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Finite-time Attitude Consensus Control of a Multi-Agent Rigid Body System

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***Abstract*— In this paper, finite-time attitude consensus control laws for multi-agent rigid body systems are presented using rotation matrices. The control objective is to stabilize the relative configurations in a finite convergence time. First, the control design is done on the kinematic level where the angular velocities are the control signals. Next, the design is conducted on the dynamic level in the framework of the tangent bundle $\mathrm{TSO}(3)$ associated with $\mathrm{SO}(3)$, where the torques implement the feedback control of relative attitudes and angular velocities. The Lyapunov-based almost global finite-time stability of the consensus subspace is demonstrated for both cases. Finally, numerical simulations are provided to verify the effectiveness of the proposed consensus control algorithms.**

I. Introduction

The consensus control of a network of dynamic agents has become an interesting topic in the past couple of decades, due to a broad application of multi-agent systems (MAS) [1]. Consensus is typically obtained asymptotically, however, various finite-time stabilizing control algorithms have been proposed in the recent years. The finite-time design is generally more complicated than an asymptotically stabilizing control problem [2]. The past few years have witnessed growing interest in finite time control of multi-agent systems.

Attitude synchronization is defined as the control of a group of rigid bodies such that their orientations are synchronized [3], [4]. In [5], almost globally convergent controllers for multiple flexible spacecraft are designed based on the rotation matrix. A distributed finite-time attitude control law is proposed in [6] for a group of spacecraft with a leader-follower architecture exploiting modified Rodriguez parameters (MRPs). In [7], a finite-time consensus protocol for strongly convex geodesic balls is proposed to solve the attitude consensus problem with switching and directed communication topology. In [8], the finite time formation control problems of second-order multi-agent systems with fixed and switching topologies are addressed. In [9], finite-time attitude synchronization protocols are proposed based on the axis-angle representations of the rotations, using discontinuous control laws.

In this paper, decentralized finite-time attitude consensus protocols are introduced on the kinematic and dynamic levels to achieve attitude synchronization in a rigid body multi-agent system. Almost global asymptotically stable feedback control laws for attitude control of a single rigid body utilize Morse-Lyapunov (M-L) functions, which have isolated non-degenerate critical points on the configuration space represented by the Lie group $\mathrm{SO}(3)$ [10]. Thus, the consensus control laws proposed here are an extension of the M-L technique to the problem of rigid body attitude consensus under a fixed and undirected communication topology among N agents on $\mathrm{SO}(3)^N$ and $\mathrm{TSO}(3)^N$ where $\mathrm{TSO}(3) = \mathrm{SO}(3) \times \mathbb{R}^3$ is the tangent bundle associated with $\mathrm{SO}(3)$. While various attitude parameters such as quaternions and MRPs are used in much of the literature for designing attitude consensus protocols, in this paper, the rotation matrices are exploited to describe rigid body attitude since using rotation matrices for the attitude representation results in a nonsingular unique representation of rigid body motion. Moreover, to achieve finite time attitude consensus, the signum function which makes the control action discontinuous is avoided in this work. Hence, the resulting control is a continuous distributed consensus protocol with feedback of the relative attitudes and angular velocities of the rigid bodies so that the finite-time consensus is achieved. The multi-agent rigid-body systems in [11] and [12] achieve consensus asymptotically while the techniques presented here are designed to control a multi-agent rigid-body system to consensus in a finite time. This work proposes an extension of the finite-time control scheme of [10] designed for controlling a single rigid body, to the case of multi-agent rigid-body system.

Since the state space associated with the rigid body attitude is the non-Euclidean tangent bundle $\mathrm{TSO}(3)$, general purpose numerical integrators like Runge-Kutta schemes, which rely on use of local coordinates to describe the motion, are not applicable since such schemes do not maintain the form of the manifold [13]. Therefore, a Lie group variational integrator is utilized to obtain numerical simulation results on the dynamic level which maintains the Lie group structure of the configuration space during the numerical discretization. This variational integrator discretizes the Lagrange-d'Alembert principle for a system before the equations of motion are obtained via variational principles with non-conservative forcing. This is constructed to conserve the Lie group geometry on the configuration space.

