Rapid Communications

Signaling and scrambling with strongly long-range interactions

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Strongly long-range interacting quantum systems—those with interactions decaying as a power law $1/r^{\alpha}$ in the distance r on a D-dimensional lattice for $\alpha \leqslant D$—have received significant interest in recent years. They are present in leading experimental platforms for quantum computation and simulation, as well as in theoretical models of quantum-information scrambling and fast entanglement creation. Since no notion of locality is expected in such systems, a general understanding of their dynamics is lacking. In a step towards rectifying this problem, we prove two Lieb-Robinson-type bounds that constrain the time for signaling and scrambling in strongly long-range interacting systems, for which no tight bounds were previously known. Our first bound applies to systems mappable to free-particle Hamiltonians with long-range hopping, and is saturable for $\alpha \leqslant D/2$. Our second bound pertains to generic long-range interacting spin Hamiltonians and gives a tight lower bound for the signaling time to extensive subsets of the system for all $\alpha < D$. This many-site signaling time lower bounds the scrambling time in strongly long-range interacting systems.

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Introduction. In nonrelativistic quantum mechanics, Lieb-Robinson bounds provide a notion of causality [1], limiting the speed of information propagation (or signaling) to a finite value in lattice systems with short-range interactions. This bounded signaling speed has strong implications for quantum information and condensed matter physics, leading to entanglement area laws [2] and the existence of topological order [3]. However, it remains an open question whether the signaling speed must be finite if interactions are long-ranged and decay as an inverse power law $1/r^{\alpha}$ in the interparticle distance r. Such power-law interacting systems arise in experimental platforms for quantum computation and quantum simulation, including Rydberg atoms [4], trapped ions [5], polar molecules [6], defect centers in solids [7], and atoms trapped along photonic crystals [8]. The lack of a bounded signaling speed in these systems makes it challenging to understand and predict their dynamics.

For power-law interacting systems with α greater than the lattice dimension D, a finite signaling speed has been shown to persist to some intermediate distance and time [9]. At long distances (or times), recent developments show that the signaling speed will diverge at most polynomially in time for $\alpha > 2D$ [10,11], ruling out the exponential divergence suggested by earlier results [12]. For $\alpha \leqslant 2D$, which is the case for most experimental long-range interacting systems, an exponentially growing signaling speed has yet to be ruled out, making the fate of causality far from settled.

In this work, we focus on the regime of *strongly long-range* interacting systems, where interaction energy per site diverges, thus implying $\alpha \leqslant D$ [13–16]. Note that even if one normalizes the interaction strength to make energy *extensive* (i.e., proportional to the number of lattice sites), these systems are still fundamentally different from those with $\alpha > D$ (as energy is in general no longer additive for subsystems [17]). To avoid confusion, we will not perform any normalization of interaction strength throughout this Rapid Communication, as such normalization can be performed later by rescaling time without changing the physics implied by our results [15].

Apart from their existence in experimental platforms [5,6,18–20], strongly long-range interacting systems have received much theoretical interest due to their applications in spin squeezing [21], novel behavior in dynamical critical scaling [22,23], divergent equilibration time [13], and close relation to fast quantum-information scrambling [24–30]. The phenomenology of these systems differs from that of their short-range counterparts at a fundamental level, and thus require new theoretical understandings. Two fundamental questions about these systems are (1) what is the shortest time t_{si} needed to send a signal from one site to a site located an extensive distance away, and (2) what is the shortest time t_{sc} needed to scramble the information stored in the system [31]?

