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Discrete Finite-time Stable Attitude Tracking Control of Unmanned Vehicles on $\mathrm{SO}(3)$

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Abstract— **This paper presents a finite-time stable (FTS) attitude tracking control scheme in discrete time for an unmanned vehicle. The attitude tracking control scheme guarantees discrete-time stability of the feedback system in finite time. This scheme is developed in discrete time as it is more convenient for onboard computer implementation and guarantees stability irrespective of sampling period. Finite-time stability analysis of the discrete-time tracking control is carried out using discrete Lyapunov analysis. This tracking control scheme ensures stable convergence of attitude tracking errors to the desired trajectory in finite time. The advantages of finite-time stabilization in discrete time over finite-time stabilization of a sampled continuous time tracking control system is addressed in this paper through a numerical comparison. This comparison is performed using numerical simulations on continuous and discrete FTS tracking control schemes applied to an unmanned vehicle model.**

I. Introduction

This paper investigates the problem of autonomous attitude trajectory tracking of an unmanned vehicle carrying out three-dimensional (3D) rotation maneuvers. This is an important problem in various applications of unmanned vehicles where remote piloting is difficult or impossible. Autonomous operations of unmanned vehicles can play an important role in these applications, which include security, inspection of civilian infrastructure, agriculture and aquaculture, space and underwater exploration, wildlife tracking, package delivery and remote sensing, all of which can benefit from reliable autonomous operations. Stable and robust autonomous guidance and control is considered a critical part of reliable operations of unmanned vehicles, particularly for operations that require safety and reliability in presence of external disturbances, e.g., those due to wind and weather. Absence of nonlinear stability and robustness in these situations can lead to failure and crash of even remotely piloted vehicles. This work presents a systematic treatment of discrete finite-time stable control for tracking attitude trajectories of unmanned vehicles, to address this problem.

Finite-time stable control has the advantage of providing guaranteed convergence to a desired state (or trajectory) in finite time, besides being more robust to bounded temporary and persistent disturbances than asymptotic stability. Furthermore, persistent disturbances are better rejected by a finite-time stable system in comparison to an asymptotically stable system, because the ultimate bound on the disturbance that can be tolerated is of larger for convergence to a given neighborhood of the desired state or state trajectory [1]. Finite-time stable (FTS) control schemes are particularly effective in applications where there are bounded disturbance inputs due to unmodeled dynamics [2]. Continuous FTS control systems have been explored in prior work, e.g., [3]–[6]. An almost global finite time stabilization of rigid body attitude motion to a desired attitude in finite time is studied in [2], [7]. Same authors designed a finite-time stable control scheme for simple mechanical systems represented in generalized coordinates, as reported in [1]. A continuous-time FTS integrated guidance and feedback tracking control scheme for pose (position and orientation) tracking of rigid bodies was reported in [8]–[10], which ensures finite-time stability of the overall tracking scheme. The continuous equations of motion were discretized in the form of a Lie Group Variational Integrator (LGVI) and the continuous time control scheme was sampled for computer implementation, by applying the discrete Lagrange-d'Alembert principle. Prior related research on LGVI discretization of rigid body dynamics includes [11]–[17].

However, implementing a sampled continuous-time stable tracking control scheme does not ensure discrete-time stability of the resulting feedback system. This was shown for the case of nonlinear observer design for attitude dynamics in some of our prior work [16]–[18]. A discrete-time stable feedback tracking control scheme was developed in [19], in which discrete-time control laws obtained guarantee asymptotic discrete-time stability of pose tracking control of underactuated vehicles on SE(3). Note that, like the continuous-time FTS control schemes in [1]- [4], [7]- [10], the discrete-time FTS control scheme proposed here maintains finite time stable convergence to the desired equilibrium or trajectory, but it does so in discrete time. In addition, this discrete-time FTS control scheme enables onboard computer implementation with any discrete-time sampling frequency. This forms the motivation of this paper: to design a *finite-time stable attitude tracking control scheme in discrete time*. To the best of our knowledge, a finite-time stable attitude tracking control scheme in discrete time as proposed in this paper has not been reported in prior literature. A finite-time stabilization scheme in discrete-time is formulated here for attitude tracking on the tangent bundle of $\mathrm{SO}(3)$, using discrete-time Lyapunov analysis which leads to the discrete time control law. The Lyapunov function designed is sum of a Morse function on $\mathrm{SO}(3)$ and a quadratic function of a vector-valued function of rotational motion tracking errors. This vector-valued function is constructed such that when

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its value is the zero vector, the rotational tracking errors converge to zero in finite time. A discrete-time control law is then obtained, that ensures that this vector converges to the zero vector in finite time, and therefore the attitude trajectory tracking errors converge to zero in a finite-time interval. Thereafter, the stability and performance of this discrete time FTS scheme is numerically compared with that of a continuous FTS scheme, and the results are discussed.