II. Preliminaries

A. Communication Graph

Suppose there are N agents in a network defined by undirected communication links between the agents. To model the communication topology among these agents, we use undirected graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_N\}$

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is the node set and $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2 \ldots, \mathcal{E}_m\} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, in which each edge $\mathcal{E}_i$, $i = 1, \ldots, m$, represents an undirected communication link between a pair of agents. Let $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ and $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{N\times N}$ be the symmetric adjacency and Laplacian matrices associated with $\mathcal{G}$, respectively. The elements of the Laplacian matrix are defined by

$$\ell_{ij} = \begin{cases} \sum_{k=1, i\neq k}^{N} a_{ik}, & i = j \\ -a_{ij}, & i \neq j \end{cases} \tag{1}$$

B. Dynamics on SO(3)

The orientation of a rigid body from the body-fixed coordinate frame $\mathcal{B}$ to the inertial frame $\mathcal{N}$ is expressed by the rotation matrix R, which is an element of the real special orthogonal group $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times3} | \ R^T R = I_3$ and $det(R) = 1\}$, where SO(3) is the 3-dimensional Lie group matrix representation of rigid body attitude and $I_3 \in \mathbb{R}^{3\times3}$ is the identity matrix.

The rotational kinematics of the rigid body are

$$\dot{R} = R\omega^{\times} \tag{2}$$

where $\omega \in \mathbb{R}^3$ denotes the angular velocity expressed in the body frame of the rigid body. For $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$,

$$x^{\times} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{so}(3) \tag{3}$$

where the space of 3×3 real skew-symmetric matrices is denoted by $\mathfrak{so}(3)$, which is the Lie algebra of the Lie group SO(3). Also, the $vec()$ operation undoes the $()^{\times}$ operation as

$$C = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \Rightarrow vec(C) = [c_1, c_2, c_3]^T \tag{4}$$

The attitude of the rigid body can also be expressed in terms of principal rotation vector $\Theta \subset \mathbb{R}^3$ using the logarithm map

$$\Theta^{\times} = \log(R) \in \mathfrak{so}(3) \tag{5}$$

in which Θ is the exponential coordinates for attitude (principal rotation vector). The set of principal rotation vectors is the closed unit ball of radius π in $\mathbb{R}^3$. Every principal rotation vector inside the ball uniquely corresponds to a certain physical attitude, while there are two principal rotation vectors on the surface of the ball corresponding to a given attitude (rotation by π about some axis).

The rotational kinetics of the rigid body can be written as

$$J\dot{\omega} = -\omega^{\times} J\omega + \tau \tag{6}$$

where $J \in D \subset \mathbb{R}^{3\times3}$ is the inertia matrix and $\tau \in \mathbb{R}^3$ denotes the control torque input.

The following two lemmas in the context of finite time control design are used throughout this paper.

Lemma 1 [14]: Consider a system described with $\dot{x} = f(x)$ with $f(0) = 0$ and there exists a continuous differential positive-definite function $V(x)$ such that

$$\dot{V}(x) + \beta V^{\eta}(x) \leq 0 \tag{7}$$

where $\beta > 0$ and $\eta \in (0, 1)$ are real numbers. Then, the origin of the system is locally finite-time stable, with the convergence time depending on the initial state $x(0)$, satisfying

$$T(x_0) \leq \frac{V^{1-\eta}(x_0)}{\beta(1-\eta)} \tag{8}$$

Lemma 2 [10]: Let a and b be non-negative real numbers and let $p \in (1, 2)$. Then

$$a^{(1/p)} + b^{(1/p)} \geq (a + b)^{(1/p)} \tag{9}$$

Note that this inequality is a strict inequality if both a and b are non-zero.

III. Problem Statement

In this section, the problem of finite-time attitude consensus control of a networked rigid body system is investigated in two cases:

Case 1: Finite-time attitude kinematic consensus steering control

Case 2: Finite-time consensus control of rigid-body attitude on $\mathrm{TSO}(3)^N$

In the subsequent discussion, the control laws and the corresponding convergence analysis are provided for each of these cases.

