$ for which $q_{\mathcal{G}}$ is below a certain threshold T_q. In such case, we have the following optimization.

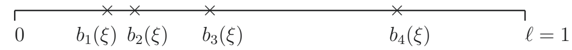

Fig. 2. Deployment of pointwise actuator-sensor pairs in $\Omega = [0, 1]$.

(O3 (*minimize performance over all topologies under a maximum number of edges*)):

$$\left\{ \text{ Find } \mathcal{G} \in \Psi_{T_q} \text{ so that } J_{\mathcal{G}} \text{ is minimized } \right\},$$

where $\Psi_{T_q} = \{\mathcal{G} = (\mathcal{V}, \mathcal{E}) \mid \mathcal{V} = \{1, 2, ..., n_a\}, |\mathcal{E}| \leq T_q\}$ is the set of all possible directed topologies among n_a nodes that have T_q or less edges, and where, for a given $\mathcal{G}$, we use $J_{\mathcal{G}}$ to denote the best performance J corresponding to the optimal $\Gamma \in \Theta(\mathcal{G})$.

The above approach was used in our analysis for the example in the next section (in the example, we also vary the threshold T_q to obtain a better picture of the tradeoffs involved). Note, however, that there are many other ways to formulate a meaningful optimization. For example, we can have the following optimization:

(O3 (*minimize weighted combination of performance and maximum number of edges over all topologies*)):

$$\left\{ \text{ Find } \mathcal{G} \in \Psi \text{ so that } (c_J J_{\mathcal{G}} + c_q q_{\mathcal{G}}) \text{ is minimized } \right\},$$

where the coefficients c_j and c_q are given weights that represent the relative importance of the performance ($J_{\mathcal{G}}$) and the number of connections ($q_{\mathcal{G}}$) in our application.

IV. Numerical results

The PDE in (16) was considered in the spatial interval $[0, \ell] = [0, 1]$ with $a = 10^{-2}$ and $n_a = 4$. The four collocated actuator-sensors were a priori selected

$$\begin{aligned} b_1(\xi) &= \delta(\xi - 0.166), \quad b_2(\xi) = \delta(\xi - 0.233), \\ b_3(\xi) &= \delta(\xi - 0.367), \quad b_4(\xi) = \delta(\xi - 0.734), \end{aligned} \tag{20}$$

and are depicted in Figure 2. For simplicity, we study the case of undirected communication topologies; the number of possible undirected communication graphs over the four nodes formed by the four actuator-sensor pairs (control units) given by $b_1(\xi), b_2(\xi), b_3(\xi), b_4(\xi)$ is 64 with

- $\binom{6}{1} = 6$ combinations representing $q = 1$ edge,
- $\binom{6}{2} = 15$ combinations representing $q = 2$ edges,
- $\binom{6}{3} = 20$ combinations representing $q = 3$ edges,
- $\binom{6}{4} = 15$ combinations representing $q = 4$ edges,
- $\binom{6}{5} = 6$ combinations representing $q = 5$ edges,
- 1 combination representing $q = 6$ edges,
- 1 combination representing $q = 0$ edges.

The initial condition for the state is $x(0, \xi) = 20 \sin(\pi(\ell - \xi)/\ell) e^{-7(\xi-\ell)^2}$. To implement the proposed optimization scheme, a Galerkin-based finite element scheme with 100 linear splines modified to account for the Dirichlet boundary conditions was employed to obtain the finite dimensional approximation of (16). The matrix representation of the semidiscretization (spatial discretization) of (16) employed a composite two-point Gauss-Legendre quadrature rule. The resulting finite dimensional state space representation of (16) was subsequently integrated numerically over the time interval $[0, 4]$s using the Matlab® stiff ODE solver `ode23s` based on a 4th order Runge-Kutta scheme. The finite dimensional

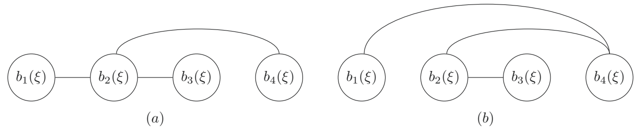

Fig. 3. Undirected graphs representing optimal communication topologies.