There have been a number of attempts to answer the above two questions, with limited success. For the first question, Refs. [32–34] show that in certain strongly long-range interacting systems with $\alpha \leqslant D$, information and correlations can spread across the entire system in a finite time that is independent of the number of sites, N. (For certain systems not engineered for fast signaling or scrambling, information propagation may even be suppressed [35].) The Lieb-Robinson-type bound derived in Ref. [15], however, suggests that the signaling time can vanish in the $N \to \infty$ limit, and does not

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rule out the possibility of $t_{\rm si}$ scaling as $\log(N)N^{2\alpha/D}/N^2$ for $\alpha < D$. No protocol that we know of comes close to achieving such fast signaling. As for scrambling, Ref. [36] shows that the scrambling time can be lower-bounded by $t_{\rm sc} \gtrsim 1/N$ for $\alpha = 0$, whereas the fastest-known scramblers are conjectured to be able to scramble in time $t_{\rm sc} \propto \log(N)/\sqrt{N}$ [28].

While the definitive answers to these two questions remain to be found, we present several advances in this Rapid Communication. First, we prove a new bound for systems that can be mapped to free bosons or fermions with $1/r^\alpha$ hopping strength, which leads to a signaling-time bound of $t_{\rm si} \gtrsim N^{\alpha/D}/\sqrt{N}$. While no previous bound has been given specifically for free-particle systems, the best existing result for interacting systems yields a significantly looser bound of $t_{\rm si} \gtrsim \log(N)N^{2\alpha/D}/N^2$ [15,37]. Notably, our free-particle bound is tight for $\alpha \leqslant D/2$, as we show that it can be saturated by a new quantum state transfer protocol.

We also prove a bound of $t_{\rm si} \gtrsim \log(N)N^{\alpha/D}/N$ for general interacting spin systems, which—while improving significantly over the previous best bound mentioned above [15]—is still not known to be tight. Building on this second result, we prove a tight bound for "many-site signaling" (from one site to an extensive part of the system). This many-site signaling bound leads to a scrambling-time bound of $t_{\rm sc} \gtrsim N^{\alpha/D}/N$, which generalizes the result in Ref. [36] of $t_{\rm sc} \gtrsim 1/N$ to all $\alpha < D$.

Tight bound for free particles. We first prove a Lieb-Robinson-type bound for noninteracting bosons/fermions on a lattice. Consider the following free-particle Hamiltonian $H(t)$ defined on a D-dimensional lattice Λ with N sites:

$$H(t) = \sum_{\substack{i,\, j \in \Lambda \\ i < j}} [J_{ij}(t)c_i^\dagger c_j + \text{H.c.}] + \sum_{i\in\Lambda} B_i(t)c_i^\dagger c_i, \qquad (1)$$

where $c_i^\dagger$ (c_i) represents the creation (annihilation) operator. The hopping strength $J_{ij}(t)$ and chemical potential $B_i(t)$ can depend on time and we do not impose any constraint on them for now. We denote an operator A at time t in the Heisenberg picture as $A(t) = U^\dagger(t)AU(t)$, where $U(t) \equiv \mathcal{T}e^{-i/\hbar\int_0^t H(t')\,dt'}$ is the time evolution operator ($\hbar = 1$). The operator norm of A will be denoted by $\|A\|$.

Theorem 1. For the Hamiltonian defined in Eq. (1) and any pair of distinct sites $X, Y \in \Lambda$,

$$\|[c_X(t), c_Y^\dagger]\| \leqslant \int_0^t d\tau \sqrt{\sum_{i\in\Lambda} |J_{iX}(\tau)|^2}. \qquad (2)$$

We use $[\cdot,\cdot]$ to denote the commutator for bosons and the anticommutator for fermions.

Roughly speaking, the quantity $\|[c_X(t), c_Y^\dagger]\|$ measures the overlap between the support of the operator $c_X(t)$ (which expands from site X due to hopping terms) and the site Y. As a result, it also quantifies the amount of information that can be sent between X and Y in a given time t. Indeed, we define the signaling time $t_{\rm si}$ as the minimal time required to achieve $\|[c_X(t), c_Y^\dagger]\| > \delta$ for some positive constant δ. Note that we do not expect the chemical potential strength $B_i(t)$ to show up in the bound, as on-site Hamiltonian terms do not change the support of $c_X(t)$.