This paper is organized as follows. Section II outlines the general formulation of rigid body attitude dynamics on $\mathrm{SO}(3)$, as well as providing the attitude kinematics and dynamics model of the vehicle and their discretizations. Section III deals with the discrete-time Lyapunov framework and finding discrete-time attitude tracking control law for FTS attitude tracking control. Numerical simulation results based on a Lie group variational integrator and the finite-time stable control laws obtained in discrete time, are presented in IV. This section also presents a comparison of the stability performance between the discrete and continuous FTS schemes, and discusses these results. The concluding section V provides a summary of results presented, and mentions related research directions to be pursued in the near future.

II. Problem Formulation

A. Coordinate Frame Definition

In order to define the attitude of an unmanned vehicle modeled as a rigid body, we consider a coordinate frame $\mathcal{B}$ fixed to its body and another coordinate frame $\mathcal{I}$ that is fixed in space and takes the role of an inertial coordinate frame. Let $R \in \mathrm{SO}(3)$ denote the orientation (attitude) of the body, defined as the rotation matrix from frame $\mathcal{B}$ to frame $\mathcal{I}$.

B. Attitude Trajectory Generation

The desired attitude trajectory for an unmanned aerial vehicle (UAV) as a rigid body is assumed to be known a prior. Such a trajectory is usually generated and available from the control law for following a position trajectory, by using the known dynamics model and body-fixed actuator orientations. The position trajectory control law gives a desired thrust direction, which is then used to generate a desired attitude trajectory, as described in [8]. Let J denotes inertia of a rigid body. The rotational dynamics of the rigid body is given by:

$$\dot{R} = R\,\Omega^{\times}, \tag{1}$$

$$J\dot{\Omega} = J\Omega \times \Omega + \tau. \tag{2}$$

where Ω is the rotational velocity of the underactuated vehicle, and τ is the input torque. The cross map:$(\cdot)^{\times} : \mathbb{R}^3 \to \mathrm{SO}(3)$ is given by [13]:

$$x^{\times} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{\times} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

C. Tracking errors kinematics and dynamics in discrete time

The attitude tracking error is defined in [13] by:

$$Q = (R^d)^{\mathrm{T}} R. \tag{3}$$

Taking the time derivative results in:

$$\dot{Q} = Q\omega^{\times}, \tag{4}$$

where $\omega = \Omega - Q^{\mathrm{T}}\Omega^d$ is the angular velocity tracking error. Consider tracking a desired attitude trajectory $R^d(t)$ as described in [8], with corresponding angular velocity $\Omega(t)$, in a time interval $[t_0,\ t_f] \in \mathbb{R}^+$ separated into N equal-length sub-intervals $[t_k,\ t_{k+1}]$ for $k = 0, 1, ..., N$, with $t_N = t_f$ and $t_{k+1} - t_k = \Delta t$ where Δt is the time step size. Therefore, one can express the discretized rotational kinematics and the real attitude dynamics of an underactuated vehicle in the form of LGVI presented in [12], [19] as

$$\begin{cases} R_{k+1} = R_k\ F_k, \\ J\,\Omega_{k+1} = F_k^{\mathrm{T}}\ J\Omega_k + u_k, \end{cases} \tag{5}$$

where $u_k = \Delta t \tau_k$ is the control input, and $F_k \approx \exp(\Delta T \Omega_k^{\times}) \in \mathrm{SO}(3)$ guarantees that R_k evolves on $\mathrm{SO}(3)$. Using the discretized rotational kinematics equation given in (5) and attitude tracking error of (3) in discrete time, one can write

$$Q_{k+1} = (R^d_{k+1})^{\mathrm{T}} R_{k+1} = (R^d_{k+1})^{\mathrm{T}} R_k\ F_k, \tag{6}$$

where $R^d_{k+1} = R^d_k\ F^d_k$. Then,

$$\begin{aligned} Q_{k+1} &= (F^d_k)^{\mathrm{T}} (R^d_k)^{\mathrm{T}} R_k\ F_k \\ &= (F^d_k)^{\mathrm{T}} Q_k\ F_k. \end{aligned} \tag{7}$$

