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Discrete Finite-time Stable Position Tracking Control of Unmanned Vehicles

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Abstract— **This paper presents a finite-time stable (FTS) position tracking control scheme in discrete time for an unmanned vehicle. The control scheme guarantees discrete-time stability of the feedback system in finite time. This scheme is developed in discrete time as it is more convenient for onboard computer implementation and guarantees stability irrespective of sampling period. Finite-time stability analysis of the discrete-time tracking control is carried out using discrete Lyapunov analysis. This tracking control scheme ensures stable convergence of position tracking errors to the desired trajectory in finite time. The advantages of finite-time stabilization in discrete time over finite-time stabilization of a sampled continuous tracking control system is addressed in this paper by a numerical comparison. This comparison is performed using numerical simulations on continuous and discrete FTS tracking control schemes applied to an unmanned vehicle model.**

I. Introduction

This paper investigates the problem of autonomous position trajectory tracking of an unmanned vehicle. In various applications where remote piloting is difficult or impossible, autonomous operations of unmanned vehicles can play an important role. Applications of unmanned aerial vehicles such as security, inspection of civilian infrastructure, agriculture and aquaculture, space and underwater exploration, wildlife tracking, package delivery and remote sensing can all benefit from reliable autonomous operations. Stable and robust autonomous guidance and control is considered a critical part of reliable operations of unmanned vehicles, particularly for operations that require safety and reliability in presence of external disturbances like wind. Absence of nonlinear stability and robustness in these situations can lead to failure and crash of even remotely piloted vehicles. This work presents a systematic treatment of discrete finite-time stable control for tracking position trajectories of unmanned vehicles, as a good solution to this problem.

Finite-time stable control has the advantage of providing a guarantee on the time it takes for the system to converge to a desired state, beside being more robust to bounded temporary and persistent disturbances than asymptotic stability. Moreover, low-level persistent disturbances are better rejected by a finite-time stable system in comparison to an asymptotically stable system, because the ultimate bound on the state is of higher order than the bound on the disturbance [1]. In particular, continuous finite-time stable (FTS) control schemes are effective in applications where there are bounded disturbance inputs due to unmodeled dynamics [2]. Early research on continuous finite-time stable (FTS) control systems can be found in [3]–[5]. An almost global finite time stabilization of rigid body attitude motion to a desired attitude in finite time is studied in [2], [6]. Same authors designed a finite-time stable control scheme for simple mechanical systems represented in generalized coordinates, as reported in [1]. A FTS integrated guidance and feedback tracking control scheme for position and attitude tracking of rigid bodies has been reported in [7]–[9], which ensures finite-time stability of the overall tracking scheme in continuous time. The continuous equation of motion were discretized in the form of Lie Group Variational Integrator (LGVI) and the continuous time control scheme was sampled for computer implementation, by applying the discrete Lagrange-d'Alembert principle. Prior related research on LGVI discretization includes [10]–[16].

However, implementing a sampled continuous-time stable tracking control scheme does not guarantee the discrete-time stability of the resulting control. This has been demonstrated convincingly for the case of nonlinear observer design for attitude dynamics, in [15]–[17]. A discrete-time stable feedback tracking control scheme was developed in [18], in which discrete-time control laws obtained ensure asymptotic discrete-time stability of pose tracking control of underactuated vehicles on $\mathrm{SE}(3)$. Note that, like the continuous time FTS control scheme in [1]- [4], [6]- [9], the discrete-time FTS control scheme proposed here maintains finite time stable convergence to the desired equilibrium or trajectory, but it does so in discrete time. In addition, discrete-time FTS control scheme enables onborad computer implementation with a variety of discrete-time input data frequencies. This forms the motivation of this paper: to design a *finite-time stable position tracking control scheme in discrete time*. To the best of our knowledge, a finite-time stable position tracking control scheme in discrete time as proposed in this paper has not been reported in prior literature. In this paper, a discrete-time Lyapunov analysis for FTS position tracking control leads to the discrete time control law. The Lyapunov function designed is quadratic in a vector-valued function that is in terms of translational motion (position and velocity) tracking errors. This vector-valued function is constructed such that when its value is the zero vector, the translational tracking errors converge to zero in finite time. A discrete-time control force vector is then designed, that ensures that this vector converges to the zero vector in finite time, and therefore the position trajectory tracking errors converge to zero in a finite-time interval. Then, the stability and performance of the proposed discrete time FTS

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scheme is numerically compared with that of a continuous FTS scheme, and the results are discussed.

This paper is organized as follows. Section II outlines the general formulation of the problem for a rigid body on SE(3), as well as providing the position kinematics and dynamics model of the vehicle. Section III deals with the discrete-time Lyapunov framework and a two-step systematic method to obtain discrete-time position tracking control law for FTS position tracking control. A continuous FTS position tracking scheme is presented in Section IV, which is the scheme first proposed in [9]. Numerical simulation results based on a Lie group variational integration scheme and the finite-time control laws obtained in discrete time, are presented in V. This section also presents a comparison of the stability performance between the discrete and continuous FTS schemes, and discussion of these results. The concluding section VI provides a summary of results presented, and mentions related research directions to be pursued in the near future.

