Computing Balanced Convex Partitions of Lines*

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Abstract. Dujmović and Langerman (2013) proved a ham-sandwich cut theorem for an arrangement of lines in the plane. Recently, Xue and Soberón (2019) generalized it to balanced convex partitions of lines in the plane. In this paper, we study the computational problems of computing a ham-sandwich cut balanced convex partitions for an arrangement of lines in the plane. We show that both problems can be solved in polynomial time.

Keywords: ham-sandwich theorem · arrangement of lines · balanced convex partitions.

1 Introduction

Dujmović and Langerman [5] proved a ham-sandwich cut theorem for an arrangement of lines in the plane.

Theorem 1 (Dujmović and Langerman [5]). *Given two finite sets A, B of lines each in the plane, if no two lines of $A \cup B$ are parallel, there exists a line ℓ such that each of the two closed half-spaces it defines encloses a subset of at least $\sqrt{|A|}$ lines of A and a subset of at least $\sqrt{|B|}$ lines of B.*

We show that the ham-sandwich line can be computed in polynomial time. The ham sandwich theorem has been generalized to convex partitions of the plane. The following theorem was proven independently by Bespamyatnikh, Kirkpatrick, and Snoeyink [1], by Ito, Uehara, and Yokoyama [9] and by Sakai [13].

Theorem 2 ([1, Theorem 10]). *Given rn red and rm blue points in the plane in general position, there exists a subdivision of the plane into r convex regions each of which contains n red and m blue points.*

A subdivision of the plane satisfying Theorem 2 is called *equitable* [1]. The main tool in the proof of Theorem 2 is equitable k-cuttings for $k = 2, 3$. A 2-cutting is simply a partition of the plane by a line. A 3-cutting is a partition of the plane into 3 wedges by 3 rays starting from the same point.

Recently, Xue and Soberón [15] generalized Theorem 1 as follows. Let L be a set of lines in the plane. We denote by $I(L)$ the set of pairwise intersection points of L. We use notation $[k] = \{1, 2, \ldots, k\}$.

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Theorem 3 (Xue and Soberón [15]). *Let A, B be two finite sets of lines in $\mathbb{R}^2$ such that $A \cup B$ is in general position, and let r be a fixed positive integer. Then, there is a convex partition $(C_1, \ldots, C_r)$ of $\mathbb{R}^2$ into r parts such that for all $j \in [r]$ there exist sets $A_j \subset A$, $B_j \subset B$ such that $I(A_j) \subset C_j, I(B_j) \subset C_j$ and*

$$|A_j| \geq r^{\ln(2/3)}|A|^{1/r} - 2r, \qquad |B_j| \geq r^{\ln(2/3)}|B|^{1/r} - 2r.$$

In this paper a convex partition satisfying the conditions of Theorem 3 is called an *equitable partition* of sets A and B.

The proof of Theorem 3 is similar to the proof of Theorem 2 [1]. For this, we need a measure μ defined as follows. For a set of lines L in the plane and a set $K \subseteq \mathbb{R}^2$, we define

$$\mu_L(K) = \max\{|L'| : L' \subseteq L, I(L') \subseteq K\}.$$

The main idea is to apply equitable k-cuttings for $k = 2, 3$ to obtain the desired partition.

Definition 1. *A 2-cutting of the plane into two parts (C_1, C_2) is called* equitable *if there exist two positive integers r_1, r_2 such that $r_1 + r_2 = r$ and*

$$\mu_A(C_i) \geq \left(\frac{2|A|}{3}\right)^{r_i/r} - 2, \quad \mu_B(C_i) \geq \left(\frac{2|B|}{3}\right)^{r_i/r} - 2 \qquad \text{for } i = 1, 2.$$

Definition 2. *A 3-cutting of the plane into three parts (C_1, C_2, C_3) is called* equitable *if there exist three positive integers r_1, r_2, r_3 such that $r_1 + r_2 + r_3 = r$ and*

$$\mu_A(C_i) \geq \left(\frac{2|A|}{3}\right)^{r_i/r} - 2, \quad \mu_B(C_i) \geq \left(\frac{2|B|}{3}\right)^{r_i/r} - 2 \qquad \text{for } i = 1, 2, 3.$$

A k-cutting is called *convex* if its parts are convex. Theorem 3 follows from the following lemma [15].