Using the definitions for F_k and F^d_k given earlier into the above expression and carrying out some algebraic simplifications, one obtains

$$\begin{aligned} Q_{k+1} &\approx Q_k \big[I + \Delta t\ (\Omega_k - Q_k^{\mathrm{T}} \Omega^d_k)^{\times}\big] \\ &= Q_k (I + \varpi_k{}^{\times}), \end{aligned} \tag{8}$$

where $\varpi_k = \Delta t\ \omega_k$, and ω_k is the angular velocity tracking error at time instant t_k.

The following section provides a finite-time stable feedback control law in discrete time to stabilize the attitude error dynamics (5).

III. Discrete Finite-time Stable Attitude Tracking Control on TSO(3)

In this section, a finite-time stable attitude tracking control scheme in discrete time is provided. The following result is a basic result on finite-time stability and convergence for discrete-time systems, and it has been reported first in [20], [21].

Lemma 1: Consider a discrete-time system with inputs $u_k \in \mathbb{R}^m$ and outputs $y_k \in \mathbb{R}^l$. Define a corresponding positive definite (Lyapunov) function $V : \mathbb{R}^l \to \mathbb{R}$ and let $V_k = V(y_k)$. Let α be a constant in the open interval $]0, 1[$, $\eta \in \mathbb{R}^+$ a constant, and let $\gamma_k := \gamma(V_k)$ where $\gamma : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is a positive definite function of V_k. Let γ_k satisfy the condition:

$$\gamma_k \geq \eta \ \text{for all}\ \ V_k \geq \varepsilon, \tag{9}$$

for some (possibly small) constant $\varepsilon \in \mathbb{R}^+$. Then, if V_k satisfies the relation

$$V_{k+1} - V_k \leq -\gamma_k V_k^{\alpha}, \tag{10}$$

the discrete system is (Lyapunov) stable at $y = 0$ and y_k converges to $y = 0$ for $k > N$, where $N \in \mathbb{W}$ is finite.

Proof: The proof of this lemma is given in [20], [21], and omitted here for brevity. ■

The following two lemmas are also used to prove the main result.

Lemma 2: Let x and y be non-negative real numbers and let $p \in (1, 2)$. Then

$$x^{(1/p)} + y^{(1/p)} \geq (x + y)^{(1/p)}. \tag{11}$$

Moreover the above inequality is a strict inequality if both x and y are non-zero.

Lemma 3: Let $K = \operatorname{diag}(k_1, k_2, k_3)$, where $k_1 > k_2 > k_3 \geq 1$. Define

$$s_K(Q) = \sum_{i=1}^{3} k_i (Q^T e_i) \times e_i, \tag{12}$$

such that $\frac{\mathrm{d}}{\mathrm{d}t}\langle K, I - Q\rangle = \omega^T s_K(Q)$. Here $\langle A, B\rangle = \operatorname{tr}(A^T B)$, which makes $\langle K, I - Q\rangle$ a Morse function defined on $\mathrm{SO}(3)$. Let $\mathcal{S} \subset \mathrm{SO}(3)$ be a closed subset containing the identity in its interior, defined by

$$\mathcal{S} = \{Q \in \mathrm{SO}(3) : Q_{ii} \geq 0 \text{ and } Q_{ij}Q_{ji} \leq 0 \\ \forall i,\ j \in \{1, 2, 3\},\ i \neq j\}. \tag{13}$$

Then for $Q \in \mathcal{S}$, we have

$$s_K(Q)^T s_K(Q) \geq \operatorname{tr}(K - KQ). \tag{14}$$

Proof: The proof of this lemma is given in [7], and omitted here for brevity. ■

The discrete finite-time attitude tracking control scheme and its proof of stability and domain of convergence are given as follows.