If the hopping terms in the Hamiltonian are short-ranged (e.g., nearest-neighbor), one might expect $\|[c_X(t), c_Y^\dagger]\|$ to decay exponentially in the distance r_{XY} between X and Y, due to the strong notion of causality that follows from the Lieb-Robinson bound [1]. Additionally, if the hopping strength decays as a power law ($|J_{ij}(t)| \leqslant 1/r^\alpha$) with $\alpha > D$, intuition would suggest that $\|[c_X(t), c_Y^\dagger]\|$ decays algebraically in r_{XY} [9,12], indicating a weak notion of causality. However, the right-hand side of Eq. (2) has no dependence on r_{XY}. This is because the bound is tailored to strongly long-range hoppings with $\alpha < D$, which makes it loose for shorter-ranged long-range hoppings.

Assuming that $|J_{ij}(t)| \leqslant 1/r^\alpha$, we can simplify Eq. (2) to

$$\|[c_X(t), c_Y^\dagger(0)]\| \leqslant t \times \begin{cases} O(1), & \alpha > D/2 \\ O(N^{1/2-\alpha/D}), & 0 \leqslant \alpha \leqslant D/2, \end{cases} \qquad (3)$$

where N is the number of lattice sites and O is the asymptotic "big-O" notation [38]. Therefore, for $\alpha \leqslant D/2$, it takes a time $t_{\rm si} = \Omega(N^{\alpha/D}/\sqrt{N})$ [38] to signal from site X to site Y, independent of the distance between X and Y.

In the next section, we show that for $\alpha \leqslant D/2$, the bound in Eq. (2) can be saturated (up to a factor of 2) by engineered free-particle Hamiltonians. This leads to the conclusion that causality can completely vanish—in the sense that signals can be sent arbitrarily fast given large enough N—for a strongly long-range hopping system with $\alpha < D/2$. It remains an open question whether such a statement can be generalized to systems with $D/2 \leqslant \alpha < D$ for either free or interacting particles.

Proof of Theorem 1. Let us first go into the interaction picture of $\sum_i B_i(t)c_i^\dagger c_i$ to eliminate the on-site terms from the Hamiltonian in Eq. (1). [This imparts a time-dependent phase $e^{i\phi_{jk}(t)}$ onto the hopping term $J_{jk}(t)$ for some $\phi_{jk}(t) \in [0, 2\pi)$ and $j \neq k$, which, since it does not change the value of $|J_{jk}(t)|$, does not affect the overall bound.] We now have a pure hopping Hamiltonian $H_I(t) = \sum_{ij} \tilde{J}_{ij}(t)c_i^\dagger c_j$ with $|\tilde{J}_{ij}(t)| \equiv |J_{ij}(t)|$. Because $H_I(t)$ is a quadratic Hamiltonian, $c_X(t)$ is a time-dependent linear combination of annihilation operators on every site, and we can write $[c_X(t), c_Y^\dagger] \equiv f_{XY}(t)\mathbb{1}$, where $f_{XY}(t)$ is a number and $\mathbb{1}$ represents the identity operator. Given that $U_I(t)|0\rangle = |0\rangle$, where $U_I(t)$ is the time-evolution operator corresponding to $H_I(t)$, and $c_X(t)|0\rangle = 0$, we have

$$f_{XY}(t) = \langle 0|[c_X(t), c_Y^\dagger]|0\rangle = \langle 0|c_X(0)U_I(t)c_Y^\dagger|0\rangle. \qquad (4)$$

For convenience, we define the (normalized) states $|\psi_X\rangle \equiv c_X^\dagger(0)|0\rangle$ and $|\psi_Y(t)\rangle \equiv U_I(t)c_Y^\dagger|0\rangle$. Taking the time derivative of Eq. (4) gives

$$\frac{df_{XY}}{dt} = -i\langle\psi_X|H_I(t)|\psi_Y(t)\rangle. \qquad (5)$$

By the Cauchy-Schwarz inequality,

$$\left\|\frac{df_{XY}}{dt}\right\| \leqslant \|H_I(t)|\psi_X\rangle\|\,\||\psi_Y(t)\rangle\| \qquad (6)$$

$$= \|H_I(t)|\psi_X\rangle\| = \sqrt{\sum_{i\in\Lambda} |\tilde{J}_{iX}(t)|^2}. \qquad (7)$$