approximation of the plant (16) is given by

$$\dot{x}(t) = Ax(t) + \mathbf{B}u(t), \quad y(t) = \mathbf{B}^T x(t) \tag{21}$$

where $\mathbf{B} = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix}$. The controller has the form $u(t) = -\Gamma y(t)$, and is chosen to minimize the cost

$$J = \int_0^\infty x^T(\tau)Qx(\tau) + x^T(\tau)\mathbf{B}\Gamma^T R\Gamma\mathbf{B}^T x(\tau)\,\mathrm{d}\tau, \tag{22}$$

where $R = R^T > 0$ is an additional matrix weight used to penalize the control effort. The optimal cost is given by

$$J = \operatorname{trace}\,[\ P\ ], \tag{23}$$

and the optimal feedback control is

$$\Gamma = -R^{-1}\mathbf{B}^T PL\mathbf{B}\left[\mathbf{B}^T L\mathbf{B}\right]^{-1} \tag{24}$$

where P is the positive semi-definite solution to the ARE

$$PA_{cl} + A_{cl}^T P + \mathbf{B}\Gamma^T R\Gamma\mathbf{B}^T + Q = 0, \tag{25}$$

with $A_{cl} = A - \mathbf{B}\Gamma\mathbf{B}^T$, and L is the positive-definite solution of the Lyapunov equation

$$LA_c^T + A_c L + I = 0. \tag{26}$$

Two different cases were considered. The first one assumes an all-to-all connectivity resulting in a fully populated gain matrix Γ with $q = 6$ edges. The optimal full matrix Γ was computed using the scheme in [11], and which was in turn based on the scheme developed for finite dimensional systems in [10] and summarized in Algorithm 1.

Algorithm 1 Optimal static output feedback

initialize: Determine a matrix Γ_0 so that it is a stabilizing feedback gain; i.e., the finite dimensional representation of the plant (21) is statically output stabilizable. Either $\Gamma_0 = 0$ giving $A_{cl} = A$ or $\Gamma_0 = \mathbf{B}^T\mathbf{B}$ giving $A_{cl} = A - \mathbf{B}(\mathbf{B}^T\mathbf{B})\mathbf{B}^T$ are good choices since the matrix representation A of $\mathcal{A}$ is a Hurwitz symmetric matrix.
iterate: $k = 0$
loop
  set $A_{cl}^k = A - \mathbf{B}\Gamma_k\mathbf{B}^T$
  solve $P^k A_{cl}^k + (A_{cl}^k)^T P^k + Q + \mathbf{B}(\Gamma^k)^T R\Gamma^k\mathbf{B}^T = 0$
  solve $L^k (A_{cl}^k)^T + A_{cl}^k L^k + I = 0$
  set $\Gamma^{k+1} = -R^{-1}\mathbf{B}^T P^k L^k\mathbf{B}\left(\mathbf{B}^T L^k\mathbf{B}\right)^{-1}$
  set $J^k = \operatorname{trace}\left[\ P^k\ \right]$
  use a gradient-based optimization update rule to determine the $k+1$ iterate
  if stopping criterion is met **then**
    set $P^{k+1} = P^k$ and $J^{k+1} = J^k$
    goto 17
  else
    $k \leftarrow k+1$
    goto 3
  end if
end loop

The fully populated gain matrix in this case is given by

$$\Gamma_{\text{full}} = \begin{bmatrix} 2.8886 & 0.3658 & 0.0341 & 0.0350 \\ 0.3669 & 2.4839 & 0.5255 & 0.0158 \\ -0.0599 & 0.3914 & 4.5266 & 0.5961 \\ -0.0354 & -0.1131 & 1.0180 & 4.7620 \end{bmatrix}. \tag{27}$$

The other case assumes that up to $T_q = 3$ edges are allowed in the graph and searches for all candidate $\Gamma \in \{\Theta(\mathcal{G})\,|\,G \in \Omega_{T_q}\}$ to find the one that minimizes (23). Restricting ourselves to a topology that has at most 3 edges (i.e. we allow $0, 1, 2$ and 3 edges), then we need to modify Algorithm 1 to enforce the sparsity condition as defined via the adjacency matrix $\mathbf{A}$. This is presented in Algorithm 2 as modified from [12]. The optimization scheme produced two solutions with

Algorithm 2 Sparsity enforcement of static output feedback

initialize: Determine a matrix $\Gamma_0 \in \Theta(\mathcal{G})$ so that it is a stabilizing feedback gain; i.e., the finite dimensional representation (21) is statically output stabilizable. The choice $\Gamma_0 = I$ giving $A_{cl} = A - \mathbf{B}\mathbf{B}^T$ is a good choice.
iterate: $k = 0$
loop
  set $A_{cl}^k = A - \mathbf{B}\Gamma_k\mathbf{B}^T$
  solve $P^k A_{cl}^k + (A_{cl}^k)^T P^k + Q + \mathbf{B}(\Gamma^k)^T R\Gamma^k\mathbf{B}^T = 0$
  solve $L^k (A_{cl}^k)^T + A_{cl}^k L^k + I = 0$
  set $\Gamma^{k+1} = -R^{-1}\mathbf{B}^T P^k L^k\mathbf{B}\left(\mathbf{B}^T L^k\mathbf{B}\right)^{-1}$
  enforce sparsity $\Gamma^{k+1} \leftarrow \Gamma^{k+1} \circ (\mathbf{I}_{n_a} + \mathbf{A})$
  set $J^k = \operatorname{trace}\left[\ P^k\ \right]$
  use a gradient-based optimization update rule to determine the $k+1$ iterate
  if stopping criterion is met **then**
    set $P^{k+1} = P^k$ and $J^{k+1} = J^k$
    goto 18
  else
    $k \leftarrow k+1$
    goto 3
  end if
end loop