A. Case 1: Finite-time attitude kinematic consensus steering control

In this part, a finite-time attitude kinematic consensus steering control protocol is designed where the angular velocities are the control signals [11]. An inner servo loop controller in each agent may be designed to track the prescribed angular velocities by implementing the required control torques. The control goal for N rigid bodies is to achieve and maintain a synchronized attitude. To this end, the leaderless consensus steering control protocol for the ith agent is obtained using the relative attitude with respect to those of its neighbors.

Assume there are N rigid bodies and the rotational kinematics of the ith agent is written as

$$\dot{R}_i = R_i \omega_i^{\times} \tag{10}$$

Suppose S_i is defined in terms of the sum over neighbors of agent i as

$$S_i = \sum_{j\sim i} vec(R_i^T R_j A_{ij} - A_{ij} R_j^T R_i) \tag{11}$$

where R_i and R_j represent the rotation matrices of the ith and jth agents, respectively. Moreover, A_{ij} is a symmetric positive definite gain matrix.

Theorem 1: Assuming that rigid body $i, i = 1, \ldots, N$ has access to the attitudes of its neighbors, then the closed-loop system reaches consensus in finite time with the attitude kinematic consensus protocol

$$\omega_i = \frac{S_i}{\left(S_i^T S_i\right)^{(1-\frac{1}{p_1})}} \qquad i = 1, \cdots, N \tag{12}$$

where $p_1 \in (1, 2)$ is a positive real number. In addition, it does so with almost global finite-time stability.

Proof: To ensure the stability of the closed-loop system, the Lyapunov analysis is used to demonstrate the stability of the closed-loop system of Eqs. (10-12). Consider the Lyapunov function as [12]

$$V_1 = \sum_{i=1}^{N} \sum_{j\sim i} \langle A_{ij}, (I_3 - R_j^T R_i)\rangle > 0 \tag{13}$$
$$\forall R_i \backslash \{R_j^T R_i = I_3\}$$

where $\langle A, B\rangle = \text{trace}(A^T B)$ denotes the trace inner product. This Lyapunov function is an extension of the Morse-Lyapunov function $V(R) = \langle A, (I_3 - R)\rangle$ used in [15] for control of a single rigid body. It can be shown that if A is diagonal with distinct diagonal elements, then $V(R)$ has the set of non-degenerate critical points

$$E_c = \{I_3, \text{diag}(-1, 1, -1), \text{diag}(1, -1, -1), \text{diag}(-1, -1, 1)\} \tag{14}$$

on SO(3). As discussed in [15], these four points consist of a minimum, a maximum, and two saddles. The critical points of the function $\langle A_{ij}, (I_3 - R_j^T R_i)\rangle$ are obtained when the relative attitude $R_{ij} = R_j^T R_i$ satisfies $vec(A_{ij} R_j^T R_i - R_i^T R_j A_{ij}) = 0$ for which $R_{ij} = I_3$ is one solution. If A_{ij} is diagonal with distinct diagonal elements, then the critical points are when R_{ij} equals one of the four points in E_c.

For an arbitrary real square matrix B and vector ω

$$\text{trace}(B\omega^\times) = \omega^T vec(B^T - B) \tag{15}$$

Differentiating Eq. (13) with respect to time, substituting Eq. (10) and using Eq. (15) yields

$$\begin{aligned}
\dot{V}_1 &= -\sum_{i=1}^{N} \sum_{j\sim i} \text{trace}(A_{ij}(\dot{R}_j^T R_i + R_j^T \dot{R}_i)) \\
&= -\sum_{i=1}^{N} \sum_{j\sim i} \text{trace}(-A_{ij}\omega_j^\times R_j^T R_i + A_{ij} R_j^T R_i \omega_i^\times) \\
&= -\sum_{i=1}^{N} \sum_{j\sim i} \omega_i{}^T vec(R_i^T R_j A_{ij} - A_{ij} R_j^T R_i) \\
&\quad -\sum_{i=1}^{N} \sum_{j\sim i} \omega_j^T vec(R_j^T R_i A_{ij} - A_{ij} R_i^T R_j)
\end{aligned} \tag{16}$$

