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Controller Synthesis subject to Logical and Structural Constraints: A Satisfiability Modulo Theories (SMT) Approach

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Abstract— **We report on a simple approach to use satisfiability modulo theories (SMT) solvers to synthesize stabilizing controllers subject to logical and structural constraints. Examples of logical/structural specifications allowed by our methodology include the transitive property of the connectivity of a networked system, and the mutually exclusive use of inputs or sensors, to name a few. The aforementioned structural constraints can also impose the sparsity pattern and linear dependency restrictions prevailing in the decentralized control literature. The main goal of this article is to discuss preliminary results and examples in which both the plant and the controller are linear time-invariant (LTI). Our approach consists of encoding the stability conditions as well as the logical and structural constraints as an SMT instance. We illustrate our methodology on two classes of problems: (i) full state feedback design for positive systems, with applications to combination drug therapy and transportation network design, and (ii) static output feedback (SOF) design. The article includes numerical examples for each of these applications computed using a freely available SMT solver. It is noteworthy that the examples of positive systems mentioned above, in particular, can be solved in less than four minutes even when the dimension of the state is one thousand.**

I. INTRODUCTION

For more than five decades, technological advances and applications have been fostering the use of distributed and networked controllers in engineered systems. It is not surprising, then, that intense research on methods to design and certify the performance of decentralized and distributed controllers is expected to continue for the foreseeable future. Unlike centralized controllers, decentralized or distributed ones consist of an assemblage of sub-controllers. The so-called information pattern [1] determines for each sub-controller which other sub-controllers it can communicate with, from which sensors it receives measurements, and which actuators it can command. Hence, the information pattern imposes structural constraints on the overall controller. Examples of structural constraints widely studied in the literature include restrictions on the sparsity pattern of linear time-invariant (LTI) controllers, imposing symmetric properties, and limiting the *memory* of the controller, such as in static output feedback (SOF).

A. Design of Structurally-Constrained Controllers: A Brief Overview of Methods and Techniques

There is no single design methodology that is capable of accounting for all of the above-mentioned structural constraints. Instead, there is an ever-increasing portfolio of methods — each tied to a specific class of structural constraints. Furthermore, available methods rely on a variety of numerical tools and theories to achieve particular goals, such as obtaining a stabilizing controller or designing a norm-optimal optimal one. For instance, the existence of a stabilizing decentralized controller for a given plant was characterized by using algebraic methods [2], [3] that determine when certain *fixed modes* are absent. Subsequent algebraic approaches also led to methods [4] to obtain a stabilizing decentralized controller when one exists. Notably, by incorporating graph-theoretic concepts and techniques, a comprehensive collection of methods [5] for the analysis and design of structured controllers has been successfully developed that introduced key concepts, such as structured controllability and observability. More recently, these techniques have been further leveraged to obtain tractable algorithms [6] for characterizing structurally controllable and observable structures with the least inputs and outputs. The analysis in [7] further characterizes the realizability of structured systems. In light of powerful convex optimization concepts and solvers, approaches proposed in the last fifteen years focused on convex parameterizations of stabilizing controllers. Notably, modifications of Youla's parameterization [8] that are compatible with structural constraints satisfying funnel-causality [9] or, more generally, quadratic invariance[1] (QI) [11] led to systematic methods to obtain norm-optimal controllers. The recent approach in [12] generalizes the aforementioned parametrizations to multidimensional systems. The methods to design sparse controllers in [13] and [14], [15] use alternating direction method of multipliers (ADMM), non-fragility, and quasi-norms, respectively.

B. Main Goal and Outline of Technical Approach

In this article, we propose a straightforward methodology to design LTI controllers that stabilize a given LTI plant, subject not only to structural constraints, as the methods described in our brief overview have been considering, but

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[1]The algorithm proposed in [10] yields the closest QI structure relative to a pre-defined one.

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also logical constraints. Examples of logical specifications allowed by our methodology include the transitive property of the connectivity of a networked system[2] and the mutually exclusive use of inputs or sensors, to name a few. The significance of logical constraints extends beyond engineering applications, such as in the case study discussed in §IV-A for combination drug therapy.

As is the case with the approaches described in the overview, our solution approach hinges on a class of solvers. In our case, we focus on solvers for satisfiability modulo theories (SMT) that were originally intended for theorem proving, and verification of the correctness of digital hardware and software. Currently, SMT solvers have been successfully used towards control-engineering-related tasks, including analysis of hybrid systems [16], controller synthesis for piece-wise linear systems [17], state estimation [18], and robot motion planning [19].