Lemma 1. *Let A, B be two finite sets of points in the plane, each in general position, and $r \geq 2$ be a positive integer. Then, there exists an equitable k-cutting for some $k \in \{2, 3\}$.*

Our results are the following.

- We show that the ham-sandwich line for two sets of lines in the plane can be computed in polynomial time (Theorem 4 in Section 3).
- An *equitable partition* of two sets of lines in the plane can be computed in polynomial time (Sections 4-7).

2 Preliminaries

The key lemma in the proof of Theorem 1 (and Theorem 3) is the following lemma [5].

Lemma 2. *For any two open halfplanes H_1 and H_2 in the plane and any finite set L of lines*

$$\mu_L(H_1 \cup H_2) \leq \mu_L(H_1) \cdot \mu_L(H_2).$$

Xue and Soberón [15] proved a key lemma for 3-cuttings.

Lemma 3. *For any convex partition of the plane into three wedges C_1, C_2, C_3 and any set A of n lines in general position in the plane*

$$\mu_A(C_1)\mu_A(C_2)\mu_A(C_3) \geq \frac{2n}{3}.$$

3 Computing Ham-Sandwich Cuts

Let $h^-(t)$ denote the set $\{(x, y) \mid x \leq t\}$ and let $h^+(t) = \{(x, y) \mid x \geq t\}$. The following problem is the basis of our algorithms.

Problem P1. Given a set L of lines in the plane and an integer $n_0 \leq n$, find smallest x_0 such that $\mu_L(h^-(x_0)) \geq n_0$.

Lemma 4. *Problem P1 can be solved in $O(n \log^2 n)$ time.*

Proof. For a given t, we can compute $\mu_L(h^-(t))$ in $O(n \log n)$ time as follows. Let $L = \{l_1, \ldots, l_n\}$. For each line l_i in L, compute its slope s_i and intercept r_i. Therefore, the equation of the line l_i is $y = s_i(x - t) + r_i$. We sort the line by the intercept and assume that $l_1, \ldots, l_n$ is the sorted order, i.e. $r_1 \leq r_2, \ldots, r_n$. Compute $l(t)$, the length of the longest increasing subsequence of $s_1, s_2, \ldots, s_n$ in $O(n \log n)$ time [8, 14].

To compute x_0, we use an algorithm for the slope selection [2–4, 10, 12]. In the dual setting this problem is the following. Given a set L of n non-vertical lines and an integer number $k \in [\binom{n}{2}]$, we want to find two lines from L such that their intersection point has kth smallest x-coordinate. This problem can be solved in $O(n \log n)$ time. We apply the binary search on the values of k and compute smallest x_0 such that $\mu_L(h^-(x_0)) \geq n_0$ using $\log \binom{n}{2} = O(\log n)$ tests. The total running time is $O(n \log^2 n)$. □

Theorem 4. *Let A and B be two finite sets of lines each in the plane such that no two lines of $A \cup B$ are parallel. A ham-sandwich line for the arrangement of $A \cup B$ can be computed in $O(n^2 \log^2 n)$ time where $n = |A| + |B|$.*

Proof. Using the algorithm for problem P1, we can compute

- x_0, the smallest t such that $\mu_A(h^-(t)) \geq \sqrt{|A|}$, and
- x_1, the largest t such that $\mu_A(h^+(t)) \geq \sqrt{|A|}$.

The value x_0 is computed by applying the algorithm to set A and the value x_1 is computed by applying the algorithm to set A' that is symmetric to A, i.e. if a line with the equation $y = ax + b$ is in A then the line with the equation $y = -ax - b$ is in A'. Then $x_1 = -t$ where t is the output value of the algorithm applied to set A'.

The interval $[x_0, x_1]$ is not empty because $\mu_A(h^-(t)) \cdot \mu_A(h^+(t)) \geq |A|$ for any value of $t \in \mathbb{R}$. In other words, $\mu_A(h^-(t)) \geq \sqrt{|A|}$ or $\mu_A(h^+(t)) \geq \sqrt{|A|}$. The running time for computing $[x_0, x_1]$ is $O(n \log^2 n)$.