II. Problem Formulation

A. Coordinate Frame Definition

The configuration of an unmanned vehicle modeled as a rigid body is given by its pose, which is the combination of its position and orientation. Let $b \in \mathbb{R}^3$ denote the position vector of the origin of the body frame $\mathcal{B}$ with respect to the inertial frame $\mathcal{I}$ represented in frame $\mathcal{I}$. Let $R \in \mathrm{SO}(3)$ denote the orientation (attitude), defined as the rotation matrix from frame $\mathcal{B}$ to frame $\mathcal{I}$. Then, the pose of the vehicle can be expressed in matrix form as

$$\mathsf{g} = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \tag{1}$$

where $\mathrm{SE}(3)$ is the six-dimensional Lie group of rigid body motions (translational and rotational) that is obtained as the semi-direct product of $\mathbb{R}^3$ with $\mathrm{SO}(3)$ [19], [20]. A smooth position trajectory that is continuous and twice differentiable (i.e., $b_d(t) = \mathrm{C}^2(\mathbb{R}, \mathbb{R}^3)$), where $b_d(t)$ gives the desired position trajectory on $\mathbb{R}^3$ will be created. Such a time trajectory for the position through given waypoints could be generated using one of several techniques.

B. Position and velocity tracking errors expressed in inertial frame

In this paper, the vehicle is assumed to have mass m. The tracking errors for translational motion are expressed with respect to the inertial frame as $\tilde{b} := b - b_d$ and $\tilde{v} := v - v_d$, which are position and velocity tracking errors, respectively. Therefore, in inertial frame $\mathcal{I}$, the position and translational velocity error dynamics are expressed as

$$\begin{cases} \dot{\tilde{b}} = \tilde{v} = v - v_d \\ m\dot{\tilde{v}} = m\, g\, e_3 - \varphi - v_d \end{cases}, \tag{2}$$

where φ is the control force vector acting on the body, expressed in inertial frame. The magnitude of this vector is the control input f, which is designed as a feedback control law, and $e_3 = [0\ 0\ 1]^{\mathrm{T}}$ is the third standard basis of $\mathbb{R}^3$.

III. Discrete-time Stable Position Tracking Control on $\mathbb{R}^3$

Consider tracking a known position trajectory $b_d(t)$, with corresponding velocity $v_d(t) = \dot{b}_d$, in a time interval $[t_0,\ t_f] \in \mathbb{R}^+$ separated into N equal-length subintervals $[t_k,\ t_{k+1}]$ for $k = 0, 1, ..., N$, with $t_N = t_f$ and $t_{k+1} - t_k = \Delta t$ where Δt is the time step size. Define the position tracking error in discrete time as

$$\tilde{b}_k = \tilde{b}(t_k) = b_k - b_k^d. \tag{3}$$

Therefore, one can express the discrete-time position and translational velocity error dynamics as

$$\begin{cases} \tilde{b}_{k+1} - \tilde{b}_k = \tilde{v}_k \Delta t \\ \tilde{v}_{k+1} = v_k + \Delta t\ g\ e_3 - \dfrac{\Delta t}{m}\varphi_k - v_{k+1}^d \end{cases}, \tag{4}$$

which is obtained by substituting $v_{k+1} = \tilde{v}_{k+1} + v_{k+1}^d$ in the discretized equations of translational motion in the form of Lie Group Variational Integrator (LGVI) presented in [11], [18].

Now define the discrete-time Lyapunov function quadratic in position tracking error as

$$V(\tilde{b}_k) = V_k = \frac{1}{2}\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k, \tag{5}$$

where $P = P^{\mathrm{T}} \in \mathbb{R}^{3\times 3}$ is a positive definite control gain matrix. The total time difference of this discrete Lyapunov function in the time interval $[t_k,\ t_{k+1}]$ for $k = 0, 1, ..., N$ is then obtained as

$$\begin{aligned} \Delta V_k = V_{k+1} - V_k &= \frac{1}{2}\tilde{b}_{k+1}^{\mathrm{T}} P \tilde{b}_{k+1} - \frac{1}{2}\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k \\ &= \frac{1}{2}(\tilde{b}_{k+1} - \tilde{b}_{k+1})^{\mathrm{T}} P (\tilde{b}_k + \tilde{b}_{k+1}). \end{aligned} \tag{6}$$

The following result is a basic result on finite-time stability and convergence for discrete-time systems, and to the best of our knowledge, it has not been reported in past research publications.