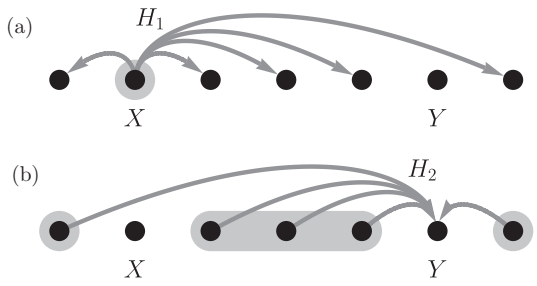

FIG. 1. A fast quantum-state transfer protocol for a long-range Hamiltonian acting on a lattice of dimension $D = 1$ with $N = 7$ sites. The strengths of the hopping terms are bounded by a power law $1/r^\alpha$ in the distance r. The active interactions in each time step are depicted as directed edges with uniform weights. (a) The site X is initially in the state $|\psi\rangle$ (gray circle), with the other (unoccupied) sites in state $|0\rangle$. Time-evolving by the Hamiltonian H_1 for time $O(N^{\alpha/D-1/2})$ (indicated by gray arrows) yields a superposition of the $|0\rangle^{\otimes N}$ state and a symmetric $|W\rangle$ state over the remaining $N-2$ sites. (b) Applying the Hamiltonian H_2 for the same duration of time completes the state transfer of $|\psi\rangle$ to the target site Y.

The last equality follows from $|\psi_X\rangle$ being a single excitation localized on site X and $H_I(t)$ consisting only of hopping terms $\tilde{J}_{ij}(t)c_i^\dagger c_j$. Applying the fundamental theorem of calculus yields the bound on $f_{XY}(t)$ and hence Theroem 1.

Saturating the free-particle bound. We now show that the bound in Theroem 1 can be saturated by engineered Hamiltonians that can also be used to perform fast quantum-state transfer. In particular, the protocol presented here has a state transfer time of $T = O(N^{\alpha/D}/\sqrt{N})$, which, for $\alpha \leqslant D/2$, improves over the fastest-known state transfer protocol using long-range interactions [34].

Our setup for the state transfer task is depicted in Fig. 1. We initialize a lattice with N sites in a tensor product of unoccupied states $|0\rangle$ and some unknown normalized bosonic/fermionic state $|\psi\rangle = a|0\rangle + b|1\rangle$ on a single site X. The goal of state transfer is to move $|\psi\rangle$ to the target site Y after the system time-evolves by a $|\psi\rangle$-independent (but possibly time-dependent) Hamiltonian $H(t)$ [39,40].

The unitary time-evolution operator $U(T)$ can be said to implement state transfer in time T if it satisfies the following condition:

$$|\langle 0|_X \langle 0|^{\otimes N-2} \langle\psi|_Y U(T) |\psi\rangle_X |0\rangle^{\otimes N-2} |0\rangle_Y| = 1. \tag{8}$$

We refer to the left-hand side of Eq. (8) as the *fidelity* of the state transfer, which can be bounded directly by a Lieb-Robinson-type bound on $H(t)$ such as Eq. (2) [40].