We also compute an interval $[x'_0, x'_1]$ for set B using μ_B. If the intervals $[x_0, x_1]$ and $[x'_0, x'_1]$ intersect then the line with equation $x = t$ is a ham-sandwich line for any $t \in [x_0, x_1] \cap [x'_0, x'_1]$. Suppose that the intervals $[x_0, x_1]$ and $[x'_0, x'_1]$ do not intersect. Wlog the interval for A is to the left of the interval for B, i.e. $x_1 < x'_0$.

Let $A(\phi)$ (resp. $B(\phi)$) be the set of lines A rotated clockwise by an angle ϕ about the origin. Let $X(\phi) = [x_0(\phi), x_1(\phi)]$ and $X'(\phi) = [x'_0(\phi), x'_1(\phi)]$ be the corresponding intervals. We want to find an angle ϕ such that the intervals $X(\phi)$ and $X'(\phi)$ intersect.

For a set of lines L, we denote by $\mathcal{A}(L)$ be the arrangement of lines L. Consider the arrangement $\mathcal{A}(A \cup B)$. Let V be the set of $\binom{|A|+|B|}{2}$ vertices of the arrangement $\mathcal{A}(A \cup B)$. Let L be the set of all lines that contain at least two points of V. Let $\phi_1, \phi_2, \ldots, \phi_{|L|}$ be the sorted sequence of the slopes of lines in L. Consider an interval $I = (\phi_i, \phi_{i+1})$. For any $\phi \in I$, the numbers $x_0(\phi), x_1(\phi), x'_0(\phi), x'_1(\phi)$ preserve the order since each of them corresponds to a vertex of V rotated clockwise by angle ϕ. Therefore the intervals $X(\phi)$ and $X'(\phi)$ preserve the relation for all $\phi \in I$, i.e. either (i) they intersect or (ii) $X(\phi)$ is to the left of $X'(\phi)$ or (iii) $X(\phi)$ is to the right of $X'(\phi)$, for all $\phi \in I$.

We want to find an interval $I = (\phi_i, \phi_{i+1})$ such that $X(\phi)$ and $X'(\phi)$ intersect for all $\phi \in I$. Note that, for all ϕ in the first interval $\phi \in (-\infty, \phi_1)$ the interval $X(\phi)$ is to the left of the interval $X'(\phi)$ since $0 \in (-\infty, \phi_1)$. Also for all ϕ in the last interval $\phi \in (\phi_{|L|}, \infty)$ the interval $X(\phi)$ is to the right of the interval $X'(\phi)$ since $\pi \in (\phi_{|L|}, \infty)$. We apply binary search on the sequence $\phi_1, \phi_2, \ldots, \phi_{|L|}$. For any interval $I = (\phi_i, \phi_{i+1})$, we pick $\phi \in I$ and compute the intervals $X(\phi)$ and $X'(\phi)$ in $O(n \log^2 n)$ time (the rotation of the lines $A \cup B$ by ϕ can be done in linear time). The total time for computing these intervals is $O(n \log^3 n)$. For binary search we use the slope selection for set of points V. Each slope selection takes $O(|V| \log |V|)$ time [2–4, 10, 12]. Then the total time is $O(n^2 \log^2 n)$. □

4 Computing an Equitable Partition

Our algorithm for computing an equitable partition of the plane is based on equitable 2- and 3-cuttings. In this section we show how 2-cuttings can be computed. In particular, we need to find the pair (r_1, r_2) for an equitable 2-cutting.

For convenience, let $M(X, i) = \left\lceil \left(\frac{2|X|}{3}\right)^{i/r} \right\rceil - 2$ for $i \in [r-1]$.

We define the sign $\sigma(i)$ for $i \in [r-1]$ as follows. As in the proof of Theorem 4, we compute

- x_0, the smallest t such that $\mu_A(h^-(t)) \geq M(A,i)$, and
- x_1, the largest t such that $\mu_A(h^+(t)) \geq M(A,r-i)$.