Lemma 3.1: Consider a discrete-time system with a corresponding positive definite Lyapunov function $V : \mathbb{R}^l \to \mathbb{R}$ and let $V_k = V(y_k)$. Let α and ϵ be a constant in the open interval $]0, 1[$, let $V_0 > 0$ be the finite initial value of the Lyapunov function along an output trajectory y_k, and let $\gamma_k := \gamma(V_k^{1-\alpha})$ where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a class-$\mathcal{K}$ function of $V_k^{1-\alpha}$ that satisfies

$$\frac{\gamma_k}{\gamma_0} \geq 1 - \epsilon \ \text{ for }\ V_k^{1-\alpha} \in]V_0^{1-\alpha} - \chi, V_0^{1-\alpha}[\tag{7}$$

for some finite positive constant $\chi < V_0^{1-\alpha}$. Then, if V_k satisfies the relation

$$V_{k+1} - V_k = -\gamma_k V_k^{\alpha}, \tag{8}$$

the system is Lyapunov stable and y_k converges to $y = 0$ for $k > N$, for some $N \in \mathbb{N}$.

Proof: Note that eq. (8) is a sufficient condition for (Lyapunov) stability of the system, as it ensures that the difference $V_{k+1} - V_k$ along trajectories of the discrete-time system is negative definite, with the right-hand side of the equality being zero if and only if $V_k = 0$, given the definition of γ_k. This equation can be expressed as:

$$V_{k+1} = V_k - \gamma_k V_k^{\alpha} = V_k\Big(1 - \frac{\gamma_k}{V_k^{1-\alpha}}\Big). \tag{9}$$

Consider an arbitrary trajectory $y_k \in \mathbb{R}^l$ of the discrete-time system. Let the initial value of the Lyapunov function along this trajectory be

$$V_0 = c_0\big(\gamma_0\big)^{\frac{1}{1-\alpha}}, \text{ where } c_0 > 0. \tag{10}$$

Note that for any finite positive value of V_0, there exists an unique positive scalar c_0 that satisfies (10). Substituting this value for V_0 in expression (8), we obtain:

$$\begin{aligned} V_1 - c_0\big(\gamma_0\big)^{\frac{1}{1-\alpha}} &= -\gamma_0 c_0^{\alpha}\big(\gamma_0\big)^{\frac{\alpha}{1-\alpha}} = -c_0^{\alpha}\big(\gamma_0\big)^{\frac{1}{1-\alpha}} \\ \Rightarrow V_1 &= (c_0 - c_0^{\alpha})\big(\gamma_0\big)^{\frac{1}{1-\alpha}}. \end{aligned} \tag{11}$$

Defining

$$c_1 := c_0 - c_0^{\alpha},$$

equation (11) can be expressed as

$$V_1 = c_1\big(\gamma_0\big)^{\frac{1}{1-\alpha}}.$$

Substituting this value for V_1 in (8), one obtains a similar expression for V_2:

$$V_2 = c_2\big(\gamma_0\big)^{\frac{1}{1-\alpha}}, \text{ where } c_2 := c_1 - a_1 c_1^{\alpha}, \text{ and } a_1 := \frac{\gamma_1}{\gamma_0}. \tag{12}$$

Continuing in this manner, we get the following expression for V_{k+1} along with a recursive relation for the c_k involving the a_k:

$$\begin{aligned} &V_{k+1} = c_{k+1}\big(\gamma_0\big)^{\frac{1}{1-\alpha}} \text{ for } k \geq 1, \text{ where} \\ &c_{k+1} := c_k - a_k c_k^{\alpha} \text{ and } a_k := \frac{\gamma_k}{\gamma_0}. \end{aligned} \tag{13}$$

If V_k is in the range given by (7), then according to eq. (13) and the inequality in (7), we have

$$\begin{aligned} c_{k+1} &\leq c_k - (1-\varepsilon)c_k^{\alpha} \\ &= \varepsilon c_k^{\alpha} - (1 - c_k^{1-\alpha})c_k^{\alpha}. \end{aligned} \tag{14}$$

As $V_{k+1} := V(y_{k+1})$ is positive definite, c_{k+1} cannot be negative according to eq. (13). From the right side of the inequality (14), we see that

$$c_{k+1} \leq 0 \Leftrightarrow \varepsilon \leq 1 - c_k^{1-\alpha} \Leftrightarrow c_k^{1-\alpha} \leq 1 - \varepsilon. \tag{15}$$

Let $k = N$ be the smallest integer for which the inequality in (15) is satisfied as an equality, i.e., $c_N = (1-\varepsilon)^{\frac{1}{1-\alpha}}$. Therefore $c_{N+1} = 0$. Consequently, using eq. (13) again, we conclude that $c_j = 0$ and $V_j = 0$ for $j > N$. As a result, y_j converges to zero for $j > N$, and we have finite-time stability. ■

A constructive method to obtain FTS position tracking control scheme in discrete time is provided here, which has two steps. In the first step, we develop a discrete vector-valued function of the position and velocity tracking errors that ensures that when this function converges to zero, the errors converge to zero as well. The following statement presents the first step of the mentioned method.

