2019 IEEE 58th Conference on Decision and Control (CDC)
Palais des Congrès et des Expositions Nice Acropolis
Nice, France, December 11-13, 2019

A Finite-Time Stable Observer for Relative Attitude Estimation

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Abstract— **Relative motion estimation of one rigid body with respect to another is a problem that has immediate applications to formations and maneuvers involving multiple unmanned vehicles or collision avoidance between vehicles. A finite-time stable observer for relative attitude estimation of a rigid object using onboard sensors on an unmanned vehicle, is developed and presented here. This observer assumes sensor inputs from onboard vision and inertial sensors, with the vision sensors measuring at least three points on the object whose relative locations with respect to a body-fixed frame on the object are also assumed to be known. In the absence of any measurement noise, the estimated relative attitude is shown to converge to the actual relative pose in a finite-time stable manner. Numerical simulations indicate that this relative attitude observer is robust to persistent measurement errors and converges to a bounded neighborhood of the true attitude.**

1. INTRODUCTION

Estimation of relative motion between rigid body objects in three-dimensional Euclidean space has several applications in formations and proximity maneuvers between unmanned aerial vehicles, spacecraft and underwater vehicles. A stable relative attitude estimation scheme that is robust to measurement noise and requires no knowledge of the dynamics model of the vehicle being observed, is presented here. This estimation scheme can enhance the autonomy and reliability of teams of unmanned vehicles operating in uncertain environments without any external navigation aids like global navigation satellite systems (GNSS). This avoids the need for measurements from external sources, which may not be available in indoor, underwater or cluttered environments [1], [2], [3].

This nonlinear relative attitude estimation scheme does not use local coordinates or quaternions to represent the relative attitude on $\mathrm{SO}(3)$. It is constructed to be finite-time stable (FTS) on the space of rigid body orientations, $\mathrm{SO}(3)$. Attitude estimation and control schemes that use generalized coordinates or quaternions for attitude representation are usually *unstable in the sense of Lyapunov*, as has been shown in prior research [4], [5], [6]. One adverse consequence of these unstable estimation and control schemes is that they end up taking longer to converge compared to stable schemes with the same initial conditions and same initial transient behavior. Attitude and pose observers and filtering schemes on $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$ have been reported in, e.g., [7], [8], [9], [10], [11], [12], [13], [14], [15].

A sample of recent work on observer design and estimation directly on the Lie groups of rigid body attitude and pose motion can be found in [16], [17],[18], [19], [20], [21], [22], [23]. These estimators do not suffer from kinematic singularities like estimators using coordinate descriptions of attitude, and they do not suffer from the unstable unwinding phenomenon encountered by continuous estimators using unit quaternions. This motivates our approach using a geometric relative attitude estimation scheme to estimate the relative attitude directly on $\mathrm{SO}(3)$.

A prior related work that obtained a relative motion estimation scheme (for relative pose and relative velocities) was reported in [24]. In that work, we obtained a relative motion estimation scheme using a variational approach based on a Lagrangian created from the estimation errors in relative pose and relative velocities. The resulting variational estimator was obtained by applying the Lagrange-d'Alembert principle to the Lagrangian, thereby dissipating the energy contained in the estimation errors and resulting in an asymptotically stable observer for the relative motion. The same approach was used for (absolute) attitude and pose estimation, respectively, in [25], [26]. For instantaneous relative attitude estimation of a rigid object observed by sensors onboard a rigid vehicle in its proximity, relative position vectors of at least three known feature points on the observed body need to be measured [24]. However, for real-time estimation of the relative attitude from continuous measurements of such feature points, the accuracy of the estimate suffers due to additive measurement noise in the feature point measurements. Therefore, for added robustness to measurement errors in relative positions of observed feature points, we use a finite-time stable estimation scheme for relative attitude estimation in this work. Like the variational attitude estimation scheme in [24], this finite time stable relative attitude estimation scheme also estimates both relative attitude and relative angular velocity from vector measurements.