Theorem 1: Consider the discretized rotational error kinematics and the real dynamics of an underactuated vehicle given in (5), with $s_K(Q_k)$ as defined in (12). Define

$$z_K(Q_k) = \frac{s_K(Q_k)}{\left(s_K^T(Q_k) s_K(Q_k)\right)^{1-1/p}} \tag{15}$$

where p is as defined in Lemma 2, and let k_l be a constant in the interval $(0, 1]$. Define

$$\psi_k(\omega_k, Q_k) = \omega_k + k_l z_K(Q_k). \tag{16}$$

Then, the discrete-time control law given by

$$u_k = J\Bigg(\left[\frac{(\psi_k^T J\psi_k)^{1-1/p} - \Gamma}{(\psi_k^T J\psi_k)^{1-1/p} + \Gamma}\right]\left(\omega_k + k_l z_K(Q_k)\right) \\ - k_l z_K(Q_{k+1}) + Q_{k+1}^T\, \Omega_{k+1}^d\Bigg) - F_k^T J\Omega_k, \tag{17}$$

stabilizes the rotational error dynamics

$$\omega_{k+1} = \left[\frac{(\psi_k^T J\psi_k)^{1-1/p} - \Gamma}{(\psi_k^T J\psi_k)^{1-1/p} + \Gamma}\right]\psi_k(\omega_k, Q_k) - k_l z_K(Q_{k+1}) \tag{18}$$

in finite time.

Proof: Consider $\psi_k(\omega_k, Q_k) = 0$ which leads to $\omega_k = -k_l z_K(Q_k)$, and discretized error kinematics given in (8) and define the discrete-time Morse-Lyapunov function $V_k = k_p \langle I - Q_k, K\rangle$ on $\mathrm{SO}(3)$ where $k_p > 1$. Then time difference of this discrete-time Morse-Lyapunov function along the attitude kinematics is given by

$$\begin{aligned} V_{k+1} - V_k &= k_p \langle Q_k - Q_{k+1}, K\rangle \\ &= k_p\langle -Q_k\, \varpi_k^{\times}, K\rangle \\ &= \frac{1}{2} k_p \langle \varpi_k^{\times}, KQ_k - Q_k^{\mathrm{T}}\, K\rangle \\ &= k_p\, \varpi_k^{\mathrm{T}}\, S_K(Q_k). \end{aligned} \tag{19}$$

Substituting $\varpi_k = \Delta t\, \omega_k$ in (19), one finds

$$\begin{aligned} k_p\, \varpi_k^{\mathrm{T}}\, S_K(Q_k) &= -\Delta t\, k_p\, k_l\, z_K(Q_k)^{\mathrm{T}} s_K(Q_k) \\ &= -k_p\, k_l\, \Delta t\, \left(s_K(Q_k)^{\mathrm{T}} s_K(Q_k)\right)^{1/p} \\ &\leq\ -k_p\, k_l\, \Delta t\, \left(\langle I - Q_k, K\rangle\right)^{1/p} \\ &\leq\ -k_l\, \Delta t\, \left(k_p \langle I - Q_k, K\rangle\right)^{1/p}. \end{aligned} \tag{20}$$

where we employed inequality (14) in lemma 3. Therefore, when $\psi_k = 0$, one can conclude that $\langle I - Q_k, K\rangle \to 0$ in finite time for all initial Q_k in the subset $\mathcal{S} \subset \mathrm{SO}(3)$ defined in Lemma 2, which yields $Q_k \to I$ in finite time once $Q_k \in \mathcal{S}$. Moreover, as ΔV_k is negative definite when $\psi_k = 0$, it keeps decreasing in time and therefore Q_k will reach $\mathcal{S}$ in finite time. Therefore, $Q_k \to I$ in finite time.

The control law is then designed to ensure that $\psi_k \to 0$ in finite time. Define the Lyapunov function

$$\mathcal{V}_k(\omega_k, Q_k) = \frac{1}{2}\psi_k^{\mathrm{T}} J\psi_k + k_p \langle I - Q_k, K\rangle. \tag{21}$$

The time difference of this discrete-time Lyapunov function can be evaluated as follows:

$$\Delta\mathcal{V}_k = \mathcal{V}_{k+1} - \mathcal{V}_k = \frac{1}{2}(\psi_{k+1} + \psi_k)^{\mathrm{T}} J(\psi_{k+1} - \psi_k) \\ + k_p \langle Q_k - Q_{k+1}, K\rangle. \tag{22}$$

One can consider

$$\psi_{k+1} = \Psi(\omega_k, Q_k)\psi_k, \tag{23}$$

where

$$\Psi(\omega_k, Q_k) = \frac{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p} - \Gamma}{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p} + \Gamma}, \tag{24}$$

and let $\Gamma > 0$.