It is seen that V_1 and $\dot{V}_1$ double count each connected (i, j) pair and are therefore symmetric with respect to the relative attitude $R_{ij} = R_j^T R_i$. Thus, for an undirected graph topology, $\dot{V}_1$ remains the same when the indices are switched in the first term. Therefore, the result of Eq. (16) can be simplified as

$$\begin{aligned}
\dot{V}_1 &= -2\sum_{i=1}^{N} \omega_i{}^T \sum_{j\sim i} vec(R_i^T R_j A_{ij} - A_{ij} R_j^T R_i) \\
&= -2\sum_{i=1}^{N} \omega_i{}^T S_i
\end{aligned} \tag{17}$$

Following the strategy of *Lemma 2* in [10], we have

$$2\sum_{i=1}^{N} S_i^T S_i \geq \sum_{i=1}^{N} \sum_{j\sim i} \langle A_{ij}, (I_3 - R_j^T R_i)\rangle = V_1 \tag{18}$$

in some neighborhood about the identity when $R_{ij} \approx I_3$. The coefficient 2 appears because V_1 double counts each connected (i, j) pair and is therefore symmetric with respect to the relative attitude R_{ij}. By inserting the proposed consensus control law of Eq. (12) into the final result of Eq. (17) while using *Lemma 2* (Eq. (9)) and Eq. (18), one has

$$\begin{aligned}
\dot{V}_1 &= -2\sum_{i=1}^{N} \left(S_i^T S_i\right)^{\frac{1}{p_1}} \leq -\left(2\sum_{i=1}^{N} S_i^T S_i\right)^{\frac{1}{p_1}} \\
&\leq -\left(\sum_{i=1}^{N} \sum_{j\sim i} \langle A_{ij}, (I_3 - R_j^T R_i)\rangle\right)^{\frac{1}{p_1}} = -V_1^{\frac{1}{p_1}}
\end{aligned} \tag{19}$$

According to *Lemma 1*, the result in Eq. (19) proves that V_1 converges to zero in finite time. Hence, the system converges to the invariant set contained in

$$E = \left\{R_i, R_j : vec(R_i^T R_j A_{ij} - A_{ij} R_j^T R_i) = 0\right\} \tag{20}$$

If A_{ij} is a symmetric positive-definite matrix, Eq. (20 implies that R_{ij} is symmetric and thus according to the Rodrigues formula for the relative attitude given by

$$R_{ij} = I_3 + \sin\theta_{ij}\left(\frac{\Theta_{ij}}{\theta_{ij}}\right)^\times + (1 - \cos\theta_{ij})\left(\left(\frac{\Theta_{ij}}{\theta_{ij}}\right)^\times\right)^2 \tag{21}$$

the corresponding principal rotation angle is $\theta_{ij} = 0$ or π. The attitude consensus subspace (where $R_{ij} = I_3$ and $\theta_{ij} = 0$) is the stable equilibrium where the exceptional set lying outside the domain of attraction forms an unstable manifold corresponding to a relative rotation of $\theta_{ij} = \pi$ about any axis, which corresponds to the manifold $\mathbb{S}^2$. Since this is a set of zero measure on SO(3), the attitudes of all rigid bodies reach consensus with almost global stability. Also, it takes a finite time for the trajectory to enter the set where Eq. (18) holds, and subsequently consensus is achieved in a finite time. Thus, the consensus subspace is almost globally finite time stable. ■

B. Finite-time consensus control of rigid-body attitude on $\text{TSO}(3)^N$

In the second case, control laws are constructed on the dynamic level for rigid body attitude synchronization using the relative attitudes for feedback. The objective is to synchronize the attitudes and set the angular velocities to zero. Assume there are N rigid bodies with inertia matrices

$J_i, i = 1, \ldots, N$. The dynamic equations for the ith rigid body attitude motion can be written as

$$\dot{R}_i = R_i \omega_i^{\times} \tag{22a}$$
$$J_i \dot{\omega}_i = J_i \omega_i \times \omega_i + \tau_i \tag{22b}$$