Lemma 3.2: Let $l(\tilde{b}_k, \tilde{v}_k)$ be as

$$l(\tilde{b}_k, \tilde{v}_k) = \tilde{v}_k \Delta t + \frac{\beta(\tilde{b}_{k+1} + \tilde{b}_k)}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p}}, \tag{16}$$

for the dynamics given in (4), where $\beta > 0$, $p \in]1, 2[$ and $\tilde{v}_k = (\tilde{b}_{k+1} - \tilde{b}_k)/\Delta t$, and let

$$\begin{aligned} &\tilde{b}_{k+1} = \mathcal{B}(\tilde{b}_k)\tilde{b}_k, \text{ where} \\ &\mathcal{B}(\tilde{b}_k) = \frac{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p} - \beta}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p} + \beta}. \end{aligned} \tag{17}$$

Then $l(\tilde{b}_k, \tilde{v}_k)$ ensures that the tracking errors $(\tilde{b}_k, \tilde{v}_k)$ converge to zero in finite time when $l(\tilde{b}_k, \tilde{v}_k) = 0$.

Proof: One can rewrite (17) as

$$\tilde{b}_{k+1} = \tilde{b}_k \frac{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p} - \beta}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p} + \beta}. \tag{18}$$

Hence, it can be simplified to

$$\tilde{b}_{k+1} - \tilde{b}_k = -\frac{\beta(\tilde{b}_{k+1} + \tilde{b}_k)}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p}}. \tag{19}$$

Note that this can be re-expressed as

$$-\frac{\beta(\tilde{b}_{k+1} + \tilde{b}_k)}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p}} = \tilde{v}_k \Delta t, \tag{20}$$

which holds when $l(\tilde{b}_k, \tilde{v}_k) = 0$.

Consider the discrete-time Lyapunov function V_k defined by (5). The difference between the values of this function at successive discrete instants is given by (6). From (19), substituting $\tilde{b}_{k+1} - \tilde{b}_k$ into (6), one gets

$$V_{k+1} - V_k = -\frac{\beta}{2}\frac{(\tilde{b}_{k+1} + \tilde{b}_k)^{\mathrm{T}} P (\tilde{b}_{k+1} + \tilde{b}_k)}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p}}. \tag{21}$$

Note that $\tilde{b}_{k+1} + \tilde{b}_k = \big(1 + \mathcal{B}(\tilde{b}_k)\big)\tilde{b}_k$, and the right side of expression (21) is zero if only if

$$\tilde{b}_{k+1} = -\tilde{b}_k,$$

which is possible if and only if $\mathcal{B}(\tilde{b}_k) = -1$ according to (17). From the expression for $\mathcal{B}(\tilde{b}_k)$ in (17), one can see that $\mathcal{B}(\tilde{b}_k) = -1$ if and only if $\tilde{b}_k = 0$. Therefore, we conclude that

$$V_{k+1} - V_k = 0 \Leftrightarrow \tilde{b}_k = 0.$$

Now substituting (18) into (21) and noting that $\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k = 2V_k$, one obtains

$$V_{k+1} - V_k = -\gamma_k (V_k)^{1/p}, \tag{22}$$

where

$$\gamma_k = 4\beta \frac{2^{1-1/p}(V_k)^{2-2/p}}{\big((2V_k)^{1-1/p} + \beta\big)^2}. \tag{23}$$

Clearly, γ_k as given by eq. (23) is a class-$\mathcal{K}$ function of V_k. From eqs. (22) and (23), one can see that V_k is monotonously decreasing if $\gamma_k > 0$ and

$$0 < \gamma_k < \frac{4\beta}{2^{1-1/p}} \quad \text{for} \quad 0 < 2V_k < \infty.$$

Therefore γ_k would lead to finite-time stability of tracking control system. Also from (23), one obtains the ratio:

$$a_k := \frac{\gamma_k}{\gamma_0} = \frac{(V_k)^{2-2/p}}{(V_0)^{2-2/p}} \frac{\big((2V_0)^{1-1/p} + \beta\big)^2}{\big((2V_k)^{1-1/p} + \beta\big)^2}. \tag{24}$$

This ratio in eq. (24) is bounded below by a positive number in the interval $]0, 1[$ for non-zero V_k and V_0. This guarantees the existence of $\varepsilon \in]0, 1[$ and $0 < \chi < (V_0)^{1-1/p}$ that satisfy the condition (7) in the statement of Lemma 3.1 for V_k. Therefore, (22) guarantees that V_k converges to zero for $k > N$ for some finite $N \in \mathbb{N}$, and this ensures the finite-time stable convergence of tracking errors to zero. ∎

In the second step of finding the FTS position tracking scheme in discrete time, one can creates a control force for the error dynamics given in (4) that ensures the convergence of the function $l(\tilde{b}_k, \tilde{v}_k)$ derived in the first step to zero in finite time. This will, in turn, ensure that $(\tilde{b}_k, \tilde{v}_k)$ converges to $(0, 0)$ in finite time. In order to fulfill this objective, a positive definite Lyapunov function in terms of the obtained vector-valued $l(\tilde{b}_k, \tilde{v}_k)$ is constructed as

$$\mathcal{V}(\tilde{b}_k, \tilde{v}_k) = \frac{1}{2} l(\tilde{b}_k, \tilde{v}_k)^{\mathrm{T}} l(\tilde{b}_k, \tilde{v}_k), \tag{25}$$

which can be used to obtain the FTS tracking control scheme in discrete time. The following statement provides the main result on finite-time position tracking control scheme.