The rest of the paper is organized as follows. The approach to determine the relative attitude and relative angular velocity from the instantaneous vector measurements is posed in Section 2. Section 3 describes the finite-time stable (FTS) attitude estimation scheme with the main results and its stability proof in details. Two numerical experiments implemented with the proposed estimation scheme are shown in Section 4 with experimental results. These experiments show the performance of the relative attitude estimator with and without measurement noise in the relative position vector measurements of feature points. Section 5 summarizes the contributions of this paper and lists possible future directions in relative motion estimation.

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2. STATIC ATTITUDE DETERMINATION FROM VECTOR MEASUREMENTS

A. Relative Pose Measurement Model

Let O denote the observed vehicle and O be the coordinate frame fixed to O. Let S denote the vehicle that is observing O and S denote a coordinate frame fixed to S. Let $R \in \mathrm{SO}(3)$ be the relative rotation matrix from frame S to frame O and b denote the position of origin of S expressed in frame O. The pose (transformation) of frame S to frame O is

$$\mathsf{g} = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3). \tag{1}$$

The positions of a fixed set of feature points or patterns on vehicle O are observed by optical sensors fixed to vehicle S. Velocities of these points are not directly measured, but may be calculated using a simple linear filter as in [27]. Assume that there are $j > 2$ feature points, which are always in the sensor field-of-view (FOV) of the sensor fixed to vehicle S, and the positions of these points are known in frame O as p_j, $j \in \{1, 2, \ldots, j\}$. These points generate $\binom{j}{2}$ unique pairwise relative position vectors, which are the vectors connecting any two of these points.

Denote the position of the optical sensor on vehicle S and the vector from that sensor to an observed point on vehicle O as $s, q_j \in \mathbb{R}^3$, $j = 1, 2, \ldots, j$, respectively, both vectors expressed in frame S. Thus, in the absence of measurement noise

$$p_j = R(q_j + s) + b = Ra_j + b,\ j \in \{1, 2, \ldots, j\}, \tag{2}$$

where $a_j = q_j + s$, are positions of these points expressed in S. In practice, the a_j are obtained from proximity optical measurements that will have additive noise; denote by a_j^m the measured vectors. The mean values of the vectors p_j and a_j^m are denoted as $\bar{p}$ and $\bar{a}^m$, and satisfy

$$\bar{a}^m = R^{\mathrm{T}}(\bar{p} - b) + \bar{\varsigma}, \tag{3}$$

where $\bar{p} = \frac{1}{j}\sum_{j=1}^{j} p_j$, $\bar{a}^m = \frac{1}{j}\sum_{j=1}^{j} a_j^m$ and $\bar{\varsigma}$ is the additive measurement noise obtained by averaging the measurement noise vectors for each of the a_j. Consider the $\binom{j}{2}$ relative position vectors from optical measurements, denoted as $d_j = p_\lambda - p_\ell$ in frame O and the corresponding vectors in frame S as $l_j = a_\lambda - a_\ell$, for $\lambda, \ell \in \{1, 2, \ldots, j\}$, $\lambda \neq \ell$. Therefore,

$$d_j = Rl_j \Rightarrow D = RL, \tag{4}$$

where $D = [d_1 \ \cdots \ d_\beta]$, $L = [l_1 \ \cdots \ l_\beta] \in \mathbb{R}^{3\times\beta}$ and $\beta = \binom{j}{2}$. Note that the matrix of known relative vectors D is assumed to be known and bounded. Denote the measured value of matrix L in the presence of measurement noise as L^m. Then,

$$L^m = R^{\mathrm{T}} D + \mathscr{L}, \tag{5}$$

where $\mathscr{L} \in \mathbb{R}^{3\times\beta}$ is the matrix of measurement errors in these vectors observed in frame S.

B. Relative Angular Velocities Model

The relative attitude can be calculated from the vector measurements provided the following assumption is satisfied.

Assumption 1: There are at least two non-collinear vectors in $d_1, \ldots d_\beta$ for attitude determination at all times. If $\beta = 2$, $d_3 = d_1 \times d_2$ is chosen as the third non-collinear vector.