To simplify the description of our approach, we assume that the controller is memoryless and, hence, is represented by a controller matrix. Our methodology is built upon the observation that conditions for the closed-loop stability, as well as the logical and structural constraints, can be cast as an SMT instance of the type accepted by freely available solvers. In particular, we recast the stability condition as a finite number of multivariate polynomial inequalities (MPIs).

Most closely to our work are the results reported in [20]–[22], where an algorithm known as Quantifier Elimination (QE) [23], [24] is used to solve robust control problems (in the frequency domain) that are encoded using quantified formulas consisting of MPIs. Interestingly, the same algorithm used to solve these QE problems, known as the Cylindrical Algebraic Decomposition (CAD), forms the base of current SMT solvers when applied to formulas that contain MPIs. Nevertheless, our approach differs from those reported in [20]–[22] in a sense it can encode logical constraints, structural constraints, and performance measures specified in the time domain, e.g., $\mathcal{H}_2$ performance measure (like LQR).

We illustrate our methodology on two classes of problems: (i) full state feedback design for positive systems, with applications to combination drug therapy and transportation network design in §IV-A, and (ii) static output feedback (SOF) design in §IV-B. The article includes numerical examples for each of these applications computed using a freely available SMT solver.

C. Notation

We denote the set of real numbers, non-negative real numbers, and positive real numbers by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$, respectively. Similarly, $\mathbb{R}^{q\times r}$ denotes the set of real-valued matrices with q rows and r columns. We denote the Boolean set $\{0, 1\}$ by $\mathbb{B}$. The upper case and lower case letters represent the matrices and vectors, respectively. To compactly express a diagonal matrix M, we use $\mathrm{diag}(\mathrm{d_M})$ wherein $\mathrm{d_M}$ denotes the vector containing all diagonal elements of M. We show the transpose of a matrix M with $\mathrm{M^T}$ and denote its $\mathrm{i^{th}}$ row and $\mathrm{j^{th}}$ column by $\mathrm{M(i,:)}$ and $\mathrm{M(:,j)}$, respectively. As usual, I and 0 denote the identity and zero matrices, respectively. We show the determinant and number of non-zero elements of a matrix M with $\det(\mathrm{M})$ and $\|\mathrm{M}\|_0$, respectively. Symbols $\dot{>}$, $\dot{<}$, $\dot{\geq}$, and $\dot{\leq}$ denote the element-wise inequality and symbol $\odot$ represents the Hadamard product of matrices (element-wise product). We denote the time derivative of x by $\dot{\mathrm{x}}$.

[2]For a networked system whose sub-components correspond to the vertices of a directed graph, transitivity would imply that if there is an edge from *a* to *b* and an edge from *b* to *c* then there is an edge from *a* to *c*. Interestingly, this property is satisfied by sparsity pattern constraints that are funnel causal or quadratically invariant.

II. Problem Formulation

Consider the following linear time-invariant (LTI) continuous-time system:

$$\dot{\mathrm{x}}(\mathrm{t}) = \mathcal{L}(\mathrm{A},\mathrm{K})\mathrm{x}(\mathrm{t}), \tag{1}$$

wherein $\mathrm{x(t)} \in \mathbb{R}^{\mathrm{n}}$, $\mathcal{L} : \mathbb{R}^{\mathrm{n\times n}} \times \mathbb{R}^{\mathrm{m\times p}} \rightarrow \mathbb{R}^{\mathrm{n\times n}}$, $\mathrm{A} \in \mathbb{R}^{\mathrm{n\times n}}$, and $\mathrm{K} \in \mathbb{R}^{\mathrm{m\times p}}$ denote the system state, closed-loop linear mapping, open-loop system matrix, and controller matrix, respectively. Before formulating our problem, we state the following preliminary definitions:

Definition 1: We call a matrix M to be *Hurwitz*, if all of its eigenvalues have negative real parts. Then, denoting the set of Hurwitz matrices by $\mathcal{H}$, we have $\mathrm{M} \in \mathcal{H}$.

Definition 2: We denote the set of imposed constraints by $\mathcal{S_{IC}}$ and if a matrix M satisfies such constraints, we denote it by $\mathrm{M} \in \mathcal{S_{IC}}$.

We formally state our problem as follows:

Problem 1: Given a closed-loop linear mapping $\mathcal{L}$ and an open-loop system matrix A in (1) and a set of imposed constraints $\mathcal{S_{IC}}$, synthesize a controller matrix K in (1) subject to the following conditions:

$$\mathcal{L}(\mathrm{A},\mathrm{K}) \in \mathcal{H}, \qquad \text{(Stability)}, \tag{2a}$$

$$\mathrm{K} \in \mathcal{S_{IC}}, \qquad \text{(Imposed Constraints)}. \tag{2b}$$

We consider two significant types of imposed constraints in Problem 1: (i) logical constraints, and (ii) structural constraints.