Before introducing the designed control law, a few variables are defined here. Let Z_i be defined as a function of S_i as

$$Z_i = \frac{S_i}{({S_i}^T S_i)^{(1-\frac{1}{p_2})}} \tag{23}$$

where S_i is defined in Eq. (11). The time derivative of S_i is defined as

$$W_i = \dot{S}_i \tag{24}$$

The variables Ψ_i and H_i are defined as

$$\Psi_i = \omega_i - Z_i \tag{25a}$$
$$H_i = I_3 - 2(1 - \frac{1}{p_2}) \frac{S_i {S_i}^T}{({S_i}^T S_i)} \tag{25b}$$

where $p_2 \in (1, 2)$ is a real constant. Using Eqs. (24-25b), the time derivative of Z_i can be written based on the aforementioned variables

$$\dot{Z}_i = \frac{\partial Z_i}{\partial S_i} . \frac{\partial S_i}{\partial t} = ({S_i}^T S_i)^{-(1-\frac{1}{p_2})} H_i W_i \tag{26}$$

Theorem 2: Consider rigid body $i, i = 1, \ldots, N$ has access to the relative attitude with respect to its neighbors. With the feedback control law for agent i as

$$\tau_i = -J_i \omega_i \times Z_i - \frac{J_i \Psi_i}{\left(\Psi_i^T J_i \Psi_i\right)^{(1-\frac{1}{p_2})}} + \frac{J_i H_i W_i}{({S_i}^T S_i)^{(1-\frac{1}{p_2})}} + 2S_i \tag{27}$$

the closed-loop system of Eqs. (22-27) reaches consensus with almost global finite time stability.

Proof: To show the finite-time stability of the closed-loop system, the Lyapunov stability theory is utilized. For this purpose, let us introduce the Lyapunov candidate as

$$V_2 = \sum_{i=1}^{N} \sum_{j \sim i} \langle A_{ij}, (I_3 - R_j^T R_i) \rangle + \frac{1}{2} \sum_{i=1}^{N} \Psi_i^T J_i \Psi_i > 0 \tag{28}$$
$$\forall \{R_i, \omega_i\} \backslash \{R_j^T R_i = I_3, \omega_i = 0\},$$

Differentiating this Lyapunov candidate with respect to time, and substituting Eq. (25) yields

$$\begin{aligned}\dot{V}_2 = &-\sum_{i=1}^{N} \sum_{j \sim i} \Big(\text{trace}(-A_{ij} \omega_j^{\times} R_j^T R_i) + \text{trace}(A_{ij} R_j^T R_i \omega_i^{\times}) \Big) \\ &+ \sum_{i=1}^{N} \Psi_i^T J_i (\dot{\omega}_i - \dot{Z}_i) = -2 \sum_{i=1}^{N} \sum_{j \sim i} \text{trace}(A_{ij} R_j^T R_i \omega_i^{\times}) \\ &+ \sum_{i=1}^{N} \Psi_i^T J_i (\dot{\omega}_i - \dot{Z}_i) = -2 \sum_{i=1}^{N} {\omega_i}^T S_i \\ &+ \sum_{i=1}^{N} \Psi_i^T \Big(J_i \omega_i \times \omega_i + \tau_i - ({S_i}^T S_i)^{-(1-\frac{1}{p_2})} J_i H_i W_i \Big)\end{aligned} \tag{29}$$

By substituting the proposed control law in Eq. (27), the result of Eq. (29) can be expressed as

$$\begin{aligned}\dot{V}_2 &= \sum_{i=1}^{N} \Big[\Psi_i^T (J_i \omega_i \times \omega_i - J_i \omega_i \times Z_i) - (\Psi_i^T J_i \Psi_i)^{\frac{1}{p_2}} - 2 Z_i^T S_i \Big] \\ &= \sum_{i=1}^{N} \Big[-(\Psi_i^T J_i \Psi_i)^{\frac{1}{p_2}} - 2(S_i^T S_i)^{\frac{1}{p_2}} \Big]\end{aligned} \tag{30}$$