$q \leq 3$ edges and corresponding adjacency matrices

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \tag{28}$$

as shown by the graphs in Figure 3a and 3b. The sparse matrix Γ for each of the two sparse topologies is given by

$$\Gamma_{\text{sparse},1} = \begin{bmatrix} 2.8889 & 0.3577 & 0 & 0 \\ 0.3769 & 1.7968 & -0.7385 & -1.4771 \\ 0 & 1.6650 & 4.1946 & 0 \\ 0 & 1.4168 & 0 & 4.2759 \end{bmatrix}, \tag{29}$$

case	# of edges q
optimal full matrix Γ_{full}	6
optimal sparse matrix $\Gamma_{\text{sparse},1}$	3
optimal sparse matrix $\Gamma_{\text{sparse},2}$	3

TABLE I

NUMBER OF EDGES (CONNECTIONS) q.

case	$\sqrt[2]{\int_0^4 \lvert x(t)\rvert^2_{L_2(0,\ell)}\,\mathrm{d}t}$
optimal full matrix Γ_{full}	2.81471
optimal sparse matrix $\Gamma_{\text{sparse},1}$	2.81043
optimal sparse matrix $\Gamma_{\text{sparse},2}$	2.81043

TABLE II

COMPUTATION OF $L_2(0,4;L_2(0,\ell))$ STATE NORM.

and

$$\Gamma_{\text{sparse},2} = \begin{bmatrix} 2.8142 & 0 & 0 & -0.6176 \\ 0 & 1.7529 & -0.7524 & -1.5278 \\ 0 & 1.6723 & 4.1830 & 0 \\ 0.6564 & 1.4692 & 0 & 4.1414 \end{bmatrix}. \quad (30)$$

To properly compare the two cases (Γ_{full} and $\Gamma_{\text{sparse},1}$, $\Gamma_{\text{sparse},2}$), we consider the number of edges q as well as the $L_1(0,4)$ norm (Table I). The sparse case requires half the connections of the fully populated Γ which translates to significant communication savings. The $L_1(0,4)$ norm (presented in Table II) indicates that no noticeable difference exists between the two cases. However, as observed above, one requires half the number of edges. The same observation can be made in Figure 4. Since no significant differences can be observed in the controller performance, then the proposed sparse gain matrix provides a significantly cheaper communication option for controlling spatially distributed processes with networked actuator/sensor pairs.

Remark 3: It should be emphasized that actuator-sensor locations, different than those in (20), will lead to a fully populated gain matrix different than (27). Similarly, they will produce an adjacency matrix different than (28), with ε

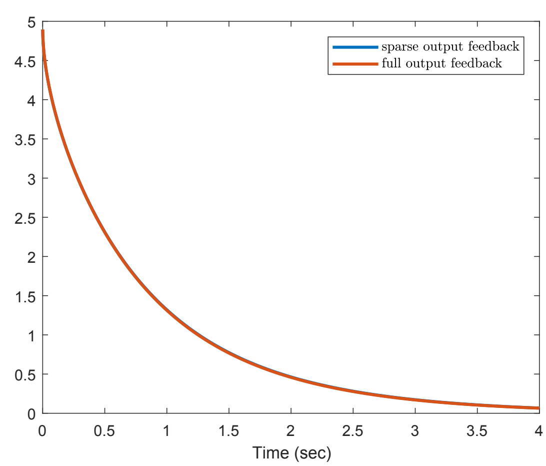

Fig. 4. Evolution of state norms.

V. CONCLUSIONS

An optimization scheme that takes into account the number of communication links between networked actuator-sensor pairs in spatially distributed processes was proposed as an inexpensive alternative to existing (centralized) approaches. More specifically, instead of using a centralized scheme utilizing a single or a small number of expensive actuating and sensing devices, a distributed alternative that employs a large number of networked actuator-sensor pairs was considered to provide the first level of cost reduction. To further reduce operating costs generated by expensive communication amongst the networked actuator-sensor pairs, the proposed optimization scheme penalized both the traditional controller performance index and the number of communication links.

A numerical example of a spatially distributed process modelled by the 1D diffusion PDE with four actuator-sensor pairs was considered to obtain an insight on the reduction in operating costs as represented by the communication links of the actuator-sensor pairs. Compared to the fully populated optimal static output feedback gain which required 6 communication links, the proposed scheme required half that and provided an almost identical controller performance.

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