We also compute an interval $[x_0', x_1']$ for set B using μ_B and the lower bounds $M(B,i), M(B,r-i)$. If the intervals $[x_0, x_1]$ and $[x_0', x_1']$ intersect then, for any $t \in [x_0, x_1] \cap [x_0', x_1']$, the line with equation $x = t$ is an equitable 2-cutting for $(r_1, r_2) = (i, r-i)$. We assign $\sigma(i) = 0$ in this case. Suppose that the intervals $[x_0, x_1]$ and $[x_0', x_1']$ do not intersect. If $x_1 < x_0'$, we set $\sigma(i) = 1$; otherwise $x_0 > x_1'$ and we set $\sigma(i) = -1$.

We apply the algorithm from Lemma 4 to compute the sign sequence $\sigma(1)$, $\sigma(2), \ldots, \sigma(r-1)$ in $O(rn \log^2 n)$ time. If there is a sign $\sigma(i) = 0$ then an equitable 2-cutting (by a vertical line) is found. Suppose that $\sigma(i) = \pm 1$ for all $i \in [r-1]$. We apply the following theorem from [1].

Theorem 5 ([1]). *For any sequence of signs $\sigma(1), \sigma(2), \ldots, \sigma(r-1)$ with $\sigma(i) = \pm 1$, there is a pair (r_1, r_2) or a triple (r_1, r_2, r_3) with sum r and the same signs such that any $1 \leq r_i \leq 2r/3$.*

The proof of Theorem 5 [1] implies that a pair (r_1, r_2) or a triple (r_1, r_2, r_3) can be computed in $O(r)$ time if the sequence of signs is known. If it is a pair (r_1, r_2) then we can compute an equitable 2-cutting for (r_1, r_2) as follows.

As in the proof of Theorem 4, we use the rotated sets $A(\phi)$ and $B(\phi)$). We can also define, for the sets $A(\phi)$ and $B(\phi)$) and any $i \in [r-1]$,
(i) the corresponding intervals $X(\phi, i) = [x_0(\phi, i), x_1(\phi, i)]$ and $X'(\phi) = [x_0'(\phi, i), x_1'(\phi, i)]$, and
(ii) the signs $\sigma(\phi, i)$.

Suppose that $\sigma(\phi, i) \neq \sigma(0, i)$ for some angle ϕ and $i \in [r-1]$. Then an equitable 2-cutting for $(i, r-i)$ can be found using binary search in the set of slopes $\phi_1, \phi_2, \ldots$ as in the proof of Theorem 4.

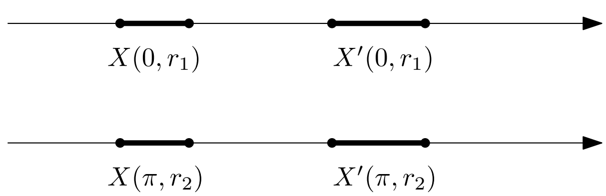

Fig. 1. The intervals for $\phi = 0, i = r_1$ and $\phi = \pi, i = r_2$ projected on the x-axis.

Note that $X(0, i) = X(\pi, r-i)$ and $X'(0, i) = X'(\pi, r-i)$ for $i \in [r-1]$. Then $\sigma(\pi, r-i) = -\sigma(0, i)$. Then $\sigma(\pi, r_2) = -\sigma(0, r_1) = -\sigma(0, r_2)$, see Figure 1. Thus, we can find an equitable 2-cutting for $(r_1, r)2)$.

Lemma 5. *If there is a pair* (r_1, r_2) *such that* $\sigma(r_1) = \sigma(r_2)$*, then an equitable 2-cutting for* (r_1, r_2) *can be computed in* $O(n^2 \log^2 n)$ *time.*

In the subsequent sections we will deal with 3-cuttings. We also may assume that, for any $i \in [r-1]$, the sign function $\sigma(\pi, i)$ is invariant for all θ as an equitable 2-cutting can be found otherwise.