We label the sites that are not X or Y by $1, \ldots, N-2$ and denote the furthest distance between any pair of sites by $L = O(N^{1/D})$. Our state transfer protocol is given by the following piecewise time-independent Hamiltonian:

$$H(t) = \begin{cases} H_1 = \frac{1}{L^\alpha} \sum_{i=1}^{N-2} c_X^\dagger c_i + \text{H.c.}, & 0 < t < \frac{T}{2}, \\ H_2 = \frac{1}{L^\alpha} \sum_{i=1}^{N-2} c_i^\dagger c_Y + \text{H.c.}, & \frac{T}{2} < t < T, \end{cases} \tag{9}$$

where $T = \pi L^\alpha/\sqrt{N-2}$ is the total time for the protocol. Note that while $H(t)$ satisfies the constraint $|J_{ij}(t)| \leqslant 1/r_{ij}^\alpha$ assumed in Eq. (3), the corresponding $J_{ij}(t)$ terms do not actually vary with the distances between sites.

Evolving the initial state $|\Psi\rangle \equiv |\psi\rangle_X |0\rangle^{\otimes N-2} |0\rangle_Y$ by H_1 for time $T/2$ yields the intermediate state

$$e^{-iH_1 T/2}|\Psi\rangle = a|0\rangle^{\otimes N} + b|0\rangle_X |W\rangle |0\rangle_Y. \tag{10}$$

Here, $|W\rangle = \frac{1}{\sqrt{N-2}} \sum_{i=1}^{N-2} c_i^\dagger |0\rangle^{\otimes N-2}$ is the W state over the $N-2$ remaining sites. Further evolving the state by H_2 for time $T/2$ yields the final state:

$$e^{-iH_2 T/2} e^{-iH_1 T/2} |\Psi\rangle = |0\rangle_X |0\rangle^{\otimes N-2} (a|0\rangle_Y + b|1\rangle_Y). \tag{11}$$

Thus we have achieved perfect quantum-state transfer in time $T = O(N^{\alpha/D}/\sqrt{N})$. Note that the distance between X and Y on the lattice does not appear in the state transfer time. Setting $b = 1$ in the above protocol leads to

$$\langle\Psi|[c_X^\dagger(T), c_Y]|\rangle\Psi = \frac{1}{2}\int_0^T d\tau \sqrt{\sum_{i\in\Lambda} |J_{iX}(\tau)|^2}. \tag{12}$$

Thus, the bound in Eq. (2) is saturated up to a factor of 2.

It should be pointed out that, for $\alpha > D/2$, the above protocol requires a time that increases with N, which is slower than for the previous result in Ref. [34]. While that protocol has a state transfer time that is constant for $\alpha \leqslant D$, it uses an engineered Hamiltonian with interactions, and therefore cannot be applied to systems of noninteracting particles. In general, allowing interactions may increase the rate of information propagation, and proving a Lieb-Robinson-type bound in these situations requires a different approach.

Improved bound for general interacting systems. We now derive bounds on the signaling time that extend beyond free-particle Hamiltonians. Without loss of generality, we study a generic interacting spin Hamiltonian $H(t) = \sum_{i<j} h_{ij}(t)$ where $\|h_{ij}(t)\| \leqslant 1/r_{ij}^\alpha$ and on-site interactions have been eliminated by going into an interaction picture. We will bound the quantity $\|[A(t), B]\|$, where A and B are arbitrary operators supported on sets of sites X and Y, respectively, using the following Lieb-Robinson series [12]:

$$\|[A(t), B]\| \leqslant 2\|A\|\|B\||X||Y| \sum_{k=1}^{\infty} \frac{(2t)^k}{k!} \mathcal{J}^k(X, Y), \tag{13}$$

$$\mathcal{J}^k(X, Y) \equiv \sum_{i_1,\ldots,i_{k-1}} J_{Xi_1} J_{i_1 i_2} \ldots J_{i_{k-1}Y}. \tag{14}$$

Here, $|X|$ stands for the number of sites X acts on. Each term in Eq. (14) represents a sequence of k directed hops in the lattice that originates at site X and ends at site Y. For distinct sites i and j, $J_{ij} = 1/r_{ij}^\alpha$ represents a directed hop from i to j. For technical reasons, we set $J_{ii} = \sum_{j\neq i} J_{ij}$ [41].