Denote the relative angular velocity of vehicle O expressed in frame S by Ω, respectively. Thus, one can write the kinematics of the rigid body as

$$\dot{R} = R\Omega^\times \tag{6}$$

where $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3) \subset \mathbb{R}^{3\times3}$ is the skew-symmetric cross-product operator that gives the vector space isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3)$. To get the angular velocity Ω with the given information l_j and d_j, following procedures can be done based on (4). Neglect the noise signal and do time derivative:

$$d_j = Rl_j \tag{7}$$

$$\Rightarrow 0 = \dot{R}l_j + R\dot{l}_j = R\Omega^\times l_j + R\dot{l}_j \tag{8}$$

$$\Rightarrow \dot{l}_j = l_j^\times \Omega \tag{9}$$

Take all of the sensor measurement into consideration, then (5) can be rewritten into the following form:

$$\dot{V}(L) = G(L)\Omega \tag{10}$$

$V(L)$ and $G(L)$ are defined as:

$$V(L) = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_\beta \end{bmatrix},\ G(L) = \begin{bmatrix} l_1^\times \\ l_2^\times \\ \vdots \\ l_\beta^\times \end{bmatrix}$$

$$V(L) \in \mathbb{R}^{3\beta}, G(L) \in \mathbb{R}^{3\beta\times3} \tag{11}$$

where $G(L)$ has full row rank. From vision-based or Doppler lidar sensors, one can measure the relative angular velocities of the observed points in frame S based on (10).

C. Relative Angular Velocity Filtering

Considering the relative angular velocity measurement and attitude estimation will both depend on the accuracy of vector measurements, the vector measurements are "pre-filtered" to improve the estimator performance. To reduce the impact of noise signal $\mathscr{L}$ from the measured L^m, a discrete-time Butterworth filter is applied here to filter the relative position vectors that are the column vectors of L^m. This discrete-time Butterworth filter was implemented in [26], and is obtained from a continuous-time second-order Butterworth filter discretized using the *Newmark-β Method*. The filtered version of matrix L^m is denoted L^f.

As this kinematics equation (10) indicates, the relative velocities of at least three points are needed to determine the observed vehicle's angular velocity uniquely at each instant. The rigid body velocities are obtained using the pseudo-inverse of $G(L^f)$, denoted $G^\#(L^f)$. Taking the measurement noise and filtering process into consideration and replacing

L^m with the filtered matrix L^f, the pre-filtered relative angular velocity Ω^f is obtained as:

$$V(\dot{L}^f) = G(L^f)\Omega^f \Rightarrow \ \Omega^m = G^{\#}(L^f)V(\dot{L}^f) \tag{12}$$

where $G^{\#}(L^f) = (G^T(L^f)G(L^f))^{-1}G^T(L^f)$.

3. REAL-TIME RELATIVE ATTITUDE ESTIMATION

This section presents the real-time relative attitude estimation scheme. Denote the estimated relative attitude and its kinematics as

$$\dot{R} = R\Omega^{\times}, \tag{13}$$

where $\Omega \in \mathbb{R}^3$ is the relative angular velocity of the observed rigid body represented in the frame S.

A. Basic Definitions For Relative Attitude Estimation

Let $(\widehat{R}, \widehat{\Omega}) \in \mathrm{SO}(3) \times \mathbb{R}^3$ be the estimates of the relative attitude and angular velocity provided by the estimation scheme, with the following kinematics,

$$\dot{\widehat{R}} = \widehat{R}\widehat{\Omega}^{\times}. \tag{14}$$

$$\widehat{\Omega} = \Omega^f - \widehat{R}^T\omega, \tag{15}$$

where $\omega \in \mathbb{R}^3$ is the error in estimating the angular velocity. Note that if the expression (15) is implemented with Ω^m instead of its filtered version Ω^f, then the measurement noise in Ω^m, which comes from the measurement noise in L^m, would not be filtered out. For this purpose, we use the discrete Butterworth filter given in [26] to filter out the noise in L^m, and use the filtered quantities L^f and $\dot{L}^f$ to create a filtered version of Ω^m, denoted as Ω^f.