5 Computing the Measure of a Wedge

In this section we present an algorithm for computing $\mu_L(W)$ for any set of lines L and a wedge W in the plane. We can assume that $W = \{(x, y) \mid x, y \geq 0\}$ by using affine transformations. Lines in L intersect both x- and y-axis. Let $x_1, \ldots, x_k$ and $y_1, \ldots, y_m$ be the sorted coordinates of the intersection points, see Figure 2.

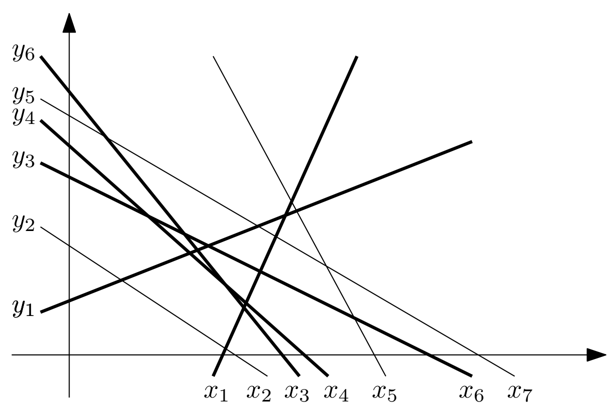

Fig. 2. 8 lines crossing the wedge $W = \{(x, y) \mid x, y \geq 0\}$. $\mu_L(W) = 5$ and the corresponding 5 lines are shown in bold.

The wedge W is between two rays $R_X = \{(x, 0) \mid x > 0\}$ and $R_X = \{(0, y) \mid y > 0\}$. There are three types of lines intersecting W. Let L_X the sets of lines intersecting R_X but not R_Y and let L_Y be the sets of lines intersecting R_Y but not R_X. Let L_{XY} be the sets of lines intersecting both R_X and R_Y.

Lemma 6. *Let* L' *be a subset of* L *such that* $I(L') \subset W$*. There exists a pair* (i, j) *such that*
(i) any line in $L' \cap L_{XY}$ *intersects* x*-axis at* $x_{i'} \geq x_i$ *and* y*-axis at* $y_{j'} \geq y_j$*,*
(ii) any line in $L' \cap L_X$ *intersects* x*-axis at* $x_{i'} < x_i$ *and*
(iii) any line in $L' \cap L_Y$ *intersects* y*-axis at* $y_{j'} < y_j$*.*

Proof. Let x_i be the smallest x-intercept of a line in $L' \cap L_{XY}$ and let l_1 be this line. Let y_j be the smallest y-intercept of a line in $L' \cap L_{XY}$ and let l_2 be this line. Clearly, the condition (i) holds. The condition (ii) holds too; otherwise line l_1 does not intersect all the lines in L_X. Similarly, the condition (iii) holds; otherwise line l_2 does not intersect all the lines in L_Y. □

We call a set of lines L' satisfying the conditions (i)-(iii) of Lemma 6, (i, j)-*set of lines*. For every (i, j), we compute $L^*_{i,j}$, a largest (i, j)-set of lines. By Lemma 6, $\mu_L(W) = |L^*_{i,j}|$.

Let $\mathbf{x}(l)$ and $\mathbf{y}(l)$ denote x- and y-intercept of a line, i.e. the equation of l can be expressed as $y = ax + \mathbf{y}(l)$ and $y = a(x - \mathbf{x}(l))$. For every pair (i, j) with $1 \leq i \leq k$ and $1 \leq j \leq m$, we show how to compute the largest set $L' \subseteq L_{XY}$ such that $I(L') \subset W$. Let $L' = \{l_1, \ldots, l_s\}$ be a set of lines in L_{XY} with $\mathbf{x}(l_1) \geq \mathbf{x}(l_2) \geq \ldots \mathbf{x}(l_s) \geq x_i$ and $\mathbf{y}(l_1), \mathbf{y}(l_2), \ldots, \mathbf{y}(l_s) \geq y_j$. Then $I(L') \subset W$ if and only if $\mathbf{y}(l_1) \leq \mathbf{y}(l_2) \leq \cdots \leq \mathbf{y}(l_s)$, for example three lines with x-intercepts x_6, x_4, and x_3 in Figure 2 have y-intercepts y_3, y_4, and y_6, respectively. Therefore, we can use an algorithm for computing the longest increasing subsequence of $\mathbf{y}(l_1), \mathbf{y}(l_2), \ldots, \mathbf{y}(l_s)$ in $O(n \log n)$ time [8, 14].