Since J_{ij} decays slowly in r_{ij} for $\alpha < D$, our improved bound on $\|[A(t), B]\|$ requires bounding each term in Eq. (14) using a new summation technique [42] absent in previous efforts [12,15]. This technique is particularly effective for tightening existing Lieb-Robinson bounds for strongly long-range interacting systems. The result (assuming $\alpha < D$) is

$$\|[A(t), B]\| \leqslant 2\|A\|\|B\||X||Y| \left(\frac{e^{\Theta(N^{1-\alpha/D})t} - 1}{\Theta(N^{1-\alpha/D}) r_{XY}^\alpha} \right). \tag{15}$$

The factor $\Theta(N^{1-\alpha/D})$ [38] comes from the total interaction energy per site given by J_{ii}.

We consider now the case of signaling between subsystems X and Y of a system Λ with $|X|, |Y| = O(1)$. We formally define t_{si}, the signaling time from X to Y, as the smallest time t such that for a fixed constant $\delta = \Theta(1)$, there exist unit-norm operators A and B supported on X and Y, respectively, such that $\|[A(t), B]\| > \delta$ [36]. If we further assume that X and Y are separated by an extensive distance $r_{XY} = \Theta(N^{1/D})$, then the following lower bound holds for the signaling time:

$$t_{\mathrm{si}} = \Omega\left(\frac{\log(N)}{N^{1-\alpha/D}}\right). \tag{16}$$

This bound supersedes the naive signaling-time bound of $t_{\mathrm{si}} = \Omega(1/N^{1-\alpha/D})$ one would obtain via normalization of interaction energy per site. While we do not know of any examples that saturate this bound, it is the tightest-known signaling-time bound for strongly long-range interacting systems. Indeed, the bound is close to being saturated in the limit of $\alpha \to D^-$, as the state transfer protocol in Ref. [34] shows that $t_{\mathrm{si}} = O(\log N)$ can be achieved at $\alpha = D$. Unfortunately, generalizing our bound in Eq. (15) to the case of $\alpha = D$ leads to $t_{\mathrm{si}} = \Omega(1)$ [42], which is not saturated by Ref. [34].

Many-site signaling and scrambling bounds. Of recent interest in the fields of theoretical high-energy and condensed matter physics has been the phenomenon of quantum information scrambling [28,36,43–51]. Previous work on scrambling in power-law interacting systems has focused primarily on numerical analysis [52,53], whereas general mathematical results are lacking. Only in all-to-all interacting systems (which can be treated as the limit $\alpha = 0$) have Lieb-Robinson-type bounds been used to bound the scrambling time [36]. Using the bound derived in Eq. (15), we can prove a scrambling-time bound for systems with $0 < \alpha < D$, a regime for which no better result is known.

To derive a bound on the scrambling time, we first derive a bound on the many-site signaling time. We define the "many-site signaling time" t_{ms} to be the smallest t required to signal from X to a Y that has extensive size. Lieb-Robinson-type bounds such as Eq. (15) naturally limit the time for many-site signaling. However, a direct application of Eq. (15) to many-site signaling leads to a loose bound. Instead, a more refined technique that sums over all sites within the subsets X and Y yields a tighter bound [54]:

$$\|[A(t), B]\| \leqslant 2\|A\|\|B\| \sum_{i\in X, j\in Y} \frac{e^{\Theta(N^{1-\alpha/D})t} - 1}{\Theta(N^{1-\alpha/D}) r_{ij}^{\alpha}}. \tag{17}$$

This bound reduces to Eq. (15) when $|X|, |Y| = 1$.

The scrambling time t_{sc} corresponds to the minimal time required for a system of N spins on a lattice Λ to evolve from a product state to a state that is nearly maximally entangled on all subsystems of size kN for some constant $0 < k < \frac{1}{2}$ [36]. From this definition, it can be shown that any information initially contained in a finite-sized subsystem $S \subset \Lambda$ is no longer recoverable from measurements on S alone [36]. That information is not lost, however, but can be recovered from the complement $\bar{S} = \Lambda \setminus S$ of S [44,55]. As a result, scrambling implies the ability to signal from S to $\bar{S}$ [36]. Thus, t_{sc} is lower bounded by the time it takes to signal from a subset S with size $\Theta(1)$ to its complement with size $\Theta(N)$, which corresponds to the many-site signaling time.