Theorem 3.1: Consider the translational kinematics and dynamics given in (4) with discrete-time force control vector given by

$$\varphi_k = \frac{m}{\Delta t}\big(v_k + \Delta t\; g\; e_3 - \tilde{v}_{k+1} - \tilde{v}^d_{k+1}\big). \tag{26}$$

Then it stabilizes the translational error dynamics

$$\begin{aligned} \tilde{v}_{k+1} &= \mathcal{F}(\tilde{b}_k, \tilde{b}_{k+1}, \tilde{v}_k, l_k) = \qquad (27) \\ &\frac{1}{\Delta t}\left[(1 + \frac{\kappa}{(l_k^{\mathrm{T}} l_k)^{1-1/p}})(1 + \frac{\beta}{(\tilde{b}_{k+1}^{\mathrm{T}} P \tilde{b}_{k+1})^{1-1/p}})\right]^{-1} \\ &\cdot \Bigg\{\Big(1 - \frac{\kappa}{(l_k^{\mathrm{T}} l_k)^{1-1/p}}\Big)\tilde{v}_k \Delta t \\ &- \frac{2\beta}{(\tilde{b}_{k+1}^{\mathrm{T}} P \tilde{b}_{k+1})^{1-1/p}}\Big(1 + \frac{\kappa}{(l_k^{\mathrm{T}} l_k)^{1-1/p}}\Big)\tilde{b}_{k+1} \\ &+ \frac{\beta(\tilde{b}_{k+1} + \tilde{b}_k)}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p}}\Big(1 - \frac{\kappa}{(l_k^{\mathrm{T}} l_k)^{1-1/p}}\Big)\Bigg\} \end{aligned}$$

in finite time, where $\kappa > 0$, p and β are as in Lemma 3.2.

Proof: Consider the Lyapunov function (25) quadratic in $l(\tilde{b}_k, \tilde{v}_k)$ as constructed in (16). Therefore, the time difference of this discrete-time Lyapunov function can be evaluated as follows:

$$\mathcal{V}_{k+1} - \mathcal{V}_k = \frac{1}{2}(l_{k+1} + l_k)^{\mathrm{T}}(l_{k+1} - l_k). \tag{28}$$

Similar to the definition for $\tilde{b}_{k+1}$ in Lemma 3.2, one can consider

$$l_{k+1} = \mathcal{L}(\tilde{b}_k, \tilde{v}_k)\; l_k, \tag{29}$$

where

$$\mathcal{L}(\tilde{b}_k, \tilde{v}_k) = \frac{(l_k^{\mathrm{T}} l_k)^{1-1/p} - \kappa}{(l_k^{\mathrm{T}} l_k)^{1-1/p} + \kappa}, \tag{30}$$

Substituting (30) in (29) gives

$$(l_{k+1} - l_k) = -\frac{\kappa}{(l_k^{\mathrm{T}} l_k)^{1-1/p}}(l_{k+1} + l_k). \tag{31}$$

Then according to lemma 3.2, one can prove similarly that

$$\mathcal{V}_{k+1} - \mathcal{V}_k = -\lambda_k \mathcal{V}_k^{1/p}, \tag{32}$$

where

$$\lambda_k = 4\kappa \frac{2^{1-1/p}(\mathcal{V}_k)^{2-2/p}}{\big((2\mathcal{V}_k)^{1-1/p} + \kappa\big)^2} \tag{33}$$

is a class-$\mathcal{K}$ function of $\mathcal{V}_k$. Also, from (32) and (33), one can see that

$$0 < \lambda_k < \frac{4\kappa}{2^{1-1/p}} \quad \text{for} \quad 0 < 2\mathcal{V}_k < \infty.$$

Therefore, λ_k would lead to finite-time stability of tracking control system.

Now, by substituting $l(\tilde{b}_k, \tilde{v}_k)$ given in (16) into (31), one can obtain

$$(\tilde{v}_{k+1} - \tilde{v}_k)\Delta t + \beta\Big[\frac{(\tilde{b}_{k+2} + \tilde{b}_{k+1})}{(\tilde{b}_{k+1}^{\mathrm{T}} P \tilde{b}_{k+1})^{1-1/p}} - \frac{(\tilde{b}_{k+1} + \tilde{b}_k)}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p}}\Big] \tag{34}$$

$$= -\frac{\kappa}{(l_k^{\mathrm{T}} l_k)^{1-1/p}}\Bigg\{(\tilde{v}_{k+1} + \tilde{v}_k)\Delta t + \beta\Big[\frac{(\tilde{b}_{k+2} + \tilde{b}_{k+1})}{(\tilde{b}_{k+1}^{\mathrm{T}} P \tilde{b}_{k+1})^{1-1/p}} + \frac{(\tilde{b}_{k+1} + \tilde{b}_k)}{(\tilde{b}_k^{\mathrm{T}} P \tilde{b}_k)^{1-1/p}}\Big]\Bigg\}.$$

Noting that $(\tilde{b}_{k+2} - \tilde{b}_{k+1})/\Delta t = \tilde{v}_{k+1}$, one can solve above expression for $\tilde{v}_{k+1}$ and obtain (27), which is the discrete-time translational error dynamics equation.