III. An SMT-Based Solution Methodology

First, presenting preliminaries on satisfiability modulo theories (SMT), we express a definition on a quantifier-free SMT fragment over the theories of nonlinear arithmetic and Boolean logic. Then, encoding the stability criteria, structural constraints, and logical constraints based on such a definition, we propose an SMT-based algorithmic solution to Problem 1 via Algorithm 1.

A. Satisfiability Modulo Theories (SMT)

A propositional logic (also known as zero-order logic) formula is a set of logical constraints $\mathrm{S(b_1,\ldots,b_N)}$ defined over the Boolean variables $(\mathrm{b_1,\ldots,b_N}) \in \mathbb{B}^{\mathrm{N}}$ using operators `AND`, `OR`, and `NOT`. The Boolean satisfiability problem (SAT) is the problem of determining if there exists an assignment for the variables $\mathrm{b_1,\ldots,b_N}$ such that the logical constraint $\mathrm{S(b_1,\ldots,b_N)}$ evaluates to `TRUE`. While the SAT problem is known to be an NP-complete problem, recent breakthroughs in the field of computer science allowed SAT solvers to scale well to problems with hundreds of Boolean variables and million of logical constraints thanks to the Davis-Putnam-Logemann-Loveland (DPLL) algorithm [25], [26].

An essential extension of the SAT problem is the satisfiability modulo theories (SMT) which is the problem of determining whether a first-order formula—where the predicate symbols have additional interpretations—is satisfiable. The following definition can capture an important sub-class of formulas that current SMT solvers can handle:

Definition 3: We can write a quantifier-free SMT fragment over the theories of multivariate polynomials (a subset of the nonlinear arithmetic theory) and Boolean logic as:

$$\begin{aligned}
&\exists(\mathrm{b}_1,\dots,\mathrm{b}_\mathrm{N},\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{M})\in\mathbb{B}^\mathrm{N}\times\mathbb{R}^\mathrm{M},\\
&\text{subject to:}\quad \mathcal{P}_\mathrm{i}(\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{M})\star_\mathrm{i} 0, && \mathrm{i}=1,\dots,\mathrm{q},\\
&\mathrm{S_j}(\mathrm{b}_1,\dots,\mathrm{b}_\mathrm{N}) \longleftrightarrow \texttt{TRUE}, && \mathrm{j}=1,\dots,\mathrm{r},\\
&\mathrm{b_l}\longleftrightarrow \big(\mathcal{P}_{\mathrm{l+q}}(\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{M})\star_{\mathrm{l+q}} 0\big), && \mathrm{l}=1,\dots,\mathrm{N},
\end{aligned}$$

wherein $\mathcal{P}_\mathrm{s}(\mathrm{k}_1,\dots,\mathrm{k}_\mathrm{M})$'s for $\mathrm{s}\in\{1,\dots,\mathrm{q}+\mathrm{N}\}$ are multivariate polynomials and $\mathrm{S_j}(\mathrm{b}_1,\dots,\mathrm{b}_\mathrm{N})$'s are propositional logic formulas, and M, q, r, and N are the number of arithmetic variables, polynomial constraints, logical constraints, and hybrid constraints, respectively. Notice that the number of hybrid constraints, i.e., N is identical to the number of Boolean variables. The comparison symbols $\star_s$'s for $\mathrm{s}\in\{1,\dots,\mathrm{q}+\mathrm{N}\}$ could be any comparison operator among $>$, $\geq$, $=$, and $\neq$.

Based on Definition 3, we propose an SMT-based encoding of the solutions to Problem 1. In particular, we encode the stability criterion (2a) as a finite number of multivariate polynomial inequality constraints obtained from different stability criteria, (e.g., Routh-Hurwitz). Similarly, we encode the imposed constraints (2b) as a combination of a finite number of multivariate polynomial structural constraints and logical constraints.

B. Stability Criteria

Motivated by Definition 3, we are interested in casting the stability criterion in (2a) as a set of finitely many multivariate polynomial inequality constraints

$$\mathcal{P}_\mathcal{H}(\mathrm{K};\mathcal{L},\mathrm{A})\ \dot{>}\ 0,\quad \text{(Stability Criterion)}, \tag{3}$$

wherein $\mathcal{P}_\mathcal{H}(\mathrm{K};\mathcal{L},\mathrm{A})$ denotes a vector containing multivariate polynomials in the elements of controller matrix K. In particular, we consider two classes of systems: 1) positive systems, and 2) general LTI systems.