One can define the trace inner product on $\mathbb{R}^{n_1\times n_2}$ as

$$\langle A_1, A_2\rangle := \mathrm{trace}(A_1^{\mathrm{T}}A_2). \tag{16}$$

Any square matrix $A \in \mathbb{R}^{n\times n}$ can be written as sum of unique symmetric and skew-symmetric matrices, that is,

$$A = \mathrm{sym}(A) + \mathrm{skew}(A), \tag{17}$$

where the symmetric and skew-symmetric components are defined as,

$$\mathrm{sym}(A) = \frac{1}{2}(A + A^{\mathrm{T}}), \mathrm{skew}(A) = \frac{1}{2}(A - A^{\mathrm{T}}). \tag{18}$$

Additionally, following properties hold. Let $A_1 \in \mathbb{R}^{n\times n}$ is a symmetric matrix and $A_2 \in \mathbb{R}^{n\times n}$ is a skew symmetric matrix, then,

$$\langle A_1, A_2\rangle = 0. \tag{19}$$

In other words, symmetric and skew matrices are orthogonal under the trace inner product. For all $a_1, a_2 \in \mathbb{R}^3$,

$$\left\langle a_1^{\times}, a_2^{\times}\right\rangle = -2a_1 \cdot a_2 \tag{20}$$

With these definitions, we proceed to lay out the attitude estimation problem.

B. Potential Function For Attitude Determination

The purpose is to obtain an estimate of the relative attitude denoted by $\widehat{R} \in \mathrm{SO}(3)$ from β known inertial vectors $d_1, \ldots, d_\beta$ and corresponding pre-filtered vectors $l_1^f, \ldots, l_\beta^f$. The static attitude estimation can be formulated as an optimization problem as follows,

$$\mathrm{Minimize}_{\widehat{R}}\, \mathcal{U} = \frac{1}{2}\sum_i^\beta w_i(d_i - \widehat{R}l_i^f)^{\mathrm{T}}(d_i - \widehat{R}l_i^f), \tag{21}$$

where $w_i > 0$ are weight factors. the rotational potential function (Wahba's cost function [28]) is expressed as

$$\mathcal{U}(\widehat{R}, L^f, D) = \frac{1}{2}\langle D - \widehat{R}L^f, (D - \widehat{R}L^f)W\rangle, \tag{22}$$

where $W = \mathrm{diag}(w_j) \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix of weight factors for the measured l_j^f. The potential function can be generalized such that W is a symmetric positive semi-definite matrix satisfying some special conditions. The structure of the generalized potential function in the absence of measurement errors, is detailed in the following lemma.

Lemma 1: Define $Q = R\widehat{R}^{\mathrm{T}}$ as the relative attitude estimation error. Let $D \in \mathbb{R}^{3\times\beta}$ be as defined in Section 2 with $rank(D) = 3$. Let the gain matrix W of the generalized Wahba potential function be given by,

$$W = D^{\mathrm{T}}\big(DD^{\mathrm{T}}\big)^{-1}K\big(DD^{\mathrm{T}}\big)^{-1}D, \tag{23}$$

where $K = \mathrm{diag}([k_1, k_2, k_3])$ and $k_1 > k_2 > k_3 \geq 1$. Then, in the absence of measurement errors,

$$\mathcal{U} = \frac{1}{2}\left\langle D - \widehat{R}L^f, (D - \widehat{R}L^f)W\right\rangle = \langle K, I - Q\rangle, \tag{24}$$

which is a Morse function on $\mathrm{SO}(3)$ whose critical points are given by the set,

$$\mathcal{C} = \{I, \mathrm{diag}([-1,-1,1]), \mathrm{diag}([1,-1,-1]), \mathrm{diag}([-1,1,-1])\}. \tag{25}$$

In addition, $\mathcal{U}$ has a global minimum at $Q = I$.