For every pair (i, j) with $1 \leq i \leq k$ and $1 \leq j \leq m$, we show how to compute the largest set $L' \subseteq L_X \cup L_Y$ such that $I(L') \subset W$. Let $L' = \{l_1, \ldots, l_s\}$ be a set of lines in $L_X \cup L_Y$ satisfying (i) and (ii). Suppose that they are sorted as follows. The lines from L_X first, then the lines from L_Y. The lines from L_X are sorted by x-intercept in decreasing order. The lines from L_Y are sorted by y-intercept in increasing order. Then $I(L') \subset W$ if and only if the slopes of he lines in L' are decreasing. Therefore, we can use an algorithm for computing the longest decreasing subsequence in $O(n \log n)$ time.

Therefore $|L^*_{i,j}| = |A^*_{i,j}| + |B^*_{i,j}|$ where $|A^*_{i,j}|$ is the maximum size of $L' \subseteq L_{XY}$ for (i, j) and $|B^*_{i,j}|$ is the maximum size of $L' \subseteq L_X \cup L_Y$ for (i, j). If we use the longest increasing/decreasing subsequence for all pairs (i, j), the total running time will be $O(n^3 \log n)$. We show that it can be reduced to $O(n^2 \log n)$.

We compute $|A^*_{i,j}|$ using $|A^*_{i+1,j}|$. Consider line l_t with $\mathbf{x}(l_t) = x_i$. If $\mathbf{y}(l_t) < y_j$, line l_t can be ignored. If $\mathbf{y}(l_t) \geq y_j$, we add it to the sequence $\mathbf{y}(l_1), \mathbf{y}(l_2), \ldots, \mathbf{y}(l_s)$ and compute the length of the longest increasing subsequence (LIS) of the new sequence. Since we add a new element to it, the length of LIS can be updated in $O(\log n)$ time. Similarly, the values of $|B^*_{i,j}|$ can be computed. Then the total running time be $O(n^2 \log n)$.

Theorem 6. *For any set L of n lines in the plane and any wedge W, the measure $\mu_L(W)$ can be computed in $O(n^2 \log n)$ time.*

6 Computing Canonical Cuttings

Let (r_1, r_2, r_3) be the triple provided by Theorem 5 (we assume that a pair (r_1, r_2) does not exist).

Similar to [1, 15] we define a *canonical cutting*. For a point p, construct three rays r_0, r_1 and r_2 starting from p. The first ray r_0 is pointing downwards. Let $C_i, i = 1, 2$ be the region defined by rays r_0 and r_i as shown in Figure 3. Let $\alpha_i, i = 1, 2$ be the angle rays r_0 and r_i. The canonical cutting is defined by choosing $\alpha_i, i = 1, 2$ to be the smallest angle such that $\mu_A(C_i) \geq M(A, r_i)$. We also denote these angles $\alpha_i(p), i = 1, 2$. Let C_3 be the region between rays r_1 and r_2, see Figure 3.

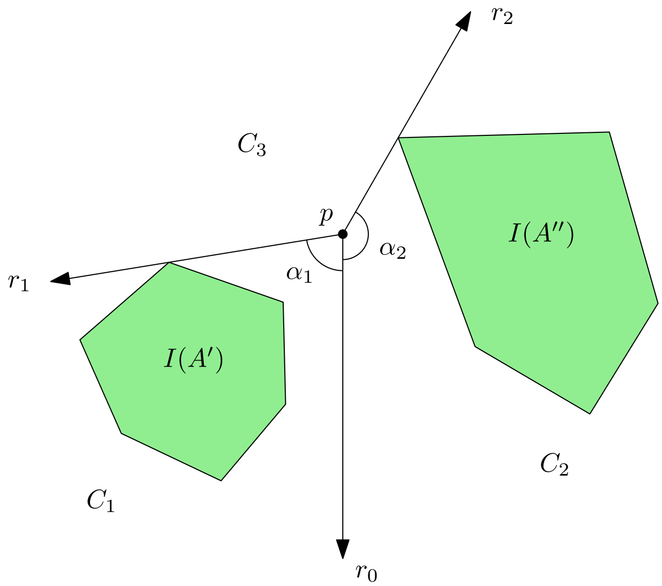

Fig. 3. Canonical 3-cutting.