1) Stability Criterion for Positive Systems: Metzler matrices play an important role in the description of positive systems. Therefore, let us recall the definition of a Metzler matrix.

Definition 4: We call a matrix M to be *Metzler* if all of its off-diagonal elements are non-negative.

For the special class, when the system is positive and hence the matrix $\mathcal{L}(\mathrm{A},\mathrm{K})$ is Metzler, we have the following stability equivalence [27]:

Proposition 1: Given a Metzler matrix $\mathrm{M}\in\mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, the following statements are equivalent:

- The matrix M is Hurwitz, i.e., $\mathrm{M}\in\mathcal{H}$.
- There exists a $\mathrm{z}\in\mathbb{R}^\mathrm{n}$ such that $\mathrm{z}\ \dot{>}\ 0$ and $\mathrm{Mz}\ \dot{<}\ 0$.

Consequently, for special case of positive systems, i.e., when the matrix $\mathcal{L}(\mathrm{A},\mathrm{K})$ is Metzler, we can rewrite the stability criterion in (2a) as a set of polynomial inequalities

$$\mathrm{z}\ \dot{>}\ 0,\quad \mathcal{L}(\mathrm{A},\mathrm{K})\mathrm{z}\ \dot{<}\ 0. \tag{4}$$

By noticing that each element of $\mathcal{L}(\mathrm{A},\mathrm{K})\mathrm{z}$ is a multivariate polynomial in both the elements of controller matrix K and elements of auxiliary variable z, we conclude that (4) is expressible as (3) which can be encoded into an SMT instance (Definition 3).

2) Stability Criterion for General LTI Systems: In case of general LTI systems, $\big($e.g., SOF design, i.e., $\mathcal{L}(\mathrm{A},\mathrm{K})=\mathrm{A}+\mathrm{BKC}\big)$, we can still write the stability criterion as a set of polynomial inequalities thanks to the Routh-Hurwitz stability criterion [28], [29]:

Proposition 2: The following statements are equivalent:

- The matrix $\mathcal{L}(\mathrm{A},\mathrm{K})$ is Hurwitz, i.e., $\mathcal{L}(\mathrm{A},\mathrm{K})\in\mathcal{H}$.
- There exists a $\mathrm{K}\in\mathbb{R}^{\mathrm{m}\times\mathrm{p}}$ such that $\mathrm{v}^{\mathrm{RH}}(\mathrm{K})\ \dot{>}\ 0$.

wherein $\mathrm{v}^{\mathrm{RH}}(\mathrm{K})$ is the first column of the Routh-Hurwitz table constructed from the corresponding characteristic polynomial $\det\big(\lambda\mathrm{I}-\mathcal{L}(\mathrm{A},\mathrm{K})\big)$. Details of constructing the Routh-Hurwitz table can be found in [28], [29].

Knowing that i^{th} element of $\mathrm{v}^{\mathrm{RH}}(\mathrm{K})$, i.e., $\mathrm{v}_\mathrm{i}^{\mathrm{RH}}(\mathrm{K})$ is of form $\mathrm{N_i}(\mathrm{K})/\mathrm{D_i}(\mathrm{K})$, $\mathrm{D_i}(\mathrm{K})^2>0$ implies that $\mathrm{v}_\mathrm{i}^{\mathrm{RHP}}(\mathrm{K}):=\mathrm{N_i}(\mathrm{K})\mathrm{D_i}(\mathrm{K})=\mathrm{v}_\mathrm{i}^{\mathrm{RH}}(\mathrm{K})\mathrm{D_i}(\mathrm{K})^2$ has the same sign as $\mathrm{v}_\mathrm{i}^{\mathrm{RH}}(\mathrm{K})$. Instead of using $\mathrm{v}^{\mathrm{RH}}(\mathrm{K})$ which is of the rational fraction form, we use $\mathrm{v}^{\mathrm{RHP}}(\mathrm{K})$ which has the polynomial form:

$$\mathrm{v}^{\mathrm{RHP}}(\mathrm{K})\ \dot{>}\ 0. \tag{5}$$

Since any element of $\mathrm{v}^{\mathrm{RHP}}(\mathrm{K})$ is a multivariate polynomial in the elements of K, (5) is also expressible as (3) which can be encoded into an SMT instance (Definition 3).