The proof of this lemma is given in [25], and is omitted here for brevity. The following four lemmas are used in the following section to prove the main result on finite time stable relative attitude estimation.

Lemma 2: Let K be as defined in Lemma 1. Then, in the absence of the measurement errors, the time derivative of $\mathcal{U}$ along the trajectories satisfying the kinematic equations (13)-(14), is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U} = \frac{\mathrm{d}}{\mathrm{d}t}\langle K, I - Q\rangle = s_K(Q)\cdot\big(\widehat{R}\widetilde{\Omega}\big) \tag{26}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{trace}(K - L^{\mathrm{T}}\widehat{R}) = -s_\Gamma(\widehat{R})\cdot\widetilde{\Omega}, \tag{27}$$

where

$$\widetilde{\Omega} = \Omega^f - \widehat{\Omega},\ s_\Gamma(\widehat{R}) = \mathrm{vex}(\Gamma^{\mathrm{T}}\widehat{R} - \widehat{R}^{\mathrm{T}}\Gamma). \tag{28}$$

Proof: Since $Q = R\widehat{R}^{\mathrm{T}}$, we obtain from eqs. (13)-(14):

$$\begin{aligned}\dot{Q} &= \frac{\mathrm{d}}{\mathrm{d}t}Q = \dot{R}\widehat{R}^{\mathrm{T}} + R\dot{\widehat{R}}^{\mathrm{T}} = R\Omega^{\times}\widehat{R}^{\mathrm{T}} - R\widehat{\Omega}^{\times}\widehat{R}^{\mathrm{T}} \\ &= R\widehat{R}^{\mathrm{T}}\big(\widehat{R}(\Omega^f - \widehat{\Omega})\big)^{\times} = Q(\widehat{R}\widetilde{\Omega})^{\times}\end{aligned} \tag{29}$$

where $\widetilde{\Omega} = \Omega^f - \widehat{\Omega}$. Substituting W as given by (23) in (24) and defining

$$\Gamma = DW(L^f)^{\mathrm{T}}, \tag{30}$$

we see that in the absence of measurement noise, $L^m = L^f = L = R^{\mathrm{T}} D$ and

$$\dot{\Gamma} = DW\dot{L}^{\mathrm{T}} = DWL^{\mathrm{T}}\Omega^{\times} = \Gamma\Omega^{\times}. \tag{31}$$

From eq. (29), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle K, I - Q\rangle = \langle K, -Q(\widehat{R}\widetilde{\Omega})^{\times}\rangle.$$

From eq. (31), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{trace}(K - \Gamma^{\mathrm{T}}\widehat{R}) = \mathrm{trace}(\Omega^{\times}\Gamma^{\mathrm{T}}\widehat{R} - \Gamma^{\mathrm{T}}\widehat{R}\widehat{\Omega}^{\times}).$$

As in the proof of Lemma 1, (19) and (20) are utilized to obtain,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle K, I - Q\rangle &= -\frac{1}{2}\mathrm{trace}\big((KQ - Q^{\mathrm{T}}K)(\widehat{R}\widetilde{\Omega})^{\times}\big)\\ &= \mathrm{vex}(KQ - Q^{\mathrm{T}}K)\cdot(\widehat{R}\widetilde{\Omega})\end{aligned} \tag{32}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{trace}(K - \Gamma^{\mathrm{T}}\widehat{R}) = -\mathrm{vex}(\Gamma^{\mathrm{T}}\widehat{R} - \widehat{R}^{\mathrm{T}}\Gamma)\cdot\widetilde{\Omega}. \tag{33}$$

As (32) is identical to (26) and (33) is identical to (27), we conclude the result. ■

Lemma 3: Let K be as defined in Lemma 1 and $s_K(Q) = \mathrm{vex}(KQ - Q^{\mathrm{T}}K)$. Let $\mathcal{S} \subset \mathrm{SO}(3)$ be a closed subset containing the identity in its interior, defined by

$$\begin{aligned}\mathcal{S} = \big\{&Q \in \mathrm{SO}(3) \,:\, Q_{ii} \geq 0 \text{ and } Q_{ij}Q_{ji} \leq 0\\ &\forall i,\ j \in \{1,2,3\},\ i \neq j\big\}.\end{aligned} \tag{34}$$