Substituting (24) in (23) gives

$$(\psi_{k+1} - \psi_k) = -\frac{\Gamma}{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}}(\psi_{k+1} + \psi_k). \tag{25}$$

Therefore, one can rewrite (22) as

$$\Delta\mathcal{V}_k = -\frac{\Gamma}{2}\frac{(\psi_{k+1}+\psi_k)^{\mathrm{T}} J(\psi_{k+1}+\psi_k)}{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}} + k_p\, \omega_k^{\mathrm{T}} S_K(Q_k). \tag{26}$$

Note that the first term on the right-hand side of expression (26) is zero if and only if $\psi_{k+1} = -\psi_k$, which is possible if and only if $\Psi = -1$ according to (23). From (24), one can see that $\Psi = -1$ if and only if $\psi_k = 0$. Therefore, from (23) and (24) we conclude that

$$-\frac{\Gamma}{2}\frac{(\psi_{k+1}+\psi_k)^{\mathrm{T}} J(\psi_{k+1}+\psi_k)}{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}} = 0 \;\Leftrightarrow\; \psi_k = 0.$$

Therefore, the first term on the right side of expression (26) can be simplified as follows:

$$-\frac{\Gamma}{2}\frac{(\psi_{k+1}+\psi_k)^{\mathrm{T}} J(\psi_{k+1}+\psi_k)}{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}} = -\rho(\psi_k)\,(\psi_k^{\mathrm{T}} J\psi_k)^{1/p}, \tag{27}$$

where

$$\rho(\psi_k) = \rho_k = 4\Gamma\frac{(0.5)^{1-1/p}(\psi_k^{\mathrm{T}} J\psi_k)^{2-2/p}}{\left((\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}+\Gamma\right)^2}. \tag{28}$$

From equations (27) and (28), one can see that the first term on the right side of expression (26) is monotonously decreasing if

$$0 < \rho_k < \frac{4\Gamma}{2^{1-1/p}} \quad \text{for} \quad 0 < \psi_k^{\mathrm{T}} J\psi_k < \infty,$$

Therefore using (20) and (27), expression (26) is evaluated as follows:

$$\begin{aligned}\Delta\mathcal{V}_k &= -\rho_k(\psi_k^{\mathrm{T}} J\psi_k)^{1/p} - k_p k_l\big(s_K(Q_k)^{\mathrm{T}} s_K(Q_k)\big)^{1/p}\\ &\le -(\psi_k^{\mathrm{T}} J\psi_k)^{1/p} - k_l\,\Delta t\,\big(k_p\,\langle I-Q_k, K\rangle\big)^{1/p}\\ &\le -k_l\,\Delta t\,\Big((\psi_k^{\mathrm{T}} J\psi_k)^{1/p} + \big(k_p\,\langle I-Q_k, K\rangle\big)^{1/p}\Big).\end{aligned} \tag{29}$$

for $(Q_k, \omega_k) \in \mathcal{S}\times\mathbb{R}^3$. Finally, using inequality (11) in Lemma 2, one obtains

$$\begin{aligned}\Delta\mathcal{V}_k &\le -k_l\,\Delta t\,\Big(\psi_k^{\mathrm{T}} J\psi_k + k_p\,\langle I-Q_k, K\rangle\Big)^{1/p}\\ &\le -k_l\,\Delta t\,\mathcal{V}_k^{1/p},\end{aligned} \tag{30}$$

where $k_p > 0$, and $0 < k_l \le 1$. Therefore, all initial states of the feedback system that start in the domain of attraction of the equilibrium $(I, 0)$ with finite value of the Lyapunov function $\mathcal{V}$, converge to $(I, 0)$ in finite time.