C. Structural Constraints

Based on Definition 3, we symbolically cast the imposed structural constraints via (2b) as a set of finitely many multivariate polynomial constraints as follows:

$$\mathcal{P}_{\mathcal{SC}}(\mathrm{K})\star 0,\quad \text{(Structural Constraint)}, \tag{6}$$

wherein $\mathcal{P}_{\mathcal{SC}}(\mathrm{K})$ denotes a matrix/vector corresponding to the structural constraints containing multivariate polynomials in the elements of controller matrix K. The comparison symbol $\star$ denotes a matrix/vector of possibly distinct comparison symbols $\star_\mathrm{i}$'s. Notice that any of those comparison symbols $\star_\mathrm{i}$'s could be any comparison operator among $>$, $\geq$, $=$, and $\neq$.

An important sub-class of structural constraints in structured control is the class of *sparsity* constraints. Given a binary structure $\mathbb{K}\in\mathbb{B}^{\mathrm{m}\times\mathrm{p}}$ that captures the imposed sparsity structure, we can simply rewrite such a sparsity structural constraint in a compact form as follows:

$$\mathrm{K} \odot \mathbb{K} = \mathrm{K}. \tag{7}$$

Obviously, the sparsity constraint (7) inherits the form of structural constraint (6).

D. Logical Constraints

We symbolically cast the logical constraints imposed via (2b) as a set of logical statements as follows:

$$\left.\begin{array}{l} \mathrm{S}(\mathrm{b}_1,\ldots,\mathrm{b}_\mathrm{N}) \longleftrightarrow \texttt{TRUE}, \\ \mathrm{b}_\mathrm{l} \longleftrightarrow \big(\mathcal{P}_{\mathrm{l+q}}(\mathrm{K}) \star_{\mathrm{l+q}} 0\big), \\ \mathrm{l} = 1,\ldots,\mathrm{N}, \end{array}\right\} \quad \text{(Logical Constraint)} \tag{8}$$

wherein $\mathrm{S}(\mathrm{b}_1,\ldots,\mathrm{b}_\mathrm{N})$ denotes a vector containing a set of logical statements over the Boolean variables, $\mathcal{P}_{\mathrm{l+q}}(\mathrm{K})$ is a polynomial in the elements of controller matrix K and $\star_{\mathrm{l+q}}$'s for $\mathrm{l} \in \{1,\ldots,\mathrm{N}\}$ could be any comparison operator among $>$, $\geq$, $=$, and $\neq$.

One of the main sub-classes of logical constraints is the class of *transitive* relations which can be defined using the following constraints:

$$(\mathrm{b}_{\mathrm{lji}} \longrightarrow \mathrm{b}_{\mathrm{li}}) \longleftrightarrow \texttt{TRUE}, \tag{9a}$$
$$\mathrm{b}_{\mathrm{lji}} \longleftrightarrow (\mathrm{k}_{\mathrm{lj}}\mathrm{k}_{\mathrm{ji}} \neq 0), \quad \mathrm{b}_{\mathrm{li}} \longleftrightarrow (\mathrm{k}_{\mathrm{li}} \neq 0), \tag{9b}$$
$$\mathrm{i} \in \{1,\ldots,\mathrm{p}\},\ \mathrm{j} \in \{1,\ldots,\min\{\mathrm{p},\mathrm{m}\}\},\ \mathrm{l} \in \{1,\ldots,\mathrm{m}\}.$$

This class of logical constraints has applications in robotics and communication networks.

E. An SMT-Based Controller Synthesis Algorithm

We summarize the previous discussion in Algorithm 1. The next Proposition captures the correctness of the proposed Algorithm 1, which is a direct consequence of the soundness and correctness of SMT solvers.

Proposition 3: Consider the linear time-invariant (LTI) dynamical system defined in (1) and the set of imposed constraints $\mathcal{S}_{\mathcal{IC}}$. A controller synthesized by Algorithm 1 is a solution to Problem 1.