Then for $Q \in \mathcal{S}$, we have

$$s_K(Q)^T s_K(Q) \geq \mathrm{trace}(K - KQ). \tag{35}$$

Proof: The proof of this result is given in [29], and is omitted here for brevity. ■

Lemma 4: Let $s_\Gamma(\widehat{R})$ and $s_K(Q)$ be as defined earlier. Then following holds:

$$s_\Gamma(\widehat{R})^{\mathrm{T}} s_\Gamma(\widehat{R}) = s_K(Q)^{\mathrm{T}} s_K(Q) \tag{36}$$

Proof: From the definition of L, it can be seen that $L = KR$. Now $s_\Gamma(\widehat{R})$ and $s_K(Q)$ can be rewritten as,

$$s_\Gamma(\widehat{R}) = \mathrm{vex}(R^{\mathrm{T}}K\widehat{R} - \widehat{R}^{\mathrm{T}}KR) =: \mathrm{vex}(A_1), \tag{37}$$

$$s_K(Q) = \mathrm{vex}(KR\widehat{R}^{\mathrm{T}} - \widehat{R}R^{\mathrm{T}}K) := \mathrm{vex}(A_2), \tag{38}$$

where A_1, A_2 are used to represent the skew symmetric matrices inside the vex(.) operator. From Eq.(20), it is clear that Eq.(36) is equivalent to the following expression,

$$tr(A_1A_1) = tr(A_2A_2). \tag{39}$$

Now, simplifying using the properties of the trace inner product, it is straightforward to see that (39), and therefore (36), holds true. The steps are omitted for conciseness. ■

C. Main Result

Here the main result of this article is described: a finite-time stable rigid body relative attitude observer. For the relative attitude estimation, Assumption 1 is required for the following theorem.

Theorem 3.1: Consider the relative attitude kinematics (6) and the relative angular velocity measurements given by (15) in the absence of measurement noise ($\mathscr{L} = 0$). Let $p \in (1, 2)$ and let $k_p > 0$ be a scalar observer gain. Let Assumption 1 be satisfied. Hence, with the relative angular velocity measured using (12), the following observer is finite-time stable for the relative attitude of the observed rigid body:

$$\omega = \frac{k_p\widehat{R}s_\Gamma(\widehat{R})}{\{s_\Gamma(\widehat{R})^T s_\Gamma(\widehat{R})\}^{1-1/p}} \tag{40}$$

$$\widehat{\Omega} = \Omega^f - \widehat{R}^T\omega \tag{41}$$

$$\dot{\widehat{R}} = \widehat{R}\widehat{\Omega}^{\times} \tag{42}$$

Proof: Consider the potential function $\mathcal{U}$ in Lemma 1, Section 3 as the Lyapunov candidate:

$$\mathcal{V} = \mathcal{U}(\widehat{R}, L^f, D) = \langle K, I - Q\rangle \tag{43}$$

Take the time derivative of this Lyapunov candidate, using (27) and (28) defined in Lemma 2 :

$$\dot{\mathcal{V}} = -s_\Gamma(\widehat{R})\widetilde{\Omega} = -s_\Gamma(\widehat{R})\widehat{R}^T\omega \tag{44}$$

Substituting the designed estimation law (40), it can be shown that:

$$\dot{\mathcal{V}} = -\frac{k_p\widehat{R}^T\widehat{R}s_\Gamma(\widehat{R})^T s_\Gamma(\widehat{R})}{\{s_\Gamma(\widehat{R})^T s_\Gamma(\widehat{R})\}^{1-1/p}} = -k_p\{s_\Gamma(\widehat{R})^T s_\Gamma(\widehat{R})\}^{1/p} \tag{45}$$