From this result and the Eq. (18) it is concluded that

$$\begin{aligned}\dot{V}_2 &= -\sum_{i=1}^{N} \Big[(\Psi_i^T J_i \Psi_i)^{\frac{1}{p_2}} + 2(S_i^T S_i)^{\frac{1}{p_2}} \Big] \\ &\leq -\Big[(\sum_{i=1}^{N} \Psi_i^T J_i \Psi_i)^{\frac{1}{p_2}} + (\sum_{i=1}^{N} \sum_{j \sim i} \langle A_{ij}, (I_3 - R_j^T R_i) \rangle)^{\frac{1}{p_2}} \Big] \\ &\leq -V_2^{\frac{1}{p_2}}\end{aligned} \tag{31}$$

which implies that the closed-loop system is almost globally finite-time stable using the same arguments as in Part III-A and according to *Lemma 1*. ■

IV. Numerical Integration

The Lie group numerical integration method is given by the following integration rule [16] to numerically obtain the states at time t_{n+1} for Case 1

$$\omega_i^n = \frac{S_i^n}{\left((S_i^T)^n S_i^n\right)^{(1-\frac{1}{p_1})}} \qquad i = 1, \cdots, N \tag{32a}$$
$$R_i^{n+1} = \exp\left(-0.5 \Delta t \left(\omega_i^n + \omega_i^{n+1}\right)^{\times}\right) R_i^n \tag{32b}$$

where the fixed time step is denoted by $\Delta t = t_{n+1} - t_n$. Note that S_i^n is derived from Eq. (11). Lie group variational integrators preserve the structure of the configuration space. Variational integration obtained from the discrete Lagrange-d'Alembert principal [13] is used for Case 2 to obtain the states at time t_{n+1} in terms of those at time t_n as

$$\mathcal{J}_i = \frac{1}{2} \text{ trace}[J_i] I_3 - J_i \tag{33a}$$
$$(J_i \omega_i^n)^{\times} = \Delta t (F_i^n \mathcal{J}_i - \mathcal{J}_i F_i^n) \tag{33b}$$
$$R_i^{n+1} = R_i^n F_i^n \tag{33c}$$
$$\begin{aligned}\tau_i^n = &\frac{1}{2}(\tau(R_i^n, \omega_i^n, R_j^n, \omega_j^n) \\ &+ \tau(R_i^{n+1}, \omega_i^n, R_j^{n+1}, \omega_j^n)), \quad j \sim i\end{aligned} \tag{33d}$$
$$J_i \omega_i^{n+1} = \{F_i^n\}^T J_i \omega_i^n + \Delta t \tau_i^n \tag{33e}$$

These equations should be used to compute the evolution of the attitude and angular velocity. Here, $\mathcal{J}_i$ is the modified inertia matrix of agent i in terms of the moment of inertia matrix J_i. Note that F_i^n is implicit, and it is solved using the Newton-Raphson method.

V. Numerical Simulations and Discussions

In this section, the performance of the proposed control schemes is studied for each of the two aforementioned cases. For simulation purposes, four (non-identical) rigid bodies ($N = 4$) with inertia tensors of $J_1 = \text{diag}(4.97, 6.16, 8.37)$ kg.m^2, $J_2 = \text{diag}(5.97, 7.16, 9.37)$

kg.m^2, $J_3 = \text{diag}(4.47, 5.66, 7.87)$ kg.m^2, and $J_4 = \text{diag}(5.47, 6.66, 8.87)$ kg.m^2 are considered. As depicted in Fig. 1, an undirected line graph communication topology is used for both scenarios. Moreover, the weight matrices between each pair of the agents are chosen as $A_{12} = \text{diag}(1.5, 1.1, 1.0)$, $A_{23} = \text{diag}(1.3, 1.2, 1.1)$, and $A_{34} = \text{diag}(1.4, 1.3, 1.0)$. In the following, each of the two described cases is examined.

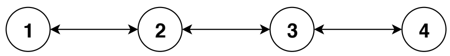

Fig. 1. Undirected information exchange topology between four rigid bodies

A. *Case 1: Finite-time attitude kinematic consensus steering control*

The first case is associated with the finite-time attitude consensus steering control problem where the angular velocities are the control signals. For this case, two examples with different initial conditions are presented.