Now, by substituting $\psi_k(\omega_k, Q_k)$ given in (16) into (23), one can obtain

$$\omega_{k+1} = \left[\frac{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}-\Gamma}{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}+\Gamma}\right]\psi_k(\omega_k, Q_k) - k_l z_K(Q_{k+1}). \tag{31}$$

From the discretized dynamics equation of rotational motion obtained in the form of LGVI given in (5) where

$$\Omega_{k+1} = \omega_{k+1} + Q_{k+1}^{\mathrm{T}}\,\Omega_{k+1}^d, \tag{32}$$

one can find the discrete-time control law u_k that guarantees the stability of the attitude tracking control in a finite time, as follows:

$$\begin{aligned}u_k = J\Bigg(&\left[\frac{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}-\Gamma}{(\psi_k^{\mathrm{T}} J\psi_k)^{1-1/p}+\Gamma}\right](\omega_k + k_l z_K(Q_k))\\ &- k_l z_K(Q_{k+1}) + Q_{k+1}^{\mathrm{T}}\,\Omega_{k+1}^d\Bigg) - F_k^{\mathrm{T}} J\Omega_k.\end{aligned} \tag{33}$$

■

The following section presents numerical comparison results obtained by implementing the proposed FTS scheme in discrete time, and a sampled finite-time continuous scheme given in [10].

IV. Simulation Results

This section presents numerical simulation results for the FTS attitude tracking control scheme in discrete time. Also, the performance of the proposed FTS tracking control scheme in discrete time is compared to a sampled continuous time FTS tracking scheme presented in [10] . The numerical simulation results are provided for an UAV quadcopter with a mass $m = 4$ kg, for different time periods of $T = 5,\ 25,$ and 50s, with different time step sizes of $\Delta t = 0.01,\ 0.05,$ and 0.1s and the same total number of time steps, using discrete-time FTS control law obtained in (33), and the sampled continuous-time control law given as equation (29) in [10]. The desired attitude (R^d) that is to be tracked by the proposed attitude tracking control scheme, is generated using the desired control force vector given by an outer loop position tracking scheme [8]. A helical desired trajectory with the following initial conditions is used:

$$\begin{aligned}b_k^d = b^d(t_k) &= \begin{bmatrix}0.4\sin\pi t_k & 0.6\cos\pi t_k & 0.4\,t_k\end{bmatrix}^{\mathrm{T}},\\ R_0 = I,\ \Omega_0 &= \begin{bmatrix}0 & 0 & 0\end{bmatrix}^{\mathrm{T}},\ \Omega_0^d = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^{\mathrm{T}}.\end{aligned}$$

The gains are selected and tuned after trial and error for FTS discrete-time attitude tracking scheme as follows:

$$k_l = 0.01,\ \Gamma = 0.5,$$

and for FTS sampled continuous scheme as follows:

$$L_\Omega = 3.5\,\mathbf{I}_{3\times 3},\ k_p = 4.5,\ \kappa = 0.04,$$

which provide desirable and similar transient response characteristics of both tracking control schemes when $\Delta t = 0.01$. The time trajectory of the UAV tracking the desired trajectory is shown in Fig. 1 and it shows that the trajectory converges to the desired values in finite time. and remains stable for all time $t > 0$.

The results of the numerical simulation for attitude and angular velocity tracking response of the discrete-time control law obtained in (33) for $\Delta t = 0.01$ and $t_f = 5s$ is shown in Fig.2. These subplots show that the attitude and angular velocity tracking errors converge to zero in finite time, and therefore the discrete-time control scheme proposed here is able to track the desired trajectory in finite time. The attitude error function Φ is parameterized as principle rotation angle, in terms of Q as given by

$$\Phi = \cos^{-1}\left(\frac{1}{2}\big(\mathrm{tr}(Q) - 1\big)\right) \tag{34}$$

The attitude tracking error Φ is shown that converges in finite time which indicates that R tracks the desired trajectory R^d as shown in Fig. 2c. The time plots of the control input u_k in Fig.2d shows that the control effort is within reasonable bounds and practically achievable for multi-rotor UAVs. Other simulation results are presented in Fig. 3 to Fig. 5 in order to compare the performance of the discrete-time FTS tracking scheme with a sampled continuous FTS tracking scheme for *different values of time step sizes*.