Then, noting that $\tilde{v}_{k+1} = v_{k+1} - v^d_{k+1}$, one can substitute (27) in (4) and obtain the discrete-time control force vector which guarantees the finite-time stability of the tracking control as (26). ∎

IV. Finite-Time Continuous Stable Position Tracking Control on $\mathbb{R}^3$

A FTS position tracking control scheme in continuous time has been reported in [9]. In this scheme, the continuous equations of error dynamics are given by (2), where $\varphi \in \mathbb{R}^3$ is the feedback control obtained in continuous time and presented as equation (17) in that paper. Then, the obtained continuous control law is sampled within a time interval $[t_0, t_f]$ and with a time step size Δt.

The following section presents numerical results obtained by implementing this paper's proposed FTS scheme in discrete time compared to the results of the a sampled finite-time continuous scheme.

V. Simulation Results

This section presents numerical simulation results for the FTS position tracking control scheme in discrete time. Also, the performance of the proposed FTS tracking control scheme in discrete time is compared to that of the sampled continuous time FTS tracking scheme presented in section IV. The numerical simulation is performed for an UAV quadcopter assumed to have a mass $m = 4$ kg, for different time periods of $T = 5, 10$, and 30s, with different time step sizes $\Delta t = 0.01, 0.05$, and 0.1s, using discrete-time FTS control law obtained in (26), and the sampled continuous-time control law given in [9]. The helical desired trajectory and the initial conditions are given as follows for both schemes

$$b_k^d = b^d(t_k) = \begin{bmatrix}0.4\sin\pi t_k & 0.6\cos\pi t_k & 0.4\,t_k\end{bmatrix}^{\mathrm{T}},$$
$$b_0 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^{\mathrm{T}},\ v_0 = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^{\mathrm{T}}.$$

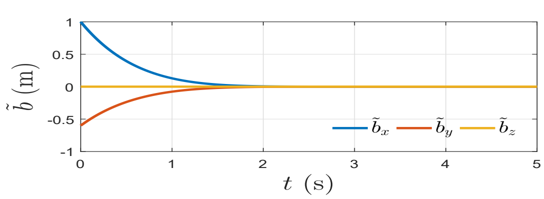
The gains are selected and tuned after trial and error for FTS discrete-time tracking scheme as follows:

$$P = 4\,\mathbf{I}_{3\times 3},\ \beta = 0.01,\ \kappa = 0.009,$$

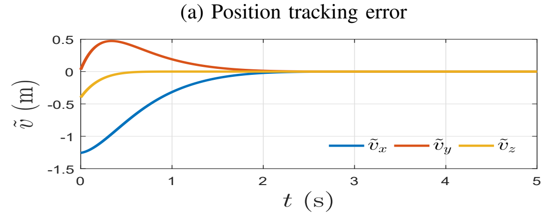
and for FTS sampled continuous scheme as follows:

$$P_r = 5\,\mathbf{I}_{3\times 3},\ \kappa_t = 0.8,$$

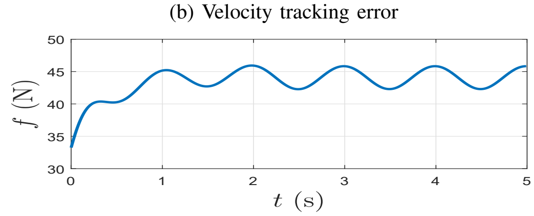
which provide desirable and similar transient response characteristics of both tracking control schemes when $\Delta t = 0.01$. The results of the numerical simulation for position and velocity tracking response of the discrete-time control law obtained in (26) for $\Delta t = 0.01$ and $t_f = 5s$ is shown in Fig.1. These subplots show that the position and translational velocity tracking errors converge to zero, and therefore the discrete-time control scheme proposed here is able to track the desired position trajectory in finite time. The time plots of the force control input $f_k = ||\varphi_k||$ shows that the total magnitude of the thrust force does not exceed 46 N, which is within the capabilities of the modeled UAV. Other simulation results are presented in Fig. 2 to Fig. 4 in order to compare the performance of the discrete-time FTS tracking scheme with a sampled continuous FTS tracking scheme for *different values of time step sizes*.



(a) Position tracking error



(b) Velocity tracking error



(c) Total thrust force

Fig. 1: Translational motion errors and force control for discrete-time FTS tracking control scheme for $\Delta t = 0.01$ and $t_f = 5s$.