Example 1: In the first example, the initial orientations of the four rigid bodies in terms of the principal rotation vectors (rad) are selected as $\Theta_1(t_0) = \begin{bmatrix} 0.50 & -0.18 & -0.68 \end{bmatrix}^T$, $\Theta_2(t_0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, $\Theta_3(t_0) = \begin{bmatrix} -0.15 & -0.07 & 0.17 \end{bmatrix}^T$, and $\Theta_4(t_0) = \begin{bmatrix} -0.57 & 0.05 & 1.53 \end{bmatrix}^T$. By implementing the proposed steering control law derived in Eq. (12), all rigid bodies converge to a synchronized attitude in a finite time. As depicted in Fig. 2, all rigid bodies achieve and maintain the same constant attitude and consequently the relative attitude between each pair of the agents converges to the identity attitude in a finite time. Note that the attitudes in Fig. 2 are expressed in terms of principal rotation angle. Since the closed-loop system is first order, the convergence rate is relatively high. The upper bound of the convergence time using Eq. (8) is calculated as 5.82s which is compliant to the simulation results in this example.

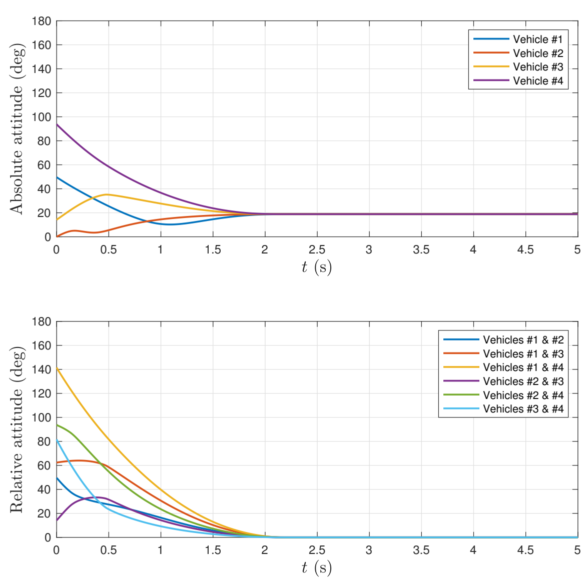

Fig. 2. Case 1 (Example 1): Absolute attitude (top) and relative attitude (principal rotation angle) (bottom) ($p_1 = 1.35$)

Example 2: As the second example, a different set of initial attitudes (rad) are considered as $\Theta_1(t_0) = \begin{bmatrix} -0.37 & -0.37 & -0.17 \end{bmatrix}^T$, $\Theta_2(t_0) = \begin{bmatrix} 0 & 0 & -2.26 \end{bmatrix}^T$, $\Theta_3(t_0) = \begin{bmatrix} -0.59 & -2.23 & 1.09 \end{bmatrix}^T$, and $\Theta_4(t_0) = \begin{bmatrix} -1.88 & 0.16 & 1.12 \end{bmatrix}^T$. It is desired to show that the convergence time depends on the initial conditions. In Fig. 3, the absolute attitudes and relative attitudes between pairs of the agents in terms of the principal rotation angle are depicted. As can be seen, the convergence time is different from that of the Example 1. The upper bound of the convergence time is calculated as 8.02s for this example.

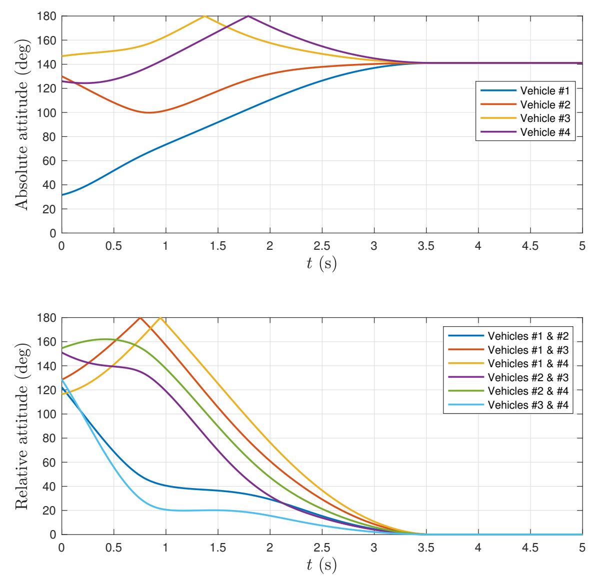

Fig. 3. Case 1 (Example 2): Absolute attitude (top) and relative attitude (principal rotation angle) (bottom) ($p_1 = 1.35$)