Using Eq. (17), we obtain the following scrambling-time bound for $0 \leqslant \alpha < D$:

$$t_{\mathrm{sc}} \geqslant t_{\mathrm{ms}} = \Omega\left(\frac{1}{N^{1-\alpha/D}}\right). \tag{18}$$

Note that this bound differs from Eq. (16) by a $\log(N)$ factor. Additionally, although the bound on t_{si} in Eq. (16) may allow further tightening, the bound on t_{ms} in Eq. (18) cannot be generically improved for $0 \leqslant \alpha < D$. To see this, we consider a long-range Ising Hamiltonian $H = \sum_{i\neq j} J_{ij}\sigma_i^z\sigma_j^z$, with $J_{ij} = 1/r_{ij}^{\alpha}$. For simplicity, we consider the subset S to be a single site indexed by i and construct operators $A = \sigma_i^+$ and $B = \bigotimes_{j\neq i}\sigma_j^+$ that are supported on S and $\bar{S}$, respectively. We can analytically calculate the expectation value of $[A(t), B]$ in an initial state $|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle^{\otimes N} + |1\rangle^{\otimes N}]$ [54]:

$$\langle\psi|[A(t), B]|\psi\rangle = \sin\left(2t\sum_{j\neq i} J_{ij}\right). \tag{19}$$

Using $J_{ij} = 1/r_{ij}^{\alpha}$, we find that the signaling time of this protocol is $t = O(N^{\alpha/D-1})$ for $0 \leqslant \alpha < D$, which saturates the many-site signaling-time bound in Eq. (18) [56]. This does not, however, imply that the corresponding scrambling-time bound is tight. In fact, previous work suggests that fast scramblers in all-to-all interacting systems ($\alpha = 0$) can scramble in time $t_{\mathrm{sc}} = O(\log(N)/\sqrt{N})$ [36,57]. This suggests that future improvements to the scrambling-time bound may be possible.

Conclusions and outlook. In this Rapid Communication, we make several advances in bounding the signaling and scrambling times in strongly long-range interacting systems. Our results suggest a number of possible future directions. One is to find the optimal signaling-time bound for general strongly long-range interacting systems. Previously, this has been an outstanding challenge; we now know of a free-particle bound that is tight for $\alpha \in [0, D/2]$ and a general bound that is nearly tight as $\alpha \to D^-$. The search for the optimal bound for $\alpha \in [0, D]$ has thus been narrowed down significantly. Another direction is to investigate how interactions can speed up signaling. We expect weakly interacting systems to possess a similar signaling-time bound to our free-particle bound, as the dynamics in such systems can often be treated using spin-wave analysis [58]. But for strongly interacting systems, it remains unclear how much speedup one can obtain over noninteracting systems.

Additionally, our bound for signaling to an extensive number of sites hints at a strategy for achieving a better scrambling bound. In particular, the protocol that saturates our many-site signaling bound relies on an initial entangled state, whereas the definition of scrambling assumes that the system begins in a product state. It may be possible to improve the scrambling-time bound by explicitly restricting our attention, when bounding t_{ms}, to initial product states.

Finally, we expect that the improved Lieb-Robinson-type bounds derived in this work may lead to a better understanding of the spreading of out-of-time-order correlators [59], the growth of entanglement entropy [54], and thermalization timescales [60] in strongly long-range interacting systems.

In addition, there are connections between Lieb-Robinson-type bounds and the critical scaling of the defects appearing in a quantum system driven across its quantum critical point

[the celebrated Kibble-Zurek (KZ) mechanism]. It remains an open question whether the KZ hypothesis can be shown to hold for strongly long-range systems.

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