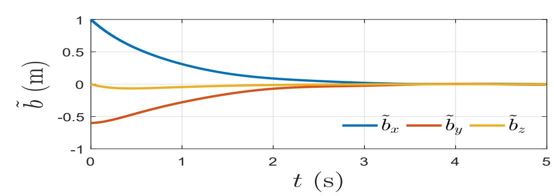
Comparing the results of these two schemes as shown in the plots, one can conclude that control force obtained by sampling the FTS continuous control input does not guarantee the stability of the position tracking when the time step size changes. The results of this comparison are given in Table I. In Table I, $\Delta\mathcal{V}_{max}$ is denoted as the maximum positive value of the time difference $\mathcal{V}_{k+1} - \mathcal{V}_k$ as:

$$\Delta\mathcal{V}_{max} = max\big[(\mathcal{V}_{k+1} - \mathcal{V}_k) > 0\big]. \tag{35}$$

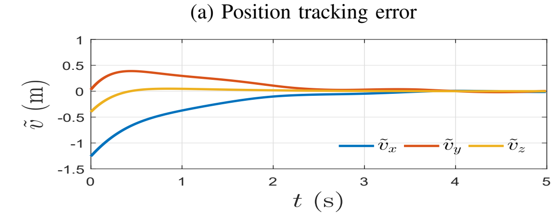
This parameter is defined to confirm whether the Lyapunov function $\mathcal{V}_k$ increases in value at certain time instants, and whether that increase is significant or is just an artifact of machine (float) precision. For a finite-time stable system one expects the value of $\mathcal{V}_{k+1} - \mathcal{V}_k$ to be negative until it converges to zero in finite time, which ensures stability of the system in finite time. On the contrary, for the sampled continuous FTS tracking scheme, one can see a significant increase in the value of $\mathcal{V}_k$ as time step size increases, whereas $\Delta\mathcal{V}_{max}$ has a negligible value (to machine precision) when the discrete-time FTS tracking control scheme is implemented.

Tracking Control Scheme	$\Delta t(s)$	$t_f(s)$	$\Delta\mathcal{V}_{max}$
Discrete-time FTS	0.01	5	3.5716×10^{-38}
	0.05	10	2.5277×10^{-38}
	0.1	30	1.9007×10^{-39}
Sampled Continuous FTS	0.01	5	5.6687×10^{-6}
	0.05	10	0.0013
	0.1	30	0.0136

TABLE I: Stability performance of discrete-time FTS vs. sampled continuous-time FTS tracking control scheme on $\mathbb{R}^3$.

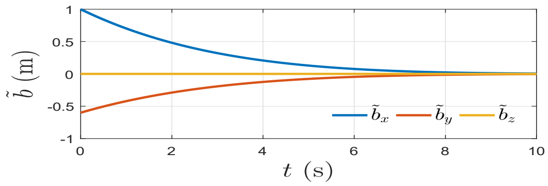


(a) Position tracking error

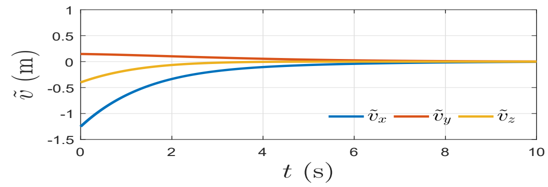


(b) Velocity tracking error

Fig. 2: Translational motion errors for sampled FTS continuous tracking control scheme for $\Delta t = 0.01$ and $t_f = 5s$.

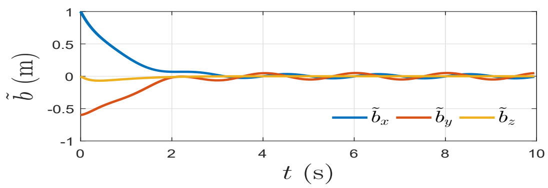


(a) Position tracking error

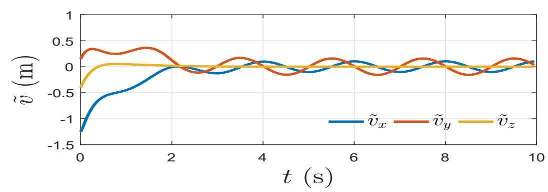


(b) Velocity tracking error

Fig. 3: Translational errors for FTS discrete-time tracking control scheme for $\Delta t = 0.05$ and $t_f = 10s$.



(a) Position tracking error



(b) Velocity tracking error

Fig. 4: Translational errors for FTS sampled continuous tracking control scheme for $\Delta t = 0.05$ and $t_f = 10s$.

VI. Conclusion

A discrete finite-time stable position tracking control scheme for unmanned vehicles is presented here. This scheme is based on using a Lyapunov framework for finite-time stabilization of a position tracking control that results in discrete-time error dynamics in terms of translational motion tracking errors. A two-step method is presented to construct a Lyapunov function quadratic in a vector-valued function that is in terms of translational motion tracking errors. When this vector-valued function vanishes, it ensures the convergence of the tracking errors to zero in finite time. Then, a discrete-time control force vector is obtained to guarantee that this vector converges to the zero vector in finite time, and therefore the system states converge to the desired trajectory in a finite time interval. The stability of the proposed scheme is also compared with that of a sampled continuous FTS scheme, and numerical results show that a discrete-time FTS position tracking control scheme is more reliable for onboard computer implementation when we need to work with a variety of input data frequencies. Future work will look at a discrete-time FTS pose tracking control scheme for underactuated vehicles on $\mathrm{SE}(3)$, and comparison of this discrete-time stable tracking control scheme with other sampled continuous time tracking control schemes.