B. *Case 2: Finite-time consensus control of rigid-body attitude on* $\text{TSO}(3)^N$

In this part, the control design is conducted on the dynamic level. Here, the initial orientations are the same as the first example in Case 1 and the initial angular velocities of the four rigid bodies (rad/s) are $\omega_1(t_0) = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T$, $\omega_2(t_0) = \begin{bmatrix} 2 & -3 & -2 \end{bmatrix}^T$, $\omega_3(t_0) = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^T$, and $\omega_4(t_0) = \begin{bmatrix} -3 & 0 & 0.3 \end{bmatrix}^T$. After applying the proposed controller given by Eq. (27), all rigid bodies converge to the same attitude in a finite time, as depicted in Fig. 4. The norms of the angular velocities of the rigid bodies are demonstrated in Fig. 5. As expected, the angular velocities of all four agents converge to zero. Furthermore, the norms of the control torques and integrated control torques are depicted in Fig. 6. Here, the upper bound of the convergence

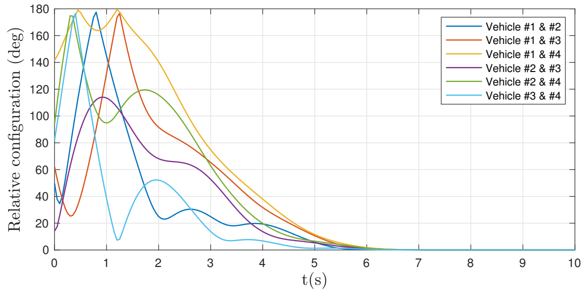

Fig. 4. Case 2: Relative attitude (principal rotation angle) between each pair of agents ($p_2 = 1.19$)

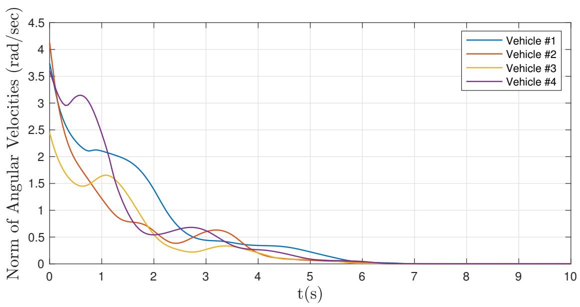

Fig. 5. Case 2: Norms of angular velocities ($p_2 = 1.19$)

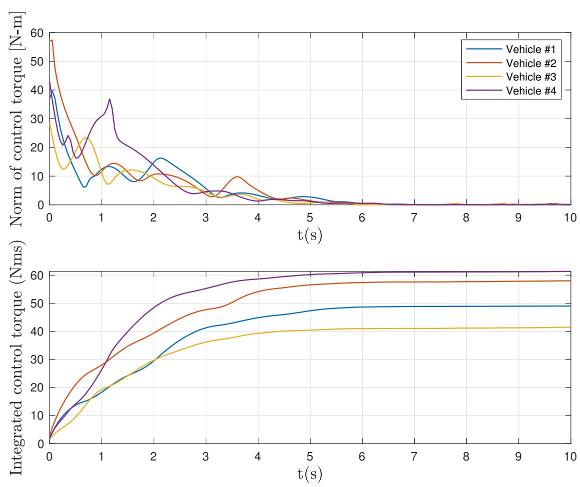

Fig. 6. Case 2: Norms of control torques (top) and integrated torques (bottom) ($p_2 = 1.19$)

time is calculated as 14.32s which is consistent with the results in this section.

VI. Conclusions

In this work, almost global finite-time consensus algorithms were introduced for the kinematic and dynamic attitude control of a multi-agent system of N heterogenous rigid bodies. The attitudes of the rigid bodies were described in terms of the rotation matrices while the communication topology was assumed to be fixed and undirected. The control objective was to achieve attitude synchronization in a finite convergence time which depends on the initial conditions. Numerical simulations were demonstrated to verify the effectiveness of these methods and it was shown the system reaches consensus in a finite time. The upper bound of the convergence times were calculated for both cases.

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