Lemma 7. *For any point p in the plane, the canonical 3-cutting can be computed in $O(n^2 \log^2 n)$ time.*

Proof. First, we compute $I(A)$ in $O(n^2)$ time. For every point $q \in I(A)$, compute the slope of the vector pq. Sort $I(A)$ by slope. Compute ray r_1 of the canonical 3-cutting at p by using binary search in the sorted set $I(A)$. For any slope s in the binary search, compute the measure $\mu_A(C_1)$ using the algorithm from Theorem 6. Similarly the ray r_2 can be computed. The total time is $O(n^2 \log^2 n)$. □

The locus of all points p defining a convex canonical 3-cutting is

$$R = \{p \in \mathbb{R}^2 \mid x_0 \leq p_x \leq x_1 \text{ and } \alpha_1(p) + \alpha_2(p) \geq \pi\}.$$

The region R contains an apex of an equitable 3-cutting [15]. It can be proven using a coloring: a point $p \in R$ has color $i \in [3]$ if $\mu_B(C_i) \geq M(B, r_i)$. Note that a point may have more than one color and a point with three colors is an apex of the equitable 3-cutting. It exists by the following theorem.

Theorem 7 (Knaster, Kuratowski, Mazurkiewicz [11]). *Let Δ be a triangle with vertices $1, 2, 3$. Suppose that Δ is colored with colors $\{1, 2, 3\}$ such* *that every vertex i has color i, and every point on a side ij has at least one of the colors i or j. If every color class is a closed set, then there is a point with all three colors.*

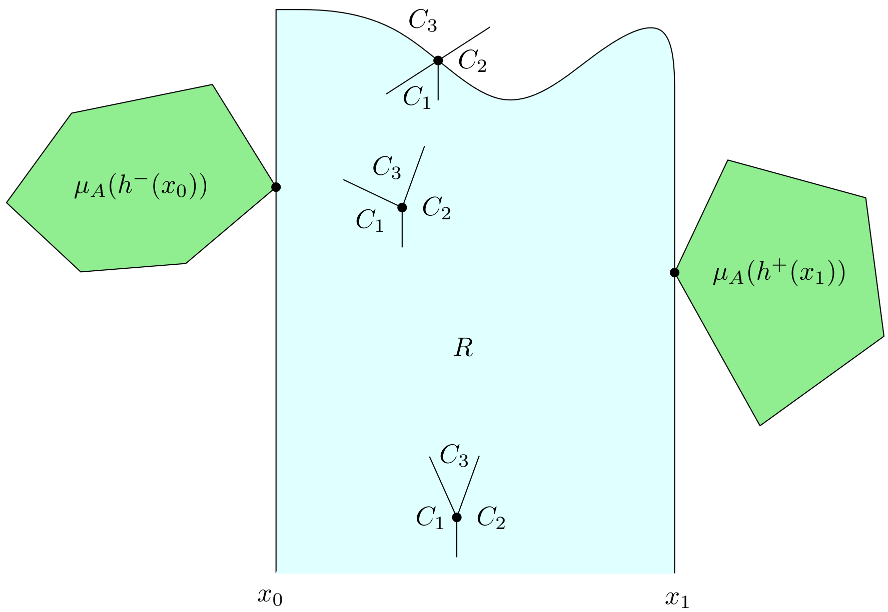

Fig. 4. Region R.

7 Computing an Equitable 3-Cutting

In this section we show that an equatable 2- or 3-cutting can be computed efficiently.