Algorithm 1 SMT-BASED-CONTROLLER-SYNTHESIS
Inputs: $\mathcal{L}$, A, and $\mathcal{S}_{\mathcal{IC}}$

Step 1: Encode the stability criterion:
if $\mathcal{L}(\mathrm{A},\mathrm{K})$ is Metzler **then**
  Encode $\mathcal{P}_{\mathcal{H}}(\mathrm{K};\mathcal{L},\mathrm{A})$ using (4),
else
  Encode $\mathcal{P}_{\mathcal{H}}(\mathrm{K};\mathcal{L},\mathrm{A})$ using (5).
end if
Step 2: Encode the imposed constraints:
Encode the structural constraints $\mathcal{P}_{\mathcal{SC}}(\mathrm{K})$ using (6),
Encode the logical constraints $\mathcal{S}(\mathrm{K})$ using (8).
Step 3: Synthesize controller
status,
K = `SMT-Solver`$\big(\mathcal{P}_{\mathcal{H}}(\mathrm{K};\mathcal{L},\mathrm{A}),\mathcal{P}_{\mathcal{SC}}(\mathrm{K}),\mathcal{S}(\mathrm{K})\big)$,
if status == `UNSAT` **then**
  Return `No stabilizing controller exists.`
else
  Return K.
end if

IV. NUMERICAL EXAMPLES

To assess the effectiveness of Algorithm 1, we consider several numerical examples of two classes of problems: (i) decentralized control of positive systems, and (ii) static output feedback (SOF) design. For all the examples in this section, we used an off-the-shelf SMT solver named Z3 [30].

A. Decentralized Control of Positive Systems

We consider two cases of decentralized control of positive systems: 1) combination drug therapy, and 2) transportation network design.

1) Case Study 1 - Combination Drug Therapy: As a significant case of decentralized control of positive systems, consider the evolution dynamics of HIV mutants in presence of a combination of drugs as follows [31]:

$$\dot{\mathrm{x}}(\mathrm{t}) = \left(\mathrm{A} - \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{k}_\mathrm{i}\mathrm{D}_\mathrm{i}\right)\mathrm{x}(\mathrm{t}), \tag{10}$$

wherein the i^{th} element of the state vector $\mathrm{x}(\mathrm{t})$, i.e., $\mathrm{x}_\mathrm{i}(\mathrm{t}) \in \mathbb{R}_+$, specifies the population of the i^{th} HIV mutant at time t and $\mathrm{A} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is a Metzler matrix in which each diagonal element denotes the net replication rate of each mutant and each off-diagonal element denotes the the rate of mutation from one mutant to another. The i^{th} control input $\mathrm{k}_\mathrm{i} \in \mathbb{R}_+$ denotes the i^{th} drug dose. The diagonal matrix $\mathrm{D}_\mathrm{i} \in \mathbb{R}_+^{\mathrm{n}\times\mathrm{n}}$ determines at what rate i^{th} drug kills each HIV mutant. The matrix $\mathrm{A} - \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{k}_\mathrm{i}\mathrm{D}_\mathrm{i}$ is a Metzler matrix which motivates the use of (4).

In addition to the stability, it is necessary to impose interference-avoidance constraints. Such constraints prevent the simultaneous injection of some drugs. Given a set of drug indices that one cannot inject simultaneously, we can encode the interference-avoidance constraint as $\prod_{\mathrm{i}\in\mathcal{I}} \mathrm{k}_\mathrm{i} = 0$.

Example 1 (Correctness of Algorithm 1): To validate Algorithm 1, we randomly generate a Metzler matrix $\mathrm{A} \in \mathbb{R}^{100\times100}$, and diagonal matrices $\{\mathrm{D}_\mathrm{i} \in \mathbb{R}_+^{100\times100}\}_{\mathrm{i}=1}^{\mathrm{m}}$ with $\mathrm{n} = 100$ HIV mutants and $\mathrm{m} = 10$ drugs. We consider the interference-avoidance constraint among the 7^{th}, 8^{th}, and 9^{th} drugs. This leads to the following SMT problem:

$$\exists(\mathrm{k}_1,\ldots,\mathrm{k}_{10},\mathrm{z}_1,\ldots,\mathrm{z}_{100}) \in \mathbb{R}^{110},$$
$$\text{subject to: } \left(\mathrm{A} - \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{k}_\mathrm{i}\mathrm{D}_\mathrm{i}\right)\mathrm{z}\dot{<}0,\ \mathrm{z}\dot{>}0,\ \mathrm{k}\dot{\geq}0,\ \mathrm{k}_7\mathrm{k}_8\mathrm{k}_9 = 0.$$

Feeding this SMT problem into Algorithm 1, we obtain the following drug doses:

$$\mathrm{k} = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 5,247 & 0 & 1 & 1 & 1\end{bmatrix}^{\mathrm{T}}.$$

Example 2 (Scalability): To assess the scalability of the proposed SMT-based controller synthesis, we randomly generate a Metzler matrix $\mathrm{A} \in \mathbb{R}^{1,000\times1,000}$, and diagonal matrices $\{\mathrm{D}_\mathrm{i} \in \mathbb{R}_+^{1,000\times1,000}\}_{\mathrm{i}=1}^{\mathrm{m}}$ with $\mathrm{n} = 1,000$ HIV mutants and $\mathrm{m} = 10$ drugs. Running Algorithm 1 for such a matrix takes less than 4 minutes to obtain drug doses. It is noteworthy mentioning that all the numerical tests were performed on a MacBook Pro with processor 3.1 GHz Intel Core i5 and memory 8 GB $2,133$ MHz.