VII. Acknowledgement

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References

[1] A. K. Sanyal and J. Bohn, "Finite-time stabilisation of simple mechanical systems using continuous feedback," *Int. J. Control*, vol. 88, no. 4, pp. 783–791, 2015.

[2] A. K. Sanyal, J. Bohn, and A. M. Bloch, "Almost global finite time stabilization of rigid body attitude dynamics," in *52nd IEEE Conference on Decision and Control*, Dec 2013, pp. 3261–3266.

[3] S. P. Bhat and D. S. Bernstein, "Continuous finite-time stabilization of the translational and rotational double integrators," *IEEE Transactions on Automatic Control*, vol. 43, no. 5, pp. 678–682, May 1998.

[4] S. P. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[5] W. M. Haddad, S. G. Nersesov, and L. Du, "Finite-time stability for time-varying nonlinear dynamical systems," in *2008 American Control Conference*, June 2008, pp. 4135–4139.

[6] J. Bohn and A. K. Sanyal, "Almost global finite-time stabilization of rigid body attitude dynamics using rotation matrices," *International Journal of Robust and Nonlinear Control*, vol. 26, no. 9, pp. 2008–2022, 2015.

[7] S. P. Viswanathan, A. K. Sanyal, and E. Samiei, "Integrated guidance and feedback control of underactuated robotics system in SE(3)," *Journal of Intelligent and Robotic Systems*, vol. 89, no. 1-2, pp. 251–263, 2018.

[8] S. P. Viswanathan, A. Sanyal, and M. Izadi, "Integrated guidance and nonlinear feedback control of underactuated unmanned aerial vehicles in SE(3)," in *AIAA Guidance, Navigation, and Control Conference, AIAA SciTech Forum*, no. AIAA 2017-1044, 2017.

[9] S. P. Viswanathan, A. K. Sanyal, and R. R. Warier, "Finite-time stable tracking control for a class of underactuated aerial vehicles in SE(3)," in *2017 American Control Conference, ACC 2017, Seattle, WA, USA, May 24-26, 2017*, 2017, pp. 3926–3931. [Online]. Available: https://doi.org/10.23919/ACC.2017.7963556

[10] T. Lee, M. Leok, and N. H. McClamroch, "A lie group variational integrator for the attitude dynamics of a rigid body with applications to the 3d pendulum," in *IEEE Conference on Control Applications*. IEEE, 2005, pp. 962–967.

[11] N. Nordkvist and A. K. Sanyal, "A lie group variational integrator for rigid body motion in $\mathrm{SE}(3)$ with applications to underwater vehicle dynamics," in *49th IEEE Conference on Decision and Control (CDC)*. IEEE, 2010, pp. 5414–5419.

[12] A. Sanyal, N. Nordkvist, and M. Chyba, "An almost global tracking control scheme for maneuverable autonomous vehicles and its discretization," vol. 56, no. 2. IEEE Transactions on Automatic Control, 2011, pp. 457 – 462.

[13] J. E. Marsden and M. West, "Discrete mechanics and variational integrators," *Acta Numerica*, vol. 10, pp. 357–514, 2001.

[14] I. I. Hussein, M. Leok, A. K. Sanyal, and A. M. Bloch, "A discrete variational integrator for optimal control problems on $\mathrm{SO}(3)$," in *IEEE Conference on Decision and Control (CDC)*, 2006, pp. 6636–6641.

[15] M. Izadi and A. K. Sanyal, "Rigid body attitude estimation based on the lagrange-d'alembert principle," *Automatica*, vol. 50, no. 10, pp. 2570–2577, 2014.

[16] ——, "Rigid body pose estimation based on the lagrange-d'alembert principle," *Automatica*, vol. 71, pp. 78–88, 2016.

[17] M. Izadi, E. Samiei, A. K. Sanyal, and V. Kumar, "Comparison of an attitude estimator based on the lagrange-d'alembert principle with some state-of-the-art filters," in *2015 IEEE International Conference on Robotics and Automation (ICRA)*, 2015.

[18] R. Hamrah, R. R. Warier, and A. K. Sanyal, "Discrete-time stable tracking control of underactuated rigid body systems on SE(3)," in *57th IEEE Conference on Decision and Control, CDC 2018, Miami, FL, USA, December 17-19, 2018*, 2018, pp. 2932–2937. [Online]. Available: https://doi.org/10.1109/CDC.2018.8618870

[19] A. Bloch, J. Baillieul, P. Crouch, and J. Marsden, *Nonholonomic Mechanics and Control, ser. Interdisciplinary Applied Mathematics*. Springer, Verlag, 2003.

[20] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems, ser. Texts in Applied Mathematics*. Springer, Verlag, 2004.