Theorem 8. *Let A, B be two finite sets of points in the plane, each in general position, and $r \geq 2$ be a positive integer. An equitable k-cutting for some $k \in \{2, 3\}$ can be computed in $O(n^6 \log^2 n)$ time.*

Proof. Let V be the set of vertices of the arrangement of lines $L = A \cup B$. Let $\mathcal{L}$ be the union of
(i) the set lines passing through two points of V, and
(ii) the vertical lines passing through V.
Then the arrangement $\mathcal{L}$ contains $O(n^2)$ lines and $O(n^4)$ faces.

We apply the topological sweep method of Edelsbrunner and Guibas [6, 7] to traverse the faces of the arrangement of $\mathcal{L}$. For each face F of the arrangement, we can check the boundary conditions of the region R using angles $\alpha_i, i \in [3]$ computed for some point $p \in F$. If the canonical 3-cutting is convex, we compute the coloring of the face. The algorithm stops if all three colors are used for p. Note that all the points in F have the same coloring. When we reach the top boundary of region R, we also check the sign of C_3. If it is opposite of $\sigma(r_3)$ we apply the algorithm for computing an equitable 2-cutting from Section 4. The total running time is $O(n^6 \log^2 n)$. □

Using the partition of r from Section 4, we conclude

Corollary 1. *Let A, B be two finite sets of lines in $\mathbb{R}^2$ such that $A \cup B$ is in general position, and let r be a positive integer. Then, an equitable partition of $\mathbb{R}^2$ into r convex regions can be computed in $O(n^6 \log^2 n \log r)$ time.*

References

1. Bespamyatnikh, S., Kirkpatrick, D., Snoeyink, J.: Generalizing ham sandwich cuts to equitable subdivisions. Discrete Comput. Geom. **24**(4), 605–622 (2000)
2. Brönnimann, H., Chazelle, B.: Optimal slope selection via cuttings. Comput. Geom. Theory Appl. **10**(1), 23–29 (1998)
3. Cole, R., Salowe, J., Steiger, W., Szemerédi, E.: An optimal-time algorithm for slope selection. SIAM J. Comput. **18**(4), 792–810 (1989)
4. Dillencourt, M.B., Mount, D.M., Netanyahu, N.S.: A randomized algorithm for slope selection. Internat. J. Comput. Geom. Appl. **2**, 1–27 (1992)
5. Dujmovic, V., Langerman, S.: A center transversal theorem for hyperplanes and applications to graph drawing. Discrete Comput. Geom. **49**(1), 74–88 (2013)
6. Edelsbrunner, H., Guibas, L.J.: Topologically sweeping an arrangement. J. Comput. Syst. Sci. **38**, 165–194 (1989), corrigendum in 42 (1991), 249–251
7. Edelsbrunner, H., Souvaine, D.L.: Computing median-of-squares regression lines and guided topological sweep. J. Amer. Statist. Assoc. **85**, 115–119 (1990)
8. Fredman, M.L.: On computing the length of longest increasing subsequences. Discrete Mathematics **11**(1), 29–35 (1975)
9. Ito, H., Uehara, H., Yokoyama, M.: 2-dimension ham sandwich theorem for partitioning into three convex pieces. In: Proc. Japan Conf. Discrete Comput. Geom.'98. Lecture Notes Comput. Sci., vol. 1763, pp. 129–157. Springer-Verlag (2000)
10. Katz, M.J., Sharir, M.: Optimal slope selection via expanders. Inform. Process. Lett. **47**, 115–122 (1993)
11. Knaster, B., Kuratowski, C., Mazurkiewicz, S.: Ein beweis des fixpunktsatzes für n-dimensionale simplexe. Fundamenta Mathematicae **14**(1), 132–137 (1929)
12. Matoušek, J.: Randomized optimal algorithm for slope selection. Inform. Process. Lett. **39**, 183–187 (1991)
13. Sakai, T.: Balanced convex partitions of measures in R^2. Graphs and Combinatorics **18**(1), 169–192 (2002)
14. Schensted, C.: Longest increasing and decreasing subsequences. Canadian Journal of Mathematics **13**, 179–191 (1961)
15. Xue, A., Soberón, P.: Balanced convex partitions of lines in the plane. CoRR **math.MG/1910.06231** (2019), https://arxiv.org/abs/1910.06231