2) Case Study 2 - Transportation Network Design: As another significant example of decentralized control of positive systems, consider the dynamics of buffers containing goods as follows [31]:

$$\dot{x}(t) = (A + EKF)x(t), \tag{11}$$

wherein the i^{th} diagonal element of diagonal matrix $A \in \mathbb{R}^{n\times n}$ denotes either produce rate or destroy rate of the i^{th} good, the j^{th} diagonal element of diagonal matrix $K \in \mathbb{R}_+^{m\times m}$ denotes the j^{th} flow of good, and we define matrices $E \in \mathbb{R}^{n\times m}$ and $F \in \mathbb{R}_+^{m\times n}$ as in [31]. It is direct to verify that $A + EKF$ is a Metzler matrix for any controller matrix $K \in \mathbb{R}_+^{m\times m}$.

In addition to the stability, the controller needs to be fully decentralized, which can be imposed using the sparsity structural constraint $K \odot I = K$. Moreover, some flows are mutually exclusive. For a set of mutually exclusive flows $\mathcal{J}$, we can impose such a constraint using the structural constraint $\prod_{j\in\mathcal{J}} k_{jj} = 0$.

Example 3 (Mutually Exclusive Flows): Let us consider a transportation network with the dynamics specifications ($A \in \mathbb{R}^{5\times5}$, $E \in \mathbb{R}^{5\times3}$, and $F \in \mathbb{R}^{3\times5}$) in [31] when the inflow k_{11} and outflow k_{22} are mutually exclusive and the outflow k_{22} and outflow k_{33} are also mutually exclusive. To synthesize a stabilizing controller under the imposed structure, we consider the following SMT problem:

$$\begin{aligned}&\exists(k_{11}, k_{12}, \ldots, k_{32}, k_{33}, z_1, \ldots, z_5) \in \mathbb{R}^{14},\\ &\text{subject to:} \qquad (A + EKF)z \dot{<} 0,\ z \dot{>} 0,\\ &K \odot I = K,\ K \dot{\geq} 0,\ k_{11}k_{22} = 0,\ k_{22}k_{33} = 0.\end{aligned}$$

Solving this SMT problem via Algorithm 1, we obtain the following flows of goods: $K = \mathrm{diag}(\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T)$.

B. Static Output Feedback (SOF) Design

Static output feedback (SOF) design has been a challenging problem in control theory and its applications. In [32], the authors prove that SOF stabilization by a bounded controller (holds even in case of state feedback), simultaneous SOF stabilization, decentralized SOF stabilization by a norm bounded controller, and decentralized SOF stabilization with identical controllers are all NP-complete problems. Therein, the authors have conjectured the general SOF stabilization is also an NP-complete problem.

We consider three cases for SOF design: (i) SOF design subject to logical constraints (Example 4), (ii) SOF design subject to structural constraints (Example 5), and (iii) SOF design subject to logical and structural constraints (Example 6). We know that any such cases could theoretically become an NP-complete problem in general. However, current breakthroughs in SMT solvers allowed them to solve several NP-complete problems with reasonable execution time.

In case of SOF design, the closed-loop $\mathcal{L}(A, K)$ has the form of $\mathcal{L}(A, K) = A + BKC$ wherein B and C denote input output matrices, respectively.

Example 4 (Transitive Relations): Considering the following randomly generated matrices A, B, and C:

$$A = \begin{bmatrix}2.4366 & -0.5741 & -1.7558\\ 0.3024 & -0.1952 & -0.2574\\ 0.0583 & -0.0505 & 0.7495\end{bmatrix},$$
$$B = \mathrm{diag}(\begin{bmatrix}0.0049 & -0.7586 & 0.4123\end{bmatrix}^T),$$
$$C = \mathrm{diag}(\begin{bmatrix}1.5724 & -0.2752 & 0.4584\end{bmatrix}^T),$$

we aim at synthesizing a stabilizing SOF subject to the transitive relations encoded by (9). To do so, we construct the following SMT problem:

$$\begin{aligned}&\exists(b_{111}, \ldots, b_{333}, b_{11}, \ldots, b_{33}, k_{11}, \ldots, k_{33}) \in \mathbb{B}^{36} \times \mathbb{R}^9,\\ &\text{subject to:} \quad v^{RHP}(K) \dot{>} 0, \quad (b_{lji} \longrightarrow b_{li}) \longleftrightarrow \mathtt{TRUE},\\ &b_{lji} \longleftrightarrow (k_{lj}k_{ji} \neq 0), \quad b_{li} \longleftrightarrow (k_{li} \neq 0),\\ &i \in \{1, 2, 3\},\ j \in \{1, 2, 3\},\ l \in \{1, 2, 3\}.\end{aligned}$$