Comparing the results of these two schemes as shown in the plots using a method similar to the one presented in our previous paper [20], one can find out that the control law obtained by sampling the FTS continuous control input does not guarantee the stability of the attitude tracking when the time step size changes. The results of this comparison are given in Table I, in which $\Delta\mathcal{V}_{\max}$ is denoted as the maximum positive value of the time difference $\mathcal{V}_{k+1} - \mathcal{V}_k$ as:

$$\Delta\mathcal{V}_{\max} = \max\big[(\mathcal{V}_{k+1} - \mathcal{V}_k) > 0\big]. \tag{35}$$

This parameter is to confirm whether the Lyapunov function $\mathcal{V}_k$ increases in value at certain time instants, and whether that increase is significant or is just an artifact of machine (float) precision. The value of $\mathcal{V}_{k+1} - \mathcal{V}_k$ is expected to be negative for a finite-time stable system until it converges to zero in finite time, which ensures stability of the system in finite time. On the contrary, a significant increase in the value of $\Delta\mathcal{V}_{\max}$ occurs for the sampled continuous FTS tracking scheme as time step size increases, whereas $\Delta\mathcal{V}_{\max}$ has a negligible value (to machine precision) when the discrete-time FTS tracking control scheme is implemented.

V. Conclusion

A discrete finite-time stable attitude tracking control scheme for unmanned vehicles is presented here. This scheme is based on using a Lyapunov framework for finite-time stabilization of an attitude tracking control that results in discrete-time error dynamics in terms of attitude motion tracking errors. A two-step method is presented to construct a Lyapunov function that is sum of a Morse-Lyapunov function and a quadratic vector-valued function in terms of rotational motion tracking errors. This is followed by a discrete-time control torque vector that guarantees that the Lyapunov function converges to zero in finite time, and therefore the attitude states converge to the desired trajectory in a finite time interval. Numerical results show that this discrete-time FTS attitude tracking control scheme is more reliable for onboard computer implementation when we need to work with a variety of input data frequencies, comparing with a sampled continuous FTS scheme. Future work will look at a discrete-time FTS pose tracking control scheme for underactuated vehicles on $\mathrm{SE}(3)$, and comparison of this discrete-time stable tracking control scheme with other sampled continuous time tracking control schemes.

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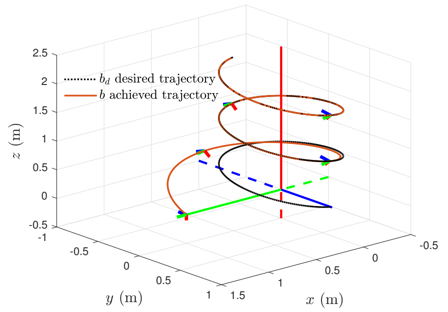
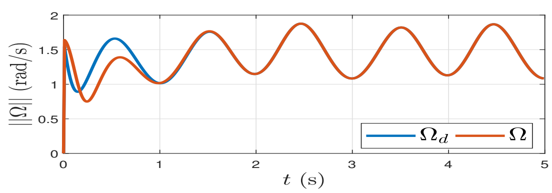


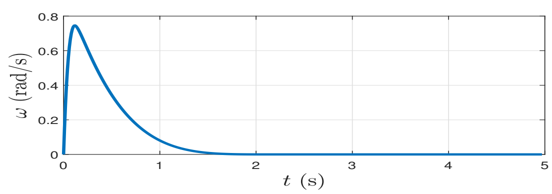
Fig. 1: Time trajectory of UAV

Tracking Control Scheme	$\Delta t(s)$	$t_f(s)$	$\Delta\mathcal{V}_{max}$
Discrete-time FTS	0.01	5	1.0991×10^{-16}
	0.05	25	4.0776×10^{-25}
	0.1	50	4.1364×10^{-25}
Sampled Continuous FTS	0.01	5	2.3153×10^{-5}
	0.05	25	0.3166
	0.1	50	1.7888

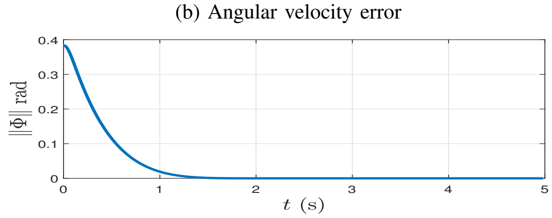
TABLE I: Stability performance of discrete-time FTS vs. sampled continuous-time FTS tracking control scheme on $\mathrm{SO}(3)$.