From the results of Lemma 3 and Lemma 4, one obtains:

$$\dot{\mathcal{V}} = -k_p\{s_\Gamma(\widehat{R})^{\mathrm{T}} s_\Gamma(\widehat{R})\}^{1/p} \tag{46}$$

$$= -k_p\{s_K(Q)^{\mathrm{T}} s_K(Q)\}^{1/p} \tag{47}$$

$$\leq -k_p\,\langle K, I - Q\rangle^{1/p} \tag{48}$$

$$\leq -k_p\mathcal{V}^{1/p} \tag{49}$$

According to [30], this proves the finite-time stability of the relative attitude estimation scheme. It means the relative attitude estimation error matrix Q will converge stably to the identity I in finite time. Further, when $Q = I$ after this finite period of time, $\widehat{R} = R$ and $s_\Gamma(\widehat{R}) = 0$. Therefore the relative angular velocity estimation error $\omega = 0$ and the relative attitude motion state estimates converge to the true states after this finite period of time. ■

4. NUMERICAL SIMULATIONS

This section provides numerical simulation results for this relative attitude estimation scheme for two cases: one without any measurement noise, and the other with measurement noise in the relative position measurements L^m of feature points observed on the object O. The numerical simulation method is that of sampling the relative attitude estimator described in Section 3 with a constant time step size. The attitude estimation is numerically implemented on

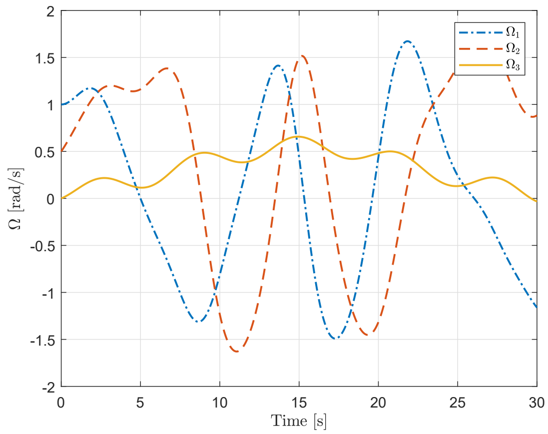

Fig. 1. Relative angular velocity profile between the two simulated vehicles.

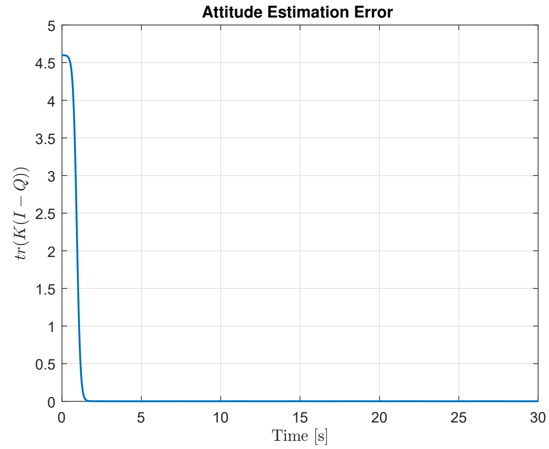

Fig. 2. Attitude error profile of the two simulated vehicles without noise signal $\mathscr{L}$.

MATLAB, using a geometric integration scheme. Unlike commonly used numerical integration methods like Runge-Kutta, geometric integration schemes preserve the geometry of the state space without any projection, parameterization.

Let $h = t_{k+1} - t_k$ be the time step size. The following discretzed equations are used for the attitude estimation scheme, which is a first order geometric integrator.

$$\widehat{R}_{i+1} = \widehat{R}_i \exp(\widehat{\Omega}_i^\times h) \tag{50}$$

$$\widehat{\Omega}_i = \Omega_i^f - \widehat{R}_i^T \omega_i \tag{51}$$

$$\widetilde{\Omega}_i = \Omega_i^f - \widehat{\Omega}_i = \widehat{R}_i^T \omega_i \tag{52}$$

$$\omega_i = \frac{k_p \widehat{R}_i s_\Gamma(\widehat{R}_i)}{\{s_\Gamma(\widehat{R}_i)^T s_\Gamma(\widehat{R}_i)\}^{1-1/p}} \tag{53}$$