Solving this SMT problem via Algorithm 1, we obtain the following stabilizing SOF: $k_{11} = 0.125$, $k_{12} = -0.5$, $k_{13} = -0.5$, $k_{21} = 1$, $k_{22} = 0.5$, $k_{23} = -1$, $k_{31} = 7$, $k_{32} = 6$, and $k_{33} = -18$.

Example 5 (Pre-Specified Binary Structure): Considering the matrices $A \in \mathbb{R}^{5\times5}$, $B \in \mathbb{R}^{5\times3}$, and $C \in \mathbb{R}^{3\times5}$ in [33], we aim at synthesizing a stabilizing SOF subject to the following pre-specified binary structure:

$$\mathbb{K} = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}.$$

Constructing the following SMT problem:

$$\begin{aligned}&\exists(k_{11}, \ldots, k_{33}) \in \mathbb{R}^9,\\ &\text{subject to:} \quad v^{RHP}(K) \dot{>} 0, \quad K \odot \mathbb{K} = K,\end{aligned}$$

and solving it via Algorithm 1, we obtain the following stabilizing SOF: $k_{13} = -7$, $k_{31} = 0.5$, and $k_{33} = 4$. Doing the similar process for the following binary structure:

$$\mathbb{K} = \begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix},$$

we get UNSAT, i.e., no stabilizing controller exists with such a binary structure. An advantage of an SMT-based solution methodology is that it provides a guarantee on the non-existence of a solution in case of unsatisfiability while non-SMT-based methods proposed by [13]–[15], [33] do not provide such a guarantee on the non-existence of a solution.

Example 6 (Logical and Structural Constraints): Considering the matrices $A \in \mathbb{R}^{6\times6}$, $B \in \mathbb{R}^{6\times5}$, and $C \in \mathbb{R}^{4\times6}$ in [34], we aim at synthesizing a stabilizing SOF subject to the following pre-specified binary structure and logical implication:

$$\mathbb{K} = \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}, \text{ if } \ k_{21}k_{31} \neq 0 \ \text{ then } \ k_{51} = 0.$$

We construct the following SMT problem:

$$\begin{aligned}
&\exists(\mathrm{b}_1, \mathrm{b}_2, \mathrm{k}_{11}, \ldots, \mathrm{k}_{54}) \in \mathbb{B}^2 \times \mathbb{R}^{20},\\
&\text{subject to: } \mathrm{v}^{\mathrm{RHP}}(\mathrm{K}) \dot{>} 0,\ \mathrm{K} \odot \mathbb{K} = \mathrm{K},\\
&(\mathrm{b}_1 \longrightarrow \mathrm{b}_2) \longleftrightarrow \mathtt{TRUE},\\
&\mathrm{b}_1 \longleftrightarrow (\mathrm{k}_{21}\mathrm{k}_{31} \neq 0), \quad \mathrm{b}_2 \longleftrightarrow (\mathrm{k}_{51} = 0),
\end{aligned}$$

Solving this SMT problem via Algorithm 1, we obtain the following stabilizing SOF: $\mathrm{k}_{21} = 0$, $\mathrm{k}_{31} = -4$, and $\mathrm{k}_{51} = 0.5$.

V. Conclusions and future Directions

We put forth a methodology that leverages SMT solvers to obtain stabilizing controllers that satisfy structural and logical constraints. In our approach, the stabilization constraint is imposed via MPIs, which, in general, lead to high-complexity SMT formulae. The fact that for positive systems, the stability MPIs have order two or less explains why our approach can solve large instances for this class of problems. This stands in contrast to other cases in which the MPIs encode the Routh-Hurtz criterion and lead to SMT formulae that can be solved only for small problems. We also observed that the execution time for the SOF problem decreases as B, C, or K become sparser. Hence, determining structural and logical constraints, as well as sub-classes of B, C, for which one can efficiently solve large-scale SOF problem is a pertinent future direction.

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