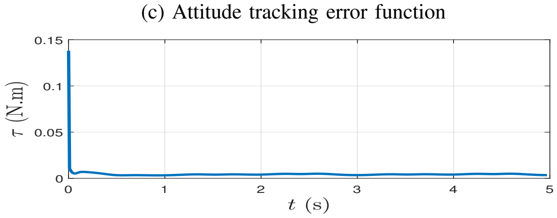
(a) Actual and desired angular velocities



(b) Angular velocity error

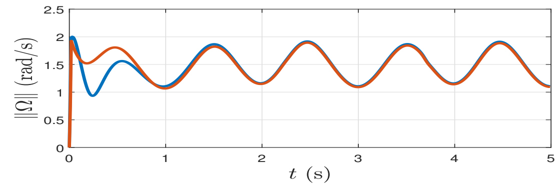


(c) Attitude tracking error function

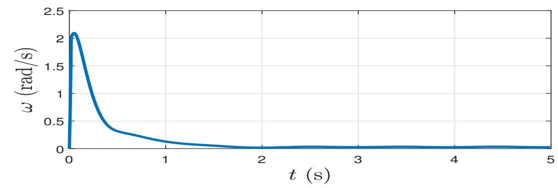


(d) Torque control

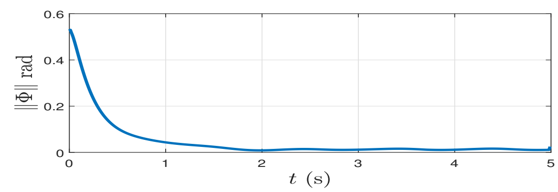
Fig. 2: Rotational motion errors and control law for discrete-time FTS tracking control scheme for $\Delta t = 0.01$ and $t_f = 5s$.



(a) Actual and desired angular velocities

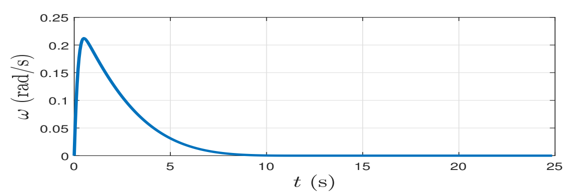


(b) Angular velocity error

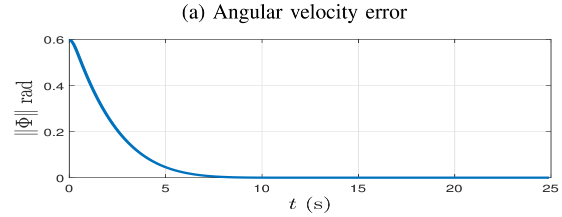


(c) Attitude tracking error function

Fig. 3: Rotational motion errors for sampled FTS continuous tracking control scheme for $\Delta t = 0.01$ and $t_f = 5s$.

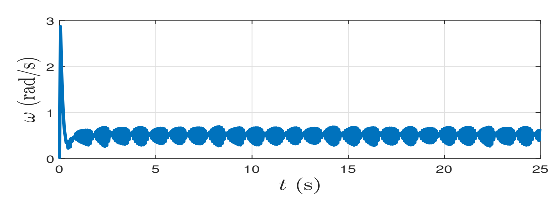


(a) Angular velocity error

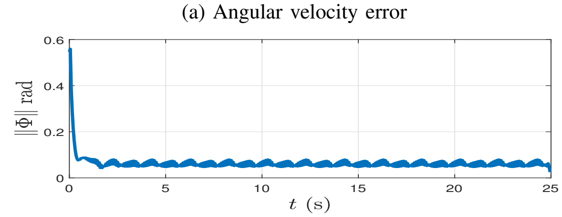


(b) Attitude tracking error function

Fig. 4: Rotational motion errors for discrete-time FTS tracking control scheme for $\Delta t = 0.05$ and $t_f = 25s$.



(a) Angular velocity error



(b) Attitude tracking error function

Fig. 5: Rotational motion errors for sampled FTS continuous tracking control scheme for $\Delta t = 0.05$ and $t_f = 25s$.