With this attitude estimation scheme, two simulations are shown here with the same initial condition, the same time-changing attitude and the same estimator gains: $k_p = 3$ and $p = 1.1$. The initial condition is given as follows:

$$R_0 = \exp(\pi([1;\ 0;\ 0]^T)^\times) \tag{54}$$

$$\Omega_0 = [1\ 0.5\ 0]^T \mathrm{rad}/s \tag{55}$$

The initial estimated states are as follows:

$$\widehat{R}_0 = I \tag{56}$$

$$\widehat{\Omega}_0 = [0\ 0\ 1]^T \mathrm{rad}/s \tag{57}$$

The known position vectors of feature points on the observed vehicle O in frame O are given by:

$$d_1 = [1\ 0\ 0]^T \tag{58}$$

$$d_2 = [0\ 1\ 0]^T \tag{59}$$

$$d_3 = [0\ 0\ 1]^T \tag{60}$$

$$d_4 = [\sqrt{6}/6\ \sqrt{6}/3\ \sqrt{6}/6]^T \tag{61}$$

The true relative angular velocity profile for both simulations is shown in 1.

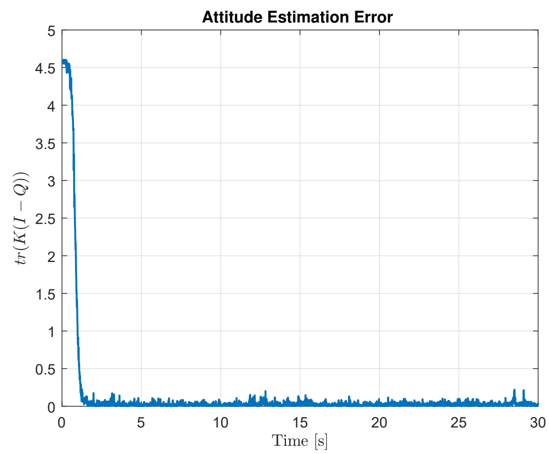

Fig. 3. Attitude error profile of the two simulated vehicles with noise signal $\mathscr{L}$.

The first simulation is for noise-free vector measurements, and the corresponding relative attitude estimation error is shown in Fig. 2. This result verifies that the estimated relative attitude converges to the true relative attitude in the case of perfect (noise-free) measurements of relative positions of feature points on the observed object.

The second simulation comes with additive noise in the measurement signal $\mathscr{L}$, which is given by a normal distributed random signal. Note again that the second-order discrete Butterworth filter is applied here to L^m to reduce the impact of the noise signal $\mathscr{L}$ and obtain the filtered L^f. A similar method has been employed in our prior work [26]. Parameters of the Buttterworth pre-filter are chosen as $\mu = 0.414$ and $\omega_n = 80\pi$.

As can be noticed from both sets of simulation results, the estimated relative attitude converges to a bounded neighborhood of the corresponding true relative attitude, where the size of this neighborhood depends on the level of measurement noise. When there is no measurement noise (ideal situation), the estimated relative attitude converges to the true relative attitude within a finite amount of time. These simulation results through this numerical experiment demonstrate the finite-time stability and robustness properties of the proposed estimation scheme.

5. CONCLUSION

This article proposes a relative attitude estimation scheme for an object observed by sensors onboard a vehicle in proximity. The relative attitude of the observed object is estimated from measurements of feature points on this object over time. For robustness to measurement errors, this estimation scheme is designed to be finite-time stable (FTS) and the measured relative position vectors of feature points on the object are pre-filtered using a discrete Butterworth filter. The stability of this relative attitude observer is theoretically proven by a Lyapunov analysis, and numerically demonstrated through a numerical simulation. Its robustness to measurement noise is also shown through a numerical simulation. Continuing work along this direction seeks to develop FTS relative motion estimation schemes and implement them in hardware experiments involving multiple UAVs operating in close formation.

ACKNOWLEDGEMENT

This research has been supported by the National Science Foundation award CISE 1739748.

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